Math. Program., Ser. A
DOI 10.1007/s10107-011-0442-6

FULL LENGTH PAPER

Validation analysis of mirror descent stochastic approximation method

Guanghui Lan · Arkadi Nemirovski · Alexander Shapiro

Received: 23 May 2008 / Accepted: 10 December 2010

Abstract The main goal of this paper is to develop accuracy estimates for stochastic programming problems by employing stochastic approximation (SA) type algorithms. To this end we show that while running a Mirror Descent Stochastic Approximation procedure one can compute, with a small additional effort, lower and upper statistical bounds for the optimal objective value. We demonstrate that for a certain class of convex stochastic programs these bounds are comparable in quality with similar bounds computed by the sample average approximation method, while their computational cost is considerably smaller.

Keywords Stochastic approximation · Sample average approximation method · Stochastic programming · Monte Carlo sampling · Mirror descent algorithm · Prox-mapping · Optimality bounds · Large deviations estimates · Asset allocation problem · Conditional value-at-risk

Mathematics Subject Classification (2000) 62L20 · 90C25 · 90C15 · 65C05

G. Lan research of this author was partly supported by the ONR Grant N000140811104 during his Ph.D. study. A. Nemirovski and A. Shapiro research of this author was partly supported by the NSF awards DMI-0619977 and DMS-0914785.

G. Lan (✉)
University of Florida, Gainesville, FL 32611, USA
e-mail: glan@ise.ufl.edu

A. Nemirovski · A. Shapiro
Georgia Institute of Technology, Atlanta, GA 30332, USA
e-mail: nemirovs@isye.gatech.edu

A. Shapiro
e-mail: ashapiro@isye.gatech.edu

Published online: 04 February 2011

1 Introduction

Consider the following Stochastic Programming (SP) problem

$$\text{Opt} = \min_{x \in X} \{ f(x) := \mathbb{E}[F(x, \xi)] \}, \tag{1.1}$$

where $X \subset \mathbb{R}^n$ is a nonempty bounded closed convex set, ξ is a random vector whose probability distribution P is supported on set $\Xi \subset \mathbb{R}^d$ and $F : X \times \Xi \to \mathbb{R}$. A basic difficulty of solving such problems is that the objective function $f(x)$ is given implicitly as the expectation and as such is difficult to compute to high accuracy. A way of solving problems (1.1) is by using randomized algorithms, based on Monte Carlo sampling. There are two competing approaches of this type, namely, the Sample Average Approximation (SAA) and the Stochastic Approximation (SA) methods. Both approaches have a long history.

The basic idea of the SAA method is to generate a sample $\xi_1, \ldots, \xi_N$, of N realizations of ξ and to approximate the "true" problem (1.1) by replacing $f(x)$ with its sample average approximation $\hat{f}_N(x) := N^{-1} \sum_{t=1}^{N} F(x, \xi_t)$. Recent theoretical studies (cf., [2,15,16]) and numerical experiments (e.g., [5,6,17]) show that the SAA method coupled with a good deterministic algorithm for minimizing the constructed SAA problem could be reasonably efficient for solving certain classes of SP problems. The SA approach originates from the pioneering work of Robbins and Monro [13] and was discussed in numerous publications since. An important improvement was developed in Polyak [11] and Polyak and Juditsky [12], where a robust version of the SA method was introduced (the main ingredients of Polyak's scheme, long steps and averaging, were in a different form proposed already in Nemirovski and Yudin [7]). Yet it was believed that the SA approach performs poorly in practice and cannot compete with the SAA method. Somewhat surprisingly it was demonstrated recently in Nemirovski et al. [9] that a proper modification of the SA approach, based on the Nemirovski and Yudin [8] mirror-descent method, can be competitive and can even significantly outperform the SAA method for a certain class of convex stochastic programs. For example, when X in (1.1) is a simplex of large dimension, the *Mirror Descent Stochastic Approximation* builds approximate solutions 10–40 times faster than an SAA based algorithm while keeping similar solution quality.

An important methodological property of the SAA approach is that, with some additional effort, it can provide an estimate of the accuracy of an obtained solution by computing upper and lower (confidence) bounds for the optimal value of the true problem (cf., [6,10]). The main goal of this paper is to show that, for a certain class of stochastic convex problems, the Mirror Descent SA method can also provide similar bounds with considerably less computational effort. More specifically we study in this paper the following aspects of the Mirror Descent SA method.

– Investigate different ways to estimate lower and upper bounds for the objective values by the Mirror Descent SA method, and thus to obtain an accuracy certificate for the attained solutions.
– Adjust the Mirror Descent SA method to solve two interesting application problems in asset allocation, namely, minimizing the expected disutility (EU) and

minimizing the conditional value-at-risk (CVaR). These models are widely used in practice, for example, by investment companies, brokerage firms, mutual funds, and any business that evaluates risks (cf., [14]).
- Understand the performance of the Mirror Descent SA algorithm for solving stochastic programs with a feasible region more complicated than a simplex. For the EU model, the feasible region is the intersection of a simplex with a box constraint and we will compare two different variants of SA methods for solving it. For the CVaR problem, the feasible region is a polyhedron and we will discuss some techniques to explore its structure.

The paper is organized as follows. In Sect. 2 we briefly introduce the Mirror Descent SA method. Section 3 is devoted to a derivation and analysis of statistical upper and lower bounds for the optimal value of the true problem. In Sect. 4 we discuss an application of the Mirror Descent SA method to the expected disutility and conditional value at risk approaches for the asset allocation problem. A discussion of numerical results is presented in Sect. 5. Finally, proofs of technical results are given in the Appendix.

We assume throughout the paper that for every $\xi \in \Xi$ the function $F(\cdot, \xi)$ is convex on X, and that the expectation

$$\mathbb{E}[F(x, \xi)] = \int_{\Xi} F(x, \xi) dP(\xi) \tag{1.2}$$

is well defined, finite valued and continuous at every $x \in X$. That is, the expectation function $f(x)$ is finite valued, convex and continuous on X. For a norm $\|\cdot\|$ on $\mathbb{R}^n$, we denote by $\|x\|_* := \sup\{x^T y : \|y\| \le 1\}$ the conjugate norm. By $\|x\|_p$ we denote the ℓ_p norm of vector $x \in \mathbb{R}^n$. In particular, $\|x\|_2 = \sqrt{x^T x}$ is the Euclidean norm of $x \in \mathbb{R}^n$. By $\Pi_X(x) := \arg\min_{y \in X} \|x - y\|_2$ we denote metric projection operator onto X. For the process $\xi_1, \xi_2, \ldots,$ we set $\xi^t := (\xi_1, \ldots, \xi_t)$, and denote by $\mathbb{E}_{|t}$ or by $\mathbb{E}[\cdot|\xi^t]$ the conditional, ξ^t being given, expectation. For a number $a \in \mathbb{R}$ we denote $[a]_+ := \max\{a, 0\}$. By $\partial\phi(x)$ we denote the subdifferential of a convex function $\phi(x)$.

2 The mirror descent stochastic approximation method

In this section, we give a brief introduction to the Mirror Descent SA algorithm as presented in [9]. We equip the embedding space $\mathbb{R}^n$, of the feasible domain X of (1.1), with a norm $\|\cdot\|$. We say that a function $\omega : X \to \mathbb{R}$ is a *distance generating function* with respect to the norm $\|\cdot\|$ and modulus $\alpha > 0$, if the following conditions hold: (i) ω is convex and continuous on X, (ii) the set

$$X^o := \{x \in X : \partial\omega(x) \neq \emptyset\} \tag{2.1}$$

is convex, and (iii) $\omega(\cdot)$ restricted to X^o is continuously differentiable and strongly convex with parameter α with respect to $\|\cdot\|$, i.e.,

$$(x' - x)^T (\nabla\omega(x') - \nabla\omega(x)) \ge \alpha\|x' - x\|^2, \quad \forall x', x \in X^o. \tag{2.2}$$

Note that the set X^o always contains the relative interior of the set X.

With the distance generating function $\omega(\cdot)$ are associated the *prox-function*[1] $V : X^o \times X \to \mathbb{R}_+$ defined as

$$V(x, z) := \omega(z) - \omega(x) - \nabla\omega(x)^T(z - x), \tag{2.3}$$

the *prox-mapping* $P_x : \mathbb{R}^n \to X^o$ defined as

$$P_x(y) := \arg\min_{z \in X} \left\{ y^T(z - x) + V(x, z) \right\}, \tag{2.4}$$

and the constant

$$D_{\omega,X} := \sqrt{\max_{x \in X} \omega(x) - \min_{x \in X} \omega(x)}. \tag{2.5}$$

Let x_1 be the minimizer of $\omega(\cdot)$ over X. This minimizer exists and is unique since X is convex and compact and $\omega(\cdot)$ is continuous and strictly convex on X. Observe that $x_1 \in X^o$, and since x_1 is the minimizer of $\omega(\cdot)$ it follows that $(x - x_1)^T \nabla\omega(x_1) \geq 0$ for all $x \in X$. Combined with the strong convexity of $\omega(\cdot)$ this implies that

$$\tfrac{1}{2}\alpha\|x - x_1\|^2 \leq V(x_1, x) \leq \omega(x) - \omega(x_1) \leq D^2_{\omega,X}, \quad \forall x \in X, \tag{2.6}$$

and hence

$$\|x - x_1\| \leq \Lambda_{\omega,X} := \sqrt{\frac{2}{\alpha}} D_{\omega,X}, \quad \forall x \in X. \tag{2.7}$$

Throughout the paper we assume existence of the following *stochastic oracle*.

It is possible to generate an iid sample $\xi_1, \xi_2, \ldots,$ of realizations of random vector ξ, and we have access to a "black box" subroutine (a stochastic oracle): given $x \in X$ and a random realization $\xi \in \Xi$, the oracle returns the quantity $F(x, \xi)$ and a *stochastic subgradient*—a vector $\mathsf{G}(x, \xi)$ such that $\mathsf{g}(x) := \mathbb{E}[\mathsf{G}(x, \xi)]$ is well defined and is a subgradient of $f(\cdot)$ at x, i.e., $\mathsf{g}(x) \in \partial f(x)$.

We also make the following assumption.

(A1) There are positive constants Q and M_* such that for any $x \in X$:

$$\mathbb{E}\left[(F(x, \xi) - f(x))^2\right] \leq Q^2, \tag{2.8}$$

$$\mathbb{E}\left[\|\mathsf{G}(x, \xi)\|_*^2\right] \leq M_*^2. \tag{2.9}$$

It could be noted that $\mathbb{E}\left[(F(x, \xi) - f(x))^2\right]$ in (2.8) is the variance of the random variable $F(x, \xi)$.

[1] It is also called Bregman distance [1].

When speaking about Stochastic Approximation as applied to minimization problem (1.1), one usually does not care about how the values of $f(\cdot)$ are observed. The only things that matter are the observations of the gradient, these being the only information used by the basic SA algorithm (2.10), see below. We, however, are interested in building upper and lower bounds on the optimal value and/or value of $f(\cdot)$ at a given solution, and in this respect, it does matter how these values are observed. Conditions (2.8)–(2.9) of assumption (A1) impose restrictions on the magnitudes of noises in the unbiased observations of the values of $f(\cdot)$ and the subgradients of $f(\cdot)$ reported by the stochastic oracle.

The description of the Mirror Descent SA algorithm is as follows. Starting from point x_1, the algorithm iteratively generates points $x_t \in X^o$ according to the recurrence

$$x_{t+1} := P_{x_t}\left(\gamma_t \mathsf{G}(x_t, \xi_t)\right), \tag{2.10}$$

where $\gamma_t > 0$ are deterministic stepsizes. Note that for $\omega(x) := \frac{1}{2}\|x\|_2^2$, we have that $P_x(y) = \Pi_X(x - y)$ and hence $x_{t+1} = \Pi_X\left(x_t - \gamma_t \mathsf{G}(x_t, \xi_t)\right)$. In that case, the Mirror Descent SA method is referred to as the Euclidean SA.

Now let N be the total number of steps. Let us set

$$\nu_t := \frac{\gamma_t}{\sum_{i=1}^N \gamma_i}, \quad t = 1, \ldots, N, \quad \text{and} \quad \tilde{x}_N := \sum_{t=1}^N \nu_t x_t. \tag{2.11}$$

Note that $\sum_{t=1}^N \nu_t = 1$, and hence $\tilde{x}_N$ is a convex combination of the iterates $x_1, \ldots, x_N$. Here $\tilde{x}_N$ is considered as the approximate solution generated by the algorithm in course of N steps. The quality of this solution can be quantified as follows (cf., [9, p. 1583]).

Proposition 1 *Suppose that condition* (2.9) *of assumption* (A1) *holds. Then for the* N*-step of Mirror Descent SA algorithm we have that*

$$\mathbb{E}\left[f(\tilde{x}_N) - \text{Opt}\right] \leq \frac{D_{\omega,X}^2 + (2\alpha)^{-1} M_*^2 \sum_{t=1}^N \gamma_t^2}{\sum_{t=1}^N \gamma_t}. \tag{2.12}$$

In implementations of the SA algorithm different stepsize strategies can be applied to (2.10) (see [9]). We discuss now the *constant stepsize* policy. That is, we assume that the number N of iterations is fixed in advance, and $\gamma_t = \gamma$, $t = 1, \ldots, N$. In that case

$$\tilde{x}_N = \frac{1}{N} \sum_{t=1}^N x_t. \tag{2.13}$$

By choosing the stepsizes as

$$\gamma_t = \gamma := \frac{\theta \sqrt{2\alpha} D_{\omega,X}}{M_* \sqrt{N}}, \quad t = 1, \ldots, N, \tag{2.14}$$

with a (scaling) constant $\theta > 0$, we have in view of (2.12) that

$$\mathbb{E}\left[f(\tilde{x}_N) - \text{Opt}\right] \le \max\{\theta, \theta^{-1}\}\Lambda_{\omega,X} M_* N^{-1/2}, \tag{2.15}$$

with $\Lambda_{\omega,X}$ given by (2.7). This shows that scaling the stepsizes by the (positive) constant θ results in updating the estimate (2.15) by the factor of $\max\{\theta, \theta^{-1}\}$ at most. By Markov's inequality it follows from (2.15) that for any $\varepsilon > 0$,

$$\text{Prob}\left\{f(\tilde{x}_N) - \text{Opt} > \varepsilon\right\} \le \frac{\sqrt{2}\max\{\theta, \theta^{-1}\}D_{\omega,X} M_*}{\varepsilon\sqrt{\alpha N}}. \tag{2.16}$$

It is possible to obtain finer bounds for the probabilities in the left hand side of (2.16) when imposing conditions more restrictive than conditions of assumption(A1). Consider the following conditions.

(A2) There are positive constants Q and M_* such that for any $x \in X$:

$$\mathbb{E}\left[\exp\left\{|F(x,\xi) - f(x)|^2/Q^2)\right\}\right] \le \exp\{1\}, \tag{2.17}$$

$$\mathbb{E}\left[\exp\left\{\|\mathsf{G}(x,\xi)\|_*^2/M_*^2\right\}\right] \le \exp\{1\}. \tag{2.18}$$

Note that conditions (2.17)–(2.18) are stronger than the respective conditions (2.8)–(2.9). Indeed, if a random variable Y satisfies $\mathbb{E}[\exp\{Y/a\}] \le \exp\{1\}$ for some $a > 0$, then by Jensen's inequality $\exp\{\mathbb{E}[Y/a]\} \le \mathbb{E}[\exp\{Y/a\}] \le \exp\{1\}$, and therefore $\mathbb{E}[Y] \le a$. Of course, conditions (2.17)–(2.18) hold if for all $(x,\xi) \in X \times \Xi$:

$$|F(x,\xi) - f(x)| \le Q \quad \text{and} \quad \|\mathsf{G}(x,\xi)\|_* \le M_*.$$

The following result has been established in [9, Proposition 2.2].

Proposition 2 *Suppose that condition* (2.18) *of assumption* (A2) *holds. Then for the constant stepsize policy, with the stepsize* (2.14)*, the following inequality holds for any $\Omega \ge 1$:*

$$\text{Prob}\left\{f(\tilde{x}_N) - \text{Opt} > \max\{\theta, \theta^{-1}\}(12 + 2\Omega)\Lambda_{\omega,X} M_* N^{-1/2}\right\} \le 2\exp\{-\Omega\}. \tag{2.19}$$

It follows from (2.19) that the number N of steps required by the algorithm to solve the problem with accuracy $\varepsilon > 0$, and a (probabilistic) confidence $1 - \beta$, is of order $O\left(\varepsilon^{-2}\log^2(1/\beta)\right)$. Note also that in practice one can modify the Mirror Descent SA algorithm so that the approximate solution $\tilde{x}_N$ is obtained by averaging over a part of the trajectory (see [9] for details).

3 Accuracy certificates for SA solutions

In this section, we discuss several ways to estimate lower and upper bounds for the optimal value of problem (1.1), which gives us an accuracy certificate for obtained solutions. Specifically, we distinguish between two types of certificates: the *online certificates* that can be computed quickly when running the SA algorithm, and the *offline certificates* obtained in a more time consuming way at the dedicated *validation step*, after a solution has been obtained.

3.1 Online certificate

Consider the numbers ν_t and solution $\tilde{x}_N$, defined in (2.11), functions

$$f^N(x) := \sum_{t=1}^{N} \nu_t \left[f(x_t) + \mathsf{g}(x_t)^T (x - x_t) \right] \text{ and}$$
$$\hat{f}^N(x) := \sum_{t=1}^{N} \nu_t [F(x_t, \xi_t) + \mathsf{G}(x_t, \xi_t)^T (x - x_t)],$$

and define

$$f_*^N := \min_{x \in X} f^N(x) \text{ and } f^{*N} := \sum_{t=1}^{N} \nu_t f(x_t). \tag{3.1}$$

Since $\nu_t > 0$ and $\sum_{t=1}^{N} \nu_t = 1$, it follows by convexity of $f(\cdot)$ that the function $f^N(\cdot)$ underestimates $f(\cdot)$ everywhere on X, and hence $f_*^N \le \text{Opt}$. Since $\tilde{x}_N \in X$ we also have that $\text{Opt} \le f(\tilde{x}_N)$, and by convexity of $f(\cdot)$ that $f(\tilde{x}_N) \le f^{*N}$. That is, for *any realization* of the random sample $\xi_1, \ldots, \xi_N$ we have that

$$f_*^N \le \text{Opt} \le f(\tilde{x}_N) \le f^{*N}. \tag{3.2}$$

It follows from (3.2) that $\mathbb{E}[f_*^N] \le \text{Opt} \le \mathbb{E}[f^{*N}]$ as well.

Of course, the bounds f_*^N and f^{*N} are unobservable since the values $f(x_t)$ are not known exactly. Therefore we consider their computable counterparts

$$\underline{f}^N = \min_{x \in X} \hat{f}^N(x) \text{ and } \overline{f}^N = \sum_{t=1}^{N} \nu_t F(x_t, \xi_t). \tag{3.3}$$

We refer to $\underline{f}^N$ and $\overline{f}^N$ as *online* bounds. The bound $\overline{f}^N$ can be easily calculated while running the SA procedure. The bound $\underline{f}^N$ involves solving the optimization problem of minimizing a linear objective function over the set X. If the set X is defined by linear constraints, this is a linear programming problem.

Since x_t is a function of $\xi^{t-1} = (\xi_1, \ldots, \xi_{t-1})$, and ξ_t is independent of ξ^{t-1}, we have that

$$\mathbb{E}\left[\overline{f}^N\right] = \sum_{t=1}^N \nu_t \mathbb{E}\left\{\mathbb{E}[F(x_t,\xi_t)|\xi^{t-1}]\right\} = \sum_{t=1}^N \nu_t \mathbb{E}\left[f(x_t)\right] = \mathbb{E}[f^{*N}]$$

and

$$\begin{aligned}\mathbb{E}\left[\underline{f}^N\right] &= \mathbb{E}\left[\mathbb{E}\left\{\min_{x\in X}\left[\sum_{t=1}^N \nu_t[F(x_t,\xi_t) + \mathsf{G}(x_t,\xi_t)^T(x-x_t)]\right] \mid \xi^{t-1}\right\}\right]\\ &\le \mathbb{E}\left[\min_{x\in X}\left\{\mathbb{E}\left[\sum_{t=1}^N \nu_t[F(x_t,\xi_t) + \mathsf{G}(x_t,\xi_t)^T(x-x_t)]\right] \mid \xi^{t-1}\right\}\right]\\ &= \mathbb{E}\left[\min_{x\in X} f^N(x)\right] = \mathbb{E}\left[f_*^N\right].\end{aligned}$$

It follows that

$$\mathbb{E}\left[\underline{f}^N\right] \le \mathrm{Opt} \le \mathbb{E}\left[\overline{f}^N\right]. \tag{3.4}$$

That is, on average $\underline{f}^N$ and $\overline{f}^N$ give, respectively, a lower and an upper bound for the optimal value of problem (1.1). In order to see how good are the bounds $\underline{f}^N$ and $\overline{f}^N$ let us estimate expectations and probabilities of the corresponding errors. Proof of the following theorem is given in the Appendix.

Theorem 1 (i) *Suppose that assumption* (A1) *holds. Then*

$$\mathbb{E}\left[f^{*N} - f_*^N\right] \le \frac{2D_{\omega,X}^2 + \frac{5}{2}\alpha^{-1}M_*^2\sum_{t=1}^N \gamma_t^2}{\sum_{t=1}^N \gamma_t}, \tag{3.5}$$

$$\mathbb{E}\left[\left|\overline{f}^N - f^{*N}\right|\right] \le Q\sqrt{\sum_{t=1}^N \nu_t^2}, \tag{3.6}$$

$$\mathbb{E}\left[\left|\underline{f}^N - f_*^N\right|\right] \le \frac{D_{\omega,X}^2 + \frac{1}{2}\alpha^{-1}M_*^2\sum_{t=1}^N \gamma_t^2}{\sum_{t=1}^N \gamma_t} + \left(Q + 8\Lambda_{\omega,X}M_*\right)\sqrt{\sum_{t=1}^N \nu_t^2}. \tag{3.7}$$

In particular, in the case of constant stepsize policy (2.14) *we have*

$$\begin{aligned}
&\mathbb{E}\left[f^{*N}-f_*^N\right]\le\left[\theta^{-1}+5\theta/2\right]\Lambda_{\omega,X}M_*N^{-1/2},\\
&\mathbb{E}\left[\left|\overline{f}^N-f^{*N}\right|\right]\le QN^{-1/2},\\
&\mathbb{E}\left[\left|\underline{f}^N-f_*^N\right|\right]\le\tfrac{1}{2}\left[\theta^{-1}+\theta\right]\Lambda_{\omega,X}M_*N^{-1/2}+\left(Q+8\Lambda_{\omega,X}M_*\right)N^{-1/2},
\end{aligned}\tag{3.8}$$

where $\Lambda_{\omega,X}$ *is given by* (2.7)*.*

(ii) *Moreover, if assumption* (A1) *is strengthened to assumption* (A2)*, then in the case of constant*[2] *stepsize policy* (2.14) *we have for any* $\Omega\ge 0$*:*

$$\begin{aligned}
&\mathrm{Prob}\left\{f^{*N}-f_*^N>N^{-1/2}\Lambda_{\omega,X}M_*\left(\left[\tfrac{5}{2}\theta+\theta^{-1}\right]+\Omega\left[4+\tfrac{5}{2}\theta N^{-1/2}\right]\right)\right\}\\
&\qquad\le 2\exp\{-\Omega^2/3\}+2\exp\{-\Omega^2/12\}+2\exp\{-3\Omega\sqrt{N}/4\},
\end{aligned}\tag{3.9}$$

$$\mathrm{Prob}\left\{\left|\overline{f}^N-f^{*N}\right|>\Omega Q\sqrt{\sum_{t=1}^N\nu_t^2}\right\}\le 2\exp\{-\Omega^2/3\},\tag{3.10}$$

$$\begin{aligned}
&\mathrm{Prob}\left\{|\underline{f}^N-f_*^N|>N^{-1/2}\left(\left[\tfrac{1}{2\theta}+2\theta\right]\Lambda_{\omega,X}M_*+\Omega\left[Q+[8+2\theta N^{-1/2}]\right.\right.\right.\\
&\qquad\left.\left.\left.\times\Lambda_{\omega,X}M_*\right]\right)\right\}\le 6\exp\{-\Omega^2/3\}+\exp\{-\Omega^2/12\}+\exp\{-3\Omega\sqrt{N}/4\}.
\end{aligned}\tag{3.11}$$

Estimates of the above theorem show that as N grows, the observable quantities $\underline{f}^N$ and $\overline{f}^N$ approach, in a probabilistic sense, their unobservable counterparts, which, in turn, approach each other and thus the optimal value of problem (1.1). For the constant stepsize policy (2.14), we have that all estimates given in the right hand side of (3.8) are of order $O(N^{-1/2})$. It follows that under assumption (A1) and for the constant stepsize policy, the difference between the upper $\overline{f}^N$ and lower $\underline{f}^N$ bounds converges on average to zero, with increase of the sample size N, at a rate of $O(N^{-1/2})$.

Note that for the constant stepsize policy (2.14) and under assumption (A2), the bounds (3.9)–(3.11) combine with (3.2) to imply that

- $\mathrm{Prob}\left\{\overline{f}_\Omega^N:=\overline{f}^N+\Omega\sigma_+N^{-1/2}\text{ is not an upper bound on }f(\tilde{x}^N)\right\}\le 2^{-\frac{\Omega^2}{3}}$, with $\sigma_+=Q$;
- $\mathrm{Prob}\left\{\underline{f}_\Omega^N:=\underline{f}^N-[\mu_-+\Omega\sigma_-]N^{-1/2}\text{ is not a lower bound on Opt}\right\}$ $\le 6e^{-\frac{\Omega^2}{3}}+e^{-\frac{\Omega^2}{12}}+e^{-\frac{3\Omega\sqrt{N}}{4}}$, with $\Lambda_{\omega,X}$ defined by (2.7) and

$$\mu_-:=\left[\frac{1}{2\theta}+2\theta\right]\Lambda_{\omega,X}M_*,\ \ \sigma_-:=Q+[8+2\theta N^{-1/2}]\Lambda_{\omega,X}M_*;$$

[2] The bounds in the Appendix cover the case of general-type stepsizes; here we restrict ourselves with the case of constant stepsizes to avoid less transparent formulas.

- $\text{Prob}\left\{\overline{f}^N_\Omega - \underline{f}^N_\Omega > [\mu + \Omega\sigma]N^{-1/2}\right\} \le 10e^{-\frac{\Omega^2}{3}} + 3e^{-\frac{\Omega^2}{12}} + 3e^{-\frac{3\Omega\sqrt{N}}{4}}$, with

$$\mu := \left[\frac{3}{2\theta} + \frac{9\theta}{2}\right]\Lambda_{\omega,X}M_*,\ \ \sigma := 2Q + \left[12 + \frac{9\theta}{2}\right]\Lambda_{\omega,X}M_*.$$

Theorem 1 shows that for large N the online observable random quantities $\overline{f}^N$ and $\underline{f}^N$ are close to the upper bound f^{*N} and lower bound f^N_*, respectively. Besides this, on average, $\overline{f}^N$ indeed overestimates Opt, and $\underline{f}_N$ indeed underestimates Opt. To save words, let us call random estimates which on average under- or overestimate a certain quantity, *on average* lower, respectively, upper bounds on this quantity. From now on, when speaking of "true" lower and upper bounds—those which always (or almost surely) under-, respectively, over-estimate the quantity, we add the adjective "valid". Thus, we refer to f^{*N} and f^N_* as *valid* upper and lower bounds on Opt, respectively. Recall that f^{*N} is also a valid upper bound on $f(\tilde{x}_N)$.

Remark 1 Recall that the SAA approach also provides a lower on average bound—the random quantity $\hat{f}^N_{\text{SAA}}$, which is the optimal value of the sample average problem (cf., [6,10]). Suppose the same sample ξ_t, $t = 1, \ldots, N$, is applied for both SA and SAA methods. Besides this, assume that the constant stepsize policy is used in the SA method, and hence $\nu_t = 1/N, t = 1, .., N$. Finally, assume (as it often is the case) that $\mathsf{G}(x, \xi)$ is a subgradient of $F(x, \xi)$ in x. By convexity of $F(\cdot, \xi)$ and since $\underline{f}^N = \min_{x\in X} \hat{f}^N(x)$, we have

$$\begin{aligned} \hat{f}^N_{\text{SAA}} &:= \min_{x\in X} N^{-1}\sum_{t=1}^N F(x, \xi_t) \\ &\ge \min_{x\in X}\sum_{t=1}^N \nu_t\left(F(x_t, \xi_t) + \mathsf{G}(x_t, \xi_t)^T(x - x_t)\right) = \underline{f}^N. \end{aligned} \tag{3.12}$$

That is, for the same sample the lower bound $\underline{f}^N$ is smaller than the lower bound obtained by the SAA method. However, it should be noted that the lower bound $\underline{f}^N$ is computed much faster than $\hat{f}^N_{\text{SAA}}$, since computing the latter one amounts to solving the sample average optimization problem associated with the generated sample. Moreover, we will discuss in the next subsection how to improve the lower bound $\underline{f}^N$. From the computational results, the improved lower bound is comparable to the one obtained by the SAA method. □

Remark 2 Similar to the SAA method, in order to estimate the variability of the lower bound $\underline{f}^N$, one can run the SA procedure M times, with independent samples, each of size N, and consequently compute the average and sample variance of M realizations of the random quantity $\underline{f}^N$. Alternatively, one can run the SA procedure once but with NM iterations, then partition the obtained trajectory into M consecutive parts, each of size N, for each of these parts calculate the corresponding SA lower bound and consequently compute the average and sample variance of the M obtained numbers.

The latter approach is similar, in spirit, to the batch means method used in simulation output analysis [3]. One advantage of this approach is that, as more iterations being run, the mirror-descent SA can output a solution $\tilde{x}_{NM}$ with much better objective value than $\tilde{x}_N$. However, this method has the same shortcoming as the batch means method, that is, the correlation among consecutive blocks will result in a biased estimation for the sample variance. $\square$

3.2 Offline certificate

Suppose now that the Mirror Descent SA method is terminated after N iterations. Given a solution $\tilde{x}_N$ obtained by this method, the objective value $f(\tilde{x}_N)$ can be estimated by Monte Carlo sampling. That is, an iid random sample ξ_j, $j = 1, \ldots, K$, (independent of the random sample used in computing $\tilde{x}_N$) is generated and $f(\tilde{x}_N)$ is estimated by $\overline{ub}^K := K^{-1} \sum_{j=1}^K F(\tilde{x}_N, \xi_j)$. Since this procedure does not require computing prox-mapping and the like, one can use here a large sample size K. Of course, we can expect that $\overline{ub}^K$ is a better upper bound on $f(\tilde{x}_N)$ than the online counterpart $\overline{f}^N$ of the valid upper bound f^{*N}.

We now demonstrate that the online lower bound $\underline{f}^N$ can also be improved in the validation step. Given an iid random sample ξ_j, $j = 1, \ldots, L$, we can estimate the (linear in x) form $\ell_L(x; \tilde{x}_N) := f(\tilde{x}_N) + \mathsf{g}(\tilde{x}_N)^T (x - \tilde{x}_N)$ by

$$\hat{\ell}_L(x; \tilde{x}_N) := \frac{1}{L} \sum_{j=1}^{L} \left[F(\tilde{x}_N, \xi_j) + \mathsf{G}(\tilde{x}_N, \xi_j)^T (x - \tilde{x}_N) \right], \tag{3.13}$$

and hence construct the following lower bound on Opt:

$$\underline{lb}^N := \min_{x \in X} \left\{ \max \left[\hat{f}^N(x), \hat{\ell}_L(x; \tilde{x}_N) \right] \right\}. \tag{3.14}$$

Clearly, by definition we have that $\underline{lb}^N \geq \underline{f}^N$.

We would also like to provide some intuition regarding how the incorporation of the linear term $\hat{\ell}_L(x; \tilde{x}_N)$ into the definition of $\underline{lb}^N$ improves the online lower bound $\underline{f}^N$. Indeed, if L is big enough, then $\hat{\ell}_L(x; \tilde{x}_N)$ will be a "close" approximation to the linear function $\ell_L(x; \tilde{x}_N)$ described above. Moreover, if N is big enough, it follows from the optimality condition that $\min_{x \in X} g(\tilde{x}_N)^T (x - \tilde{x}_N)$ should not be too negative and hence that $\min_{x \in X} \ell_L(x; \tilde{x}_N)$ will be close to $f(\tilde{x}_N)$. As a result, if both L and N are large, we can expect that the value of $\underline{\tilde{lb}}^N := \min_{x \in X} \hat{\ell}_L(x; \tilde{x}_N)$ will be close to $\sum_{j=1}^L F(\tilde{x}_N, \xi_j)$ and thus gives us a tight lower bound. On the other hand, if N is not big enough, then $\tilde{x}_N$ will stay far away from x^* and the bound $\underline{\tilde{lb}}^N$ can not be tight. In that case, the incorporation of the $\hat{\ell}_L(x; \tilde{x}_N)$ into the definition of $\underline{lb}^N$ may not be significant. Nevertheless, our numerical results indicate that the off-line bound $\underline{\tilde{lb}}^N$ significantly outperforms the on-line bound $\underline{f}^N$ for almost every instance. Our numerical results also indicate that, even with large L, using $\underline{lb}^N$ is superior to both $\underline{f}^N$ and $\underline{\tilde{lb}}^N$.

Remark 3 It should be noted that although $\mathbb{E}\left[\hat{f}^N(x)\right] \leq f(x)$ and $\mathbb{E}\left[\hat{\ell}_L(x;\tilde{x}_N)\right] \leq f(x)$, the expected value of the maximum of these two quantities is not necessarily $\leq f(x)$. Therefore the expected value of $\underline{lb}^N$ is not necessarily $\leq$ Opt, i.e., we cannot claim that $\underline{lb}^N$ is a lower *on average* bound on Opt. Theoretical justification of the lower bound $\underline{lb}^N$ is provided by the following theorem showing that $\underline{lb}^N$ is "statistically close" to a valid lower bound on Opt, provided that N and L are sufficiently large. □

Proof of Theorem 2 is given in the Appendix.

Theorem 2 *Suppose that assumption* (A1) *holds and let the constant stepsizes* (2.14) *be used. Then*

$$\sqrt{\mathbb{E}\left\{\left(\left[\underline{lb}^N - \text{Opt}\right]_+\right)^2\right\}} \leq \sqrt{2Q^2 + 32\Lambda_{\omega,X}^2 M_*^2}\left[\frac{1}{\sqrt{N}} + \frac{1}{\sqrt{L}}\right]. \tag{3.15}$$

Moreover, under assumption (A2)*, we have that for all* $\Omega \geq 0$*:*

$$\text{Prob}\left\{\underline{lb}^N - \text{Opt} > [Q + 4\Lambda_{\omega,X}M_*]\left[\frac{1}{\sqrt{N}} + \frac{1}{\sqrt{L}}\right]\right\} \leq 4\exp\{-\Omega^2/3\}. \tag{3.16}$$

4 Applications in asset allocation

In this section, we discuss an application of the Mirror Descent SA method to solving asset allocation problems based on the expected disutility (EU) and the conditional value-at-risk (CVaR) models.

4.1 Minimizing the expected disutility

We consider the following stochastic utility[3] model:

$$\min_{x\in X}\left\{f(x) := \mathbb{E}\left[\phi\left(\sum_{i=1}^n (a_i + \xi_i)x_i\right)\right]\right\}. \tag{4.1}$$

Here $X := X' \cap X''$, where

$$X' := \left\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i \leq r\right\} \quad \text{and} \quad X'' := \left\{x \in \mathbb{R}^n : l_i \leq x_i \leq u_i,\ i = 1,\dots,n\right\},$$

[3] Since we deal here with minimization rather than maximization formulation, we refer to it as disutility minimization.

$r > 0$, a_i and $0 \le l_i < u_i$, $i = 1, \dots, n$, are given numbers, $\xi_i \sim \mathcal{N}(0, 1)$ are independent random variables having standard normal distribution and $\phi(\cdot)$ is a piecewise linear convex function given by

$$\phi(t) := \max\{c_1 + b_1 t, \dots, c_m + b_m t\}, \tag{4.2}$$

where c_j and b_j, $j = 1, \dots, m$, are certain constants. Note that by varying parameters r and l_i, u_i we can change the feasible region from a simplex to a box, or the intersection of a simplex with a box. Note that since the set X is compact and $f(x)$ is continuous, the set of optimal solutions of (4.1) is nonempty, provided that X is nonempty. A simpler version of problem (4.1), in which X is assumed to be a standard simplex, has been considered in [9].

For solving this problem, we consider two variants of the Mirror Descent SA algorithm: *Non-Euclidean SA (N-SA)* and Euclidean SA (E-SA), which differ from each other in how the norm $\|\cdot\|$ and the distance generating function $\omega(\cdot)$ are chosen.

4.1.1 Non-Euclidean SA

In N-SA for solving the EU model, the entropy distance generating function

$$\omega(x) := \sum_{i=1}^{n} \frac{x_i}{r} \ln \frac{x_i}{r}, \tag{4.3}$$

coupled with the $\|\cdot\|_1$ norm is employed. Note that here $X^o = \{x \in X : x > 0\}$ and for $n \ge 3$,

$$D^2_{\omega,X} = \max_{x\in X} \omega(x) - \min_{x\in X} \omega(x) \le \max_{x\in X'} \omega(x) - \min_{x\in X'} \omega(x) \le \ln n.$$

Also observe that for any $x \in X'$, $x > 0$, and $h \in \mathbb{R}^n$,

$$\begin{aligned}\left(\sum_{i=1}^{n} |h_i|\right)^2 &= \left(\sum_{i=1}^{n} x_i^{1/2} |h_i| x_i^{-1/2}\right)^2 \le \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} h_i^2 x_i^{-1}\right)\\ &\le r\left(\sum_{i=1}^{n} h_i^2 x_i^{-1}\right) = r^2 h^T \nabla^2 \omega(x) h,\end{aligned}$$

where the first inequality follows by Cauchy's inequality. Therefore the modulus of ω, with respect to the $\|\cdot\|_1$ norm, satisfies $\alpha \ge r^{-2}$. Note that here $D_{\omega,X}$ can be overestimated while α being underestimated since $X \subseteq X'$, therefore, the stepsizes computed according to (2.14) in view of these estimates may not be optimal. Of course, the quantity $D_{\omega,X}$ can be estimated more accurately, for example, by computing $\min_{x\in X}\omega(x)$ explicitly. We will also discuss a few different ways to fine-tune the stepsizes in Sect. 5.

For the entropy distance generating function (4.3), the prox-mapping $P_v(z)$ (defined in (2.4)) is r times the optimal solution to the optimization problem

$$\begin{aligned} \min_x \; & \sum_{i=1}^n (s_i x_i + x_i \ln x_i), \\ \text{s.t.} \; & \sum_{i=1}^n x_i \le 1, \\ & \tilde{l}_i \le x_i \le \tilde{u}_i, \; i = 1, \ldots, n, \end{aligned} \tag{4.4}$$

where $s_i = rz_i - \ln(v_i/r) - 1$, $\tilde{l}_i = l_i/r$, $\tilde{u}_i = u_i/r$.

In some cases problem (4.4) has an explicit solution, e.g., if $l_i = 0$ and $u_i \ge r$, $i = 1, \ldots, n$ (in that case the constraints $z_i \le u_i$ are redundant). In general, we can solve (4.4) as follows. Let $\lambda \ge 0$ denote the Lagrange multiplier associated with the constraint $\sum_{i=1}^n x_i \le 1$ and consider the corresponding Lagrangian relaxation of (4.4):

$$\begin{aligned} \min_x \; & \sum_{i=1}^n (s_i x_i + x_i \ln x_i) + \lambda \left(\textstyle\sum_{i=1}^n x_i\right), \\ \text{s.t.} \; & \tilde{l}_i \le x_i \le \tilde{u}_i, \; i = 1, \ldots, n. \end{aligned} \tag{4.5}$$

This is a separable problem. Since $s_i x_i + x_i \ln x_i + \lambda x_i$ is monotonically decreasing for x_i less than $\exp[-(s_i + 1 + \lambda)]$ and is monotonically increasing after, we have that the i-th coordinate $\bar{x}_i(\lambda)$ of the optimal solution of (4.5) is given by the projection of $\exp[-(s_i + 1 + \lambda)]$ onto the interval $[\tilde{l}_i, \tilde{u}_i]$. Then, to solve problem (4.4) is equivalent to find $\lambda \ge 0$ such that

$$\sum_{i=1}^n \bar{x}_i(\lambda) = 1, \quad \text{if } \lambda > 0, \tag{4.6}$$

$$\sum_{i=1}^n \bar{x}_i(\lambda) \le 1, \quad \text{if } \lambda = 0. \tag{4.7}$$

While inequality (4.7) can be easily checked, the root-finding problem (4.6) is usually solved to certain precision by using bisection, and each bisection step requires $\mathcal{O}(n)$ operations.

4.1.2 Euclidean SA

In the E-SA approach to order to solve the EU model, the Euclidean distance generating function $\omega(x) := \frac{1}{2}x^T x$, coupled with the $\|\cdot\|_2$ norm is employed. Clearly here $X^o = X$ and $\alpha = 1$. We have

$$D^2_{\omega,X} = \max_{x\in X} \omega(x) - \min_{x\in X} \omega(x) \le \tfrac{1}{2}\left(\min\{r^2, \|u\|_2^2\} - \|l\|_2^2\right).$$

Moreover a procedure similar to the one given in Subsect. 4.1.1 can be developed for computing the prox mapping $P_x(y)$, which is given here by the metric projection $\Pi_X(x-y)$.

As it was noted in [9, Example 2.1], if X is a standard simplex, N-SA can be potentially $\mathcal{O}(\sqrt{n/\log n})$ times faster than E-SA. The same conclusion seems to be applicable to our current situation, although certain caution should be taken since the error estimate (2.14) now also depends on l, u and r.

4.2 Minimizing the conditional value-at-risk

The idea of minimizing CVaR in place of Value-at-Risk (VaR) is due to Rockafellar and Uryasev [14]. Recall that VaR and CVaR of a random variable Z are defined as

$$\mathrm{VaR}_{1-\beta}(Z) := \inf\{\tau : \mathrm{Prob}(Z \le \tau) \ge 1-\beta\}, \tag{4.8}$$

$$\mathrm{CVaR}_{1-\beta}(Z) := \inf_{\tau\in\mathbb{R}} \left\{\tau + \beta^{-1}\mathbb{E}[Z-\tau]_+\right\}. \tag{4.9}$$

Note that $\mathrm{VaR}_{1-\beta}(Z) \in \mathrm{Argmin}_{\tau\in\mathbb{R}}\left\{\tau + \beta^{-1}\mathbb{E}[Z-\tau]_+\right\}$, and hence

$$\mathrm{VaR}_{1-\beta}(Z) \le \mathrm{CVaR}_{1-\beta}(Z). \tag{4.10}$$

The problem of interest in this subsection is:

$$\min_{y\in Y} \mathrm{CVaR}_{1-\beta}\left(-\xi^T y\right), \tag{4.11}$$

where ξ is a random vector with mean $\bar{\xi} := \mathbb{E}[\xi]$ and covariance matrix Σ, and

$$Y := \left\{ y \in \mathbb{R}^n_+ : \sum_{i=1}^n y_i = 1,\ \bar{\xi}^T y \ge R \right\}.$$

We assume that Y is nonempty and, moreover, contains a positive point. For simplicity we assume in the remaining part of the paper that ξ has continuous distribution, and hence $\xi^T y$ has continuous distribution for any $y \in Y$.

In view of the definition of CVaR in (4.9), our problem becomes:

$$\min_{x\in X} f(x) := \tau + \frac{1}{\beta}\mathbb{E}\left\{[-\xi^T y - \tau]_+\right\}, \tag{4.12}$$

where $X := Y \times \mathbb{R}$ and $x := (y, \tau)$. Apparently, there exists one difficulty to apply the Mirror Descent SA for solving the above problem—in (4.12), the variables are y *and* τ, so that the feasible domain $Y \times \mathbb{R}$ of the problem is unbounded, while our Mirror Descent SA requires a bounded feasible domain. However, we will alleviate this problem by showing that the variable τ can actually be restricted into a bounded interval and thus the Mirror Descent SA method can be applied.

Noting that $\mathrm{VaR}_{1-\beta}(Z) \in \mathrm{Argmin}_{\tau\in\mathbb{R}}\left[\tau + \beta^{-1}\mathbb{E}\{[Z-\tau]_+\}\right]$, all we need is to find an interval which covers all points $\mathrm{VaR}_{1-\beta}(-\xi^T y)$, $y \in Y$. Now, let Z be a random variable with finite mean μ and variance σ^2. By Cantelli's inequality (also called the one-sided Tschebyshev inequality) we have

$$\mathrm{Prob}\{Z \geq t) \leq \frac{\sigma^2}{(t-\mu)^2 + \sigma^2}.$$

Assuming that Z has continuous distribution, we obtain

$$\beta = \mathrm{Prob}\{Z \geq \mathrm{VaR}_{1-\beta}(Z)\} \leq \frac{\sigma^2}{[\mathrm{VaR}_{1-\beta}(Z) - \mu]^2 + \sigma^2},$$

which implies that

$$\mathrm{VaR}_{1-\beta}(Z) \leq \mu + \sqrt{\frac{1-\beta}{\beta}}\sigma. \tag{4.13}$$

Similarly, if $\mathrm{VaR}_{1-\beta}(Z) \leq \mu$, then

$$1-\beta = \mathrm{Prob}\{-Z \geq -\mathrm{VaR}_{1-\beta}(Z)\} \leq \frac{\sigma^2}{[-\mathrm{VaR}_{1-\beta}(Z) + \mu]^2 + \sigma^2},$$

which implies that

$$\mathrm{VaR}_{1-\beta}(Z) \geq \mu - \sqrt{\frac{\beta}{1-\beta}}\sigma. \tag{4.14}$$

Combining inequality (4.13) and (4.14) we obtain

$$\mathrm{VaR}_{1-\beta}(Z) \in \left[\mu - \sqrt{\tfrac{\beta}{1-\beta}}\sigma,\ \mu + \sqrt{\tfrac{1-\beta}{\beta}}\sigma\right]. \tag{4.15}$$

Note also that if Z is symmetric and $\beta \leq 0.5$, then the previous inclusion can be strengthened to

$$\mathrm{VaR}_{1-\beta}(Z) \in \left[\mu,\ \mu + \sqrt{\tfrac{1-\beta}{\beta}}\sigma\right]. \tag{4.16}$$

From this analysis it clearly follows that we lose nothing when restricting τ in (4.12) to vary in the segment

$$\tau \in \mathcal{T} := \left[\underline{\mu} - \sqrt{\tfrac{\beta}{1-\beta}}\overline{\sigma},\ \overline{\mu} + \sqrt{\tfrac{1-\beta}{\beta}}\overline{\sigma}\right], \tag{4.17}$$

where

$$\underline{\mu} := \min_{y \in Y}\{-\bar{\xi}^T y\},\ \ \overline{\mu} := \max_{y \in Y}\{-\bar{\xi}^T y\},\ \ \overline{\sigma}^2 := \max_{y \in Y} y^T \Sigma\, y. \tag{4.18}$$

In the case when ξ is symmetric and $\beta \le 0.5$, this segment can be can be further reduced to:

$$\tau \in \mathcal{T}' := \left[\underline{\mu},\ \overline{\mu} + \sqrt{\tfrac{1-\beta}{\beta}}\overline{\sigma}\right]. \tag{4.19}$$

Note that the quantities $\underline{\mu}$ and $\overline{\mu}$ can be easily computed by solving the corresponding linear programs in (4.18). Moreover, although $\overline{\sigma}$ can be difficult to compute exactly, it can be replaced with its easily computable upper bound $\max_i \Sigma_{ii}$.

It is worth noting that an alternative upper bound for τ can be obtained in some cases: given an initial point $y_0 \in Y$, we have

$$\mathrm{CVaR}_{1-\beta}(-\xi^T y_0) \ge \mathrm{CVaR}_{1-\beta}(-\xi^T y^*) \ge \mathrm{VaR}_{1-\beta}(-\xi^T y^*),$$

where y^* is an optimal solution of problem (4.11) and the second inequality follows from (4.10). Therefore, if the value of $\mathrm{CVaR}_{1-\beta}(-\xi^T y_0)$ can be computed or estimated (e.g., by Monte-Carlo simulation), we can restrict the variable τ in (4.12) to be $\le \mathrm{CVaR}_{1-\beta}(-\xi^T y_0)$.

To apply the Mirror Descent SA to problem (4.11), we set $X = Y \times \mathcal{T}$ and define the stochastic oracle by setting

$$\begin{aligned} F(x,\xi) &\equiv F(y,\tau,\xi) = \tau + \tfrac{1}{\beta}\max[-\xi^T y - \tau, 0],\\ \mathsf{G}(x,\xi) &\equiv [\mathsf{G}_y(y,\tau,\xi); \mathsf{G}_\tau(y,\tau,\xi)] = \begin{cases} [-\beta^{-1}\xi; 1-\beta^{-1}], & -\xi^T y - \tau > 0\\ [0;\ldots;0;1], & \text{otherwise}\end{cases}\end{aligned}$$

Further, we choose D_y and D_τ from the relations

$$D_y \ge \max\left[1/2, \sqrt{\max_{y\in Y}\sum_i y_i \ln y_i - \min_{y\in Y}\sum_i y_i \ln y_i}\right],\ \ D_\tau = \frac{1}{2}\left[\max_{\tau\in\mathcal{T}}\tau^2 - \min_{\tau\in\mathcal{T}}\tau^2\right]$$

(we always can take $D_y = \max[1/2, \sqrt{\ln(n)}]$) and equip X and its embedding space $\mathbb{R}^n_y \times \mathbb{R}_\tau \supset X$ with the distance generating function and the norm as follows:

$$\|(y,\tau)\| = \sqrt{\|y\|_1^2/(2D_y^2) + \tau^2/(2D_\tau^2)} \qquad \left[\Leftrightarrow \|(z,\rho)\|_* = \sqrt{2D_y^2\|z\|_\infty^2 + 2D_\tau^2\rho^2}\right]$$

$$\omega(x) \equiv \omega(y,\tau) = \frac{1}{2D_y^2}\sum_{i=1}^n y_i \ln y_i + \frac{1}{2D_\tau^2}\tau^2$$

Note that with this setup, $X^o = \{(y,\tau) \in X : y > 0\}$. Besides this, it is easily seen that $\sum_{i=1}^n y_i \ln y_i$, restricted on Y, is strongly convex, modulus 1, w.r.t. $\|\cdot\|_1$, whence

ω is strongly convex, modulus $\alpha = 1$, on X. An immediate computation shows that $D_{\omega,X} = 1$, and therefore $\Lambda_{\omega,X} = \sqrt{2}$. Finally, we set

$$M_* = \sqrt{2D_y^2\beta^{-2}\mathbb{E}\left[\|\xi\|_\infty^2\right] + 2D_\tau^2 \max[1, (\beta^{-1} - 1)^2]}. \tag{4.20}$$

It is easy to verify that with this M_*, our stochastic oracle satisfies (2.9).

Indeed, from the formula for $\mathsf{G}(x, \xi)$ we have

$$\mathbb{E}\left[\|\mathsf{G}(x,\xi)\|_*^2\right] = \mathbb{E}\left[2D_y^2\beta^{-2}\|\xi\|_\infty^2 + 2D_\tau^2 \max[1, \beta^{-1} - 1]^2\right] = M_*^2,$$

as required in (2.9). Further, for $x \in X$ we have $|F(x,\xi) - \tau - \beta^{-1}\max[-\tau, 0]| \leq \beta^{-1}|\xi^T y| \leq \beta^{-1}|\xi\|_\infty$, whence

$$\begin{aligned}\mathbb{E}[(F(x,\xi) - f(x))^2] &= \mathbb{E}[(F(x,\xi) - \mathbb{E}[F(x,\xi)])^2]\\ &\leq \mathbb{E}[(F(x,\xi) - \tau - \beta^{-1}\max[-\tau, 0])^2]\\ &\leq \beta^{-2}\mathbb{E}[\|\xi\|_\infty^2] \leq \Lambda_{\omega,X}^2 M_*^2,\end{aligned}$$

where the concluding inequality is due to $D_y \geq 1/2$ and $\Lambda_{\omega,X} = \sqrt{2}$. We see that assumption (A1) is satisfied with M_* given by (4.20) and $Q = \Lambda_{\omega,X}M_* = \sqrt{2}M_*$.

5 Numerical results

5.1 More implementation details

- **Fine-tuning the stepsizes:** In Sect. 2, we specified the constant stepsize policy for the Mirror Descent SA method up to the "scaling parameter" θ. In our experiments, this parameter was chosen as a result of pilot runs of the Mirror Descent SA algorithm with several trial values of θ and a very small sample size N (namely, $N = 100$). From these values of θ, we chose for the actual run the one resulting in the smallest online upper bound $\overline{f}^N$ on the optimal value.
- **Bundle-level method for solving SAA problem:** We also compare the results obtained by the Mirror Descent SA method with those obtained by the SAA coupled with the bundle-level method (SAA-BL) [4]. Note that the SAA problem is to be solved by the Bundle-level method; in our experiments, the SAA problems were solved within relative accuracy 1.e-4 through 1.e-6, depending on the instance.

5.2 Computational results for the EU model

In our experiments, we fix $l_i = 0$ and $u_i = u$ for all $1 \leq i \leq n$. The experiments were conducted for ten random instances which have the same dimension $n = 1000$ but differ in the parameters u and r, and the function $\phi(\cdot)$. A detailed description of these

Table 1 The test instances for EU model

Name	r	u	Name	r	u
EU-1	100	0.05	EU-6	1	$+\infty$
EU-2	100	0.20	EU-7	10	$+\infty$
EU-3	100	0.40	EU-8	100	$+\infty$
EU-4	100	10.00	EU-9	1,000	$+\infty$
EU-5	100	50.00	EU-10	5,000	$+\infty$

Table 2 The stepsize factors

Name	Best θ	Inferred θ	Name	Best θ	Inferred θ
EU-1	0.005	0.005	EU-6	5.000	5.000
EU-2	1.000	5.000	EU-7	10.000	10.000
EU-3	1.000	5.000	EU-8	10.000	10.000
EU-4	5.000	10.000	EU-9	10.000	10.000
EU-5	5.000	5.000	EU-10	5.000	5.000

Table 3 Changing u

Name	N-SA ($\hat{f}(x_*)/f(x_*)$)	E-SA ($\hat{f}(x_*)/f(x_*)$)	SAA ($\hat{f}(x_*)/f(x_*)$)	Opt
EU-1	−19.3558/−19.3279	−19.1311/−19.0953	−19.2700/−19.2435	−19.3307
EU-2	−61.4004/−61.3332	−61.7670/−61.6979	−62.8794/−62.7962	−62.9636
EU-3	−81.5215/−81.4339	−80.5735/−80.4873	−83.0845/−82.9732	−83.2145
EU-4	−100.1597/−99.6734	−92.1313/−92.0161	−99.3096/−99.0400	−102.6819
EU-5	−99.5680/−99.2872	−91.2051/−91.0923	−98.5458/−98.2697	−101.9112

instances is shown in Table 1. Observe that for the first five instances, we fix $r = 100$ but change u from 0.05 to 50. For the next five instances, we assume $u = +\infty$ but change r from 1.0 to $5,000.0$.

Here we highlight some interesting findings based on our computational results. More numerical results can be found the end of this paper.

– **The effect of stepsize factor θ:** Our first test is to verify that we can fine-tune the stepsizes by using a small pilot. In this test, we chose between eight different stepsize factors, namely, 0.005, 0.01, 0.05, 0.1, 0.5, 1.0, 5, 10 for both N-SA and E-SA. First, we used short pilot runs ($M = 100$) to select the "most promising" value of the stepsize factor θ, see the beginning of Sect. 5.1. Second, we directly tested which one of the outlined eight values of θ results in the highest quality solution for the sample size $N = 2,000$. The results are presented in the columns "Inferred θ," resp., "Best θ," of Table 2. As we can see from this table, the inferred θ's are very close to the best ones for all test instances and the same conclusion also holds for the E-SA.
– **The effect of changing u:** In Table 3, we report the objective values of EU-1–EU-5 evaluated at the solutions obtained by N-SA, E-SA and SAA when the sample size

Table 4 Changing r

Name	N-SA $(\hat{f}(x_*)/f(x_*))$	E-SA $(\hat{f}(x_*)/f(x_*))$	SAA $(\hat{f}(x_*)/f(x_*))$	Opt
EU-6	−6.2999/−6.2864	−6.2211/−6.2186	−6.3073/ −6.3027	−6.3460
EU-7	−16.2514/−16.2294	−15.3818/−15.3717	−16.1474/−16.1226	−16.4738
EU-8	−97.3613/−97.1581	−89.2032/−89.0897	−96.5163/−96.2450	−99.8824
EU-9	−9.540e+2/−9.513e+2	−8.686e+2/−8.675e+2	−9.419e+2/−9.393e+2	−9.757e+2
EU-10	−4.730e+3/−4.717e+3	−4.322e+3/−4.316e+3	−4.689e+3/−4.675e+3	−4.857e+3

is $N = 2,000$. In this table, $\hat{f}(x_*)$ denotes the estimated objective value (using sample size $K = 10,000$) at the obtained solution x_*. Due to the assumption that ξ is normally distributed, the actual objective value $f(x_*)$ can be also computed. Moreover, a close examination reveals that the optimal value of problem (4.1) can be computed efficiently (see [9]); it is shown in the last column of Table 3.
One interesting observation from this table is that the performance of N-SA is slightly better than that of E-SA even for EU-1 whose feasible region is actually a box instead of a simplex, so that there are no theoretical reasons to prefer N-SA to E-SA.
One other observation from this table is that the solution quality of N-SA significantly outperforms that of E-SA for the two largest values of u. The possible explanation is that the feasible region appears more like a simplex when u is big.

- **The effect of changing** r**:** Table 4 shows the objective values of EU-6 to EU-10 evaluated at the solutions obtained by N-SA, E-SA and SAA when the sample size is $N = 2,000$. In this table, $\hat{f}(x_*)$ and $f(x_*)$, respectively, denote the estimated objective value (using sample size $K = 10,000$) and the actual objective value at the obtained solution x_*, and "opt" denotes the optimal value of problem (4.1).
Recall that the feasible regions for these five instances are simplices. So, as expected, N-SA consistently outperforms E-SA for all these instances. It is interesting to observe that the objective values achieved by N-SA can be smaller than those by SAA for large r. Note that the SAA problem has been solved to a relatively high accuracy by using the Bundle-level method. For example, for EU-10, the SAA problem was solved to accuracy 0.7e-005.
- **The lower bounds:** Table 5 shows the lower bounds on the objective values of EU-1 to EU-10 obtained by N-SA, E-SA and SAA when the sample size is $N = 2,000$. In Table 5, the lower bounds $\underline{f}^N$ and $\underline{lb}^N$ are the online and offline bounds defined in Sect. 3. The lower bound for SAA is defined as the optimal value of the corresponding SAA problem. As we can see from this table, the lower bound for SAA is always better than the online lower bound $\underline{f}^N$ for the SA methods (as it should be in the case of constant stepsizes, see Remark 1). However, the offline lower bound $\underline{lb}^N$ can be close or even better than the lower bound obtained from SAA.
Moreover, we estimate the variability of the online lower bounds in the way discussed in Sect. 3.1 and the results are reported in Table 6. In particular, the second and third column of this table show the mean and the standard deviation obtained from $M = 10$ independent replications of N-SA, each of which has the same

Table 5 Lower bounds on optimal values and true optimal values

Name	N-SA		E-SA		SAA	Opt
	$\underline{f}^N$	$\underline{lb}^N$	$\underline{f}^N$	$\underline{lb}^N$	$\hat{f}^N_{\mathrm{SAA}}$	
EU-1	−19.4063	−19.2994	−19.4063	−19.2994	−19.4063	−19.3307
EU-2	−62.9984	−62.8754	−62.9984	−62.8758	−62.9984	−62.9367
EU-3	−83.0039	−82.9730	−83.0039	−82.9730	−83.0039	−83.2145
EU-4	−107.5820	−104.5046	−107.2058	−104.4072	−105.0890	−102.6819
EU-5	−107.5745	−104.0644	−108.4063	−104.3577	−104.3214	−101.9112
EU-6	−6.6111	−6.5288	−6.9171	−6.5849	−6.3658	−6.3460
EU-7	−17.0130	−16.7060	−17.1800	−16.7605	−16.7027	−16.4378
EU-8	−106.7958	−102.6311	−106.5921	−102.2588	−102.2914	−99.8824
EU-9	−1029.0530	−997.7217	−1042.7008	−1000.6626	−999.9114	−9.757e2
EU-10	−5192.0409	−4967.9144	−5192.0409	−4981.8515	−4978.2333	−4.857e3

sample size $N = 1000$. The third and fourth column show the mean and standard deviation computed for the lower bounds associated with the $M = 10$ consecutive partitions of the trajectory of N-SA with a sample size $NM = 10,000$. The last column reports the online lower bound $\underline{f}^{NM}$. The results indicate that the bounds obtained from independent replications have relatively smaller variability in general.

– **The computation times:** For all instances, the computation times of generating a solution for SA were $10 - 30$ times smaller than that for SAA.
– **The standard deviations:** For the generated solution x_*, we evaluate the corresponding objective value $f(x_*)$ by generating an independent large sample $\xi_1, \ldots, \xi_K$, of size $K = 10,000$, and computing the estimate $\hat{f}(x_*) = K^{-1} \sum_{j=1}^{K} F(x_*, \xi_j)$ of $f(x_*)$. We also computed an estimate of the standard deviation of $F(x_*, \xi)$:

$$\hat{\sigma} = \sqrt{\sum_{j=1}^{K} \left(F(x_*, \xi_j) - \hat{f}(x_*) \right)^2 / (K-1)}.$$

Note that the standard deviation of $\hat{f}(x_*)$, as an estimate of $f(x_*)$, is estimated by $\frac{\hat{\sigma}}{\sqrt{K}}$. Table 7 compares the deviations for N-SA and SAA computed in the above way. From this table, we observe that for instances with either a larger u or larger r, the values of $\hat{\sigma}$ corresponding to the solutions obtained by N-SA can be significantly smaller than those by SAA. One possible explanation is that, if the true problem has a large set of optimal (nearly optimal) solutions (which is typical for high dimensional problems), the solutions produced by the mirror-descent SA method tend to have less variability. Indeed, after a closer examination, we observe that the solutions computed by the mirror descent SA algorithm typically have a larger number of non-zero entries than those computed by the SAA approach, possibly due to the averaging operation (See Columns 4 and 7 in Table 7). As a result,

Table 6 Variability of the lower bounds for N-SA

Name	Ind. repl.		Dep. repl.		Whole Traj.
	Mean	Deviation	Mean	Deviation	$\underline{f}^{NM}$
EU-1	−19.5681	0.0857	−19.5387	0.0842	−19.3461
EU-2	−63.3898	0.2372	−63.3786	0.3502	−63.0444
EU-3	−83.6973	0.3121	−83.7339	0.3098	−83.2649
EU-4	−112.2483	1.5616	−114.1652	2.7470	−105.5543
EU-5	−113.7526	1.5951	−115.3103	2.8232	−104.4565
EU-6	−6.7812	0.0265	−6.8969	0.1374	−6.4522
EU-7	−17.7911	0.2326	−18.3881	0.5519	−16.8022
EU-8	−113.5263	2.1348	−117.4176	4.6588	−102.3509
EU-9	−1091.2836	20.2804	−1140.23774	61.1979	−1006.1846
EU-10	−5466.1266	124.5894	−5553.80221	144.6298	−5048.5643

Table 7 Standard deviations

Name	N-SA			SAA		
	$\hat{f}(x_*)$	$\hat{\sigma}$	NNZ	$\hat{f}(x_*)$	$\hat{\sigma}$	NNZ
EU-1	−19.3558	3.1487	1000	−19.2700	3.0019	910
EU-2	−61.4004	8.4178	893	−62.8749	8.9099	501
EU-3	−81.5215	11.7493	447	−83.0845	12.6015	251
EU-4	−100.1597	38.6309	179	−99.3096	61.1053	31
EU-5	−99.5680	35.1278	447	−98.5458	60.8440	31
EU-6	−6.2999	0.6798	303	−6.3073	0.7030	107
EU-7	−16.2514	3.5233	254	−16.1474	5.7941	33
EU-8	−97.3613	36.3939	280	−96.5163	61.0974	31
EU-9	−953.9882	383.8223	318	−941.9854	611.0414	31
EU-10	−4729.8534	1746.7144	788	−4688.9239	3053.7409	31

the mirror descent SA generates more diversified portfolios which are known to be more robust against uncertainty.

5.3 Computational results for the CVaR model

In this subsection, we report some numerical results on applying the Mirror Descent SA method for the CVaR model (4.11). Here the return ξ is assumed to be a normal random vector. In that case random variable $-\xi^T y$ has normal distribution with mean $-\bar{\xi}^T y$ and variance $y^T \Sigma y$, and

$$\mathrm{CVaR}_{1-\beta}\{-\xi^T y\} = -\bar{\xi}^T y + \rho\sqrt{y^T \Sigma y}, \tag{5.1}$$

Table 8 The test instances for CVaR model

Name	n	β	R	Opt
CVaR-1	95	0.05	1.0000	−0.9841
CVaR-2	1,000	0.10	1.0500	1.5272

Table 9 Comparing SA and SAA for the CVaR model

Name	N	SA					SAA			
		$\hat{f}(x_*)$	$f(x_*)$	$\underline{f}_N$	$\underline{lb}^N$	Time	$\hat{f}(x_*)$	$f(x_*)$	$\hat{f}^N_{\text{SAA}}$	Time
CVaR-1	1000	−0.9807	−0.9823	−1.0695	−1.0136	0	−0.9823	−0.9828	−0.9854	15
	2000	−0.9824	−0.9832	−1.0518	−0.9877	1	−0.9832	−0.9835	−0.9852	27
CVaR-2	1000	1.6048	1.5896	1.1301	1.4590	20	1.6396	1.5795	1.3023	928
	2000	1.5766	1.5633	1.3696	1.4973	39	1.5835	1.5557	1.4780	2784

where $\rho := \frac{\exp(-z_\beta^2/2)}{\beta\sqrt{2\pi}}$ and $z_\beta := \Phi^{-1}(1-\beta)$ with $\Phi(\cdot)$ being the cdf of the standard normal distribution. Consequently the optimal solution for (4.11) can be easily obtained by replacing the objective function of (4.11) with the right hand side of (5.1). Clearly, the resulting problem can be reformulated as a conic-quadratic programming program, and its optimal value thus gives us a benchmark to compare the SA and SAA methods.

Two instances for the CVaR model are considered in our experiments. The first instance (CVaR-1) is obtained from [18]. This instance consists of the 95 stocks from S&P100 (excluding SBC, ATI, GS, LU, and VIA-B) and the mean $\bar{\xi}$ and covariance Σ_ξ were estimated using historical monthly prices from 1996 to 2002. The second one (CVaR-2), which contains $1,000$ assets, was randomly generated by setting the random return $\xi = \bar{\xi} + Q\zeta$, where ζ is the standard Gaussian vector, $\bar{\xi}_i$ is uniformly distributed in [0.9, 1.2], and Q_{ij} is uniformly distributed in [0, 0.1] for $1 \le i, j \le 1,000$. The reliability level β, the bound for expected return R, and the optimal value for these two instances are reported in Table 8.

The computational results for the CVaR model are reported in Table 9, where $\hat{f}(x_*)$ and $f(x_*)$, respectively, denote the estimated objective value (using sample size $K = 10,000$) and the actual objective value at the obtained solution x_*. We conclude from the results in Table 9 that the Mirror Descent SA method can generate good solutions much faster than SAA. The lower bounds derived for the SA method are also comparable to those for the SAA method.

Appendix

We will need the following result (cf., [9, Lemma 6.1]).

Lemma 1 *Let $\zeta_t \in \mathbb{R}^n$, $v_1 \in X^o$ and $v_{t+1} = P_{v_t}(\zeta_t)$, $t = 1, \ldots, N$. Then*

$$\sum_{t=1}^{N} \zeta_t^T (v_t - u) \le V(v_1, u) + (2\alpha)^{-1} \sum_{t=1}^{N} \|\zeta_t\|_*^2, \quad \forall u \in X. \tag{5.1}$$

We denote here $\delta_t := F(x_t, \xi_t) - f(x_t)$ and $\Delta_t := \mathsf{G}(x_t, \xi_t) - \mathsf{g}(x_t)$. Since x_t is a function of ξ^{t-1} and ξ_t is independent of ξ^{t-1}, we have that the conditional expectations

$$\mathbb{E}_{|t-1}[\delta_t] = 0 \text{ and } \mathbb{E}_{|t-1}[\Delta_t] = 0, \tag{5.2}$$

and hence the unconditional expectations $\mathbb{E}[\delta_t] = 0$ and $\mathbb{E}[\Delta_t] = 0$ as well.

Part (i) of Theorem 1: Proof

Proof of (3.5). If in Lemma 1 we take $v_1 := x_1$ and $\zeta_t := \gamma_t \mathsf{G}(x_t, \xi_t)$, then the corresponding iterates v_t coincide with x_t. Therefore, we have by (5.1) and since $V(x_1, u) \leq D^2_{\omega,X}$ that

$$\sum_{t=1}^{N} \gamma_t (x_t - u)^T \mathsf{G}(x_t, \xi_t) \leq D^2_{\omega,X} + (2\alpha)^{-1} \sum_{t=1}^{N} \gamma_t^2 \|\mathsf{G}(x_t, \xi_t)\|_*^2, \quad \forall u \in X. \tag{5.3}$$

It follows that for any $u \in X$:

$$\begin{aligned} &\sum_{t=1}^{N} \nu_t \left[-f(x_t) + (x_t - u)^T \mathsf{g}(x_t) \right] + \sum_{t=1}^{N} \nu_t f(x_t) \\ &\leq \frac{D^2_{\omega,X} + (2\alpha)^{-1} \sum_{t=1}^{N} \gamma_t^2 \|\mathsf{G}(x_t, \xi_t)\|_*^2}{\sum_{t=1}^{N} \gamma_t} + \sum_{t=1}^{N} \nu_t \Delta_t^T (x_t - u). \end{aligned}$$

Since

$$f^{*N} - f_*^N = \sum_{t=1}^{N} \nu_t f(x_t) + \max_{u \in X} \sum_{t=1}^{N} \nu_t \left[-f(x_t) + (x_t - u)^T \mathsf{g}(x_t) \right],$$

it follows that

$$f^{*N} - f_*^N \leq \frac{D^2_{\omega,X} + (2\alpha)^{-1} \sum_{t=1}^{N} \gamma_t^2 \|\mathsf{G}(x_t, \xi_t)\|_*^2}{\sum_{t=1}^{N} \gamma_t} + \max_{u \in X} \sum_{t=1}^{N} \nu_t \Delta_t^T (x_t - u). \tag{5.4}$$

Let us estimate the second term in the right hand side of (5.4). Let

$$\begin{aligned} u_1 = v_1 = x_1; u_{t+1} = P_{u_t}(-\gamma_t \Delta_t),\ t = 1, 2, \ldots, N; v_{t+1} \\ = P_{v_t}(\gamma_t \Delta_t),\ t = 1, 2, \ldots N. \end{aligned} \tag{5.5}$$

Observe that Δ_t is a deterministic function of ξ^t, whence u_t and v_t are deterministic functions of ξ^{t-1}. By using Lemma 1 we obtain

$$\sum_{t=1}^{N} \gamma_t \Delta_t^T (v_t - u) \le D_{\omega,X}^2 + (2\alpha)^{-1} \sum_{t=1}^{N} \gamma_t^2 \|\Delta_t\|_*^2, \quad \forall u \in X. \tag{5.6}$$

Moreover,

$$\Delta_t^T (v_t - u) = \Delta_t^T (x_t - u) + \Delta_t^T (v_t - x_t),$$

and hence it follows by (5.6) that

$$\max_{u \in X} \sum_{t=1}^{N} \nu_t \Delta_t^T (x_t - u) \le \sum_{t=1}^{N} \nu_t \Delta_t^T (x_t - v_t) + \frac{D_{\omega,X}^2 + (2\alpha)^{-1} \sum_{t=1}^{N} \gamma_t^2 \|\Delta_t\|_*^2}{\sum_{t=1}^{N} \gamma_t}. \tag{5.7}$$

Observe that by similar reasoning applied to $-\Delta_t$ in the role of Δ_t we get

$$\max_{u \in X} \left[-\sum_{t=1}^{N} \nu_t \Delta_t^T (x_t - u) \right] \le \left[-\sum_{t=1}^{N} \nu_t \Delta_t^T (x_t - u_t) \right] + \frac{D_{\omega,X}^2 + (2\alpha)^{-1} \sum_{t=1}^{N} \gamma_t^2 \|\Delta_t\|_*^2}{\sum_{t=1}^{N} \gamma_t}. \tag{5.8}$$

Moreover, $\mathbb{E}_{|t-1}[\Delta_t] = 0$ and u_t, v_t and x_t are functions of ξ^{t-1}, while $\mathbb{E}_{|t-1}\Delta_t = 0$ and hence

$$\mathbb{E}_{|t-1}\left[(x_t - v_t)^T \Delta_t \right] = \mathbb{E}_{|t-1}\left[(x_t - u_t)^T \Delta_t \right] = 0. \tag{5.9}$$

We also have that $\mathbb{E}_{|t-1}\left[\|\Delta_t\|_*^2\right] \le 4M_*^2$, and hence in view of condition (2.9) it follows from (5.7) and (5.9) that

$$\mathbb{E}\left[\max_{u \in X} \sum_{t=1}^{N} \nu_t \Delta_t^T (x_t - u) \right] \le \frac{D_{\omega,X}^2 + 2\alpha^{-1} M_*^2 \sum_{t=1}^{N} \gamma_t^2}{\sum_{t=1}^{N} \gamma_t}. \tag{5.10}$$

Therefore, by taking expectation of both sides of (5.4) and using (2.9) together with (5.10) we obtain the estimate (3.5).

Proof of (3.6). In order to prove (3.6) let us observe that $\overline{f}^N - f^{*N} = \sum_{t=1}^{N} \nu_t \delta_t$, and that for $1 \le s < t \le N$,

$$\mathbb{E}[\delta_s \delta_t] = \mathbb{E}\{\mathbb{E}_{|t-1}[\delta_s \delta_t]\} = \mathbb{E}\{\delta_s \mathbb{E}_{|t-1}[\delta_t]\} = 0.$$

Therefore

$$\mathbb{E}\left[\left(\overline{f}^N - f^{*N}\right)^2\right] = \mathbb{E}\left[\left(\sum_{t=1}^N \nu_t \delta_t\right)^2\right] = \sum_{t=1}^N \nu_t^2 \mathbb{E}\left[\delta_t^2\right] = \sum_{t=1}^N \nu_t^2 \mathbb{E}\left\{\mathbb{E}_{|t-1}\left[\delta_t^2\right]\right\}.$$

Moreover, by condition (2.8) of assumption (A1) we have that $\mathbb{E}_{|t-1}\left[\delta_t^2\right] \le Q^2$, and hence

$$\mathbb{E}\left[\left(\overline{f}^N - f^{*N}\right)^2\right] \le Q^2 \sum_{t=1}^N \nu_t^2. \tag{5.11}$$

Since $\sqrt{\mathbb{E}[Y^2]} \ge \mathbb{E}|Y|$ for any random variable Y, inequality (3.6) follows from (5.11).

Proof of (3.7). Let us now look at (3.7). We have

$$\begin{aligned} \left|\underline{f}^N - f_*^N\right| &= \left|\min_{x\in X} \hat{f}^N(x) - \min_{x\in X} f^N(x)\right| \le \max_{x\in X}\left|\hat{f}^N(x) - f^N(x)\right| \\ &\le \left|\sum_{t=1}^N \nu_t \delta_t\right| + \max_{x\in X}\left|\sum_{t=1}^N \nu_t \Delta_t^T(x_t - x)\right|. \end{aligned} \tag{5.12}$$

We already showed above (see (5.11)) that

$$\mathbb{E}\left[\left|\sum_{t=1}^N \nu_t \delta_t\right|\right] \le Q\sqrt{\sum_{t=1}^N \nu_t^2}. \tag{5.13}$$

Invoking (5.7), (5.8), we get

$$\begin{aligned} \max_{x\in X}\left|\sum_{t=1}^N \nu_t \Delta_t^T(x_t - x)\right| &\le \left|\sum_{t=1}^N \nu_t \Delta_t^T(x_t - v_t)\right| + \left|\sum_{t=1}^N \nu_t \Delta_t^T(x_t - u_t)\right| \\ &\quad + \frac{D_{\omega,X}^2 + (2\alpha)^{-1}\sum_{t=1}^N \gamma_t^2 \|\Delta_t\|_*^2}{\sum_{t=1}^N \gamma_t}. \end{aligned} \tag{5.14}$$

Moreover, for $1 \le s < t \le N$ we have that $\mathbb{E}\left[\left(\Delta_s^T(x_s - v_s)\right)\left(\Delta_t^T(x_t - v_t)\right)\right] = 0$, and hence

$$\mathbb{E}\left[\left|\sum_{t=1}^N \nu_t \Delta_t^T(x_t - v_t)\right|^2\right] = \sum_{t=1}^N \nu_t^2 \mathbb{E}\left[\left|\Delta_t^T(x_t - v_t)\right|^2\right]$$
$$\le 4M_*^2 \sum_{t=1}^N \nu_t^2 \mathbb{E}\left[\|x_t - v_t\|^2\right]$$
$$\le 32 M_*^2 \alpha^{-1} D_{\omega,X}^2 \sum_{t=1}^N \nu_t^2,$$

where the last inequality follows by (2.7). It follows that

$$\mathbb{E}\left[\left|\sum_{t=1}^N \nu_t \Delta_t^T(x_t - v_t)\right|\right] \le 4\sqrt{2\alpha^{-1}} D_{\omega,X} \sqrt{\sum_{t=1}^N \nu_t^2}.$$

By similar reasons,

$$\mathbb{E}\left[\left|\sum_{t=1}^N \nu_t \Delta_t^T(x_t - u_t)\right|\right] \le 4\sqrt{2\alpha^{-1}} D_{\omega,X} \sqrt{\sum_{t=1}^N \nu_t^2}.$$

These two inequalities combine with (5.13), (5.14) and (5.12) to imply (3.7). This completes the proof of part (i) of Theorem 1. □

Preparing to prove part (ii) of Theorem 1: To prove part (ii) of Theorem 1 we need the following known result; we give its proof for the sake of completeness.

Lemma 2 *Let $\xi_1, \xi_2, \ldots$ be a sequence of iid random variables, $\sigma_t > 0$, μ_t, $t = 1, \ldots,$ be a sequence of deterministic numbers and $\phi_t = \phi_t(\xi^t)$ be deterministic (measurable) functions of $\xi^t = (\xi_1, \ldots, \xi_t)$ such that either*

Case A: $\mathbb{E}_{|t-1}[\phi_t] = 0$ *w.p.1 and* $\mathbb{E}_{|t-1}\left[\exp\{\phi_t^2/\sigma_t^2\}\right] \le \exp\{1\}$ *w.p.1 for all t, or*

Case B: $\mathbb{E}_{|t-1}\left[\exp\{|\phi_t|/\sigma_t\}\right] \le \exp\{1\}$ *for all t.*

Then for any $\Omega \ge 0$ we have the following. In the case of A:

$$\text{Prob}\left\{\sum_{t=1}^N \phi_t > \Omega \sqrt{\sum_{t=1}^N \sigma_t^2}\right\} \le \exp\{-\Omega^2/3\}. \tag{5.15}$$

In the case of B, setting $\sigma^N := (\sigma_1, \ldots, \sigma_N)$*:*

$$\begin{aligned} &\mathrm{Prob}\left\{\sum_{t=1}^N \phi_t > \|\sigma^N\|_1 + \Omega\|\sigma^N\|_2\right\} \le \exp\{-\Omega^2/12\} + \exp\left\{-\tfrac{3\|\sigma^N\|_2}{4\|\sigma^N\|_\infty}\Omega\right\} \\ &\le \exp\{-\Omega^2/12\} + \exp\{-3\Omega/4\}. \end{aligned} \tag{5.16}$$

Proof Let us set $\bar{\phi}_t := \phi_t/\sigma_t$.
Case A: By the respective assumptions about ϕ_t we have that $\mathbb{E}_{|t-1}[\bar{\phi}_t] = 0$ and $\mathbb{E}_{|t-1}\left[\exp\{\bar{\phi}_t^2\}\right] \le \exp\{1\}$ w.p.1. By Jensen's inequality it follows that for any $a \in [0, 1]$:

$$\mathbb{E}_{|t-1}\left[\exp\{a\bar{\phi}_t^2\}\right] = \mathbb{E}_{|t-1}\left[(\exp\{\bar{\phi}_t^2\})^a\right] \le \left(\mathbb{E}_{|t-1}\left[\exp\{\bar{\phi}_t^2\}\right]\right)^a \le \exp\{a\}.$$

We also have that $\exp\{x\} \le x + \exp\{9x^2/16\}$ for all x (this can be verified by direct calculations), and hence

$$\mathbb{E}_{|t-1}\left[\exp\{\lambda\bar{\phi}_t\}\right] \le \mathbb{E}_{|t-1}\left[\exp\{(9\lambda^2/16)\bar{\phi}_t^2\}\right] \le \exp\{9\lambda^2/16\}, \quad \forall \lambda \in [0, 4/3]. \tag{5.17}$$

Besides this, we have $\lambda x \le \frac{3}{8}\lambda^2 + \frac{2}{3}x^2$ for any λ and x, and hence

$$\mathbb{E}_{|t-1}\left[\exp\{\lambda\bar{\phi}_t\}\right] \le \exp\{3\lambda^2/8\}\mathbb{E}_{|t-1}\left[\exp\{2\bar{\phi}_t^2/3\}\right] \le \exp\{2/3 + 3\lambda^2/8\}.$$

Combining the latter inequality with (5.17), we get

$$\mathbb{E}_{|t-1}\left[\exp\{\lambda\bar{\phi}_t\}\right] \le \exp\{3\lambda^2/4\}, \quad \forall \lambda \ge 0.$$

Going back to ϕ_t, the above inequality reads

$$\mathbb{E}_{|t-1}\left[\exp\{\kappa\phi_t\}\right] \le \exp\{3\kappa^2\sigma_t^2/4\}, \quad \forall \kappa \ge 0. \tag{5.18}$$

Now, since ϕ_τ is a deterministic function of ξ^τ and using (5.18), we obtain for any $\kappa \ge 0$:

$$\begin{aligned} \mathbb{E}\left[\exp\left\{\kappa\sum_{\tau=1}^t \phi_\tau\right\}\right] &= \mathbb{E}\left[\exp\left\{\kappa\sum_{\tau=1}^{t-1} \phi_\tau\right\}\mathbb{E}_{|t-1}\exp\{\kappa\phi_t\}\right] \\ &\le \exp\left\{3\kappa^2\sigma_t^2/4\right\}\mathbb{E}\left[\exp\left\{\kappa\sum_{\tau=1}^{t-1}\phi_\tau\right\}\right], \end{aligned}$$

and hence

$$\mathbb{E}\left[\exp\left\{\kappa\sum_{t=1}^{N}\phi_t\right\}\right] \le \exp\left\{3\kappa^2\sum_{t=1}^{N}\sigma_t^2/4\right\}. \tag{5.19}$$

By Markov's inequality, we have for $\kappa > 0$ and $\Omega \ge 0$:

$$\begin{aligned}\text{Prob}\left\{\sum_{t=1}^{N}\phi_t > \Omega\sqrt{\sum_{t=1}^{N}\sigma_t^2}\right\} &= \text{Prob}\left\{\exp\left[\kappa\sum_{t=1}^{N}\phi_t\right] > \exp\left[\kappa\Omega\sqrt{\sum_{t=1}^{N}\sigma_t^2}\right]\right\}\\ &\le \exp\left[-\kappa\Omega\sqrt{\sum_{t=1}^{N}\sigma_t^2}\right]\mathbb{E}\left\{\exp\left[\kappa\sum_{t=1}^{N}\phi_t\right]\right\}.\end{aligned}$$

Together with (5.19) this implies for $\Omega \ge 0$:

$$\begin{aligned}\text{Prob}\left\{\sum_{t=1}^{N}\phi_t > \Omega\sqrt{\sum_{t=1}^{N}\sigma_t^2}\right\} &\le \inf_{\kappa>0}\exp\left\{\tfrac{3}{4}\kappa^2\sum_{t=1}^{N}\sigma_t^2 - \kappa\Omega\sqrt{\sum_{t=1}^{N}\sigma_t^2}\right\}\\ &= \exp\left\{-\Omega^2/3\right\}.\end{aligned}$$

Case B: Observe first that if η is a random variable such that $\mathbb{E}[\exp\{|\eta|\}] \le \exp\{1\}$, then

$$0 \le t \le \frac{1}{2} \Rightarrow \mathbb{E}[\exp\{t\eta\}] \le \exp\{t + 3t^2\}. \tag{5.20}$$

Indeed, let $f(t) = \mathbb{E}[\exp\{t\eta\}]$. Then $f(0) = 1$, $f'(0) = \mathbb{E}[\eta] \le \ln(\mathbb{E}[\exp\{\eta\}]) \le 1$. Besides this, when $0 \le t \le 1/2$, invoking Cauchy's and the Hölder's inequalities we have

$$\begin{aligned}f''(t) &= \mathbb{E}[\exp\{t\eta\}\eta^2] \le \left[\mathbb{E}[\exp\{2t|\eta|\}]\right]^{1/2}\left[\mathbb{E}[\eta^4]\right]^{1/2} \le \left[\mathbb{E}[\exp\{|\eta|\}]\right]^{t}\left[\mathbb{E}[\eta^4]\right]^{1/2}\\ &\le \exp\{1/2\}\left[\mathbb{E}[\eta^4]\right]^{1/2}.\end{aligned}$$

It is immediately seen that $s^4 \le (4/\mathrm{e})^4\exp\{|s|\}$ for all s, whence $\left[\mathbb{E}[\eta^4]\right]^{1/2} \le (4/\mathrm{e})^2\mathrm{e}^{1/2}$ due to $\mathbb{E}[\exp\{|\eta|\}] \le \mathrm{e}$. Thus, $f''(t) \le 16/\mathrm{e}$ when $0 \le t \le 1/2$, and thus $f(t) \le 1 + t + (8/\mathrm{e})t^2 \le \exp\{t + (8/\mathrm{e})t^2\} \le \exp\{t + 3t^2\}$, and (5.20) follows.

Let $\gamma \geq 0$ be such that $\gamma\sigma_t \leq 1/2$, $1 \leq t \leq N$. When $t \leq N$, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\sum_{\tau=1}^{t}\gamma\phi_\tau\right\}\right] &= \mathbb{E}\left[\exp\left\{\sum_{\tau=1}^{t}\gamma\sigma_\tau\bar{\phi}_\tau\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{\sum_{\tau=1}^{t-1}\gamma\sigma_\tau\bar{\phi}_\tau\right\}\mathbb{E}_{|t-1}\left[\exp\{\gamma\sigma_t\bar{\phi}_t\}\right]\right] \\
&\leq \exp\{\gamma\sigma_t + 3\gamma^2\sigma_t^2\}\mathbb{E}\left[\exp\left\{\sum_{\tau=1}^{t-1}\gamma\sigma_\tau\bar{\phi}_\tau\right\}\right],
\end{aligned}$$

where the concluding inequality is given by (5.20) (note that we are in the case when $\mathbb{E}_{|t-1}[\exp\{|\bar{\phi}_t|\}] \leq \exp\{1\}$ w.p.1). From the resulting recurrence we get

$$0 \leq \gamma\|\sigma^N\|_\infty \leq 1/2 \Rightarrow \mathbb{E}\left[\exp\left\{\sum_{t=1}^{N}\gamma\phi_t\right\}\right] \leq \exp\{\gamma\|\sigma^N\|_1 + 3\gamma^2\|\sigma^N\|_2^2\}.$$

whence for every $\Omega \geq 0$, denoting $\beta_s = \|\sigma^N\|_s$,

$$0 \leq \gamma\beta_\infty \leq 1/2 \Rightarrow p := \text{Prob}\left\{\sum_{t=1}^{N}\phi_t > \beta_1 + \Omega\beta_2\right\} \leq \exp\{3\gamma^2\beta_2^2 - \gamma\Omega\beta_2\}. \tag{5.21}$$

When $\Omega \leq \bar{\Omega} := 3\beta_2/\beta_\infty$, $\gamma = \Omega/(6\beta_2)$ satisfies the premise in (5.21), and this implication then says that $p \leq \exp\{-\Omega^2/12\}$. When $\Omega > \bar{\Omega}$, we can use the implication with $\gamma = (2\beta_\infty)^{-1}$, thus getting $p \leq \exp\{\frac{\beta_2}{2\beta_\infty}\left[\frac{3\beta_2}{2\beta_\infty} - \Omega\right]\} \leq \exp\{-\frac{3\beta_2}{4\beta_\infty}\Omega\}$. Thus (5.16) is proved. □

Part (ii) of Theorem 1:

Proof Recall that in part (ii) of Theorem 1 assumption (A1) is strengthened to assumption (A2). Then, in addition to (5.2), we have that

$$\mathbb{E}_{|t-1}\left[\exp\{\delta_t^2/Q^2\}\right] \leq \exp\{1\} \text{ and } \mathbb{E}_{|t-1}\left[\exp\{\|\Delta_t\|_*^2/(2M_*)^2\}\right] \leq \exp\{1\}. \tag{5.22}$$

Let us also make the following simple observation. If Y_1 and Y_2 are random variables and a_1, a_2, a are numbers such that $a_1 + a_2 \geq a$, then the event $\{Y_1 + Y_2 > a\}$ is included in the union of the events $\{Y_1 > a_1\}$ and $\{Y_2 > a_2\}$, and hence $\text{Prob}\{Y_1+Y_2 > a\} \le \text{Prob}\{Y_1 > a_1\} + \text{Prob}\{Y_2 > a_2\}$.

Proof of (3.10). Recall that $\overline{f}^N - f^{*N} = \sum_{t=1}^N \nu_t \delta_t$, and hence it follows by case A of Lemma 2 together with the first equality in (5.2) and (5.22) that for any $\Omega \ge 0$:

$$\text{Prob}\left\{\overline{f}^N - f^{*N} > \Omega Q \sqrt{\sum_{t=1}^N \nu_t^2}\right\} \le \exp\{-\Omega^2/3\}. \tag{5.23}$$

In the same way, by considering $-\delta_t$ instead of δ_t, we have that

$$\text{Prob}\left\{f^{*N} - \overline{f}^N > \Omega Q \sqrt{\sum_{t=1}^N \nu_t^2}\right\} \le \exp\{-\Omega^2/3\}, \tag{5.24}$$

The assertion (3.10) follows from (5.23) and (5.24).

Proof of (3.11). Now by (5.12) and (5.14) we have

$$\left|\underline{f}^N - f_*^N\right| \le \left|\sum_{t=1}^N \nu_t \delta_t\right| + \left|\sum_{t=1}^N \nu_t \Delta_t^T (x_t - v_t)\right| + \left|\sum_{t=1}^N \nu_t \Delta_t^T (x_t - u_t)\right| + \frac{D_{\omega,X}^2 + (2\alpha)^{-1}\sum_{t=1}^N \gamma_t^2 \|\Delta_t\|_*^2}{\sum_{t=1}^N \gamma_t}. \tag{5.25}$$

As it was shown above (see (5.23),(5.24)):

$$\text{Prob}\left\{\left|\sum_{t=1}^N \nu_t \delta_t\right| > \Omega Q \sqrt{\sum_{t=1}^N \nu_t^2}\right\} \le 2\exp\{-\Omega^2/3\}. \tag{5.26}$$

Moreover, by (2.7) we have that $\|x_t - v_t\| \le \|x_t - x_1\| + \|v_t - x_1\| \le 2\sqrt{2\alpha^{-1}} D_{\omega,X}$, and hence

$$\mathbb{E}_{|t-1}\left[\exp\{|\Delta_t^T (x_t - v_t)|^2/(4\sqrt{2\alpha^{-1}} D_{\omega,X} M_*)^2\}\right] \le \exp\{1\}.$$

It follows by case A of Lemma 2 that

$$\text{Prob}\left\{\left|\sum_{t=1}^N \nu_t \Delta_t^T (x_t - v_t)\right| > 4\Omega\sqrt{2\alpha^{-1}} D_{\omega,X} M_* \sqrt{\sum_{t=1}^N \nu_t^2}\right\} \le 2\exp\{-\Omega^2/3\}. \tag{5.27}$$

and similarly

$$\text{Prob}\left\{\left|\sum_{t=1}^{N}\nu_t\Delta_t^T(x_t-u_t)\right|>4\Omega\sqrt{2\alpha^{-1}}D_{\omega,X}M_*\sqrt{\sum_{t=1}^{N}\nu_t^2}\right\}\le 2\exp\{-\Omega^2/3\}. \tag{5.28}$$

Furthermore, invoking (5.22), the random variables $\phi_t = (2\alpha)^{-1}\gamma_t^2\|\Delta_t\|_*^2 (\sum_{t=1}^N \gamma_t)^{-1}$ satisfy the premise of case B in Lemma 2 with $\sigma_t = 2\alpha^{-1}M_*^2\gamma_t^2 (\sum_{t=1}^N \gamma_t)^{-1}$. Invoking case B of Lemma, we get

$$\begin{aligned}\text{Prob}&\left\{\frac{(2\alpha)^{-1}\sum_{t=1}^N\gamma_t^2\|\Delta_t\|_*^2}{\sum_{t=1}^N\gamma_t}>\frac{2\alpha^{-1}M_*^2\sum_{t=1}^N\gamma_t^2}{\sum_{t=1}^N\gamma_t}+\Omega\frac{2\alpha^{-1}M_*^2\sqrt{\sum_{t=1}^N\gamma_t^4}}{\sum_{t=1}^N\gamma_t}\right\}\\&\le\exp\{-\Omega^2/12\}+\exp\{-\Gamma_N\Omega\},\\ \Gamma_N&=\frac{3\|(\gamma_1^2,\dots,\gamma_N^2)\|_2}{4\|(\gamma_1^2,\dots,\gamma_N^2)\|_\infty}\end{aligned} \tag{5.29}$$

Combining this bound with (5.27), (5.28) and taking into account (5.25), we arrive at (3.11).

Proof of (3.9). It remains to prove (3.9). To this end note by (5.4) and (5.7) we have

$$\begin{aligned}f^{*N}-f_*^N\le&\frac{2D_{\omega,X}^2+(2\alpha)^{-1}\sum_{t=1}^N\gamma_t^2(\|\mathsf{G}(x_t,\xi_t)\|_*^2+\|\Delta_t\|_*^2)}{\sum_{t=1}^N\gamma_t}\\&+\sum_{t=1}^N\nu_t\Delta_t^T(x_t-v_t),\end{aligned} \tag{5.30}$$

Completely similar to (5.29), we have

$$\begin{aligned}\text{Prob}&\left\{\frac{(2\alpha)^{-1}\sum_{t=1}^N\gamma_t^2\|\mathsf{G}(x_t,\xi_t)t\|_*^2}{\sum_{t=1}^N\gamma_t}>\frac{(2\alpha)^{-1}M_*^2\sum_{t=1}^N\gamma_t^2}{\sum_{t=1}^N\gamma_t}\right.\\&\left.+\Omega\frac{(2\alpha)^{-1}M_*^2\sqrt{\sum_{t=1}^N\gamma_t^4}}{\sum_{t=1}^N\gamma_t}\right\}\le\exp\{-\Omega^2/12\}+\exp\{-\Gamma_N\Omega\}\end{aligned} \tag{5.31}$$

This bound combines with (5.29) and (5.27) to imply (3.9). □

Theorem 2: proof Let $x_1, \ldots, x_N$ be the trajectory of Mirror Descent SA, and let $x_{N+t} := \tilde{x}_N$, $t = 1, \ldots, L$. Then we can write

$$\hat{\ell}_L(x; \tilde{x}_N) = \frac{1}{L} \sum_{t=N+1}^{N+L} \left[F(x_t, \xi_t) + \mathsf{G}(x_t, \xi_t)^T (x - x_t) \right].$$

Let x_* be an optimal solution to (1.1), and let us set $\eta_t := \Delta_t^T (x_* - x_t)$, $t = 1, \ldots, N+L$. By (2.7) we have $\|x_t - x_*\| \le 2\Lambda_{\omega,X}$, and since x_t is a deterministic function of ξ^{t-1}, $1 \le t \le N+L$, and the oracle is unbiased, under assumption (A1) we have for $1 \le t \le N+L$,

$$\begin{aligned} &\mathbb{E}_{|t-1}[\delta_t] = 0, \mathbb{E}_{|t-1}\left[\delta_t^2\right] \le Q^2, \\ &\mathbb{E}_{|t-1}[\eta_t] = 0, \mathbb{E}_{|t-1}\left[\eta_t^2\right] \le 4\Lambda_{\omega,X}^2 \mathbb{E}_{|t-1}\left[\|\Delta_t\|_*^2\right] \le 16\Lambda_{\omega,X}^2 M_*^2. \end{aligned} \tag{5.32}$$

Consequently

$$\begin{aligned} \hat{f}^N(x_*) &= \underbrace{\frac{1}{N} \sum_{t=1}^{N} [f(x_t) + \mathsf{g}(x_t)^T (x_* - x_t)]}_{\le f(x_*) = \text{Opt}} + \underbrace{\frac{1}{N} \sum_{t=1}^{N} [\delta_t + \eta_t]}_{\zeta_1}, \\ \hat{\ell}_L(x_*; \tilde{x}_N) &= \underbrace{\frac{1}{L} \sum_{t=N+1}^{N+L} [f(x_t) + \mathsf{g}(x_t)^T (x_* - x_t)]}_{\le f(x_*) = \text{Opt}} + \underbrace{\frac{1}{L} \sum_{t=N+1}^{N+L} [\delta_t + \eta_t]}_{\zeta_2}. \end{aligned}$$

It follows that

$$lb^N - \text{Opt} \le \max\left\{ \hat{f}^N(x_*), \hat{\ell}_L(x_*; \tilde{x}_N) \right\} - \text{Opt} \le \max\{\zeta_1, \zeta_2\} \le |\zeta_1| + |\zeta_2|. \tag{5.33}$$

From (5.32) it follows that

$$\begin{aligned} \mathbb{E}[\zeta_1^2] &\le N^{-1}\left(2\mathbb{E}\left[\delta_t^2\right] + 2\mathbb{E}\left[\eta_t^2\right]\right) \le \left(2Q^2 + 32\Lambda_{\omega,X}^2 M_*^2\right) N^{-1}, \\ \mathbb{E}[\zeta_2^2] &\le L^{-1}\left(2\mathbb{E}\left[\delta_t^2\right] + 2\mathbb{E}\left[\eta_t^2\right]\right) \le \left(2Q^2 + 32\Lambda_{\omega,X}^2 M_*^2\right) L^{-1}, \end{aligned}$$

which combines with (5.33) to imply (3.15).

Under assumption (A2), along with (5.32) we also have that

$$\mathbb{E}_{|t-1}\left[\exp\{\delta_t^2/Q^2\}\right] \le \exp\{1\}, \quad \mathbb{E}_{|t-1}\left[\exp\{\eta_t^2/(4\Lambda_{\omega,X} M_*)^2\}\right] \le \exp\{1\},$$

and hence

$$\mathbb{E}_{|t-1}\left[\delta_t + \eta_t\right] = 0, \quad \mathbb{E}_{|t-1}\left[\exp\{[\delta_t + \eta_t]^2/(Q + 4\Lambda_{\omega,X} M_*)^2\}\right] \le \exp\{1\}.$$

Invoking case A of Lemma 2, we conclude that for all $\Omega \geq 0$:

$$\text{Prob}\left\{|\zeta_1| > \Omega[Q + 4\Lambda_{\omega,X}M_*]N^{-1/2}\right\} \leq 2\exp\{-\Omega^2/3\},$$

$$\text{Prob}\left\{|\zeta_2| > [Q + 4\Lambda_{\omega,X}M_*]L^{-1/2}\right\} \leq 2\exp\{-\Omega^2/3\},$$

which combines with (5.33) to imply (3.16). □

References

1. Bregman, L.M.: The relaxation method of finding the common points of convex sets and its application to the solution of problems in convex programming. Comput. Math. Math. Phys. **7**, 200–217 (1967)
2. Kleywegt, A.J., Shapiro, A., Homem-de-Mello, T.: The sample average approximation method for stochastic discrete optimization. SIAM J. Optim. **12**, 479–502 (2001)
3. Law, A.M.: Simulation Modeling and Analysis. McGraw Hill, New York (2007)
4. Lemarechal, C., Nemirovski, A., Nesterov, Yu.: New variants of bundle methods. Math. Program. **69**, 111–148 (1995)
5. Linderoth, J., Shapiro, A., Wright, S.: The empirical behavior of sampling methods for stochastic programming. Ann. Oper. Res. **142**, 215–241 (2006)
6. Mak, W.K., Morton, D.P., Wood, R.K.: Monte Carlo bounding techniques for determining solution quality in stochastic programs. Oper. Res. Lett. **24**, 47–56 (1999)
7. Nemirovskii, A., Yudin, D.: On Cezari's convergence of the steepest descent method for approximating saddle point of convex-concave functions. (in Russian)—Doklady Akademii Nauk SSSR, **239**, 5 (1978) (English translation: Soviet Math. Dokl. **19**, 2 (1978))
8. Nemirovski A., Yudin D.: Problem complexity and method efficiency in optimization. Wiley-Interscience Series in Discrete Mathematics, Wiley, XV (1983)
9. Nemirovski, A., Juditsky, A., Lan, G., Shapiro, A.: Robust stochastic approximation approach to stochastic programming. SIAM J. Optim. **19**, 1574–1609 (2009)
10. Norkin, V.I., Pflug, G.Ch., Ruszczyński, A.: A branch and bound method for stochastic global optimization. Math. Program. **83**, 425–450 (1998)
11. Polyak, B.T.: New stochastic approximation type procedures. Automat. i Telemekh. **7**, 98–107 (1990). (English translation: Automation and Remote Control)
12. Polyak, B.T., Juditsky, A.B.: Acceleration of stochastic approximation by averaging. SIAM J. Control Optim. **30**, 838–855 (1992)
13. Robbins, H., Monro, S.: A stochastic spproximation method. Ann. Math. Stat. **22**, 400–407 (1951)
14. Rockafellar, R.T., Uryasev, S.P.: Optimization of conditional value-at-risk. The Journal of Risk **2**, 21–41 (2000)
15. Shapiro, A.: Monte Carlo sampling methods. In: Ruszczyński, A., Shapiro, A. (eds.) Stochastic Programming, Handbook in OR & MS, vol. 10, North-Holland Publishing Company, Amsterdam (2003)
16. Shapiro, A., Nemirovski, A. : On complexity of stochastic programming problems. In: Jeyakumar, V., Rubinov, A.M. (eds.) Continuous Optimization: Current Trends and Applications, pp. 111–144. Springer, New York (2005)
17. Verweij, B., Ahmed, S., Kleywegt, A.J., Nemhauser, G., Shapiro, A.: The sample average approximation method applied to stochastic routing problems: a computational study. Comput. Optim. Appl. **24**, 289–333 (2003)
18. Wang, W., Ahmed, S.: Sample average approximation of expected value constrained stochastic programs, E-print available at: http://www.optimization-online.org (2007)