MATHEMATICS OF OPERATIONS RESEARCH
Vol. 26, No. 2, May 2001, pp. 193–205
Printed in U.S.A.

ON POLYHEDRAL APPROXIMATIONS OF THE SECOND-ORDER CONE

AHARON BEN-TAL AND ARKADI NEMIROVSKI

We demonstrate that a conic quadratic problem,

$$\text{(CQP)} \qquad \min_x \left\{ e^T x \,\middle|\, Ax \geq b,\ \|A_\ell x - b_\ell\|_2 \leq c_\ell^T x - d_\ell,\ \ell = 1, \dots, m \right\}, \quad \|y\|_2 = \sqrt{y^T y},$$

is "polynomially reducible" to Linear Programming. We demonstrate this by constructing, for every $\epsilon \in (0, \frac{1}{2}]$, an LP program (explicitly given in terms of ϵ and the data of (CQP))

$$\text{(LP)} \qquad \min_{x,\,u} \left\{ e^T x \,\middle|\, P\begin{pmatrix} x \\ u \end{pmatrix} + p \geq 0 \right\}$$

with the following properties:

(i) the number $\dim x + \dim u$ of variables and the number $\dim p$ of constraints in (LP) do not exceed

$$O(1)\left[\dim x + \dim b + \sum_{\ell=1}^{m} \dim b_\ell\right] \ln \frac{1}{\epsilon};$$

(ii) every feasible solution x to (CQP) can be extended to a feasible solution (x, u) to (LP);

(iii) if (x, u) is feasible for (LP), then x satisfies the "ϵ-relaxed" constraints of (CQP), namely,

$$Ax \geq b,\ \|A_\ell x - b_\ell\|_2 \leq (1+\epsilon)\left[c_\ell^T x - d_\ell\right],\ \ell = 1, \dots, m.$$

1. Introduction. The initial motivation for the question posed and resolved in this paper originated from the practical need to solve numerically a conic quadratic problem

$$\text{(CQP)} \qquad \min_x \left\{ e^T x \,\middle|\, Ax \geq b,\ \|A_\ell x - b_\ell\|_2 \leq c_\ell^T x - d_\ell,\ \ell = 1, \dots, m \right\},$$

$\|y\|_2 = \sqrt{y^T y}$ being the Euclidean norm. There are two major sources of recent interest in these problems:

• (CQP) is a natural form of several important applied problems, e.g., problems with Coulomb friction in contact mechanics, see Christensen and Pang (1999), Lo (1996), Lobo et al. (1998), Pang and Stewart (1999);

• Conic quadratic constraints have very powerful "expressive abilities," which allow one to cast a wide variety of nonlinear convex optimization problems in the form of (CQP), see, e.g., Lobo et al. (1998), Nesterov and Nemirovski (1994), and Examples 1–5 below.

As a matter of fact, the case of (CQP) is covered by the existing general theory of polynomial time interior point (IP) methods (e.g., the general theory for "well-structured" convex programs (Nesterov and Nemirovski 1994) and the Nesterov-Todd theory of IP algorithms for conic problems on self-scaled cones (Nesterov and Todd 1997, 1998)). These theories yield both algorithms and complexity bounds; in view of these bounds, (CQP)

Received July 2, 1999; revised June 10, 2000.
MSC 2000 subject classification. Primary: 90C25, 90C20, 90C05.
OR/MS subject classification. Primary: Programming/quadratic.
Key words. Second-order cone programming, linear programming, polynomial-time reducibility.

0364-765X/01/2602/0193/$05.00
1526-5471 electronic ISSN,

with m constraints and n variables is not more difficult than a linear programming problem of similar sizes (in both cases, the problem can be solved within an accuracy δ in $O(1)\sqrt{m}\ln(O(\delta^{-1}))$ steps of a polynomial time IP method, with no more than $O(n^2(m+n))$ arithmetic operations per step). Moreover, there exists software that implements the theoretical schemes. However, to the best of our knowledge the IP software for (CQP) available at the moment, although capable of handling problems with tens of thousands of conic quadratic constraints, imposes severe restrictions on the design dimension of the problem (a few thousand variables). In this respect, the state of the art in numerical processing of CQPs is incomparable to that in LP, where we can solve routinely problems with even hundreds of thousands of variables and constraints. Given this huge difference, the question arises: can we process CQPs via LP techniques? Mathematically, the question is whether we can approximate a CQP problem by an LP one, without increasing dramatically the sizes of the problem, and this is the question we address in this paper.

We start with a precise formulation of the question. Let $\epsilon > 0$, and let

$$\mathbf{L}^k = \{(y, t) \in \mathbf{R}^k \times \mathbf{R} \mid t \geq \|y\|_2\}$$

be the $(k+1)$-dimensional Lorentz cone. We define a *polyhedral ϵ-approximation* of $\mathbf{L}^k$ as a linear mapping

$$\Pi^k(y, t, u) : \mathbf{R}^k \times \mathbf{R} \times \mathbf{R}^{p_k} \to \mathbf{R}^{q_k}$$

such that

(i) If $(y, t) \in \mathbf{L}^k$, then there exists $u \in \mathbf{R}^{p_k}$ such that $\Pi^k(y, t, u) \geq 0$;

(ii) If $(y, t) \in \mathbf{R}^k \times \mathbf{R}$ is such that $\Pi^k(y, t, u) \geq 0$ for some u, then $\|y\|_2 \leq (1+\epsilon)t$.

Geometrically: The system of homogeneous linear constraints $\Pi^k(y, t, u) \geq 0$ defines a polyhedral cone $\mathbf{K}$ in the space of (y, t, u)-variables. We say that $\Pi^k(\cdot)$ is a polyhedral ϵ-approximation of $\mathbf{L}^k$, if the projection of $\mathbf{K}$ on the space of (y, t)-variables contains $\mathbf{L}^k$ and is contained in the "$(1+\epsilon)$-extension" of $\mathbf{L}^k$, that is, in the image of $\mathbf{L}^k$ under linear mapping $(x, t) \mapsto (x, (1+\epsilon)^{-1}t)$ (note that for ϵ close to 0 the mapping is "nearly the identity").

Let k_ℓ, $\ell = 1, \ldots, m$, be the row sizes of the matrices A_ℓ in (CQP). Given polyhedral ϵ-approximations $\Pi^{k_\ell}(\cdot)$ of the Lorentz cones $\mathbf{L}^{k_\ell}$, $\ell = 1, \ldots, m$, we can approximate (CQP) by the linear programming problem,

$$\text{(LP)} \qquad \min_{x, \{u_\ell\}_{\ell=1}^m} \left\{ e^T x \middle| Ax \geq b, \Pi^{k_\ell}\left(A_\ell x - b_\ell, c_\ell^T x - d_\ell, u^\ell\right) \geq 0, \ell = 1, \ldots, m \right\}.$$

From the definition of a polyhedral approximation it follows that (CQP) and (LP) are linked as follows: Both problems have the same objectives, and the projection $\widehat{X}$ of the feasible set of (LP) onto the x-space is in between the feasible set of (CQP) and that of its ϵ-relaxation:

$$(\text{CQP}_\epsilon) \qquad \min_x \left\{ e^T x \middle| Ax \geq b,\ \|A_\ell x - b_\ell\|_2 \leq (1+\epsilon)[c_\ell^T x - d_\ell],\ \ell = 1, \ldots, m \right\}.$$

We see that if ϵ is close to 0, then (LP) is a good approximation of (CQP).

Note that the sizes of (LP) are larger than those of (CQP); specifically, the design dimension of (LP) is

$$N = n + \sum_{\ell=1}^m p_{k_\ell} \quad [n = \dim x],$$

and the total number of linear constraints in (LP) is

$$M = k_0 + \sum_{\ell=1}^m q_{k_\ell},$$

where k_0 is the row size of A; recall that p_k and q_k are the dimensions of the u-vector and the image dimensions of the polyhedral approximation $\Pi^k(\cdot,\cdot,u)$, respectively. The weakest necessary requirement for the proposed scheme to be computationally meaningful is that the size $M+N$ of (LP) should be polynomial in the size $n+\sum_{\ell=0}^m k_\ell$ of (CQP) and should grow moderately as ϵ approaches 0. We arrive at the following question:

Can we construct a polyhedral ϵ-approximation of $\mathbf{L}^k$ with the sizes p_k+q_k growing moderately as k grows and as ϵ approaches 0?

The straightforward approach, when $\mathbf{L}^k$ is approximated by a circumscribed polyhedral cone with a sufficiently large number of facets, does not work: The required number of facets blows up exponentially as k grows (even with $\epsilon = 1$, a rough *lower* bound on the number of required facets is $\exp\{k/8\}$). Surprisingly, the situation is not as bad as suggested by the latter observation. We prove the following result:

THEOREM 1.1. *For every k and every $\epsilon \in (0,1]$, $\mathbf{L}^k$ admits a polyhedral ϵ-approximation with*

$$p_k+q_k \le O(1)k\ln\frac{2}{\epsilon} \tag{1}$$

(*from now on, all $O(1)$s are absolute constants*).

Theorem 1.1 establishes an interesting (and, to the best of our knowledge, new) geometric fact, and as such, seems to be important by its own right. As far as the "computational consequences" are concerned, the situation is as follows. It was already mentioned that *theoretically* problem (CQP) with n variables and m constraints is not more difficult than a linear programming problem of the same size. When replacing (CQP) with its polyhedral approximation, we increase both m and n, thus losing in the complexity. Note, however, that theoretical complexity analysis and computational practice are not the same: Performance and reliability of modern commercial LP software are much better than those of CQP-solvers available at the moment, and this (hopefully temporary) difference can make the use of polyhedral approximation "practically meaningful," especially in the case when this approximation does not increase that dramatically the sizes of the problem. The latter happens, e.g., when the row sizes k_ℓ of the conic quadratic constraints in (CQP) are small (recall that we need $O(k_\ell \ln(\epsilon^{-1}))$ linear constraints and extra variables to approximate the constraint $\|A_\ell x - b_\ell\|_2 \le c_\ell^T x - d_\ell$). In this respect it should be mentioned that there are important sources of conic quadratic problems with fairly small k_ℓ's, e.g., problems with Coulomb friction ($k_\ell \le 3$) (Christensen and Pang 1999, Lo 1996, Lobo et al. 1998, Pang and Stewart 1999), and truss topology design problems ($k_\ell \le 2$) (Ben-Tal and Nemirovskii 1994). We list below several other examples where the polyhedral approximation may be of practical use.

EXAMPLE 1: QUADRATICALLY CONSTRAINED CONVEX QUADRATIC PROGRAM.

$$\min_x \left\{ x^T Q_0^T Q_0 x - 2b_0^T x : \quad x^T Q_i^T Q_i x - 2b_i^T x \le c_i,\ i = 1, \dots, m \right\}$$

can be posed equivalently as the conic quadratic program

$$\min_{x,t}\left\{ t : \left\| \begin{pmatrix} Q_0 x \\ \frac{t+2b_0^T x - 1}{2} \end{pmatrix} \right\|_2 \le \frac{t+2b_0^T x+1}{2},\ \left\| \begin{pmatrix} Q_i x \\ \frac{2b_i^T x + c_i - 1}{2} \end{pmatrix} \right\|_2 \le \frac{2b_i^T x + c_i + 1}{2},\ 1 \le i \le m \right\}.$$

If the number of "truly quadratic" (i.e., with $Q_i \ne 0$) constraints is relatively small, a polyhedral approximation of these constraints does not significantly increase the sizes of the problem. We see that linear programming software can be easily adapted to handle mixed linearly-quadratically constrained convex quadratic problems with few quadratic constraints.

The accuracy of the resulting approximation, as well as those of the approximations in Examples 2–4 below, will be discussed in §4.

EXAMPLE 2: "GEOMETRIC MEANS." Let $\pi_i = p_i/p,\ p_i, p \in \mathbf{N}_+$, be positive rationals, $i = 1, \dots, n$, with $\sum_{i=1}^n \pi_i \le 1$. The hypograph

$$\mathscr{H} = \left\{(x, t) \in \mathbf{R}^{n+1} \mid x \ge 0, t \le \prod_{i=1}^n x_i^{\pi_i}\right\}$$

of the concave function $f(x) = \prod_{i=1}^n x_i^{\pi_i} : \mathbf{R}^n_+ \to \mathbf{R}$ can be represented via conic quadratic inequalities. Indeed, let us choose the smallest integer k such that $2^k \ge p$ and consider the following system of constraints in variables $y = (x, t, \tau, \{y_j^\ell\})$:

$$y^0 \equiv (y_1^0, \dots, y_{2^k}^0) = (\underbrace{x_1, \dots, x_1}_{p_1}, \dots, \underbrace{x_n, \dots, x_n}_{p_n}, \underbrace{\tau, \dots, \tau}_{2^k - p}, \underbrace{1, \dots, 1}_{p - \sum_1^n p_i}),$$

$$\begin{gathered} y^0 \ge 0, \\ 0 \le y_j^\ell \le \sqrt{y_{2j-1}^{\ell-1} y_{2j}^{\ell-1}}, \quad j = 1, \dots, 2^{k-\ell}, \quad \ell = 1, \dots, k-1; \\ 0 \le \tau \le \sqrt{y_1^{k-1} y_2^{k-1}}, \\ t \le \tau. \end{gathered} \tag{2}$$

Note that (2) is a system of conic quadratic inequalities, since

$$\left\{u, v, w \ge 0 : u \le \sqrt{vw}\right\} = \left\{u, v, w \ge 0 : \left\|\begin{pmatrix} u \\ \frac{v-w}{2} \end{pmatrix}\right\|_2 \le \frac{v+w}{2}\right\}.$$

Moreover, it is immediately seen that the projection of the solution set $\mathscr{X}$ of (2) on the x, t-space is exactly the set $\mathscr{H}$.

Replacing every conic quadratic constraint in (2) by its "tight" polyhedral approximation, let us denote by $\mathscr{Y}$ the solution set of the resulting system of linear inequalities. This set "lives" in a certain $\mathbf{R}^N$, which contains, as a subspace, the space of y variables, and the projection of $\mathscr{Y}$ on this space is a tight outer approximation of $\mathscr{X}$. It follows that the projection of $\mathscr{Y}$ on the subspace of x, t variables is a tight approximation of the hypograph $\mathscr{H}$ of f. As a result, we get the possibility to process via LP optimization programs with linear objectives and constraints of the form

$$\sum_j a_j \prod_\ell x_\ell^{\pi_{\ell j}} \ge b \quad \left[a_j \ge 0,\ \sum_\ell \pi_{\ell j} \le 1,\ \pi_{\ell j} \text{ are positive rationals}\right].$$

EXAMPLE 3: "INVERSE GEOMETRIC MEANS." Let $\pi_i = p_i/p,\ p_i, p \in \mathbf{N}_+$, be positive rationals, $i = 1, \dots, n$, The epigraph

$$\mathscr{E} = \left\{(x, t) \in \mathbf{R}^{n+1} \mid x > 0, t \ge \prod_{i=1}^n x_i^{-\pi_i}\right\}$$

of the function $f(x) = \prod_{i=1}^n x_i^{-\pi_i}$ can be represented via conic quadratic inequalities. Indeed, choosing the smallest integer k such that $2^k \ge p + \sum_i p_i$, consider the following system of constraints in variables $y = (x, t, \{y_j^\ell\})$:

$$
\begin{gathered}
y^0 \equiv (y_1^0, \dots, y_{2^k}^0) = (\underbrace{x_1, \dots, x_1}_{p_1}, \dots, \underbrace{x_n, \dots, x_n}_{p_n}, \underbrace{t, \dots, t}_{p}, \underbrace{1, \dots, 1}_{2^k - p\sum_i p_i}), \\
y^0 \geq 0, \\
0 \leq y_j^\ell \leq \sqrt{y_{2j-1}^{\ell-1} y_{2j}^{\ell-1}}, \quad j = 1, \dots, 2^{k-\ell},\ \ell = 1, \dots, k-1; \\
1 \leq \sqrt{y_1^{k-1} y_2^{k-1}}.
\end{gathered} \tag{3}
$$

As in (2), the system (3) is in fact a system of conic quadratic inequalities; it is immediately seen that the projection of the solution set $\mathscr{X}$ of this system onto the space of x, t variables is exactly the epigraph $\mathscr{E}$ of f. As in Example 2, we can use "tight" polyhedral approximations of the conic quadratic inequalities in (3) to get a tight polyhedral approximation of the epigraph of f. Thus, we get the possibility to process via linear programming optimization programs with linear objectives and constraints of the form,

$$\sum_j a_j \prod_\ell x_\ell^{-\pi_{\ell j}} \leq b \quad [a_j \geq 0,\ \pi_{\ell j} \text{ are positive rationals}].$$

Example 4: Geometric Programming. To process Geometric Programming problems via LP, it suffices to build a tight polyhedral approximation of the epigraph of the exponential function. Assume that the exponents $a_i^T x$ of the exponential monomials $\exp\{a_i^T x\}$ arising in the problem vary in a given segment $-R \leq a_i^T x \leq R$ ($R < \infty$). Note that one can enforce this assumption by adding to the original problem the constraints $|a_i^T x| \leq R$, and that from the computational viewpoint these constraints are indeed a "must": e.g., a SUN computer believes that $\exp\{750\} = \infty$, and $\exp\{-750\} = 0$, so that in actual computations one can safely set $R = 750$. We come to the problem as follows:

Given a set

$$\mathscr{E}_R = \{(x, t) \in \mathbf{R}^2 \mid |x| \leq R, \exp\{x\} \leq t\},$$

and an $\epsilon > 0$, find an $(1+\epsilon)$-polyhedral approximation of the set, i.e., a system

$$P(x, t, u) \leq 0$$

of affine constraints in variables x, t and additional variables u such that
(i) *If $(x, t) \in \mathscr{E}_R$, then there exists u such that $P(x, t, u) \leq 0$;*
(ii) *If $(x, t, u) \in \mathscr{E}_R$, then $|x| \leq R$ and $\exp\{x\} \leq (1+\epsilon)t$.*

A reasonably good solution to this problem is given by Theorem 1.1. First we observe that $\mathscr{E}_R$ can be approximated by the following system of quadratic constraints:

$$
\begin{array}{rcll}
|x| & \leq & R, & \\
\sqrt{6\left(2^{-q}x + \frac{5}{3}\right)^2 + \frac{19}{3}} & \leq & u_1; & \\
\sqrt{4(1+2^{-q}x)^2 + (u_2-1)^2} & \leq & u_2 + 1 & [\Leftrightarrow (1+2^{-q}x)^2 \leq u_2] \\
\sqrt{4u_1^2 + 4u_2^2 + (24u_3 - 1)^2} & \leq & 24u_3 + 1 & [\Leftrightarrow \frac{1}{24}(u_1^2 + u_2^2) \leq u_3] \\
\sqrt{4u_{2+i}^2 + (u_{3+i} - 1)^2} & \leq & u_{3+i} + 1 & [\Leftrightarrow u_{2+i}^2 \leq u_{3+i}], \\
 & & i = 1, \dots, q, & \\
u_{3+q} & \leq & t, &
\end{array} \tag{4}
$$

with properly chosen $q = O(\ln R/\epsilon)$. Indeed, the system in question, after eliminating the variables u_1 and u_2, reduces to

$$\begin{aligned} |x| &\le R, \\ g_q(x) \equiv 1 + (2^{-q}x) + \tfrac{(2^{-q}x)^2}{2} + \tfrac{(2^{-q}x)^3}{6} + \tfrac{(2^{-q}x)^4}{24} &\le u_3, \\ u_{2+i}^2 &\le u_{3+i}, \qquad i = 1, \dots, q, \\ u_{3+q} &\le t, \end{aligned}$$

and the projection of the solution set of the latter system on the plane of (x, t)-variables is the set

$$\mathscr{E}_R^q = \{(x, t) \in \mathbf{R}^2 \mid |x| \le R, g_q^{(2^q)}(x) \le t\};$$

which, for large q, is a good approximation of $\mathscr{E}_R$ due to $g_q^{(2^q)}(x) \approx (\exp\{2^{-q}x\})^{(2^q)} = \exp\{x\}$.

Replacing the conic quadratic constraints in the system (4) by their tight polyhedral approximations, given by Theorem 1.1, one obtains a "tight" polyhedral approximation of $\mathscr{E}_R$.

EXAMPLE 5: ROBUST COUNTERPART OF UNCERTAIN LP PROGRAMS WITH ELLIPSOIDAL UNCERTAINTY. See Ben-Tal and Nemirovski (1998). This is a conic quadratic program, obtained from an original LP program by replacing every inequality constraint affected by data uncertainty with a conic quadratic constraint. The motivation behind this procedure implies that the "stabilizing effect" of the procedure remains essentially the same when the conic quadratic constraints are further replaced with their polyhedral ϵ-approximations with "a moderate" (not necessarily close to 0) value of ϵ, say, $\epsilon = 1$, thus allowing a moderate growth of the number of constraints, as compared to the original uncertain LP problem.

The rest of the paper is organized as follows. The simple construction of a concrete polyhedral approximation leading to Theorem 1.1 is presented in the §2. In §3 we demonstrate that the upper bound in (1) is, in a sense, the best possible. §4 contains some concluding remarks.

2. The construction. Let $\epsilon \in (0, 1]$ and a positive integer k be given. We intend to build a polyhedral ϵ-approximation of the Lorentz cone $\mathbf{L}^k$. Without loss of generality, we may assume that k is an integer power of 2: $k = 2^\theta$, $\theta \in \mathbf{N}$.

"Tower of variables." The first step of our construction is to represent a conic quadratic constraint,

$$\text{(CQI)} \qquad \sqrt{y_1^2 + \cdots + y_k^2} \le t,$$

of dimension $k+1$ by a system of conic quadratic constraints of dimension 3 each. Namely, let us call the original y-variables, "variables of generation 0," and let us split them into pairs $(y_1, y_2), \dots, (y_{k-1}, y_k)$. We associate with each pair its "successor"—an additional variable of "generation 1." We split the resulting $2^{\theta-1}$ variables of generation 1 into pairs and associate with every pair its successor—an additional variable of "generation 2," and so on; after $\theta - 1$ steps we end up with two variables of the generation $\theta - 1$. Finally, the only variable of generation θ is the variable t from (CQI).

To introduce convenient notation, let us denote by y_i^ℓ the ith variable of generation ℓ, so that $y_1^0, \dots, y_k^0$ are our original variables y-variables $y_1, \dots, y_k$, $y_1^\theta \equiv t$ is the original t variable, and the "parents" of y_i^ℓ are the two variables $y_{2i-1}^{\ell-1}, y_{2i}^{\ell-1}$.

Note that the total number of all variables in this "tower of variables" is $2k-1$.

It is clear that the system of constraints

$$\sqrt{[y^{\ell-1}_{2i-1}]^2+[y^{\ell-1}_{2i}]^2} \le y^\ell_i, \quad i=1,\dots,2^{\theta-\ell}, \quad \ell=1,\dots,\theta \tag{5}$$

is a representation of (CQI) in the sense that a collection $(y^0_1 \equiv y_1,\dots,y^0_k \equiv y_k, y^\theta_1 \equiv t)$ can be extended to a solution of (5) if and only if (y,t) solves (CQI). Moreover, let $\Pi_\ell(x_1,x_2,x_3,u^\ell)$, $\ell=1,\dots,\theta$, be polyhedral ϵ_ℓ approximations of the cone

$$\mathbf{L}^2 = \left\{(x_1,x_2,x_3) \,\middle|\, \sqrt{x_1^2+x_2^2} \le x_3\right\}.$$

Consider the system of linear constraints in variables y^ℓ_i, u^ℓ_i:

$$\Pi_\ell(y^{\ell-1}_{2i-1}, y^{\ell-1}_{2i}, y^\ell_i, u^\ell_i) \ge 0, \quad i=1,\dots,2^{\theta-\ell}, \quad \ell=1,\dots,\theta. \tag{6}$$

Writing down this system of linear constraints as $\Pi(y,t,u)\ge 0$, where Π is linear in its arguments, $y=(y^0_1,\dots,y^0_k)$, $t=y^\theta_1$, and u is the collection of all u^ℓ_i, $\ell=1,\dots,\theta$ and all y^ℓ_i, $\ell=1,\dots,\theta-1$, we immediately conclude that Π is a polyhedral ϵ approximation of $\mathbf{L}^k$ with

$$\epsilon = \prod_{\ell=1}^{\theta}(1+\epsilon_\ell)-1. \tag{7}$$

In view of this observation, we may focus on building polyhedral approximations of the Lorentz cone $\mathbf{L}^2$.

A polyhedral approximation of $\mathbf{L}^2$. We are about to show that this is given by the following system of linear inequalities (the positive integer ν is a parameter of the construction):

$$\begin{array}{ll}
(a) & \begin{cases} \xi^0 \ge |x_1| \\ \eta^0 \ge |x_2| \end{cases} \\
(b) & \begin{cases} \xi^j = \cos\left(\frac{\pi}{2^{j+1}}\right)\xi^{j-1} + \sin\left(\frac{\pi}{2^{j+1}}\right)\eta^{j-1} \\ \eta^j \ge \left|-\sin\left(\frac{\pi}{2^{j+1}}\right)\xi^{j-1} + \cos\left(\frac{\pi}{2^{j+1}}\right)\eta^{j-1}\right|, \end{cases} \quad j=1,\dots,\nu \\
(c) & \begin{cases} \xi^\nu \le x_3 \\ \eta^\nu \le \tan\left(\frac{\pi}{2^{\nu+1}}\right)\xi^\nu \end{cases}
\end{array} \tag{8}$$

Note that (8) can be straightforwardly written as a system of linear homogeneous inequalities $\Pi^{(\nu)}(x_1,x_2,x_3,u)\ge 0$, where u is the collection of the $2(\nu+1)$ variables ξ^j, η^i, $j=0,\dots,\nu$.

PROPOSITION 2.1. *$\Pi^{(\nu)}$ is a polyhedral $\delta(\nu)$ approximation of $\mathbf{L}^2=\{(x_1,x_2,x_3)\mid \sqrt{x_1^2+x_2^2}\le x_3\}$ with*

$$\delta(\nu) = \frac{1}{\cos\left(\frac{\pi}{2^{\nu+1}}\right)} - 1 = O\left(\frac{1}{4^\nu}\right). \tag{9}$$

PROOF. We should prove that

(i) if $(x_1,x_2,x_3)\in\mathbf{L}^2$, then the triple (x_1,x_2,x_3) can be extended to a solution to (8);

(ii) if a triple (x_1,x_2,x_3) can be extended to a solution to (8), then $\|(x_1,x_2)\|_2 \le (1+\delta(\nu))x_3$. $\square$

PROOF OF (i). Given $(x_1, x_2, x_3) \in \mathbf{L}^2$, let us set $\xi^0 = |x_1|$, $\eta^0 = |x_2|$, thus ensuring (8.a). Note that $\|(\xi^0, \eta^0)\|_2 = \|(x_1, x_2)\|_2$ and that the point $P^0 = (\xi^0, \eta^0)$ belongs to the first quadrant.

Now, for $j = 1, \ldots, \nu$ let us set

$$\xi^j = \cos\left(\frac{\pi}{2^{j+1}}\right)\xi^{j-1} + \sin\left(\frac{\pi}{2^{j+1}}\right)\eta^{j-1}$$
$$\eta^j = \left|-\sin\left(\frac{\pi}{2^{j+1}}\right)\xi^{j-1} + \cos\left(\frac{\pi}{2^{j+1}}\right)\eta^{j-1}\right|,$$

thus ensuring (8.b), and let $P^j = (\xi^j, \eta^j)$. The point P^j is obtained from P^{j-1} by the following construction: we rotate clockwise P^{j-1} by the angle $\phi_j = \pi/2^{j+1}$, thus getting a point Q^{j-1}; if this point is in the upper half-plane, we set $P^j = Q^{j-1}$, otherwise P^j is the reflection of Q^{j-1} with respect to the x-axis. From this description it is clear that

(I) $\|P^j\|_2 = \|P^{j-1}\|_2$, so that all vectors P^j are of the same Euclidean norm as P^0, i.e., of the norm $\|(x_1, x_2)\|_2$;

(II) Since the point P^0 is in the first quadrant, the point Q^0 is in the angle $-\pi/4 \le \arg(P) \le \pi/4$, so that P^1 is in the angle $0 \le \arg(P) \le \pi/4$. The latter relation, in turn, implies that Q^1 is in the angle $-\pi/8 \le \arg(P) \le \pi/8$, whence P^2 is in the angle $0 \le \arg(P) \le \pi/8$. Similarly, P^3 is in the angle $0 \le \arg(P) \le \pi/16$, and so on: P^j is in the angle $0 \le \arg(P) \le \pi/(2^{j+1})$.

By (I), $\xi^\nu \le \|P^\nu\|_2 = \|(x_1, x_2)\|_2 \le x_3$, so that the first inequality in (8.c) is satisfied. By (II), P^ν is in the angle $0 \le \arg(P) \le \pi/(2^{\nu+1})$, so that the second inequality in (8.c) is also satisfied. We have extended a point from $\mathbf{L}^2$ to a solution to (8). □

PROOF OF (ii). Assume that (x_1, x_2, x_3) can be extended to a solution $(x_1, x_2, x_3, \{\xi^j, \eta^j\}_{j=0}^{\nu})$ to (8). Let us set $P^j = (\xi^j, \eta^j)$. From (8.a, b) it follows that all vectors P^j are nonnegative. We have $\|P^0\|_2 \ge \|(x_1, x_2)\|_2$ by (8.a). Now, (8.b) implies $\|P^j\|_2 \ge \|P^{j-1}\|_2$. Thus, $\|P^\nu\|_2 \ge \|(x_1, x_2)\|_2$. On the other hand, by (8.c) one has $\|P^\nu\|_2 \le (1/\cos(\pi/2^{\nu+1}))\xi^\nu \le (1/\cos(\pi/2^{\nu+1}))x_3$, so that $\|(x_1, x_2)\|_2 \le (1 + \delta(\nu))x_3$, as claimed. □

PROOF OF THEOREM 1.1. Specifying in (6) the mappings $\Pi_\ell(\cdot)$ as $\Pi^{(\nu_\ell)}(\cdot)$, we conclude that for every collection of positive integers $\nu_1, \ldots, \nu_\theta$, the following system of linear inequalities describes a polyhedral approximation $\Pi_{\nu_1,\ldots,\nu_\theta}(y, t, u)$ of $\mathbf{L}^k$, $k = 2^\theta$:

$$\begin{array}{lll}
(a_{\ell,i}) & \begin{cases} \xi^0_{\ell,i} \ge |y^{\ell-1}_{2i-1}| \\ \eta^0_{\ell,i} \ge |y^{\ell-1}_{2i}| \end{cases} & \\
(b_{\ell,i}) & \begin{cases} \xi^j_{\ell,i} = \cos\left(\frac{\pi}{2^{j+1}}\right)\xi^{j-1}_{\ell,i} + \sin\left(\frac{\pi}{2^{j+1}}\right)\eta^{j-1}_{\ell,i} \\ \eta^j_{\ell,i} \ge \left|-\sin\left(\frac{\pi}{2^{j+1}}\right)\xi^{j-1}_{\ell,i} + \cos\left(\frac{\pi}{2^{j+1}}\right)\eta^{j-1}_{\ell,i}\right| \end{cases}, & j = 1, \ldots, \nu_\ell \\
(c_{\ell,i}) & \begin{cases} \xi^{\nu_\ell}_{\ell,i} \le y^\ell_i \\ \eta^{\nu_\ell}_{\ell,i} \le \tan\left(\frac{\pi}{2^{\nu_\ell+1}}\right)\xi^{\nu_\ell}_{\ell,i} \end{cases} & \\
& i = 1, \ldots, 2^{\theta-\ell}, \qquad \ell = 1, \ldots, \theta. &
\end{array} \tag{10}$$

The approximation possesses the following properties:

1. The dimension of the u vector (comprised of all variables in (10) except $\{y_i = y_i^0\}_{i=1}^k$ and $t = y_1^\theta$) is

$$p(k, \nu_1, \ldots, \nu_\theta) \le k + O(1)\sum_{\ell=1}^{\theta} 2^{\theta-\ell}\nu_\ell;$$

2. The image dimension of $\Pi_{\nu_1,\dots,\nu_\theta}(\cdot)$ (i.e., the number of linear inequalities plus twice the number of linear equations in (10)) is

$$q(k,\nu_1,\dots,\nu_\theta) \le O(1)\sum_{\ell=1}^{\theta} 2^{\theta-\ell}\nu_\ell;$$

3. The quality β of the approximation is

$$\beta = \beta(\nu_1,\dots,\nu_\theta) = \prod_{\ell=1}^{\theta} \frac{1}{\cos\left(\frac{\pi}{2^{\nu_\ell+1}}\right)} - 1.$$

Given $\epsilon \in (0,1]$ and setting

$$\nu_\ell = \left\lfloor O(1)\ell \ln\frac{2}{\epsilon}\right\rfloor, \qquad \ell = 1,\dots,\theta,$$

with properly chosen absolute constant $O(1)$, we ensure that

$$\begin{aligned} \beta(\nu_1,\dots,\nu_\theta) &\le \epsilon, \\ p(k,\nu_1,\dots,\nu_\theta) &\le O(1)k\ln\frac{2}{\epsilon}, \\ q(k,\nu_1,\dots,\nu_\theta) &\le O(1)k\ln\frac{2}{\epsilon}, \end{aligned}$$

as claimed in Theorem 1.1.

Simplifying the polyhedral approximation. It is easily seen that the approximation (10) can be "economized" by reducing the image dimension $q_k = q(k,\nu_1,\dots,\nu_\theta)$ and the number $p_k = p(k,\nu_1,\dots,\nu_\theta)$ of extra variables without influencing the quality of the approximation. Specifically,

(A) The variables $\xi^1_{\ell,i},\dots,\xi^{\nu_\ell}_{\ell,i}$ in (10) are given by linear equations; one can use these equations to represent $\xi^j_{\ell,i}$, $j \ge 1$, as linear combinations of $\xi^0_{\ell,i}$ and the $\eta_{\ell,i}$ variables, thus eliminating the variables $\xi^1_{\ell,i},\dots,\xi^{\nu_\ell}_{\ell,i}$, along with the corresponding linear equations;

(B) For $\ell = 2,\dots,\theta$, one can replace inequalities $(10.a_{\ell,i})$ with

$$\xi^0_{\ell,i} \ge y^{\ell-1}_{2i-1}, \qquad \eta^0_{\ell,i} \ge y^{\ell-1}_{2i}, \tag{11}$$

since the system we end up with after this modification ensures the nonnegativity of $y^{\ell-1}_i$, $\ell = 2,\dots,\theta$;

(C) In the system of linear inequalities obtained from (10), after modifications (A) and (B), every variable y^ℓ_i, $\ell = 1,\dots,\theta-1$, enters the inequalities

$$\begin{aligned} (c_{\ell,i}) &: \xi^{\nu_\ell}_{\ell,i} \le y^\ell_i, \\ (11) &: y^\ell_i \le \begin{cases} \xi^0_{\ell+1,(i+1)/2}, & i \text{ odd} \\ \eta^0_{\ell+1,i/2}, & i \text{ even} \end{cases}; \end{aligned} \tag{12}$$

it is immediately seen that we preserve the approximation property and the quality of approximation by converting these inequalities into equations. After this modification, we can eliminate from the system of constraints the Inequalities (11), and eliminate the variables $\xi^0_{\ell,i}$ and $\eta^0_{\ell,i}$, $\ell = 2,\dots,\theta$ (these variables become linear forms of the remaining ones).

To get an impression of the complexity of our polyhedral approximation scheme, we present a table showing the numbers of extra variables p_k and linear constraints q_k in a polyhedral ϵ-approximation of a conic quadratic constraint $\|y\|_2 \le t$ with dim $y = k$:

	$\epsilon = 10^{-1}$		$\epsilon = 10^{-3}$		$\epsilon = 10^{-6}$		$\epsilon = 10^{-12}$	
k	$p_k \approx 5.2k \ln \frac{1}{\epsilon}$	$q_k \approx 13.7k \ln \frac{1}{\epsilon}$	$p_k \approx 4.7k \ln \frac{1}{\epsilon}$	$q_k \approx 10.2k \ln \frac{1}{\epsilon}$	$p_k \approx 4.1k \ln \frac{1}{\epsilon}$	$q_k \approx 9.0k \ln \frac{1}{\epsilon}$	$p_k \approx 4.0k \ln \frac{1}{\epsilon}$	$q_k \approx 8.3k \ln \frac{1}{\epsilon}$
4	6	17	16	39	31	69	60	128
16	30	83	84	196	159	345	308	643
64	133	623	352	829	677	1458	1301	2711
256	543	1486	1436	3368	2711	5916	5273	10994
1024	2203	6006	5784	13566	10899	23758	21065	44154
4096	9083	24246	23288	54320	43635	95206	84329	176850

3. A lower bound. We have demonstrated that, for every $\epsilon \in (0, 1]$, the Lorentz cone $\mathbf{L}^k$ admits a polyhedral ϵ-approximation of the "size" $p_k + q_k \le O(1)k \ln 2/\epsilon$. We are about to prove that the order of this upper bound in fact cannot be improved:

PROPOSITION 3.1. *Let k be a positive integer, $\epsilon \in (0, 0.5]$, and let*

$$\Pi(y, t, u) : \mathbf{R}^k \times \mathbf{R} \times \mathbf{R}^p \to \mathbf{R}^q$$

be a polyhedral ϵ approximation of $\mathbf{L}^k$. Then

$$q \ge O(1)k \ln \frac{1}{\epsilon}, \tag{13}$$

with positive absolute constant $O(1)$.

PROOF. The projection $\widehat{\mathbf{L}}^k$ of the polyhedral cone

$$\mathbf{K} = \{(y, t, u) \mid \Pi(y, t, u) \ge 0\}$$

on the plane of (y, t) variables does not contain lines; replacing, if necessary, u with its projection on a properly chosen subspace in $\mathbf{R}^p$, we may assume that the cone $\mathbf{K}$ itself does not contain lines, so that $\mathbf{K}$ is a conic hull of its extreme rays. Let N be the number of extreme rays of $\mathbf{K}$; one clearly has $N \le 2^q$. Now, since $\mathbf{K}$ is the conic hull of N rays, $\widehat{\mathbf{L}}^k$ is the conic hull of N rays $R_1, \ldots, R_N$, and every one of these rays intersects the hyperplane $H = \{t = 1\}$ in the (y, t) space. Now let us look at the set

$$G = \{y \mid (y, 1) \in \widehat{\mathbf{L}}^k\}.$$

Since $\Pi(\cdot)$ is a polyhedral α approximation of $\mathbf{L}^k$, the set G contains the unit k-dimensional Euclidean ball B and is contained in the ball $(1+\epsilon)B$. On the other hand, G is the convex hull of N points $y_1, \ldots, y_N$—the y components of the intersections $R_i \cap H$. We claim that the N Euclidean balls B_i centered at y_i of the radius $\sqrt{2\epsilon(1+\epsilon)}$ cover the boundary of $(1+\epsilon)B$. Indeed, if for some $\rho \ge 0$ there exists a point $y \in \partial((1+\epsilon)B)$ which is at the distance $\ge \rho$ from every y_i, $i = 1, \ldots, N$, then the convex hull of $y_1, \ldots, y_N$ is contained in the convex hull of the set

$$\{z \mid \|z\|_2 \le 1+\epsilon, \|z-y\|_2 \ge \rho\},$$

so that the latter convex hull contains B; this observation via elementary geometry implies that $\rho \le \sqrt{2\epsilon(1+\epsilon)}$. For $\epsilon \le 0.5$, the fact that the balls B_i cover $\partial((1+\epsilon)B)$ implies that $N \ge \exp\{O(1)k \ln 1/\epsilon\}$ with a positive absolute constant $O(1)$. Combining this observation with the inequality $N \le 2^q$, we deduce that $q \ge O(1)k \ln 1/\epsilon$, as claimed. □

4. Concluding remarks. With our approach, we are able to associate with a given conic quadratic problem (CQP) and a given $\epsilon \in (0, 1]$ an LP program (LP$_\epsilon$) of the same (up to factor $O(\ln 1/\epsilon)$) size, such that the projection of the feasible set of the latter problem on the x space is in between the feasible set of (CQP) and the feasible set of the ϵ relaxation (CQP$_\epsilon$) of (CQP). Thus, (LP$_\epsilon$) is a good approximation of (CQP), provided that the feasible sets of (CQP) and (CQP$_\epsilon$) are close to each other. Note that the latter is not necessarily the case; one can easily find an example where (CQP) is infeasible, while all problems (CQP$_\epsilon$), $\epsilon > 0$, are feasible. There is, however, a simple sufficient condition ensuring that the feasible sets of (CQP) and (CQP$_\epsilon$) are "$O(\epsilon)$-close" to each other.

PROPOSITION 4.1. *Assume that* (CQP) *is*
(i) *strictly feasible: there exist* $\bar{x}$ *and* $r > 0$ *such that*

$$A\bar{x} \geq b,\ \|A_\ell \bar{x} - b_\ell\|_2 \leq [c_\ell^T x - d_\ell] - r, \qquad \ell = 1, \ldots, m;$$

(ii) *"semibounded:" there exists R, such that*

$$Ax \geq b,\ \|A_\ell x - b_\ell\|_2 \leq c_\ell^T x - d_\ell,\ \ell = 1, \ldots, m \Rightarrow c_\ell^T x - d_\ell \leq R, \qquad \ell = 1, \ldots, m.$$

Then for every $\epsilon > 0$, *such that* $\gamma(\epsilon) \equiv R\epsilon/r < 1$, *one has*

$$\gamma(\epsilon)\bar{x} + (1-\gamma(\epsilon))\mathrm{Feas}(\mathrm{CQP}_\epsilon) \subset \mathrm{Feas}(\mathrm{CQP}) \subset \mathrm{Feas}(\mathrm{CQP}_\epsilon) \tag{14}$$

where Feas(P) *denotes the feasible set of a problem* (P).

PROOF. We already know that the right inclusion in (14) holds true. To prove the left inclusion, let y be a feasible solution to (CQP$_\epsilon$); we should prove that the point $x = (1-\gamma)y + \gamma\bar{x}$ is feasible for (CQP). Since both $\bar{x}$ and y satisfy the linear constraints $Ax \geq b$, all we should prove is that

$$\|A_\ell x - b_\ell\|_2 \leq c_\ell^T x - d_\ell, \qquad \ell = 1, \ldots, m. \tag{15}$$

For $\delta \in [0, 1]$, let $x_\delta = (1-\delta)y + \delta\bar{x}$. By (i) and since y is feasible for (CQP$_\epsilon$), we have

$$\begin{aligned} \|A_\ell \bar{x} - b_\ell\|_2 &\leq c_\ell^T \bar{x} - d_\ell - r \\ \|A_\ell y - b_\ell\|_2 &\leq (1+\epsilon)\big[\underbrace{c_\ell^T y - d_\ell}_{t_\ell}\big], \qquad \ell = 1, \ldots, m; \end{aligned}$$

setting $\delta = \max_\ell \epsilon t_\ell/(r + \epsilon t_\ell)$, we conclude that

$$\|A_\ell x_\delta - b_\ell\|_2 \leq \big[c_\ell^T x_\delta - d_\ell\big] + [(1-\delta)\epsilon t_\ell - \delta r] \leq \big[c_\ell^T x_\delta - d_\ell\big], \qquad \ell = 1, \ldots, m,$$

and clearly $Ax_\delta \geq b$. Thus, x_δ is feasible for (CQP). It follows from (ii) that

$$\big[c_\ell^T x_\delta - d_\ell\big] = \delta\big[c_\ell^T \bar{x} - d_\ell\big] + (1-\delta)t_\ell \leq R, \qquad \ell = 1, \ldots, m.$$

Since $\bar{x}$ is feasible for (CQP), we have $[c_\ell^T \bar{x} - d_\ell] \geq 0$, whence $(1-\delta)t_\ell \leq R$ for all ℓ. Recalling the origin of δ, we conclude that $\delta = \epsilon t/(r + \epsilon t)$ for some t satisfying $(1-\delta)t \leq R$. From these relations and in view of $\gamma(\epsilon) \equiv R\epsilon/r < 1$ we conclude that $t \leq R/(1-\gamma(\epsilon))$, whence $\delta \leq \gamma(\epsilon)$. Since $\delta \leq \gamma(\epsilon) \leq 1$ and both x_δ and $x_1 = \bar{x}$ are feasible for (CQP), so is $x = x_\gamma(\epsilon)$. □

Proposition 4.1 allows to quantify the "actual" quality of polyhedral approximations of conic quadratic problems, in particular, those mentioned in Examples 1–4 of the introduction. In these examples, we are interested in building a polyhedral approximation of the epigraph

$$\mathrm{Epi}(f) = \{(x,t) \mid x \in \mathrm{Dom}\, f, t \geq f(\infty)\}$$

of a particular convex function $f(x)$, i.e., in building a system of linear inequalities,

$$\text{(S)} \qquad Px + tp + Qu + q \geq 0,$$

in such a way that the projection of the solution set of the system onto the (y,t) space is the epigraph of certain function $\hat{f}$ "close" to f. Specifically, Examples 1–4 deal with the following functions f:

- Example 1: $f(x) = x^T Q^T Q x : \mathbf{R}^n \to \mathbf{R}$;
- Example 2: $f(x) = -\prod_{i=1}^n x_i^{\pi_i} : \mathbf{R}_+^n \to \mathbf{R}$, with $\pi_i = \frac{p_i}{p}$, $p_i, p \in \mathbf{N}_+$, $\sum_i \pi_i \leq 1$;
- Example 3: $f(x) = \prod_{i=1}^n x_i^{-\pi_i} : \mathrm{int}\, \mathbf{R}_+^n \to \mathbf{R}$, with $\pi_i = \frac{p_i}{p}$, $p_i, p \in \mathbf{N}_+$;
- Example 4: $f(x) = \exp\{x\}$.

In all these examples, the closeness between f and $\hat{f}$ we can efficiently ensure is the uniform closeness on *compact* subsets of the domain of the "target" function f. Combining straightforwardly Theorem 1.1, Proposition 4.1, and the approximation schemes mentioned in the introduction, in connection with Examples 1–4, we come to the following results:

EXAMPLE 1. For every $\epsilon \in (0,1]$, $R > 0$, one can explicitly point out System (S) of the size

$$\mathrm{Size(S)} \equiv \dim x + \dim u + \dim q \leq O(n) \ln \frac{(1 + R\sqrt{n})(1 + n \max_{i,j} |Q_{ij}|)}{\epsilon} \qquad [n = \dim x]$$

in such a way that the associated function $\hat{f}$ satisfies

$$|f(x) - \hat{f}(x)| \leq \epsilon \quad \forall (x : \|x\|_\infty \leq R).$$

EXAMPLE 2. For every $\epsilon \in (0,1]$, $R > 0$, one can explicitly point out System (S) of the size

$$\mathrm{Size(S)} \leq O(p) \ln \frac{p(1+R)}{\epsilon}$$

in such a way that the associated function $\hat{f}$ satisfies

$$|f(x) - \hat{f}(x)| \leq \epsilon \quad \forall (x \geq 0 : \|x\|_\infty \leq R).$$

EXAMPLE 3. For every $\epsilon \in (0,1]$, $R > 0$, one can explicitly point out System (S) of the size

$$\mathrm{Size(S)} \leq O(p + \sum_i p_i) \ln \frac{(p + \sum_i p_i)(1+R)}{\epsilon}$$

in such a way that the associated function $\hat{f}$ satisfies

$$|f(x) - \hat{f}(x)| \leq \epsilon \quad \forall (x : R^{-1} \leq x_i \leq R, \qquad i = 1, \ldots, n).$$

EXAMPLE 4. For every $\epsilon \in (0,1]$, $R > 0$, one can explicitly point out System (S) of the size

$$\mathrm{Size(S)} \leq O(1) \left[R + \ln \frac{2}{\epsilon} \right] \ln \frac{(1+R)}{\epsilon}$$

in such a way that the associated function $\hat{f}$ satisfies

$$\hat{f}(x) \leq \exp\{x\} \leq (1+\epsilon)\hat{f}(x) \quad \forall (x : |x| \leq R).$$

Acknowledgments. This research was supported by the German-Israeli Foundation for R&D Contract No. I-0455-214.06/95 and the Israel Ministry of Science Grant # 0200-1-98. Part of this work was done while the author was a visiting professor at the Faculty of Information Technology and Systems, Delft University of Technology, The Netherlands.

References

Ben-Tal, A., A. Nemirovskii. 1994. Potential reduction polynomial time method for Truss topology design. *SIAM J. Optim.* **4** 596–612.

———. 1998. Robust convex optimization. *Math. Oper. Res.* **23** 769–805.

Christensen, P. W., J. S. Pang. 1999. Frictional contact algorithms based on semismooth Newton methods. M. Fukushima, L. Qi, eds. *Reformulation—Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods*. Kluwer Academic Publishers, Boston, MA, 81–116.

Lo, G. 1996. *Complementarity problems in robotics*. Ph.D. dissertation, Department of Mathematical Sciences, The John Hopkins University, Baltimore, MD.

Lobo, M., L. Vandenberghe, S. Boyd, H. Lebret. 1998. Applications of second-order cone programming. *Linear Algebra Appl.* **284** 193–228.

Nesterov, Yu., A. Nemirovski. 1994. *Interior-Point Polynomial Algorithms in Convex Programming*. SIAM Series in Applied Mathematics, SIAM, Philadelphia, PA.

Nesterov, Yu., M. J. Todd. 1997. Self-scaled barriers and interior-point met hods for convex programming. *Math. Oper. Res.* **22** 1–42.

———. 1998. Primal-dual interior-point methods for self-scaled cones. *SIAM J. Optim.* **8** 324–363.

Pang, J. S., D. E. Stewart. 1999. A unified approach to frictional contact problems. *Int. J. Eng. Sc.* **37** 1747–1768.

A. Ben-Tal: Faculty of Industrial Engineering and Management, Technion—Israel Institute of Technology, Haifa 32000, Israel; e-mail: abental@ie.technion.ac.il

A. Nemirovski: Faculty of Industrial Engineering and Management, Technion—Israel Institute of Technology, Haifa 32000, Israel; e-mail: nemirovs@ie.technion.ac.il