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Selected topics in robust convex optimization

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Abstract Robust Optimization is a rapidly developing methodology for handling optimization problems affected by non-stochastic "uncertain-but-bounded" data perturbations. In this paper, we overview several selected topics in this popular area, specifically, (1) recent extensions of the basic concept of *robust counterpart* of an optimization problem with uncertain data, (2) tractability of robust counterparts, (3) links between RO and traditional chance constrained settings of problems with stochastic data, and (4) a novel generic application of the RO methodology in Robust Linear Control.

Keywords Optimization under uncertainty · Robust optimization · Convex programming · Chance constraints · Robust linear control

Mathematics Subject Classification (2000) 90C34 · 90C05 · 90C20 · 90C22 · 90C15

1 Introduction

The goal of this paper is to overview recent progress in *Robust Optimization* — one of the methodologies aimed at optimization under uncertainty. The entity

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of interest is an *uncertain* optimization problem of the form

$$\min_{x,t}\{t : f_0(x,\zeta) - t \le 0, f_i(x,\zeta) \in K_i,\ i = 1,\dots,m\}, \tag{1}$$

where $x \in \mathbf{R}^n$ is the vector of decision variables, $\zeta \in \mathbf{R}^d$ is the vector of *problem's data*, $f_0(x,\zeta) : \mathbf{R}^n \times \mathbf{R}^d \to \mathbf{R}$, $f_i(x,\zeta) : \mathbf{R}^n \times \mathbf{R}^d \to \mathbf{R}^{k_i}$, $1 \le i \le m$, are given functions, and $K_i \subset \mathbf{R}^{k_i}$ are given nonempty sets. Uncertainty means that *the data vector ζ is not known exactly at the time when the solution has to be determined.* As a result, it is unclear what does it mean "to solve" an uncertain problem. In Stochastic Programming — historically, the first methodology for handling data uncertainty in optimization — one assumes that the data are of stochastic nature with known distribution and seeks for a solution which minimizes the expected value of the objective over candidate solutions which satisfy the constraints with a given (close to 1) probability. In Robust Optimization (RO), the data is assumed to be "uncertain but bounded", that is, varying in a given *uncertainty set* $\mathcal{Z}$, rather than to be stochastic, and the aim is to choose the best solution among those "immunized" against data uncertainty. The most frequently used interpretation of what "immunized" means is as follows: *a candidate solution to* (1) *is "immunized" against uncertainty if it is robust feasible, that is, remains feasible for all realizations of the data from the uncertainty set.* With this approach, one associates with the uncertain problem (1) its *Robust Counterpart* (RC) — the semi-infinite problem

$$\min_{x,t}\{t : f_0(x,\zeta) \le t, f_i(x,\zeta) \in K_i,\ i = 1,\dots,m,\ \forall \zeta \in \mathcal{Z}\} \tag{2}$$

of minimizing the *guaranteed value* $\sup_{\zeta\in\mathcal{Z}} f_0(x,\zeta)$ of the objective over robust feasible solutions. The resulting optimal solutions, called *robust optimal* solutions of (1), are interpreted as the recommended for use "best immunized against uncertainty" solutions of an uncertain problem. Note that the uncertainty set plays the role of "a parameter" of this construction. The outlined approach originates from Soyster [62]. Associated in-depth developments started in mid-1990s [5–7,9,34,35] and initially were mainly focused on motivating the approach and on its theoretical development, with emphasis on the crucial issue of computational tractability of the RC. An overview of these developments was the subject of the semi-plenary lecture "Robust Optimization — Methodology and Applications" delivered by A. Ben-Tal at XVII ISMP, Atlanta, 2000, see [13]. Since then, the RO approach has been rapidly gaining popularity. Extensive research on the subject in the recent years was aimed both at developing the basic RO theory (see [10,14–17,19,23,24,26,40,44,32,61] and references therein) and at applications of the RO methodology in various areas, including, but not restricted to, Discrete optimization [2,3,21,22,48], Numerical Linear Algebra [37], Dynamic Programming [43,54], Inventory Management [1,18,25], Pricing [1,55], Portfolio selection [11,36,39], Routing [53], Machine Learning [27,47], Structural design [5,45], Control [31,20,38,46], Signal processing and estimation [4,29,33].[1] It would be impossible to outline, even briefly, this broad research in a single paper; our intention here is to overview several selected RO-related topics, primarily, those related to (a) extensions of the RO paradigm, (b) its links with Stochastic Optimization, and (c) computational tractability of RO models. In the sequel we restrict our considerations solely to the case of *convex bi-affine* uncertain optimization problems, that is, problems (1) with closed convex K_i and $f_i(x,\zeta)$, $i = 0,\ldots,m$, bi-affine in x and in ζ:

$$f_i(x,\zeta) = f_{i0}(x) + \sum_{\ell=1}^{d} \zeta_\ell f_{i\ell}(x) = \phi_{i0}(\zeta) + \sum_{j=1}^{n} x_j \phi_{ij}(\zeta), \quad i = 0,\ldots,m, \tag{3}$$

where all $f_{i\ell}(x), \phi_{i\ell}(\zeta)$ are *affine* scalar ($i = 0$) or vector-valued ($i > 0$) functions of x, ζ, respectively. The reason for this restriction comes from the fact that at the end of the day we should be able to process the RC numerically and thus want it to be computationally tractable. At our present level of knowledge, this sine qua non ultimate goal requires, generically, *at least* convexity and bi-affinity of the problem.[2] Note that the bi-affinity requirement is satisfied, in particular, by conic problems $\min_x \left\{c^{\mathrm{T}}x : Ax - b \in \mathbf{K}\right\}$, $\mathbf{K}$ being a closed convex cone, with the "natural data" (c, A, b) affinely parameterized by ζ. Thus, our bi-affinity restriction does not rule out the most interesting generic convex problems like those of Linear, Conic Quadratic and Semidefinite Programming.

The rest of the paper is organized as follows. Section 2 is devoted to two recent extensions of the RO paradigm, specifically, to the concepts of *affinely adjustable* and *globalized* Robust Counterparts. Section 3 is devoted to results on computational tractability of Robust Counterparts. Section 4 establishes some instructive links with Chance Constrained Stochastic Optimization. Concluding Sect. 5 is devoted to a novel application of the RO methodology in Robust Linear Control.

2 Extending the scope of robust optimization: affinely adjustable and globalized robust counterparts

2.1 Adding adjustability: motivation

On a closest inspection, the concept of Robust Counterpart of an uncertain optimization problem is based on the following three tacitly accepted assumptions:

[1] More information on RO-related publications can be found in the references in cited papers and in the section "Robust optimization" at www.optimization-online.org.

[2] In some of the situations to be encountered, bi-affinity can be weakened to affinity of f_i, $i = 1,\ldots,m$, in ζ and convexity (properly defined for $i > 0$) of these functions in x. However, in order to streamline the presentation and taking into account that the extensions from affine to convex case, when they are possible, are completely straightforward, we prefer to assume affinity in x.

A.1. All decision variables in (1) represent "here and now" decisions which should get specific numerical values as a result of solving the problem and *before* the actual data "reveal itself";

A.2. The constraints in (1) are "hard", that is, we cannot tolerate violations of constraints, even small ones;

A.3 The data are "uncertain but bounded" — we can specify an appropriate *uncertainty set* $\mathcal{Z} \subset \mathbf{R}^d$ of possible values of the data and are fully responsible for consequences of our decisions when, and only when, the actual data is within this set.

With all these assumptions in place, the only meaningful candidate solutions of the uncertain problem (1) are the robust feasible ones, and the RC (2) seems to be the only possible interpretation of "optimization over uncertainty-immunized solutions". However, in many cases assumption **A.1** is *not* satisfied, namely, only part of the decision variables represent "here and now" decisions to be fully specified when the problem is being solved. Other variables can represent "wait and see" decisions which should be made when the uncertain data partially or completely "reveal itself", and decisions in question can "adjust" themselves to the corresponding portions of the data. This is what happens in dynamical decision-making under uncertainty, e.g., in multi-stage inventory management under uncertain demand, where the replenishment orders of period t can depend on actual demands at the preceding periods. Another type of "adjustable" decision variables is given by *analysis variables* — those which do not represent decisions at all and are introduced in order to convert the problem into a desired form, e.g., the LP one. For example, consider the constraint

$$\sum_{i=1}^{I} |a_i^T x - b_i| \le t \tag{4}$$

along with its LP representation

$$-y_i \le a_i^T x - b_i \le y_i,\ 1 \le i \le I,\ \sum_i y_i \le t. \tag{5}$$

With uncertain data $\zeta = \{a_i, b_i\}_{i=1}^{I}$ varying in given set $\mathcal{Z}$ and x, t representing "here and now" decisions, there is absolutely no reason to think of y_i as of here and now decisions as well: in fact, y_i's do not represent decisions at all and as such can "adjust" themselves to the actual values of the data.

2.2 Adjustable and affinely adjustable RC

A natural way to capture the situations where part of the decision variables can "adjust" themselves, to some extent, to actual values of the uncertain data, is to assume that every decision variable x_j in (1) is allowed to depend on a prescribed portion $P_j\zeta$ of the uncertain data, where P_j are given in advance matrices. With this assumption, a candidate solution to (1) becomes a collection of *decision rules* $x_j = X_j(P_j\zeta)$ rather than a collection of fixed reals, and the natural candidate to the role of (2) becomes the *adjustable robust counterpart* (ARC) of (1):

$$\min_{\{X_j(\cdot)\}_{j=1}^n, t} \left\{ t : \left. \begin{array}{l} f(X_1(P_1\zeta), \ldots, X_n(P_n\zeta), \zeta) - t \leq 0 \\ f_i(X_1(P_1\zeta), \ldots, X_n(P_n\zeta), \zeta) \in K_i,\ 1 \leq i \leq m \end{array} \right\} \forall \zeta \in \mathcal{Z} \right\}. \quad (6)$$

The nature of candidate solutions to the ARC (decision rules rather than fixed vectors) and the constraints in this problem resembles those of a multi-stage Stochastic Programming problem; essentially, the only difference is that in (6) we intend to minimize the worst case value of the objective rather than its expectation w.r.t. a given distribution of ζ. While in our current "decision environment" the ARC seems to be a completely natural entity, there are pretty slim chances to make this concept "workable"; the problem is that the ARC usually is "severely computationally intractable". Indeed, (6) is an infinite-dimensional problem, and in general it is absolutely unclear even how to store candidate solutions to this problem, not speaking of how to optimize over these solutions. Seemingly the only optimization technique which under appropriate structural assumptions could handle ARC's is Dynamic Programming; this technique, however, heavily suffers from "course of dimensionality".

Note that the RC of (1) is a semi-infinite problem and as such may also be computationally intractable; there are, however, important generic cases, most notably, uncertain Linear Programming with computationally tractable uncertainty set (see below), where this difficulty does not occur. In contrast to this, even in the simplest case of uncertain LP (that is, bi-affine problem (1) with $K_i = \mathbf{R}_-$, $i = 1, \ldots, m$), just two generic (both not too interesting) cases where the ARC is tractable are known [42]. In both these cases, there are just two types of decision variables: "non-adjustable" (those with $P_j = 0$) and "fully adjustable" ($P_j = I$), and the problem has "fixed recourse": for all j with $P_j \neq 0$ all the functions $\phi_{ij}(\zeta)$ in (3) are independent of ζ ("coefficients of all adjustable variables are certain"). In the first case, the uncertainty set is the direct product of uncertainty sets in the spaces of data of different constraints ("constraint-wise uncertainty"); here, under mild regularity assumptions (e.g., when all the variables are subject to finite upper and lower bounds), the ARC is equivalent to the RC. The second case is the one of "scenario uncertainty" $\mathcal{Z} = \mathrm{Conv}\{\zeta^1, \ldots, \zeta^S\}$. In this case, assuming w.l.o.g. that the non-adjustable variables are $x_1, \ldots, x_k$ and the adjustable ones are $x_{k+1}, \ldots, x_n$, the ARC of (1) is equivalent to the explicit convex problem

$$\min_{\substack{t, x_1, \ldots, x_k, \\ x^s_{k+1}, \ldots, x^s_n}} \left\{ t : \begin{array}{l} \phi_{00}(\zeta^s) + \sum_{j=1}^k x_j \phi_{0j}(\zeta^s) + \sum_{j=k+1}^n \phi_{0j} x^s_j - t \leq 0 \\ \phi_{i0}(\zeta^s) + \sum_{j=1}^k x_j \phi_{ij}(\zeta^s) + \sum_{j=k+1}^n \phi_{ij} x^s_j \in K_i, \end{array} , \begin{array}{l} 1 \leq i \leq I \\ 1 \leq s \leq S \end{array} \right\}$$

(for notation, see (3)); this equivalence remains valid for the case of general (convex!) sets K_i as well.
Finally, to give an instructive example to the dramatic increase in complexity when passing from the RC to the ARC, consider the uncertain ℓ_1-approximation problem where one is interested to minimize t in t, x linked by the constraint (4) (both x and t are non-adjustable) and the data $\zeta = \{a_i, b_i\}_{i=1}$ runs through a pretty simple uncertainty set $\mathcal{Z}$, namely, a_i are fixed, and $b = (b_1, \ldots, b_I)^T$ runs through an ellipsoid. In other words, we are speaking about the uncertain Linear Programming problem of minimizing t under the constraints (5) on variables x, t, y where x, t are non-adjustable and y_i are fully adjustable. The ARC of this uncertain problem is clearly equivalent to the semi-infinite convex program

$$\mathrm{Opt} = \min_{x,t} \left\{ t : \sum_{i=1}^{I} |a_i^T x - b_i| \le t \, \forall (b = Qu : u^T - u \le 1) \right\}.$$

This simple-looking problem is NP-hard (one can reduce to it the well-known MAXCUT problem); it is even NP-hard to approximate Opt within a close enough to 1 absolute constant factor. In contrast to this, the RC of our uncertain LP (5) with the outlined uncertainty set is clearly equivalent to the explicit LP program

$$\min_{x,t,y} \left\{ |a_i^T x| \le y_i, \sum_i (y_i + d_i) \le t \right\}, \quad d_i = \max_u \left\{ (Qu)_i : u^T u \le 1 \right\}$$

and is therefore easy.

The bottom line is as follows: when the decision variables in (1) can "adjust themselves", to some extent, to the actual values of the uncertain data, the RC (2) of the uncertain problem cannot be justified by our "decision-making environment" and can be too conservative. A natural remedy — passing to the ARC (6) — typically requires solving a severely computationally intractable problem and thus is not an actual remedy. The simplest way to resolve the arising difficulty is to restrict the type of decision rules we allow in the ARC, and the "strongest" restriction here (aside of making the decision rules constant and thus coming back to the RC) is to enforce these rules to be affine:

$$x_j = \eta_j^0 + \eta_j^T P_j \zeta, \quad j = 1, \ldots, n. \tag{7}$$

With this dramatic simplification of the decision rules, (6) becomes an optimization problem in the variables η_j^0, η_j — the coefficients of our decision rules. The resulting problem, called the *affinely adjustable robust counterpart* (AARC) of

(1), is the semi-infinite problem

$$\min_{\{\eta_j^0,\eta_j\},t}\left\{t:\left.\begin{array}{r}\phi_{00}(\zeta)+\sum\limits_{j=1}^{n}\phi_{0j}(\zeta)[\eta_j^0+\eta_j^TP_j\zeta]-t\leq 0\\ \phi_{i0}(\zeta)+\sum\limits_{j=1}^{n}\phi_{ij}(\zeta)[\eta_j^0+\eta_j^TP_j\zeta]\in K_i\\ 1\leq i\leq m\end{array}\right\}\forall\zeta\in\mathcal{Z}\right\}. \tag{8}$$

(cf. (3)). In terms of conservatism, the AARC clearly is "in-between" the RC and the ARC, and few applications of the AARC reported so far (most notably, to Inventory Management under uncertainty [18]) demonstrate that passing from the RC to the AARC can reduce dramatically the "built-in" conservatism of the RO methodology. Note also that with $P_j = 0$ for all j, the AARC becomes exactly the RC.

Note that the RC of (1) is the semi-infinite problem

$$\min_{\{x_j\},t}\left\{t:\left.\begin{array}{r}\phi_{00}(\zeta)+\sum\limits_{j=1}^{n}\phi_{0j}(\zeta)x_j-t\leq 0\\ \phi_{i0}(\zeta)+\sum\limits_{j=1}^{n}\phi_{ij}(\zeta)x_j\in K_i\\ 1\leq i\leq m\end{array}\right\}\forall\zeta\in\mathcal{Z}\right\}, \tag{9}$$

and its structure is not too different from the one of (8) — both problems are semi-infinite convex programs with constraints which depend affinely on the respective decision variables. The only — although essential — difference is that the constraints of the RC are affine in the uncertain data ζ as well, while the constraints in AARC are, in general, quadratic in ζ. There is, however, an important particular case where this difference disappears; this is the previously mentioned case of *fixed recourse*, that is, the case where the functions $\phi_{ij}(\zeta)$, $i = 0, 1, \ldots, m$ associated with *adjustable* variables x_j — those with $P_j \neq 0$ — are in fact constants. In this case, both ARC and AARC are of exactly the same structure — they are semi-infinite convex programs with bi-affine constraints. We shall see in Sect. 3 that bi-affinity makes both RC and AARC computationally tractable, at least in the case where all K_i are polyhedral sets given by explicit lists of linear inequalities ("uncertain Linear Programming").

In principle, the AARC (8) always can be thought of as a semi-infinite *bi-affine* convex problem; to this end, is suffices to "lift" quadratically the data — to treat as the data the matrix $Z(\zeta) = \begin{bmatrix} 1 & \zeta^T \\ \zeta & \zeta\zeta^T \end{bmatrix}$ rather than ζ itself. Note that the left hand sides of the constraints of both RC and AARC can be thought of as bi-affine functions of the corresponding decision variables and $Z(\zeta)$, and this bi-affinity implies that the RC and the AARC remain intact when we replace the original uncertainty set $\mathcal{Z}$ in the space of "actual data" ζ with the uncertainty set $\widehat{\mathcal{Z}} = \mathrm{Conv}\{Z(\zeta) : \zeta \in \mathcal{Z}\}$. Note, however, that in order for a semi-infinite convex problem of the form

$$\min_y \left\{ c^T y : F_\ell(y,u) \in Q_\ell,\ \ell = 1,\ldots,L\ \forall u \in \mathcal{U} \right\}$$

(the sets Q_ℓ are closed and convex, $F_\ell(\cdot,\cdot)$ are bi-affine) to be computationally tractable, we need more than mere convexity; "tractability results" here, like those presented in Sect. 3, require from the sets $\mathrm{Conv}(\mathcal{U})$ and Q_ℓ to be computationally tractable, and, moreover, somehow "match" each other. While the requirement of computational tractability of Q_ℓ and of the convex hull $\mathrm{Conv}\{\mathcal{Z}\}$ of the "true" uncertainty set $\mathcal{Z}$ are usually non-restricting, the quadratic lifting $\mathcal{Z} \mapsto \widehat{\mathcal{Z}}$ generally destroys computational tractability of the corresponding convex hull and the "matching" property. There are, however, particular cases when this difficulty does not occur, for example, the trivial case of finite $\mathcal{Z}$ ("pure scenario uncertainty"). A less trivial case is the one where $\mathcal{Z}$ is an ellipsoid. This case immediately reduces to the one where $\mathcal{Z}$ is the unit Euclidean ball $\{\zeta : \zeta^T\zeta \le 1\}$, and here $\mathrm{Conv}\{\widehat{\mathcal{Z}}\} = \mathrm{Conv}\{Z(\zeta) : \zeta^T\zeta \le 1\}$ is the computationally tractable set $\{Z \in \mathbf{S}^{d+1}_+ : Z_{11} = 1, \sum_{\ell=2}^{d+1} Z_{\ell\ell} \le 1\}$. Whether this set "matches" Q_ℓ or not, this depends on the geometry of the latter sets, and the answer, as we shall see, is positive when Q_ℓ are polyhedral sets given by explicit lists of linear inequalities.

Convention. From now on, unless it is explicitly stated otherwise, we restrict attention to the case of fixed recourse, so that both the RC (9) and the AARC (8) of the uncertain problem (1) (which always is assumed to be bi-affine with convex K_i) are bi-affine semi-infinite problems. In this case, due to bi-affinity of the left hand sides in the constraints of RC and AARC and to the convexity of K_i, both the RC and the AARC remain intact when the uncertainty set $\mathcal{Z}$ is extended to its closed convex hull. By this reason, from now on this set is assumed to be closed and convex.

2.3 Controlling global sensitivities: globalized robust counterpart

The latest, for the time being, extension of the RO paradigm was proposed in [19] and is motivated by the desire to relax, to some extent, assumptions **A.2**, **A.3**. Specifically, it may happen that some of the constraints in the uncertain problem are "soft" — their violation, while undesirable, can however be tolerated. With respect to such constraints, it does not make much sense to follow the "black and white" policy postulated in assumption **A.3**; instead of taking full care of feasibility when the data is in the uncertainty set and not bothering at all what happens when the data is outside of this set, it is more natural to ensure feasibility when the data is in their "normal range" and to allow for *controlled* violation of the constraint when the data runs out of this normal range. The simplest way to model these requirements is as follows. Consider a *bi-affine* "soft" semi-infinite constraint

$$f(y,\zeta) \equiv \phi_0(\zeta) + \sum_{j=1}^{N} \phi_j(\zeta) y_j \in K \quad \left[\phi_j(\zeta) = \phi_j^0 + \Phi_j \zeta,\, 0 \le j \le N\right], \tag{10}$$

where y are the design variables, $\phi_j(\zeta), j = 0, \ldots, N$ are affine in $\zeta \in \mathbf{R}^d$ vector-valued functions taking values in certain $\mathbf{R}^k$, and K is a closed convex set in $\mathbf{R}^k$. In our context, this constraint may come from the RC (9) of the original uncertain problem, or from its AARC (8),[3] this is why we choose "neutral" notation for the design variables. Assume that the set of all "physically possible" values of ζ is of the form

$$\mathcal{Z}\mathcal{L} = \mathcal{Z} + \mathcal{L},$$

where $\mathcal{Z} \subset \mathbf{R}^d$ is a closed and convex set representing the "normal range" of the data, and $\mathcal{L} \subset \mathbf{R}^d$ is a closed convex cone. Let us say that y is a *robust feasible solution* to (10) *with global sensitivity* $\alpha \ge 0$, if, first, y remains feasible for the constraint whenever $\zeta \in \mathcal{Z}$, and, second, the violation of the constraint when $\zeta \in \mathcal{Z}\mathcal{L} \backslash \mathcal{Z}$ can be bounded in terms of the distance of ζ to its normal range, specifically,

$$\mathrm{dist}(f(y,\zeta), K) \le \alpha \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L}) \;\; \forall \zeta \in \mathcal{Z}\mathcal{L} = \mathcal{Z} + \mathcal{L}; \tag{11}$$

here

$$\mathrm{dist}(u, K) = \min_{v \in K} \|u - v\|_K, \;\; \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L}) = \min_z \left\{ \|\zeta - z\|_Z : \begin{array}{c} z \in \mathcal{Z} \\ \zeta - z \in \mathcal{L} \end{array} \right\}$$

and $\|\cdot\|_K, \|\cdot\|_Z$ are given norms on the respective spaces. In the sequel, we refer to the setup of this construction — the collection $(\mathcal{Z}, \mathcal{L}, \|\cdot\|_K, \|\cdot\|_Z)$ — as to the *uncertainty structure* associated with uncertain constraint (10). Note that since K is closed, (11) automatically ensures that $f(y,\zeta) \in K$ whenever $\zeta \in \mathcal{Z}$, so that the outlined pair of requirements in fact reduces to the single requirement (11).

We refer to (11) as to *Globalized* RC (GRC) of uncertain constraint (10) associated with the uncertainty structure in question. Note that with $\mathcal{L} = \{0\}$, the GRC recovers the usual RC/AARC.

In order to build the Globalized RC (Globalized AARC) of uncertain problem (1), we replace the semi-infinite constraints of the RC (9), respectively, those of the AARC (8), with their modifications (11). In general, both the global sensitivity and the uncertainty structure can vary from constraint to constraint.

[3] Recall that we have once for ever postulated fixed recourse.

The following simple statement is the key to successful processing of globalized robust counterparts; the assumption on $\mathcal{L}$ to be a cone rather than an arbitrary convex set is instrumental in achieving this goal.

Proposition 1 [19] *The semi-infinite constraint* (11) *is equivalent to the pair of semi-infinite constraints* (*see* (10))

$$f(y,\zeta) := \phi_0(\zeta) + \sum_{j=1}^{N} \phi_j(\zeta) y_j \in K \;\forall \zeta \in \mathcal{Z}, \tag{12}$$

$$\begin{aligned} &\mathrm{dist}(\underbrace{[\Phi_0 + \sum_{j=1}^{N} y_j \Phi_j]}_{\Phi[y]} \zeta, \mathrm{Rec}(K)) \equiv \min_{u \in \mathrm{Rec}(K)} \|\Phi[y]\zeta - u\|_K \le \alpha \\ &\forall \zeta \in \mathcal{L}_{\|\cdot\|_Z} \equiv \{\zeta \in \mathcal{L} : \|\zeta\|_Z \le 1\}, \end{aligned} \tag{13}$$

where $\mathrm{Rec}(K)$ *is the recessive cone of* K.

Remark 1 Sometimes (e.g., in Control applications to be considered in Sect. 5) it makes sense to add some structure to the construction of Globalized RC, specifically, to assume that the space $\mathbf{R}^d$ where ζ lives is given as a direct product: $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\nu}$, and both $\mathcal{Z}$, $\mathcal{L}$ are direct products as well: $\mathcal{Z} = \mathcal{Z}^1 \times \cdots \times \mathcal{Z}^\nu$, $\mathcal{L} = \mathcal{L}^1 \times \cdots \times \mathcal{L}^\nu$, where $\mathcal{Z}^i$, $\mathcal{L}^i$ are closed convex sets/cones in $\mathbf{R}^{d_i}$. Given norms $\|\cdot\|_{Z^i}$ on $\mathbf{R}^{d_i}$, $i = 1, \ldots, \nu$, we can impose requirement (11) in the structured form

$$\mathrm{dist}(f(y,\zeta), K) \le \sum_{i=1}^{\nu} \alpha_i \mathrm{dist}(\zeta^i, \mathcal{Z}^i | \mathcal{L}^i) \;\forall \zeta \in \mathcal{Z}\mathcal{L} = \mathcal{Z} + \mathcal{L}, \tag{14}$$

where ζ^i is the projection of ζ onto $\mathbf{R}^{d_i}$, $\mathrm{dist}(\zeta^i, \mathcal{Z}^i|\mathcal{L}^i)$ is defined in terms of $\|\cdot\|_{Z^i}$, and $\alpha_i \ge 0$ are "partial global sensitivities". The associated "structured" version of Proposition 1, see [19], states that (14) is equivalent to the system of semi-infinite constraints

$$\begin{aligned} &f(y,\zeta) := \phi_0(\zeta) + \textstyle\sum_{j=1}^{N} \phi_j(\zeta) y_j \in K \;\forall \zeta \in \mathcal{Z}, \\ &\mathrm{dist}(\Phi[y] E_i \zeta^i, \mathrm{Rec}(K)) \le \alpha_i \;\; \forall i \; \forall \zeta^i \in \mathcal{L}^i_{\|\cdot\|_{Z^i}} \equiv \{\zeta^i \in \mathcal{L}^i : \|\zeta^i\|_{Z^i} \le 1\}, \end{aligned}$$

where E_i is the natural embedding of $\mathbf{R}^{d_i}$ into $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\nu}$.

3 Tractability of robust counterparts

Here we address the crucial issue of computational tractability of robust counterparts of an uncertain problem. The "tractability framework" we use here (what "computational tractability" actually means) is standard for continuous optimization; its description can be found, e.g., in [12]. In our context, a reader will lose nearly nothing when interpreting computational tractability of a convex set X as the fact that X is given by *semidefinite representation* $X = \{x : \exists u : \mathcal{A}(x,u) \succeq 0\}$, where $\mathcal{A}(x,u)$ is a symmetric matrix affinely depending on x, u. "Computational tractability" of a system of convex constraints is then the fact that we can point out a semidefinite representation of its solution set. "Efficient solvability" of a convex optimization problem $\min_{x\in X} c^T x$ means that the feasible set X is computationally tractable.

As we have seen in the previous section, under our basic restrictions (bi-affinity and convexity of the uncertain problem plus fixed recourse when speaking about affinely adjustable counterparts) the robust counterparts are semi-infinite convex problems with linear objective and bi-affine semi-infinite constraints of the form

$$f(y,\zeta) \equiv f(y) + F(y)\zeta \in Q \ \forall \zeta \in \mathcal{U}, \tag{15}$$

where $f(y), F(y)$ are vector and matrix affinely depending on the decision vector y and $Q, \mathcal{U}$ are closed convex sets. In order for such a problem to be computationally tractable, it suffices to build an explicit finite system $\mathcal{S}_{\mathrm{f}}$ of efficiently computable convex constraints, e.g., Linear Matrix Inequalities (LMI's) in our original design variables y and, perhaps, additional variables u which *represents* the feasible set Y of (15) in the sense that

$$Y = \{y : \exists u : (y,u) \text{ satisfies } \mathcal{S}_{\mathrm{f}}\}.$$

Building such a representation is a constraint-wise task, so that we can focus on building computationally tractable representation of a single semi-infinite constraint (15). Whether this goal is achievable, it depends on the tradeoff between the geometries of Q and $\mathcal{U}$: the simpler is $\mathcal{U}$, the more complicated Q can be. We start with two "extreme" cases which are good in this respect. The first of them is not too interesting; the second is really important.

3.1 "Scenario uncertainty": $\mathcal{Z} = \mathrm{Conv}\{\zeta^1, \ldots, \zeta^S\}$

In the case of scenario uncertainty, (15) is computationally tractable, provided that Q is so. Indeed, here (15) is equivalent to the finite system of tractable constraints

$$f(y) + F(y)\zeta^s \in Q, \quad s = 1, \ldots, S.$$

3.2 Uncertain linear programming: Q is a polyhedral set given by an explicit finite list of linear inequalities

In the case described in the title of this section, semi-infinite constraint (15) clearly reduces to a finite system of *scalar* bi-affine semi-infinite inequalities of the form

$$f(y) + F^T(y)\zeta \le 0 \ \forall \zeta \in \mathcal{U}. \tag{16}$$

with real-valued f and vector-valued F affinely depending on y, and all we need is computational tractability of such a scalar inequality. This indeed is the case when the set $\mathcal{U}$ is computationally tractable. We have, e.g., the following result.

Theorem 1 [7] *Assume that the closed convex set $\mathcal{U}$ is given by conic representation*

$$\mathcal{U} = \{\zeta : \exists u : P\zeta + Qu + p \in \mathbf{K}\}, \tag{17}$$

where $\mathbf{K}$ is either (i) *a nonnegative orthant $\mathbf{R}^k_+$, or* (ii) *a direct product of the Lorentz cones $\mathbf{L} = \{x \in \mathbf{R}^k : x_k \ge \sqrt{\sum_{i=1}^{k-1} x_i^2}\}$, or* (iii) *a semidefinite cone $\mathbf{S}^k_+$* (*the cone of positive semidefinite $k \times k$ matrices in the space $\mathbf{S}^k$ of symmetric $k \times k$ matrices equipped with the Frobenius inner product $\langle A, B\rangle = \mathrm{Tr}(AB)$*). *In the cases* (ii–iii), *assume that the representation in question is strictly feasible: $P\bar{\zeta} + Q\bar{u} + p \in \mathrm{int}\mathbf{K}$ for certain $\bar{\zeta}, \bar{u}$. Then the semi-infinite scalar constraint* (16) *can be represented by the following explicit and tractable system of convex constraints*

$$P^T w + F(y) = 0, Q^T w = 0, p^T w + f(y) \le 0, w \in \mathbf{K} \tag{18}$$

in variables y and additional variables w.

Proof We have

$$\begin{aligned}&\{y \text{ satisfies } (16)\} \Leftrightarrow \left\{\max_{\zeta,u}\left\{F^T(y)\zeta : P\zeta + Qu + p \in \mathbf{K}\right\} \le -f(y)\right\}\\ &\Leftrightarrow \left\{\exists w \in \mathbf{K} : P^T w + F(y) = 0, Q^T w = 0, p^T w + f(y) \le 0\right\}\end{aligned}$$

where the concluding $\Leftrightarrow$ is given by LP/Conic Duality. □

Corollary 1 *The RC* (9) *of an uncertain Linear Programming problem* (1) (*i.e., problem with bi-affine constraints and polyhedral sets K_i given by explicit finite lists of linear inequalities*) *is computationally tractable, provided that the uncertainty set $\mathcal{Z}$ is so. In particular, with $\mathcal{Z}$ given by representation* (17), *the RC is equivalent to an explicit Linear Programming* (*case* (i)) *or Conic Quadratic* (*case* (ii)), *or Semidefinite* (*case* (iii)) *problem.*

In the case of fixed recourse, the same is true for the AARC (8) *of the uncertain problem.*

Now consider the Globalized RC/AARC of an uncertain LP with computationally tractable perturbation structure, say, with the normal range $\mathcal{Z}$ of the data and the set $\mathcal{L}_{\|\cdot\|_Z}$, see (13), satisfying the premise of Theorem 1. By Proposition 1, the Globalized RC is tractable when the semi-infinite inclusions (12), (13) associated with the uncertain constraints of (1) are tractable. In the situation in question, tractability of (12) is readily given by Theorem 1, so that we may focus solely on the constraint (13). Tractability of the semi-infinite inclusion (13) coming from i-th constraint of the uncertain problem depends primarily on the structure of the recessive cone $\mathrm{Rec}(K_i)$ and the norm $\|\cdot\|_{K_i}$ used to measure the distance in the left hand side of (11). For example,

- Invoking Theorem 1, the semi-infinite inclusion (13) is computationally tractable under the condition that the set $\{u : \exists v \in \mathrm{Rec}(K_i) : \|u - v\|_{K_i} \leq \alpha\}$ in the right hand side of the inclusion can be described by an explicit finite list of linear inequalities. This condition is satisfied, e.g., when $\|\cdot\|_{K_i} = \|\cdot\|_\infty$ and the recessive cone $\mathrm{Rec}(K_i) \subset \mathbf{R}^{k_i}$ of K_i is given by "sign restrictions", that is, is comprised of all vectors with given restrictions on the signs of every one of the coordinates ("≥ 0", "≤ 0", "$= 0$", or no restriction at all).
- Another "good case" is the one where $K_i \subset \mathbf{R}^{k_i}$ is bounded, the associated cone $\mathcal{L}$ is the entire space $\mathbf{R}^d$ and the norms $\|\cdot\|_Z$, $\|\cdot\|_{K_i}$ form a "good pair" in the sense that one can compute efficiently the associated norm $\|A\|_{ZK_i} = \max\{\|A\zeta\|_{K_i} : \|\zeta\|_Z \leq 1\}$ of a $k_i \times d$ matrix. Indeed, in the case in question (13) becomes the efficiently computable convex constraint

$$\|\Phi_0 + \sum_j y_j \Phi_j\|_{ZK_i} \leq \alpha.$$

Examples of "good pairs" of norms include (a) $\|\cdot\|_Z = \|\cdot\|_1$, $\|\cdot\|_{K_i}$ efficiently computable, (b) $\|\cdot\|_Z$ efficiently computable, $\|\cdot\|_{K_i} = \|\cdot\|_\infty$, and (c) both $\|\cdot\|_Z$, $\|\cdot\|_{K_i}$ are Euclidean norms on the respective spaces.

We have presented sufficient tractability conditions for the Globalized RC of uncertain LP's; note that the first of these conditions is automatically satisfied in the case of "LP proper", where all K_i are one-dimensional. Note also that in the case of fixed recourse, exactly the same conditions ensure tractability of the Globalized AARC.

3.3 Uncertain conic quadratic programming

Now let us pass to uncertain *Conic Quadratic problems* (also called *Second Order Conic problems*). These are problems (1) with bi-affine objective and left hand sides of the constraints, and with the right hand sides sets K_i in the constraints given by finitely many *conic quadratic inequalities:*

$$K_i = \left\{u \in \mathbf{R}^{k_i} : \|A_{i\nu}u + b_{i\nu}\|_2 \leq c_{i\nu}^T u + d_{i\nu}, \nu = 1, \dots, N_i\right\}.$$

Here the issue of computational tractability of the RC clearly reduces to tractability of a *single* semi-infinite conic quadratic inequality

$$\|A[\zeta]y + b[\zeta]\|_2 \leq c^T[\zeta]y + d[\zeta]\ \forall \zeta \in \mathcal{Z}, \tag{19}$$

where $A[\zeta],\ldots,d[\zeta]$ are affine in ζ. Tractability of the constraint (19) depends on the geometry of $\mathcal{Z}$ and is a "rare commodity": already pretty simple uncertainty sets (e.g., boxes) can lead to intractable constraints. Aside from the trivial case of scenario uncertainty (Sect. 3.1), we know only two generic cases where (19) is computationally tractable [34,6,15]:

- the case where $\mathcal{Z}$ is an ellipsoid. Here (19) is computationally tractable, although we do not know a "well-structured", e.g., semidefinite, representation of this constraint;
- the case of "side-wise" uncertainty with ellipsoidal uncertainty set for the left hand side data. "Side-wise" uncertainty means that the uncertainty set is given as $\mathcal{Z} = \mathcal{Z}^l \times \mathcal{Z}^r$, the left hand side data $A[\zeta], b[\zeta]$ in (19) depend solely on the $\mathcal{Z}^l$-component of $\zeta \in \mathcal{Z}$, while the right hand side data $c[\zeta], d[\zeta]$ depend solely on the $\mathcal{Z}^r$-component of ζ. If, in addition, $\mathcal{Z}^l$ is an ellipsoid, and $\mathcal{Z}^r$ is a set satisfying the premise in Theorem 1, the semi-infinite constraint (19) can be represented by an explicit system of LMI's [34,6]; this system can be easily extracted from the system (22) below.

A natural course of actions in the case when a constraint in an optimization problem is intractable is to replace this constraint with its *safe tractable approximation* — a tractable (e.g., admitting an explicit semidefinite representation, see the beginning of Sect. 3) constraint with the feasible set contained in the one of the "true" constraint. When some of the constraints in the RC (9) are intractable, we can replace them with their safe tractable approximations, thus ending up with tractable problem which is "on the safe side" of the RC — all feasible solutions of the problem are robust feasible solutions of the underlying uncertain problem. Exactly the same approach can be used in the case of affinely adjustable and globalized RC's.

Now, there exist quite general ways to build safe approximations of semi-infinite conic (in particular, conic quadratic) inequalities [26]. These general techniques, however, do not specify how conservative are the resulting approximations. Here and in the next section we focus on a more difficult case of "tight" approximations — safe approximations with quantified level of conservatism which is (nearly) independent of the size and the values of the data. We start with quantifying the level of conservatism.

3.3.1 Level of conservatism

A simple way to quantify the level of conservatism of a safe approximation is as follows. In applications the uncertainty set $\mathcal{Z}$ is usually given as $\mathcal{Z} = \zeta_{\rm n} + \Delta\mathcal{Z}$, where $\zeta_{\rm n}$ is the nominal data and $\Delta\mathcal{Z} \ni 0$ is the set of "data perturbations".

Such an uncertainty set can be included in the single-parametric family

$$\mathcal{Z}_\rho = \zeta_{\rm n} + \rho\Delta\mathcal{Z}, \tag{20}$$

where $\rho \geq 0$ is the level of perturbations. In this case, a semi-infinite bi-affine constraint with uncertainty set $\mathcal{Z}$

$$A[\zeta]y + b[\zeta] \in Q \;\forall \zeta \in \mathcal{Z}$$

($A[\cdot], b[\cdot]$ are affine in ζ, Q is convex) becomes a member of the parametric family

$$A[\zeta]y + b[\zeta] \in Q \;\forall \zeta \in \mathcal{Z}_\rho = \zeta_{\rm n} + \rho\Delta\mathcal{Z}, \tag{U_ρ}$$

both the original uncertainty set and the original constraint corresponding to $\rho = 1$. Note that the feasible set Y_ρ of (U_ρ) shrinks as ρ grows. Now assume that the family of semi-infinite constraints (U_ρ), $\rho \geq 0$, is equipped with safe approximation, say, a semidefinite one:

$$\mathcal{A}_\rho(y, u) \succeq 0, \tag{A_ρ}$$

where $\mathcal{A}_\rho(y, u)$ is a symmetric matrix affinely depending on y and additional variables u. The fact that (A_ρ) is a safe approximation of (U_ρ) means that the projection $\widehat{Y}_\rho = \{y : \exists u : \mathcal{A}_\rho(y, u) \succeq 0\}$ of the feasible set of (A_ρ) onto the space of y-variables is contained in Y_ρ. We now can measure the conservatism of the approximation by its *tightness factor* defined as follows:

Definition 1 Consider a parametric family of semi-infinite constraints (U_ρ), and let (A_ρ) be its safe approximation. We say that the approximation is tight within factor $\vartheta \geq 1$, if, for every uncertainty level ρ, the feasible set $\widehat{Y}_\rho$ of the approximation is in-between the feasible set Y_ρ of the true constraint and the feasible set $Y_{\vartheta\rho}$ of the true constraint with increased by factor ϑ uncertainty level:

$$Y_{\vartheta\rho} \subset \widehat{Y}_\rho \subset Y_\rho \;\; \forall \rho \geq 0,$$

in which case we refer to ϑ as to level of conservatism (or tightness factor) of the approximation.

3.3.2 Tight tractable approximations of semi-infinite conic quadratic constraints

To the best of our knowledge, the strongest known result on tight tractable approximation of semi- infinite conic quadratic constraint (19) is as follows.

Theorem 2 [15] *Consider the semi-infinite conic quadratic constraint with side-wise uncertainty*

$$\|\underbrace{A[\zeta^l]y+b[\zeta^l]}_{\equiv p(y)+P(y)\Delta\zeta^l}\|_2 \le \underbrace{c^T[\zeta^r]y+d[\zeta^r]}_{\equiv q(y)+r^T(y)\Delta\zeta^r}\ \forall \begin{pmatrix} \zeta^l\equiv\zeta^l_{\rm n}+\Delta\zeta^l,\ \Delta\zeta^l\in\rho\Delta\mathcal{Z}^l,\\ \zeta^r=\zeta^r_{\rm n}+\Delta\zeta^r,\Delta\zeta^r\in\rho\Delta\mathcal{Z}^r\end{pmatrix}, \tag{21}$$

where $A[\cdot],\ldots,d[\cdot]$ (and, consequently, $p(\cdot),\ldots,r(\cdot)$) are affine in their arguments. Assume that the left hand side perturbation set is the intersection of ellipsoids centered at 0:

$$\Delta\mathcal{Z}^l=\left\{\zeta^l:[\zeta^l]^TQ_\ell\zeta^l\le 1,\ \ell=1,\ldots,L\right\}\quad\left[Q_\ell\succeq 0,\sum_\ell Q_\ell\succ 0\right]$$

while the right hand side perturbation set is given by strictly feasible semidefinite representation by an $N\times N$ LMI:

$$\Delta\mathcal{Z}^r=\left\{\zeta^r:\exists u:\mathcal{C}\zeta^r+\mathcal{D}u+E\succeq 0\right\}.$$

Consider the following system of semidefinite constraints in variables $\tau\in\mathbf{R},\lambda\in\mathbf{R}^L,V\in\mathbf{S}^N,y$:

$$\begin{array}{c}\left[\begin{array}{c|c|c}\tau-\sum_\ell\lambda_\ell & 0 & p^T(y)\\ \hline 0 & \sum_\ell\lambda_\ell Q_\ell & \rho P^T(y)\\ \hline p(y) & \rho P(y) & \tau I\end{array}\right]\succeq 0,\ \lambda\ge 0,\\ \mathcal{C}^*V=r(y),\mathcal{D}^*V=0,\tau\le q(y)-\mathrm{Tr}(VE),V\succeq 0,\end{array} \tag{22}$$

where $V\mapsto\mathcal{C}^*V$, $V\mapsto\mathcal{D}^*V$ are the linear mappings adjoint to the mappings $\zeta^r\mapsto\mathcal{C}\zeta^r$, $u\mapsto\mathcal{D}u$, respectively. Then (22) *is a safe approximation of* (21)*, and the tightness factor ϑ of this approximation can be bounded as follows:*

1. *In the case $L=1$* (*simple ellipsoidal uncertainty*) *the approximation is exact: $\vartheta=1$;*
2. *In the case of box uncertainty in the left hand side data* ($L=\dim\zeta^l$ *and* $[\zeta^l]^TQ_\ell\zeta^l\equiv(\zeta^l)^2_\ell$, $\ell=1,\ldots,L$) *one has $\vartheta=\frac{\pi}{2}$;*
3. *In the general case, one has $\vartheta=\left(2\ln\left(6\sum_{\ell=1}^L\mathrm{Rank}(Q_\ell)\right)\right)^{1/2}$.* [4]

Theorem 2 provides sufficient conditions, expressed in terms of the geometry of the uncertainty set, for the RC of an uncertain conic quadratic problems to be tractable or to admit tight tractable approximations. In the case of fixed recourse, exactly the same results are applicable to the AARC's of uncertain conic quadratic problems. As about the Globalized RC/AARC, these results

[4] With recent results on large deviations of vector-valued martingales from [50], this bound on ϑ can be improved to $\vartheta=\sqrt{O(1)\ln(L+1)}$.

cover the issue of tractability/tight tractable approximations of the associated semi-infinite constraints of the form (12). The remaining issue – the one of tractability of the semi-infinite constraints (13) — has to do with the geometry of the *recessive cones* of the sets K_i rather than these sets themselves (and, of course, with the geometry of the uncertainty structure). The sufficient conditions for tractability of the constraints (13) presented at the end of Sect. 3.2 work for uncertain conic quadratic problems (same as for uncertain semidefinite problems to be considered in the Sect. 3.4).

Back to uncertain LP: *tight approximations of the AARC in absence of fixed recourse.* When there is no fixed recourse, the only positive tractability result on AARC is the one where the uncertainty set is an ellipsoid (see the discussion on quadratic lifting in Sect. 2.2). What we intend to add here, is that the AARC of uncertain LP without fixed recourse admits a tight tractable approximation, provided that the perturbation set $\Delta\mathcal{Z}$ is the intersection of L ellipsoids centered at 0 [17]. Specifically, the semi-infinite constraints comprising the AARC of an uncertain LP with uncertainty set (20) are of the form

$$\Delta\zeta^T\Gamma(y)\Delta\zeta + 2\gamma^T(y)\Delta\zeta \le c(y)\ \forall\zeta \in \rho\Delta\mathcal{Z} \tag{23}$$

with $\Gamma(\cdot)$, $\gamma(\cdot)$, $c(\cdot)$ affine in the decision vector y of the AARC. Assuming

$$\Delta\mathcal{Z} = \{\Delta\zeta : \Delta\zeta^T Q_\ell \Delta\zeta \le 1,\ \ell = 1,\dots,L\} \quad \left[Q_\ell \succeq 0, \sum_\ell Q_\ell \succ 0\right] \tag{24}$$

and applying the standard semidefinite relaxation, the system of LMI's

$$\left[\begin{array}{c|c} \sum_{\ell=1}^L \lambda_\ell Q_\ell - \Gamma(y) & -\gamma(y) \\ \hline -\gamma^T(y) & \lambda_0 \end{array}\right] \succeq 0,\ \lambda_0 + \rho^2\sum_{\ell=1}^L \lambda_\ell \le c(y),\ \lambda_0,\dots,\lambda_L \ge 0,$$

in variables $\lambda_0,\dots,\lambda_L, y$, is a safe approximation of (23). By the $\mathcal{S}$-Lemma, this approximation is exact when $L = 1$, which recovers the "quadratic lifting" result. In the case of $L > 1$, by "approximate $\mathcal{S}$-Lemma" [15], the tightness factor of the approximation is at most $\vartheta = 2\ln(6\sum_\ell \mathrm{Rank}(Q_\ell))$; here again [50] allows to improve the factor to $\vartheta = O(1)\ln(L+1)$. Thus, with perturbation set (24), the AARC of uncertain LP admits a safe approximation with "nearly data-independent" level of conservatism $O(\ln(L+1))$.

3.4 Uncertain semidefinite problems

Finally, consider *uncertain semidefinite problems* — problems (1) with constraints having bi-affine left hand sides and the sets K_i given by explicit finite lists of LMI's:

$$K_i = \{u \in \mathbf{R}^{k_i} : \mathcal{A}_{i\nu}u - B_{i\nu} \succeq 0,\ \nu = 1,\dots,N_i\}.$$

Here the issue of tractability of the RC reduces to the same issue for an *uncertain LMI*

$$A(y,\zeta) \succeq 0 \;\forall \zeta \in \mathcal{Z}_\rho = \zeta_{\rm n} + \rho\Delta\mathcal{Z}. \tag{25}$$

Aside from the trivial case of scenario uncertainty (see Sect. 3.1), seemingly the only generic case where (25) is tractable is the case of *unstructured norm-bounded perturbation*, where

$$\begin{array}{l}\Delta\mathcal{Z} = \{\Delta\zeta \in \mathbf{R}^{p_i \times q_i} : \|\Delta\zeta\| \le 1\},\\ A(y,\zeta_{\rm n} + \Delta\zeta) = A_{\rm n}(y) + [L^T\Delta\zeta R(y) + R^T(y)\Delta\zeta^T L];\end{array}$$

here $\|\cdot\|$ is the usual matrix norm (maximal singular value), and $A_{\rm n}(y)$, $R(y)$ are affine in y. This is a particular case of what in Control is called a *structured* norm-bounded perturbation, where

$$\begin{array}{l}\Delta\mathcal{Z} = \big\{\Delta\zeta = (\Delta\zeta_1,\ldots,\Delta\zeta_P) \in \mathbf{R}^{d_1\times d_1} \times\cdots\times \mathbf{R}^{d_P\times d_P} : \|\Delta\zeta_p\| \le 1,\\ \qquad\qquad p = 1,\ldots,P,\; \Delta\zeta_p = \delta_p I_{d_p},\; p \in I_{\rm s}\big\},\\ A(y,\zeta_{\rm n} + \Delta\zeta) = A_{\rm n}(y) + \sum_{p=1}^P [L_p^T\Delta\zeta_p R_p(y) + R_p^T(y)\Delta\zeta_p^T L_p].\end{array} \tag{26}$$

Note that uncertain semidefinite problems with norm-bounded and structured norm-bounded perturbations are typical for Robust Control applications, e.g., in Lyapunov Stability Analysis/Synthesis of linear dynamical systems with uncertain dynamics (see, e.g., [28]). Another application comes from robust settings of the *obstacle-free Structural Design problem* with uncertain external load [5,8]. The corresponding RC has a single uncertainty-affected constraint of the form

$$A(y) + [e\zeta^T + \zeta e^T] \succeq 0 \;\forall (\zeta \in \mathcal{Z}),$$

where ζ represents external load and $\mathcal{Z} = \Delta\mathcal{Z}$ is an ellipsoid.

To the best of our knowledge, the strongest result on tractability/tight tractable approximation of a semi-infinite LMI with norm-bounded structured perturbation is the following statement:

Theorem 3 ([16], see also [14]) *Consider semi-infinite LMI with structured norm-bounded perturbations* (25), (26) *along with the system of LMI's*

$$\begin{array}{c} Y_p \pm \left[L_p^T R_p(y) + R_p^T(y) L_p\right] \succeq 0,\; p \in I_{\rm s},\; \begin{bmatrix} Y_p - \lambda_p L_p^T L_p & R_p^T(y)\\ R_p(y) & \lambda_p I_{d_p}\end{bmatrix} \succeq 0,\; p \notin I_{\rm s}\\ A_{\rm n}(y) - \rho\sum_{p=1}^P Y_p \succeq 0 \end{array} \tag{27}$$

in variables Y_p, λ_p, y. Then system (27) *is a safe tractable approximation of* (25), (26)*, and the tightness factor ϑ of this approximation can be bounded as follows:*

1. *in the case of $P = 1$ (unstructured norm-bounded perturbation), the approximation is exact: $\vartheta = 1$;*[5]
2. *In the case $P > 1$, let*

$$\mu = \begin{cases} 0, & I_s \text{ is empty or } d_p = 1 \text{ for all } p \in I_s \\ \max\{d_p : p \in I_s\}, & \text{otherwise} \end{cases}.$$

Then $\vartheta \le \vartheta_*(\mu)$, where $\vartheta_*(\mu)$ is a certain universal function satisfying

$$\vartheta_*(0) = \pi/2, \vartheta_*(2) = 2, \mu > 2 \Rightarrow \vartheta_*(\mu) \le \pi\sqrt{\mu/2}.$$

In particular, if $P > 1$ and there are no scalar perturbation blocks ($I_s = \emptyset$), the tightness factor is $\le \pi/2$.

For extensions of Theorem 3 to the Hermitian case and its applications in Control, see [14,16].

4 Robust optimization and chance constraints

4.1 Chance constrained uncertain LP

Robust Optimization does not assume the uncertain data to be of stochastic nature; however, if this is the case, the corresponding information can be used to define properly the uncertainty set for the RC and the AARC of the uncertain problem, or the normal range of the data for the Globalized RC/AARC. We intend to consider this issue in the simplest case of "uncertain LP proper", that is, the case of uncertain problem (1) with bi-affine left hand sides of the constraints and with the nonpositive rays $\mathbf{R}_-$ in the role of K_i. Assume that we solve (1) in affine decision rules (7) (which includes as special case non-adjustable x_j as well). Assuming fixed recourse, the constraints of the resulting uncertain problem are of the form

$$f_{i0}(y) + \sum_{\ell=1}^{d} \zeta_\ell f_{i\ell}(y) \le 0, \quad i = 0, \ldots, m, \tag{28}$$

where the real-valued functions $f_{i\ell}(y)$ are affine in the decision vector $y = (t, \{\eta_j^0, \eta_j\}_{j=1}^n)$ (see (1), (7)). Assuming that the uncertain data ζ are random with a partially known probability distribution P, a natural way to "immunize" the constraints w.r.t. data uncertainty is to pass to the *chance constrained* version of the uncertain problem, where the original objective t is minimized over

[5] This fact was established already in [28].

the feasible set of chance constraints

$$\mathrm{Prob}_{\zeta\sim P}\left\{f_{i0}(y)+\sum_{\ell=1}^{d}\zeta_\ell f_{i\ell}(y)\leq 0\right\}\geq 1-\epsilon,\quad i=0,\ldots,m\ \forall P\in\mathcal{P},\tag{29}$$

where $\epsilon << 1$ is a given tolerance and $\mathcal{P}$ is the family of all probability distributions compatible with our a priori information. This approach was proposed as early as in 1958 by Charnes et al. [30] and was extended further by Miller and Wagner [49] and Prékopa [57]. Since then it was discussed in numerous publications (see Prékopa [58–60] and references therein). While being quite natural, this approach, unfortunately, has a too restricted field of applications, due to severe computational difficulties. First, in general it is difficult already to check the validity of a chance constraint at a given candidate solution, especially when ϵ is small (like 1.e−4 or less). Second, the feasible domain of a chance constraint, even as simple looking as (29), is usually nonconvex. While these difficulties can sometimes be avoided (most notably, when P is a Gaussian distribution), in general chance constraints (29), even those with independent ζ_ℓ and with exactly known distributions, are severely computationally intractable. Whenever this is the case, the natural course of actions is to replace the chance constraints with their safe tractable approximations. We are about to consider a specific *Bernstein* approximation originating from [56] and significantly improved in [51].

4.1.1 Bernstein approximations of chance constraints

Consider a chance constraint of the form of (29):

$$\mathrm{Prob}_{\zeta\sim P}\left\{f_0(y)+\sum_{\ell=1}^{d}\zeta_\ell f_\ell(y)\leq 0\right\}\geq 1-\epsilon,\quad \forall P\in\mathcal{P}\tag{30}$$

and let us make the following assumption

B.1. $\mathcal{P}=\{P=P_1\times\cdots\times P_d: P_\ell\in\mathcal{P}_\ell\}$ (that is, the components $\zeta_1,\ldots,\zeta_d$ of ζ are known to be independent of each other with marginal distributions P_ℓ belonging to given families $\mathcal{P}_\ell$ of probability distributions on the axis), where every $\mathcal{P}_\ell$ is a *-compact convex set, and all distributions from $\mathcal{P}_\ell$ have a common bounded support.

Replacing, if necessary, the functions $f_\ell(\cdot)$ with their appropriate linear combinations, we can w.l.o.g. normalize the situation by additional assumption

B.2. The distributions from $\mathcal{P}_\ell$ are supported on $[-1,1]$, that is, ζ_ℓ are known to vary in the range $[-1,1]$, $1\leq\ell\leq d$.

Let us set

$$\Lambda_\ell(z)=\max_{P_\ell\in\mathcal{P}_\ell}\ln\left(\int\exp\{zs\}\mathrm{d}P_\ell(s)\right):\mathbf{R}\to\mathbf{R}.$$

It is shown in [51] that the function

$$\Psi(t,y) = f_0(y) + t\sum_{\ell=1}^{d} \Lambda_\ell(t^{-1}f_\ell(y)) + t\ln(1/\epsilon)$$

is convex in $(t > 0, y)$, and the *Bernstein approximation* of (30) — the convex inequality

$$\inf_{t>0} \Psi(t,y) \le 0 \tag{31}$$

— is a *safe approximation* of the chance constraint: if y satisfies (31), then y satisfies the chance constraint. Note that this approximation is tractable, provided that $\Lambda_\ell(\cdot)$ are efficiently computable.

Now consider the case when

$$\Lambda_\ell(z) \le \max[\mu_\ell^- z, \mu_\ell^+ z] + \frac{\sigma_\ell^2}{2} z_\ell^2, \quad \ell = 1,\dots,d \tag{32}$$

with appropriately chosen parameters $-1 \le \mu_\ell^- \le \mu_\ell^+ \le 1$, $\sigma_\ell \ge 0$. Then the left hand side in (31) can be bounded from above by

$$\begin{aligned}&\inf_{t>0}\left[f_0(y) + \textstyle\sum_{\ell=1}^d \max[\mu_\ell^- f_\ell(y), \mu_\ell^+ f_\ell(y)] + \frac{t^{-1}}{2}\sum_{\ell=1}^d \sigma_\ell^2 f_\ell^2(y) + t\ln(1/\epsilon)\right]\\ &\quad = f_0(y) + \textstyle\sum_{\ell=1}^d \max[\mu_\ell^- f_\ell(y), \mu_\ell^+ f_\ell(y)] + \sqrt{2\ln(1/\epsilon)}\left(\sum_{\ell=1}^d \sigma_\ell^2 f_\ell^2(y)\right)^{1/2}\end{aligned}$$

so that the explicit convex constraint

$$f_0(y) + \sum_{\ell=1}^d \max[\mu_\ell^- f_\ell(y), \mu_\ell^+ f_\ell(y)] + \sqrt{2\ln(1/\epsilon)}\left(\sum_{\ell=1}^d \sigma_\ell^2 f_\ell^2(y)\right)^{1/2} \le 0 \tag{33}$$

is a safe approximation of (30), somewhat more conservative than (31).

In fact we can reduce slightly the conservatism of (33):

Proposition 2 *Let assumptions* **B.1-2** *and relation* (32) *be satisfied. Then, for every* $\epsilon \in (0,1)$*, the system of constraints*

$$\begin{aligned}&f_0(y) + \sum_{\ell=1}^d |z_\ell| + \sum_{\ell=1}^d \max[\mu_\ell^- w_\ell, \mu_\ell^+ w_\ell] + \sqrt{2\ln(1/\epsilon)}\left(\sum_{\ell=1}^d \sigma_\ell^2 w_\ell^2\right)^{1/2} \le 0,\\ &f_\ell(y) = z_\ell + w_\ell, \quad \ell = 1,\dots,d\end{aligned} \tag{34}$$

in variables y, z, w *is a safe approximation of the chance constraint* (30).

Note that (34) is less conservative than (33); indeed, whenever y is feasible for the latter constraint, the collection $y, \{z_\ell = 0, w_\ell = f_\ell(y)\}_{\ell=1}^d$ is feasible for the former system of constraints.

Proof of Proposition 2 Let y, z, w be a solution to (34). For $P = P_1 \times \cdots \times P_d \in \mathcal{P}$ we have

$$\begin{aligned}
&\mathrm{Prob}_{\zeta\sim P}\left\{f_0(y) + \sum_{\ell=1}^d \zeta_\ell f_\ell(y) > 0\right\} \\
&\quad = \mathrm{Prob}_{\zeta\sim P}\left\{f_0(y) + \sum_{\ell=1}^d \zeta_\ell z_\ell + \sum_{\ell=1}^d \zeta_\ell w_\ell > 0\right\} \\
&\quad \le \mathrm{Prob}_{\zeta\sim P}\{\underbrace{[f_0(y) + \sum_{\ell=1}^d |z_\ell|]}_{w_0} + \sum_{\ell=1}^d \zeta_\ell w_\ell > 0\} \quad [\text{by } \mathbf{B.2}].
\end{aligned}$$

On the other hand, from (34) it follows that

$$w_0 + \sum_{\ell=1}^d \max[\mu_\ell^- w_\ell, \mu_\ell^+ w_\ell] + \sqrt{2\ln(1/\epsilon)}\left(\sum_{\ell=1}^d \sigma_\ell^2 w_\ell^2\right)^{1/2} \le 0$$

(cf. (33)), whence $\mathrm{Prob}_{\zeta\sim P}\left\{w_0 + \sum_{\ell=1}^d \zeta_\ell w_\ell > 0\right\} \le \epsilon$ by arguments preceding the formulation of the proposition. □

Corollary 2 *Given $\epsilon \in (0,1)$, consider the system of constraints*

$$\begin{aligned}
&f_0(y) + \sum_{\ell=1}^d |z_\ell| + \sum_{\ell=1}^d \max[\mu_\ell^- w_\ell, \mu_\ell^+ w_\ell] + \sqrt{2d\ln(1/\epsilon)} \max_{1\le\ell\le d} \sigma_\ell |w_\ell| \le 0, \\
&f_\ell(y) = z_\ell + w_\ell, \quad \ell = 1, \dots, d
\end{aligned} \tag{35}$$

in variables y, z, w. Under the premise of Proposition 2, this system is a safe approximation of the chance constraint (30).

Indeed, for a d-dimensional vector e we clearly have $\|e\|_2 \le \sqrt{d}\|e\|_\infty$, so that the feasible set of (34) is contained in the one of (34).

4.2 Approximating chance constraints via robust optimization

An immediate follow-up to Proposition 2 is the following observation (we skip its straightforward proof):

Proposition 3 *A given y can be extended to a solution (y, z, w) of (34) iff*

$$\max_{\zeta \in \mathcal{Z}} \left[f_0(y) + \sum_{\ell=1}^{d} \zeta_\ell f_\ell(y) \right] \leq 0, \tag{36}$$

where

$$\mathcal{Z} = \mathcal{B} \cap [\mathcal{M} + \mathcal{E}] \tag{37}$$

and $\mathcal{B}$ is the unit box $\{\zeta \in \mathbf{R}^d : \|\zeta\|_\infty \leq 1\}$, $\mathcal{M}$ is the box $\{\zeta : \mu_\ell^- \leq \zeta_\ell \leq \mu_\ell^+, 1 \leq \ell \leq d\}$, and $\mathcal{E}$ is the ellipsoid $\{\zeta : \sum_{\ell=1}^d \zeta_\ell^2/\sigma_\ell^2 \leq 2\ln(1/\epsilon)\}$.

Similarly, y can be extended to a feasible solution of (35) iff y satisfies (36) with

$$\mathcal{Z} = \mathcal{B} \cap [\mathcal{M} + \mathcal{D}], \tag{38}$$

where $\mathcal{B}$, $\mathcal{M}$ are as above and $\mathcal{D}$ is the scaled $\|\cdot\|_1$–ball $\{\zeta : \sum_{\ell=1}^d |\zeta_l|/\sigma_\ell \leq \sqrt{2d\ln(1/\epsilon)}\}$.

In other words, (34) represents the RC of the uncertain constraint

$$f_0(y) + \sum_{\ell=1}^{d} \zeta_\ell f_\ell(y) \leq 0 \tag{39}$$

equipped with the uncertainty set (37), and (35) represents the RC of the same uncertain constraint equipped with the uncertainty set (38).

4.2.1 Discussion

A. We see that in the case of uncertain LP with random data ζ satisfying **B.1-2** and (32), there exists a way to associate with the problem an “artificial” uncertainty set $\mathcal{Z}$ (given either by (37), or by (38)) in such a way that the resulting robust solutions — those which remain feasible for *all* realizations $\zeta \in \mathcal{Z}$ — remain feasible for “nearly all”, up to probability ϵ, realizations of the random data of every one of the constraints. Note that this result holds true both when solving our uncertain LP in non-adjustable decision variables and when solving the problem in affine decision rules, provided fixed recourse takes place. As a result, we get a possibility to build computationally tractable safe approximations of chance constrained LP problems in the forms of RC’s/AARC’s taken with respect to a properly defined simple uncertainty sets $\mathcal{Z}$. By itself, this fact is not surprising — in order to “immunize” a constraint (39) with random data ζ against “$(1-\epsilon)$–part” of realizations of the data, we could take a convex set $\mathcal{Z}$ such that $\text{Prob}\{\zeta \in \mathcal{Z}\} \geq 1-\epsilon$ and then “immunize” the constraint against *all* data $\zeta \in \mathcal{Z}$ by passing to the associated RC. What is surprising, is that this naive approach has nothing in common with (37), (38) — these relations can produce uncertainty sets which are incomparably smaller than those given by the naive approach, and thus result in essentially less conservative approximations of chance constraints than those given by the naive approach.

> Here is an instructive example. Assume that all we know on the random data ζ is that $\zeta_1, \ldots, \zeta_d$ are mutually independent, take their values in $[-1, 1]$ and have zero means. It is easily seen that in this case one can take in (32) $\mu_\ell^\pm = 0$, $\sigma_\ell = 1$, $\ell = 1, \ldots, d$. With these parameters, the set $\mathcal{Z}_I$ given by (37) is the intersection of the unit box with the (centered at the origin) Euclidean ball of radius $\Omega = \sqrt{2\ln(1/\epsilon)}$ ("ball-box" uncertainty, cf. [7,9]), while the set $\mathcal{Z}_{II}$ given by (38) is the intersection of the same unit box and the $\|\cdot\|_1$-ball of the radius $\Omega\sqrt{d}$ ("budgeted uncertainty" of Bertsimas and Sim). Observe that when, say, ζ is uniformly distributed on the vertices of the unit box (i.e., ζ_ℓ take, independently of each other, the values ± 1 with probabilities 0.5) and the dimension d of this box is large, the probability for ζ to take its value in $\mathcal{Z}_I$ or $\mathcal{Z}_{II}$ is *exactly zero*, and both $\mathcal{Z}_I$ and $\mathcal{Z}_{II}$ become incomparably smaller, w.r.t. all natural size measures, than the natural domain of ζ – the unit box.

B. A natural question arising in connection with the safe tractable approximations (34), (35) of the chance constraint (30) is as follows: since the argument used in justification of Corollary 2 shows that the second approximation is more conservative than the first one, then why should we use (35) at all? The answer is, that the second approximation can be represented by a short system of *linear* inequalities, so that the associated safe approximation of the chance constrained LP of interest is a usual LP problem. In contrast to this, the safe approximation of the chance constrained problem given by (34) is a conic quadratic program, which is more computationally demanding (although still tractable) than an LP program of similar sizes. For this reason, "budgeted uncertainty" may be more appealing for large scale applications than the "ball-box" one.

C. A good news about the outlined safe approximations of chance constrained LP's is that under assumptions **B.1-2**, it is usually easy to point out explicitly the parameters $\mu_\ell^\pm$, σ_ℓ required by (32). In Table 1, we present these parameters for a spectrum of natural families $\mathcal{P}_\ell$.

5 An application of RO methodology in robust linear control

This section is devoted to a novel application of the RO methodology, recently developed in [19,20], to Robust Linear Control.

Table 1 Parameters $\mu_\ell^\pm, \sigma_\ell$ for "typical" families $\mathcal{P}_\ell$

$\mathcal{P}_\ell$ is given by	μ_ℓ^-	μ_ℓ^+	σ_ℓ
$\mathrm{supp}(P) \subset [-1,1]$	-1	1	0
$\mathrm{supp}(P) \subset [-1,1]$ P is uniomodal w.r.t. 0	$-1/2$	$1/2$	$\sqrt{1/12}$
$\mathrm{supp}(P) \subset [-1,1]$ P is uniomodal w.r.t. 0 P is symmetric w.r.t. 0	0	0	$\sqrt{1/3}$
$\mathrm{supp}(P) \subset [-1,1]$ $[-1 <]\ \mu^- \leq \mathrm{Mean}[P] \leq \mu^+\ [< 1]$	μ^-	μ^+	$\Sigma_1(\mu^-, \mu^+, 1)$, see (40$a$)
$\mathrm{supp}(P) \subset [-1,1]$ $[-\nu \leq]\ \mu^- \leq \mathrm{Mean}[P] \leq \mu^+\ [\leq \nu]$ $\mathrm{Var}[P] \leq \nu^2\ [\leq 1]$	μ^-	μ^+	$\Sigma_1(\mu^-, \mu^+, \nu)$, see (40$a$)
$\mathrm{supp}(P) \subset [-1,1]$ P is symmetric w.r.t. 0 $\mathrm{Var}[P] \leq \nu^2\ [\leq 1]$	0	0	$\Sigma_2(\nu)$, see (40b)
$\mathrm{supp}(P) \subset [-1,1]$ P is symmetric w.r.t. 0 P is unimodal w.r.t. 0 $\mathrm{Var}[P] \leq \nu^2\ [\leq 1/3]$	0	0	$\Sigma_3(\nu)$, see (40c)

For a probability distribution P on the axis, we set $\mathrm{Mean}[P] = \int s \mathrm{d}P(s)$ and $\mathrm{Var}[P] = \int s^2 \mathrm{d}P(s)$. The functions $\Sigma_\ell(\cdot)$ are as follows:

$$
\begin{array}{ll}
\text{(a)} & \Sigma_1(\mu^-, \mu^+, \nu) = \min\left\{c \geq 0 : h_{\mu,\nu}(t) \leq \max[\mu^- t, \mu^+ t] + \frac{c^2}{2}t^2\ \forall \left(\begin{smallmatrix}\mu\in[\mu^-,\mu^+]\\ t\end{smallmatrix}\right)\right\}, \\
& h_{\mu,\nu}(t) = \ln\left(\left\{\begin{array}{ll} \frac{(1-\mu)^2\exp\{t\frac{\mu-\nu^2}{1-\mu}\}+(\nu^2-\mu^2)\exp\{t\}}{1-2\mu+\nu^2}, & t \geq 0 \\ \frac{(1+\mu)^2\exp\{t\frac{\mu+\nu^2}{1+\mu}\}+(\nu^2-\mu^2)\exp\{-t\}}{1+2\mu+\nu^2}, & t \leq 0 \end{array}\right.\right) \\
\text{(b)} & \Sigma_2(\nu) = \min_c\left\{c \geq 0 : \ln\left(\nu^2\cosh(t) + 1 - \nu^2\right) \leq \frac{c^2}{2}t^2\ \forall t\right\} \\
\text{(c)} & \Sigma_3(\nu) = \min\left\{c \geq 0 : \ln\left(1 - 3\nu^2 + 3\nu^2\frac{\sinh(t)}{t}\right) \leq \frac{c^2}{2}t^2\ \forall t\right\}
\end{array}
\tag{40}
$$

5.1 Robust affine control over finite time horizon

Consider a discrete time linear dynamical system

$$
\begin{aligned}
x_0 &= z, \\
x_{t+1} &= A_t x_t + B_t u_t + R_t d_t, \quad t = 0, 1, \ldots, \\
y_t &= C_t x_t + D_t d_t,
\end{aligned}
\tag{41}
$$

where $x_t \in \mathbf{R}^{n_x}$, $u_t \in \mathbf{R}^{n_u}$, $y_t \in \mathbf{R}^{n_y}$ and $d_t \in \mathbf{R}^{n_d}$ are the state, the control, the output and the exogenous input (disturbance) at time t, and A_t, B_t, C_t, D_t, R_t are known matrices of appropriate dimensions.

A typical problem of (finite-horizon) Linear Control associated with the "open loop" system (41) is to "close" the system by a non-anticipative affine output-based control law

$$u_t = g_t + \sum_{\tau=0}^{t} G_{t\tau} y_\tau \tag{42}$$

(where the vectors g_t and matrices $G_{t\tau}$ are the parameters of the control law) in order for the closed loop system (41), (42) to meet prescribed design specifications. We assume that these specifications are represented by a system of linear inequalities

$$Aw^{\mathcal{T}} \leq b_t \tag{43}$$

on the *state-control trajectory* $w^{\mathcal{T}} = (x_0, \dots, x_{\mathcal{T}+1}, u_0, \dots, u_{\mathcal{T}})$ over a given finite time horizon $t = 0, 1, \dots, \mathcal{T}$.

An immediate observation is that for a given control law (42), the dynamics (41) specifies the trajectory as an affine function of the initial state z and the sequence of disturbances $d^{\mathcal{T}} = (d_0, \dots, d_{\mathcal{T}})$:

$$w^{\mathcal{T}} = w_0^{\mathcal{T}}[\gamma] + W^{\mathcal{T}}[\gamma]\zeta, \ \zeta = (z, d^{\mathcal{T}}),$$

where $\gamma = \{g_t, G_{t\tau}, 0 \leq \tau \leq t \leq \mathcal{T}\}$, is the "parameter" of the underlying control law (42). Substituting this expression for $w^{\mathcal{T}}$ into (43), we get the following system of constraints on the decision vector γ:

$$A\left[w_0^{\mathcal{T}}[\gamma] + W^{\mathcal{T}}[\gamma]\zeta\right] \leq b. \tag{44}$$

If the disturbances $d^{\mathcal{T}}$ and the initial state z are certain, (44) is "easy" — it is a system of constraints on γ with certain data. Moreover, in the case in question we lose nothing by restricting ourselves with "off-line" control laws (42) – those with $G_{t\tau} \equiv 0$; when restricted onto this subspace, let it be called Γ, in the γ-space, the function $w_0^{\mathcal{T}}[\gamma] + W^{\mathcal{T}}[\gamma]\zeta$ turns out to be bi-affine in γ and in ζ, so that (44) reduces to a system of explicit linear inequalities on $\gamma \in \Gamma$. Now, when the disturbances and/or the initial state are *not* known in advance (which is the only case of interest in Robust Control), (44) becomes an uncertainty-affected system of constraints, and we could try to solve the system in a robust fashion, e.g., to seek for a solution γ which makes the constraints feasible for all realizations of $\zeta = (z, d^{\mathcal{T}})$ from a given uncertainty set $\mathcal{ZD}^{\mathcal{T}}$, thus arriving at the system of semi-infinite scalar constraints

$$A\left[w_0^{\mathcal{T}}[\gamma] + W^{\mathcal{T}}[\gamma]\zeta\right] \leq b \ \forall \zeta \in \mathcal{ZD}^{\mathcal{T}}. \tag{45}$$

Unfortunately, the semi-infinite constraints in this system are *not* bi-affine, since the dependence of $w_0^{\mathcal{T}}$, $W^{\mathcal{T}}$ on γ is highly nonlinear, unless γ is restricted to vary in Γ. Thus, when seeking for "on-line" control laws (those where $G_{t\tau}$ can be nonzero) (45) becomes a system of highly nonlinear semi-infinite constraints and as such seems to be severely computationally intractable. A good news is, that we can overcome the resulting difficulty, the remedy being an appropriate re-parameterization of affine control laws.

5.2 Purified-output-based representation of affine control laws and efficient design of finite-horizon linear controllers

Imagine that in parallel with controlling (41) with the aid of a whatever non-anticipating output-based control law $u_t = U_t(y_0, \dots, y_t)$, we run the *model* of (41) as follows:

$$\begin{aligned} \widehat{x}_0 &= 0, \\ \widehat{x}_{t+1} &= A_t \widehat{x}_t + B_t u_t, \\ \widehat{y}_t &= C_t \widehat{x}_t, \\ v_t &= y_t - \widehat{y}_t, \end{aligned} \qquad . \tag{46}$$

Since we know past controls, we can run this system in an "on-line" fashion, so that the *purified output* v_t becomes known when the decision on u_t should be made. An immediate observation is, that *the purified outputs are completely independent of the control law in question — they are affine functions of the initial state and the disturbances* $d_0, \dots, d_t$, *and these functions are readily given by the dynamics of* (41). Now, it was mentioned that $v_0, \dots, v_t$ are known when the decision on u_t should be made, so that we can consider *purified-output-based* (POB) affine control laws

$$u_t = h_t + \sum_{\tau=0}^{t} H_{t\tau} v_\tau. \tag{47}$$

A simple and fundamental fact proved in [19] (and independently, for the special case when $y_t \equiv x_t$, in [41]) is that (47), (42) *are equivalent representations of non-anticipating affine control laws: for every controller of the form* (41)*, there exists controller* (42) *which results in exactly the same state-control behaviour of the closed loop system* (e.g., exactly the same dependence of $w^{\mathcal{T}}$ on the initial state and the disturbances), *and vice versa.* At the same time, the representation (47) is incomparably better suited for design purposes than the representation (42) — with controller (47), *the state-control trajectory* $w^{\mathcal{T}}$ *becomes bi-affine in* $\zeta = (z, d^{\mathcal{T}})$ *and in the parameters* $\eta = \{h_t, H_{t\tau}, 0 \le \tau \le t \le \mathcal{T}\}$ *of the controller:*

$$w^{\mathcal{T}} = \omega^{\mathcal{T}}[\eta] + \Omega^{\mathcal{T}}[\eta]\zeta \tag{48}$$

with vector- and matrix-valued functions $\omega^{\mathcal{T}}[\eta]$*,* $\Omega^{\mathcal{T}}[\eta]$ *affinely depending on* η *and readily given by the dynamics* (41). Substituting (48) into (43), we arrive at the system of semi-infinite *bi-affine* scalar inequalities

$$A\left[\omega^{\mathcal{T}}[\eta] + \Omega^{\mathcal{T}}[\eta]\zeta\right] \le b \tag{49}$$

in variables η, and can use the tractability results from Sect. 3.2 in order to solve efficiently the robust counterpart of this uncertain system. For example, we can process efficiently the GRC setting of the semi-infinite constraints (48)

$$\begin{array}{l} a_i^T\left[\omega^{\mathcal{T}}[\eta] + \Omega^{\mathcal{T}}[\eta](z, d^{\mathcal{T}})\right] - b_i \le \alpha_z^i \mathrm{dist}(z, \mathcal{Z}) + \alpha_d^i \mathrm{dist}(d^{\mathcal{T}}, \mathcal{D}^{\mathcal{T}}) \\ \quad \forall (z, d^{\mathcal{T}})\ \forall i = 1, \ldots, I \end{array} \tag{50}$$

where $\mathcal{Z}, \mathcal{D}^{\mathcal{T}}$ are "good" (e.g., given by strictly feasible semidefinite representations) closed convex normal ranges of z, $d^{\mathcal{T}}$, respectively, and the distances are defined via the $\|\cdot\|_\infty$-norms (this setting corresponds to the structured GRC, see Remark 1). By the results presented in Sect. 3.2, system (50) is equivalent to the system of constraints

$$\begin{array}{l} \forall (i, 1 \le i \le I): \\ (a)\ a_i^T\left[\omega^{\mathcal{T}}[\eta] + \Omega^{\mathcal{T}}[\eta](z, d^{\mathcal{T}})\right] - b_i \le 0\ \forall (z, d^{\mathcal{T}}) \in \mathcal{Z} \times \mathcal{D}^{\mathcal{T}} \\ (b)\ \|a_i^T \Omega_z^{\mathcal{T}}[\eta]\|_1 \le \alpha_z^i \qquad (c)\ \|a_i^T \Omega_d^{\mathcal{T}}[\eta]\|_1 \le \alpha_d^i, \end{array} \tag{51}$$

where $\Omega^{\mathcal{T}}[\eta] = \left[\Omega_z^{\mathcal{T}}[\eta], \Omega_d^{\mathcal{T}}[\eta]\right]$ is the partition of the matrix $\Omega^{\mathcal{T}}[\eta]$ corresponding to the partition $\zeta = (z, d^{\mathcal{T}})$. Note that in (51), the semi-infinite constraints (a) admit explicit semidefinite representations (Theorem 1), while constraints $(b–c)$ are, essentially, just linear constraints on η and on α_z^i, α_d^i. As a result, (51) can be thought of as a computationally tractable system of convex constraints on η *and on the sensitivities* α_z^i, α_d^i, and we can minimize under these constraints a "nice" (e.g., convex) function of η and the sensitivities. Thus, after passing to the POB representation of affine control laws, we can process efficiently specifications expressed by systems of linear inequalities, to be satisfied in a robust fashion, on the (finite-horizon) state-control trajectory.

5.2.1 Example: controlling finite-horizon gains

Natural design specification pertaining to finite-horizon Robust Linear Control are bounds on finite-horizon *gains* $z2x^{\mathcal{T}}$, $z2u^{\mathcal{T}}$, $d2x^{\mathcal{T}}$, $d2u^{\mathcal{T}}$ defined as follows: with a linear (i.e., with $h_t \equiv 0$) control law (47), the states x_t and the controls u_t are linear functions of z and $d^{\mathcal{T}}$:

$$x_t = X_t^z[\eta]z + X_t^d[\eta]d^T, \quad u_t = U_t^z[\eta]z + U_t^d[\eta]d^T$$

with matrices $X_t^z[\eta],\ldots,U_t^d[\eta]$ affinely depending on the parameters η of the control law. Given t, we can define the *z-to-x_t gains* and the *finite-horizon z*-to-*x* gain as $z2x_t(\eta) = \max_z\{\|X_t^z[\eta]z\|_\infty : \|z\|_\infty \le 1\}$ and $z2x^{\mathcal{T}}(\eta) = \max_{0 \le t \le \mathcal{T}} z2x_t(\eta)$.

The definitions of the z-to-u gains $z2u_t(\eta)$, $z2u^{\mathcal{T}}(\eta)$ and the "disturbance-to-x/u" gains $d2x_t(\eta)$, $d2x^{\mathcal{T}}(\eta)$, $d2u_t(\eta)$, $d2u^{\mathcal{T}}(\eta)$ are completely similar, e.g., $d2u_t(\eta) = \max\limits_{d^{\mathcal{T}}}\{\|U_t^d[\eta]d^{\mathcal{T}}\|_\infty : \|d^{\mathcal{T}}\|_\infty \leq 1\}$ and $d2u^{\mathcal{T}}(\eta) = \max\limits_{0\leq t\leq \mathcal{T}} d2u_t(\eta)$. The finite-horizon gains clearly are non-increasing functions of the time horizon $\mathcal{T}$ and have a transparent Control interpretation; e.g., $d2x^{\mathcal{T}}(\eta)$ ("peak-to-peak d-to-x gain") is the largest possible perturbation in the states x_t, $t = 0, 1, \ldots, \mathcal{T}$, caused by a unit perturbation of the sequence of disturbances $d^{\mathcal{T}}$, both perturbations being measured in the $\|\cdot\|_\infty$ norms on the respective spaces. Upper bounds on $^{\mathcal{T}}$-gains (and on *global* gains like $d2x^\infty(\eta) = \sup_{\mathcal{T}\geq 0} d2x^{\mathcal{T}}(\eta)$) are natural Control specifications. With our purified-output-based representation of linear control laws, the finite-horizon specifications of this type result in explicit systems of linear constraints on η and thus can be processed routinely via LP. Indeed, an upper bound on, say, $d2x^{\mathcal{T}}$-gain $d2x^{\mathcal{T}}(\eta) \leq \lambda$ is exactly equivalent to the requirement $\sum_j |(X_t^d[\eta])_{ij}| \leq \lambda$ for all i and all $t \leq \mathcal{T}$; since X_t^d is affine in η, this is just a system of linear constraints on η and on appropriate slack variables. Note that imposing bounds on the gains can be interpreted as passing to the GRC (50) in the case where the "desired behaviour" merely requires $w^{\mathcal{T}} = 0$, and the normal ranges of the initial state and the disturbances are the origins in the corresponding spaces: $\mathcal{Z} = \{0\}$, $\mathcal{D}^{\mathcal{T}} = \{0\}$.

5.3 Handling infinite-horizon design specifications

One might think that the outlined reduction of (discrete time) Robust Linear Control problems to Convex Programming, based on passing to the POB representation of affine control laws and tractable reformulations of semi-infinite bi-affine scalar inequalities is intrinsically restricted to the case of finite-horizon control specifications. In fact our approach is well suited for handling infinite-horizon specifications — those imposing restrictions on the asymptotical behaviour of the closed loop system. Specifications of the latter type usually have to do with *time-invariant* open loop system (41) — system of the form

$$\begin{aligned} x_0 &= z, \\ x_{t+1} &= Ax_t + Bu_t + Rd_t, \quad t = 0, 1, \ldots, \\ y_t &= Cx_t + Dd_t. \end{aligned} \tag{52}$$

The presentation to follow is based on [20]. From now on we assume that *the open loop system* (52) *is stable*, that is, the spectral radius of A is < 1 (in fact this restriction can be somehow circumvented, see [20]). Imagine that we "close" (52) by a *nearly time-invariant* POB control law *of order k*, that is, a law of the form

$$u_t = h_t + \sum_{\nu=0}^{k-1} H_\nu^t v_{t-\nu}, \tag{53}$$

where $h_t = 0$ for $t \geq \mathcal{T}_*$ and $H^t_\tau = H_\tau$ for $t \geq \mathcal{T}_*$ for certain *stabilization time* $\mathcal{T}_*$; from now on, all entities with negative indices are set to 0. While the "time-varying" part $\{h_t, H^t_\tau, 0 \leq t < \mathcal{T}_*\}$ of the control law can be used to adjust the finite-horizon behaviour of the closed loop system, its asymptotical behaviour is as if the law were time-invariant: $h_t \equiv 0$ and $H^t_\tau \equiv H_\tau$ for all $t \geq 0$. Setting $\delta_t = x_t - \widehat{x}_t$, $H^t = [H^t_0, \ldots, H^t_{k-1}]$, $H = [H_0, \ldots, H_{k-1}]$, the dynamics (52), (46), (53) for $t \geq k-1$ is given by

$$
\begin{array}{l}
\overbrace{\begin{bmatrix} x_{t+1} \\ \delta_{t+1} \\ \delta_t \\ \vdots \\ \delta_{t-k+2} \end{bmatrix}}^{\omega_{t+1}} = \overbrace{\left[\begin{array}{c|cccc} A & BH^t_0C & BH^t_1C & \ldots & BH^t_{k-1}C \\ \hline & A & & & \\ & & A & & \\ & & & \ddots & \\ & & & & A \end{array}\right]}^{A_+[H^t]} \omega_t \\
\qquad + \overbrace{\left[\begin{array}{c|cccc} R & BH^t_0D & BH^t_1D & \ldots & BH^t_{k-1}D \\ \hline & R & & & \\ & & R & & \\ & & & \ddots & \\ & & & & R \end{array}\right]}^{R_+[H^t]} \begin{bmatrix} d_t \\ d_t \\ d_{t-1} \\ \vdots \\ d_{t-k+1} \end{bmatrix} + \begin{bmatrix} Bh_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\
u_t = h_t + \sum_{\nu=0}^{k-1} H^t_\nu [C\delta_{t-\nu} + Dd_{t-\nu}].
\end{array}
\tag{54}
$$

We see, in particular, that starting with time $\mathcal{T}_*$, dynamics (54) is exactly as if the underlying control law were the time invariant POB law with the parameters $h_t \equiv 0$, $H^t \equiv H$. Moreover, since A is stable, we see that *system* (54) *is stable independently of the parameter H of the control law, and the resolvent $\mathcal{R}_H(s) := (sI - A_+[H])^{-1}$ of $A_+[H]$ is the affine in H matrix*

$$
\left[\begin{array}{c|c|c|c|c}
\mathcal{R}_A(s) & \mathcal{R}_A(s)BH_0C\mathcal{R}_A(s) & \mathcal{R}_A(s)BH_1C\mathcal{R}_A(s) & \ldots & \mathcal{R}_A(s)BH_{k-1}C\mathcal{R}_A(s) \\ \hline
 & \mathcal{R}_A(s) & & & \\ \hline
 & & \mathcal{R}_A(s) & & \\ \hline
 & & & \ddots & \\ \hline
 & & & & \mathcal{R}_A(s)
\end{array}\right], \tag{55}
$$

where $\mathcal{R}_A(s) = (sI - A)^{-1}$ is the resolvent of A.

Now imagine that the sequence of disturbances d_t is of the form $d_t = s^t d$, where $s \in \mathbf{C}$ differs from 0 and from the eigenvalues of A. From the stability of (54) it follows that as $t \to \infty$, the solution ω_t of the system, independently of the initial state, approaches, as $t \to \infty$, the "steady-state" solution $\widehat{\omega}_t = s^t \mathcal{H}(s) d$, where $\mathcal{H}(s)$ is certain matrix. In particular, the state-control vector $w_t = \begin{bmatrix} x_t \\ u_t \end{bmatrix}$ approaches, as $t \to \infty$, the steady-state trajectory $\widehat{w}_t = s^t \mathcal{H}_{xu}(s) d$. The associated *disturbance-to-state/control transfer matrix* $\mathcal{H}_{xu}(s)$ is easily computable:

$$\mathcal{H}_{xu}(s) = \left[\begin{array}{c} \overbrace{\mathcal{R}_A(s)\left[R + \sum_{\nu=0}^{k-1} s^{-\nu} B H_\nu \left[D + C\mathcal{R}_A(s)R\right]\right]}^{\mathcal{H}_x(s)} \\ \hline \underbrace{\left[\sum_{\nu=0}^{k-1} s^{-\nu} H_\nu\right]\left[D + C\mathcal{R}_A(s)R\right]}_{\mathcal{H}_u(s)} \end{array} \right] \tag{56}$$

The crucial fact is *that the transfer matrix $\mathcal{H}_{xu}(s)$ is affine in the parameters $H = [H_0, \ldots, H_{k-1}]$ of the nearly time invariant control law* (53). As a result, *design specifications representable as explicit convex constraints on the transfer matrix $\mathcal{H}_{xu}(s)$* (these are typical specifications in infinite-horizon design of linear controllers) *are equivalent to explicit convex constraints on the parameters H of the underlying POB control law and therefore can be processed efficiently via Convex Optimization.*

5.3.1 Example: discrete time H_∞ control

Discrete time H_∞ design specifications impose constraints on the behaviour of the transfer matrix along the unit circumference $z = \exp\{\imath\phi\}$, $0 \leq \phi \leq 2\pi$, that is, on the steady state response of the closed loop system to a disturbance in the form of a harmonic oscillation. A rather general form of these specifications is a system of constraints

$$\|Q_i(s) - M_i(s)\mathcal{H}_{xu}(s)N_i(s)\| \leq \tau_i \quad \forall (s = \exp\{\imath\omega\} : \omega \in \Delta_i), \tag{57}$$

where $Q_i(s)$, $M_i(s)$, $N_i(s)$ are given rational matrix-valued functions with no singularities on the unit circumference $\{s : |s| = 1\}$, $\Delta_i \subset [0, 2\pi]$ are given segments, and $\|\cdot\|$ is the standard matrix norm (the largest singular value). From the results of [52] on semidefinite representation of the cone of Hermitian-matrix-valued trigonometric polynomials which are $\succeq 0$ on a given segment it follows that constraints (57) can be represented by an explicit finite system of LMI's (for details, see [20]); as a result, specifications (57) can be efficiently processed numerically.

We see that the purified-output-based reformulation of affine control laws, combined with the results of RO on tractable reformulations of semi-infinite bi-affine convex constraints, allow to handle efficiently design of linear controllers for uncertainty-affected linear dynamical systems with known dynamics. The corresponding design problems can include rather general specifications on the finite-horizon state-control behaviour of the closed loop systems, and in the case of time-invariant open loop system these constraints can be coupled with restrictions on the asymptotical behaviour of the state-control trajectory, provided that these restrictions can be expressed by convex constraints on the transfer matrix of the closed loop system. The outlined approach seems to be a valuable complement to the existing Convex Optimization-based Control techniques. For instructive illustrations and comparison with the usual time-invariant linear feedback controllers, see [20].

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