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# On maximization of quadratic form over intersection of ellipsoids with common center

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**Abstract**. We demonstrate that if $A_1, \ldots, A_m$ are symmetric positive semidefinite $n \times n$ matrices with positive definite sum and $A$ is an arbitrary symmetric $n \times n$ matrix, then the relative accuracy, in terms of the optimal value, of the semidefinite relaxation

$$\max_X \{\mathrm{Tr}(AX) \mid \mathrm{Tr}(A_i X) \le 1,\ i = 1, \ldots, m;\ X \succeq 0\} \tag{SDP}$$

of the optimization program

$$x^T Ax \to \max \mid x^T A_i x \le 1,\ i = 1, \ldots, m \tag{P}$$

is not worse than $1 - \frac{1}{2\ln(2m^2)}$. It is shown that this bound is sharp in order, as far as the dependence on $m$ is concerned, and that a feasible solution $x$ to (P) with

$$x^T Ax \ge \frac{\mathrm{Opt(SDP)}}{2\ln(2m^2)} \tag{$*$}$$

can be found efficiently. This somehow improves one of the results of Nesterov [4] where bound similar to $(*)$ is established for the case when all $A_i$ are of rank 1.

**Key words.** semidefinite relaxations – quadratic programming

## 1. Introduction

Let $A_i$, $i = 1, \ldots, m$, be positive semidefinite $n \times n$ matrices with positive definite sum, and $A$ be a $n \times n$ symmetric matrix. Consider the optimization problem

$$x^T Ax \to \max \mid x^T A_i x \le 1,\ i = 1, \ldots, m. \tag{P}$$

This problem, in general, is NP-hard (take, e.g., $m = n$ and $A_i = e_i e_i^T$, where $e_i$ are the standard basic orths in $\mathbf{R}^n$; then (P) becomes the problem of maximizing a homogeneous quadratic form over the unit cube, which is known to be NP-hard even in the case of positive semidefinite $A$).

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In view of NP-hardness of (P), it makes sense to look at the standard semidefinite relaxation of the problem. To get this relaxation, we rewrite all $x^T A_i x$ as $\mathrm{Tr}(A_i x x^T)$, thus coming to the equivalent problem

$$\mathrm{Tr}(AX) \to \max \mid \mathrm{Tr}(A_i X) \leq 1, \ i = 1, ..., m, X \succeq 0, \mathrm{Rank}(X) = 1,$$

and then discard the rank restriction, thus coming to the relaxation

$$\mathrm{Tr}(AX) \to \max \mid \mathrm{Tr}(A_i X) \leq 1, \ i = 1, ..., m, \ X \succeq 0. \tag{SDP}$$

By its origin, the optimal value in the relaxation is not less that the one in the original problem:

$$\mathrm{Opt(SDP)} \geq \mathrm{Opt(P)} \tag{1}$$

The main goal of this note is to demonstrate that the "gap" in (1) is "not too big", namely, that

$$\mathrm{Opt(P)} \geq \frac{\mathrm{Opt(SDP)}}{2\ln(2m\mu)}, \ \mu = \min[m; \max_i \mathrm{Rank}(A_i)]. \tag{2}$$

In the standard terminology (2) says that (SDP) approximates (P) within the relative accuracy $\theta = 1 - \frac{1}{2\ln(2m\mu)}$, i.e., $0 \leq \mathrm{Opt(SDP)} - \mathrm{Opt(P)} \leq \theta \mathrm{Opt(SDP)}$. This result complements, in a sense, stronger results known from literature and dealing with maximization of a quadratic form over the unit cube and cube-like sets:

A) It was shown by Goemans and Williamson [2] that if $m = n$, $A_i = e_i e_i^T$ and $A$ is positive semidefinite matrix with nonpositive non-diagonal entries and row sums 0 (which corresponds to the Maximum Cut problem), then

$$\frac{\mathrm{Opt(P)}}{\mathrm{Opt(SDP)}} \geq 0.87856...$$

(approximation of the relative accuracy 0.12143...).

B) Nesterov [3] shows that if $A_i$ are as in A), and $A$ is an arbitrary positive semidefinite matrix, then

$$\frac{\mathrm{Opt(P)}}{\mathrm{Opt(SDP)}} \geq \frac{2}{\pi} = 0.6366... \tag{3}$$

(approximation of the relative accuracy 0.3633...). For closely related results, see Ye [5] and Bertsimas and Ye [1].

C) It is known that

C.1) (3) holds if $A_i$ commute with each other and $A \succeq 0$ (Ye [5], Nesterov [4]);

C.2) If $A_i = a_i a_i^T$ are of rank 1, then, for certain efficiently solvable convex optimization program (Cnv) (different from (SDP)) it holds

$$\mathrm{Opt(Cnv)} \geq \mathrm{Opt(P)} \geq \frac{1}{e \ln m} \mathrm{Opt(Cnv)}. \tag{4}$$

(approximation of the relative accuracy $1 - \frac{1}{e \ln m}$, Nesterov [4]).

The bounds mentioned in A), B), C.1) are significantly better than (2) – the quality of semidefinite relaxation there is independent of problem's dimensions. We shall prove that this phenomenon is possible only in the "special cases" of (P); specifically, we demonstrate that for every positive integer $m$ there exists an instance of (P) with $n = O(\ln m)$ and positive definite $A$ such that

$$\frac{\mathrm{Opt(P)}}{\mathrm{Opt(SDP)}} \leq O(1)\frac{1}{\ln m} \tag{5}$$

with a positive absolute constant $O(1)$.

As compared to (4), the progress in (2) is that in our setting $A_i \succeq 0$ may have arbitrary ranks, and, more essentially, that every feasible solution $X$ to (SDP) with $\mathrm{Tr}(AX) > 0$ for every $\alpha \geq \frac{1}{2\ln(2m\mu)}$ can be efficiently converted to a feasible solution of (P) with the value of the objective at least $\alpha^{-1}\mathrm{Tr}(AX)$, which is not exactly so for the construction leading to (4) (quite different from the one we use).

The rest of the note is organized as follows. Inequality (2) is proved in Sect. 2, where we present a simple randomized algorithm which allows to pass from a feasible solution $X$ of (SDP) to a feasible solution $x$ of (P) such that

$$x^T Ax \geq \frac{\mathrm{Tr}(AX)}{2\ln(2m\mu)}.$$

In Sect. 3, we demonstrate that the ratio $\frac{\mathrm{Opt(P)}}{\mathrm{Opt(SDP)}}$ indeed can be of order of $\frac{1}{\ln m}$. In Sect. 4, we use the standard derandomization technique to get a simple polynomial time deterministic algorithm with the same properties as those of the randomized algorithm from Sect. 2. In the concluding Sect. 5, we extend our main result to the case when the objective is an inhomogeneous quadratic form.

## 2. Main result

We restrict ourselves with the only nontrivial case when $A$ is not negative semidefinite (otherwise the optimal values in (SDP) and (P) both are equal to 0, and (2) is trivially true). Note that our "nontrivial" case can be efficiently recognized, and that in this case the optimal values in (P) and (SDP) are positive.

We start with presenting a randomized algorithm $\mathcal{R}$ which, given on input a feasible solution $X$ to (SDP) with positive value of the objective, generates random feasible solutions to (P), namely, as follows.

Preprocessing. 1) Observe that $X$ can be efficiently converted to a feasible solution $X'$ of (SDP) with at least the same value of the objective as at $X$ and with rank not exceeding $m$. Indeed, since $X \succeq 0$, we can efficiently represent $X$ as

$$X = \sum_{j=1}^{n} g_j g_j^T \quad [g_j \in \mathbf{R}^n].$$

Now let us look at the polyhedral set

$$\mathcal{X} = \Big\{\lambda \in \mathbf{R}^n_+ \mid \mathrm{Tr}(A_i \sum_{j=1}^n \lambda_j g_j g_j^T) \le 1, i = 1, ..., m\Big\}.$$

This set is nonempty (it contains the point $\lambda^0 = (1, ..., 1)^T$), and the linear form

$$c^T\lambda \equiv \mathrm{Tr}\Big(A \sum_{j=1}^n \lambda_j g_j g_j^T\Big)$$

is bounded above on the set (because (SDP) is above bounded due to $\sum_i A_i \succ 0$). Applying the usual purification technique, we can efficiently pass from $\lambda^0$ to an extreme point $\lambda^*$ of $\mathcal{X}$ such that $c^T\lambda^* \ge c^T\lambda^0$; in other words, the matrix $X' = \sum_{j=1}^m \lambda_j^* g_j g_j^T$ is a feasible solution to (SDP) with the value of the objective at least as at $X$. It remains to note that since $\lambda^*$ is an extreme point of $\mathcal{X}$, the number of nonzero weights $\lambda_j^*$ is at most $m$, so that $X'$ is of rank $\le m$.

In view of the outlined construction, we may without loss of generality assume that our input feasible solution $X$ to (SDP) is of rank $\le m$.

2) We can efficiently decompose $X$ as $X = \Delta^T\Delta$ with $\mathrm{Rank}(\Delta) \le m$. Let us set

$$B_i = \Delta A_i \Delta^T, \;\; B = \Delta A \Delta^T.$$

Since $X$ is feasible for (SDP), we have

$$\begin{array}{ll} (a) & B_i \succeq 0, \;\; i = 1, ..., m; \\ (b) & \mathrm{Tr}(B_i) = \mathrm{Tr}(A_i X) \le 1, \;\; i = 1, ..., m; \\ (c) & \mathrm{Rank}(B_i) \le \mu, \;\; i = 1, ..., m; \\ (d) & \mathrm{Tr}(B) = \mathrm{Tr}(AX). \end{array} \tag{6}$$

3) Finally, we can efficiently pass to the orthonormal basis where $B$ is diagonal[1]. Thus, we may assume that an orthogonal matrix $U$ and matrices $\widehat{B}_i$, $i = 1, ..., m$, $\widehat{B}$ are available such that

$$B_i = U\widehat{B}_i U^T, \; B = U\widehat{B}U^T$$

and $\widehat{B}$ is diagonal.

Generation of feasible solutions to (P) after preprocessing is extremely simple. We generate at random a vector $\xi$ with independent entries taking with equal probabilities the values $\pm 1$ and convert the realized $\xi$ into a feasible solution $x$ of (P) according to

$$x = x(\xi) = \frac{1}{\sqrt{\max_i \xi^T \widehat{B}_i \xi}} \Delta^T U \xi. \tag{7}$$

[1] In fact, of course, we cannot find this basis exactly in finite, not saying polynomial in $n, m$, time. We can, however, find in time polynomial in $n$ and $\ln(1/\epsilon)$ (for every $\epsilon \in (0, 1)$) an orthonormal basis where an $\epsilon$-perturbation $A'$ of $A$ ($\|A - A'\| \le \epsilon\|A\|$) is diagonal, which, in our context, is essentially the same as the possibility to find an orthonormal eigenbasis of $A$. Note also that to prove (2) – i.e., to establish the "existence" part of our result – we should not bother at all whether this eigenbasis can or cannot be found efficiently; all which is important for us is that such a basis exists.

**Proposition 1.** *For the outlined randomized algorithm as applied to a feasible solution $X$ of* (SDP) *with positive value of the objective $x$ always is well-defined and is feasible for* (P). *Moreover, for every $\alpha > 0$ one has*

$$\mathrm{Prob}\left\{x^T Ax \geq \frac{1}{\alpha}\mathrm{Tr}(AX)\right\} > 1 - 2m\mu \exp\{-\alpha/2\}, \quad \mu = \min[m, \max_i \mathrm{Rank} A_i]. \tag{8}$$

**Corollary 1.** *One has*

$$\mathrm{Opt(P)} \geq \frac{\mathrm{Opt(SDP)}}{2\ln(2m\mu)}.$$

*Proposition ⇒ Corollary:* Problem (SDP) has a nonempty and bounded (since $\sum_i A_i \succ 0$) feasible set and is therefore solvable. Let us apply $\mathcal{R}$ to the optimal solution $X_*$ of the problem and specify $\alpha$ in (8) as $\alpha_* = 2\ln(2m\mu)$. According to (8), $\mathcal{R}$ with positive probability generates a feasible solution to (P) with the value of the objective $\geq \frac{1}{\alpha_*}\mathrm{Tr}(AX_*) = \frac{1}{\alpha_*}\mathrm{Opt(SDP)}$, whence $\mathrm{Opt(P)} \geq \frac{1}{\alpha_*}\mathrm{Opt(SDP)}$.

□

*Proof of Proposition.* $1^0$. First let us prove that $x(\xi)$ always is well-defined, i.e., that $\max_i \xi^T \widehat{B}_i \xi > 0$ for every vector $\xi$ with coordinates $\pm 1$. Indeed, assuming opposite, there exists a vector $\xi$ with coordinates $\pm 1$ such that $\xi^T \widehat{B}_i \xi = 0$ for all $i$ (recall that the matrices $\widehat{B}_i = U^T B_i U = U^T \Delta A_i \Delta^T U$ are positive semidefinite). On the other hand,

$$\mathrm{Tr}(\widehat{B}) = \mathrm{Tr}(B) = \mathrm{Tr}(AX) > 0 \qquad [\text{by } (6.d)]$$

and since $\xi$ is with coordinates $\pm 1$ and $\widehat{B}$ is diagonal, we have

$$\xi^T \widehat{B} \xi = \mathrm{Tr}(\widehat{B}) > 0. \tag{9}$$

Now let

$$\tilde{x}(\xi) = \Delta^T U \xi.$$

We have

$$\begin{aligned} \tilde{x}^T(\xi) A_i \tilde{x}(\xi) &= \xi^T U^T \Delta A_i \Delta^T U \xi = \xi^T U^T B_i U \xi = \xi^T \widehat{B}_i \xi, \\ \tilde{x}^T(\xi) A \tilde{x}(\xi) &= \xi^T U^T \Delta A \Delta^T U \xi = \xi^T U^T B U \xi = \xi^T \widehat{B} \xi. \end{aligned} \tag{10}$$

Thus, assuming that $\xi^T \widehat{B}_i \xi = 0$ for all $i = 1, ..., m$ and a vector $\xi$ with coordinates $\pm 1$, we conclude that for $z = \tilde{x}(\xi)$ it holds $z^T A_i z = 0$, $i = 1, ..., m$, $z^T Az > 0$, which contradicts the assumption that $\sum_i A_i \succ 0$.

We see that $\max_i \xi^T \widehat{B}_i \xi > 0$ for all realizations $\xi$, so that $x(\xi)$ is always well-defined; combining (7) and (10), we conclude that for all realizations of $\xi$ the vector $x(\xi)$ is feasible for (P), and

$$\begin{aligned} x^T(\xi) A x(\xi) &= \frac{1}{\max_i \xi^T \widehat{B}_i \xi} \mathrm{Tr}(\xi^T \widehat{B} \xi) \\ &= \frac{1}{\max_i \xi^T \widehat{B}_i \xi} \mathrm{Tr}(\widehat{B}) \\ &= \frac{1}{\max_i \xi^T \widehat{B}_i \xi} \mathrm{Tr}(B) \\ &= \frac{1}{\max_i \xi^T \widehat{B}_i \xi} \mathrm{Tr}(AX). \end{aligned} \tag{11}$$

$2^0$. It remains to prove (8); in view of (11), all we should prove is that

$$\text{Prob}\left\{\max_i \xi^T \widehat{B}_i \xi > \alpha\right\} < 2m\mu \exp\{-\alpha/2\} \quad \forall \alpha > 0. \tag{12}$$

$2^0$.a) Since the matrices $\widehat{B}_i$ are positive semidefinite of the same ranks as $B_i$, i.e., of ranks $\leq \mu$ (see (6.$c$)), we have

$$\widehat{B}_i = \sum_{j=1}^{\mu} f^{ij}(f^{ij})^T.$$

with certain vectors $f^{ij} \in \mathbf{R}^n$. Given $\alpha > 0$, consider the events

$$\begin{aligned} \mathcal{A}_{ij} &= \left\{\xi \mid |\xi^T f^{ij}| > \sqrt{\alpha}\| f^{ij}\|_2\right\}, \\ \mathcal{A} &= \bigcup_{i,j} \mathcal{A}_{ij}, \end{aligned} \tag{13}$$

$\| f\|_2 = \sqrt{f^T f}$ being the standard Euclidean norm.

$2^0$.b) Note that if $\mathcal{A}$ does not take place, then

$$\begin{aligned} &|\xi^T f^{ij}| \leq \sqrt{\alpha}\| f^{ij}\|_2 \quad \forall i, j \\ \Rightarrow &\textstyle\sum_{j=1}^m \xi^T [f^{ij}(f^{ij})^T]\xi \leq \alpha \sum_{j=1}^m \| f^{ij}\|_2^2 = \alpha \text{Tr}(\widehat{B}_i) \\ &= \alpha \text{Tr}(B_i) \leq \alpha \quad \forall i \qquad\qquad [\text{by } (6.b)] \end{aligned}$$

i.e., $\max_i \xi^T \widehat{B}_i \xi \leq \alpha$. We see that in order to prove (12) it suffices to demonstrate that for every $i, j$ it holds

$$\text{Prob}\left\{|\xi^T f^{ij}| > \sqrt{\alpha}\| f^{ij}\|_2\right\} < 2\exp\{-\alpha/2\}. \tag{14}$$

$2^0$.c) (14) is readily given by Bernstein's theorem on large deviations. For our further purposes, let us reproduce the proof.

Whenever $\theta \geq 0$, we have

$$\begin{aligned} \text{Prob}\{\xi^T f^{ij} > \sqrt{\alpha}\| f^{ij}\|_2\} &< \mathcal{E}\left\{\exp\{\theta \textstyle\sum_{k=1}^n f_k^{ij}\xi_k\}\right\} \exp\{-\theta\sqrt{\alpha}\| f^{ij}\|_2\} \\ &= \left(\prod_{k=1}^n \cosh(\theta f_k^{ij})\right) \exp\{-\theta\sqrt{\alpha}\| f^{ij}\|_2\} \\ &\leq \left(\prod_{k=1}^n \exp\{\theta^2 (f_k^{ij})^2/2\}\right) \exp\{-\theta\sqrt{\alpha}\| f^{ij}\|_2\} \\ &= \exp\{\tfrac{1}{2}\theta^2 \| f^{ij}\|_2^2 - \theta\sqrt{\alpha}\| f^{ij}\|_2\}. \end{aligned}$$

Setting $\theta = \sqrt{\alpha}/\| f^{ij}\|_2$, we get

$$\text{Prob}\{\xi^T f^{ij} > \sqrt{\alpha}\| f^{ij}\|_2\} < \exp\{-\alpha/2\}. \tag{15}$$

Similarly, if $\theta \leq 0$, then

$$\begin{aligned}
\text{Prob}\{\xi^T f^{ij} < -\sqrt{\alpha}\| f^{ij}\|_2\} &< \mathcal{E}\left\{\exp\{\theta \textstyle\sum_{k=1}^n f_k^{ij}\xi_k\}\right\}\exp\{\theta\sqrt{\alpha}\| f^{ij}\|_2\} \\
&= \left(\prod_{k=1}^n \cosh(\theta f_k^{ij})\right)\exp\{\theta\sqrt{\alpha}\| f^{ij}\|_2\} \\
&\leq \left(\prod_{k=1}^n \exp\{\theta^2(f_k^{ij})^2/2\}\right)\exp\{\theta\sqrt{\alpha}\| f^{ij}\|_2\} \\
&= \exp\{\tfrac{1}{2}\theta^2\| f^{ij}\|_2^2 + \theta\sqrt{\alpha}\| f^{ij}\|_2\}.
\end{aligned}$$

Setting $\theta = -\sqrt{\alpha}/\| f^{ij}\|_2$, we get

$$\text{Prob}\{\xi^T f^{ij} < -\sqrt{\alpha}\| f^{ij}\|_2\} < \exp\{-\alpha/2\}. \tag{16}$$

Combining (15), (16), we come to (14).

□

## 3. Sharpness of (2)

Here we demonstrate that

**Proposition 2.** *For every positive integer $m \geq 3$, there exists problem* (P) *with positive definite $A$ and $n = O(\ln m)$ such that*

$$\frac{\text{Opt(SDP)}}{\text{Opt(P)}} \geq \kappa \ln m \tag{17}$$

*with positive absolute constant $\kappa$ (for large $m$, one can take $\kappa = 0.55$).*

*Proof.* Let us fix $\phi \in (0, \pi/2)$, and let, for positive integer $n$, $\Gamma_\phi$ be a maximal, with respect to inclusion, set of unit vectors from $\mathbf{R}^n$ such that the angle between every two distinct vectors from the set is $> \phi$. Denoting $M(n, \phi)$ the cardinality of $\Gamma_\phi$ and taking into account that the "spherical hats" $S_v = \{x \in \mathbf{R}^n \mid \|x\|_2 = 1, x^T v \geq \cos(\phi/2)\}$ associated with distinct $v \in \Gamma_\phi$ have no points in common, we get

$$M(n, \phi)\sigma_n(S^\phi) \leq \sigma_n\left(\{x \in \mathbf{R}^n \mid \|x\|_2 = 1\}\right), \tag{18}$$

where $\sigma_n(\cdot)$ is the $(n-1)$-dimensional area of a set on the unit sphere in $\mathbf{R}^n$ and $S^\phi$ is the spherical hat

$$\{x \in \mathbf{R}^n \mid \|x\|_2 = 1, x^T e \geq \cos(\phi/2)\},$$

$e$ being a once for ever fixed unit vector in $\mathbf{R}^n$. Rough estimation of the areas in (18) implies the upper bound

$$\ln(M(n, \phi)) \leq \left[n \ln\left(\frac{1}{\sin(\phi/2)}\right)\right](1 + o_\phi(1)), \tag{19}$$

where $o_\phi(1) \to 0$ as $n \to \infty$.

On the other hand, $\Gamma_\phi$ clearly possesses the property that for every $x \in \mathbf{R}^n$ with $\|x\|_2 = 1$ there exists $v \in \Gamma_\phi$ such that $x^T v \geq \cos(\phi)$ (otherwise we could extend $\Gamma_\phi$ by adding $x$, thus increasing the cardinality of the set without violating the property $v^T u < \cos(\phi)$ for all distinct $v, u \in \Gamma_\phi$). Now let us look at the following instance of (P) with $m = M(n, \phi)$:

$$x^T I x \to \max \mid \; x^T[vv^T]x \leq 1 \quad \forall v \in \Gamma_\phi. \tag{$\mathrm{P}_{n,\phi}$}$$

The matrix $X = I$ clearly is a feasible solution for the associated (SDP), and for this solution $\mathrm{Tr}(AX) = \mathrm{Tr}(I^2) = n$, so that $\mathrm{Opt(SDP)} \geq n$. On the other hand, the feasible set of $(\mathrm{P}_{n,\phi})$ is contained in the Euclidean ball $\{x \mid \; \|x\|_2 \leq \frac{1}{\cos(\phi)}\}$. Indeed, if $x \neq 0$ is feasible for $(\mathrm{P}_{n,\phi})$ and $e = \|x\|_2^{-1}x$, then there exists $v \in \Gamma_\phi$ with $v^T e \geq \cos(\phi)$, i.e., with $v^T x \geq \|x\|_2 \cos(\phi)$, and since $|v^T x| \leq 1$ by constraints of $(\mathrm{P}_{n,\phi})$, we conclude that $\|x\|_2 \leq \frac{1}{\cos(\phi)}$. It follows that

$$\mathrm{Opt}(\mathrm{P}_{n,\phi}) \leq \frac{1}{\cos^2(\phi)},$$

whence

$$\frac{\mathrm{Opt(SDP)}}{\mathrm{Opt}(\mathrm{P}_{n,\phi})} \geq n\cos^2(\phi) \geq \frac{\cos^2(\phi)}{\ln \sin^{-1}(\phi/2)}(1 + o(1)) \ln M(n, \phi)$$

(we have used (19)). Specifying $\phi = \frac{3}{16}\pi$, we get

$$\frac{\mathrm{Opt(SDP)}}{\mathrm{Opt}(\mathrm{P}_{n,\phi})} \geq 0.55 \ln M(n, \phi)$$

for all large enough values of $n$, and the statement follows.

□

## 4. Derandomization

Here we demonstrate that given a feasible solution $X$ to (SDP) with certain value $\gamma > 0$ of the objective and a real $\alpha \geq 2\ln(2m\mu)$, one can explicitly point out, in an efficient deterministic fashion, a feasible solution $x$ to (P) with the value of the objective at least $\gamma/\alpha$.

Indeed, in view of the proof of Proposition 1, we can reduce the situation to the following one:

(*) Given $m\mu$ vectors $g^\ell \in \mathbf{R}^n$, find a vector $\xi$ with coordinates $\pm 1$ such that

$$|[g^\ell]^T \xi| \leq \sqrt{\alpha}\|g^\ell\|_2 \;\; \ell = 1, ..., m\mu; \tag{20}$$

in our context, $g^\ell$ are the vectors $f^{ij}$ from item $2^0$.a) of the aforementioned proof, and a vector $\xi$ satisfying (20) produces a feasible solution $x(\xi)$ of (P) with the value of the objective $\geq \gamma/\alpha$.

To build $\xi$, let us apply the standard derandomization technique. Namely, let

$$\theta_\ell^\pm = \pm\sqrt{\alpha}/\|g^\ell\|_2,$$

and let

$$\begin{aligned} F_k(x_1,...,x_k) = &\sum_{\ell=1}^{m\mu} \exp\left\{\theta_\ell^+ \sum_{\nu=1}^{k} g_\nu^\ell x_\nu\right\}\left[\prod_{\nu=k+1}^{n} \cosh(\theta_\ell^+ g_\nu^\ell)\right]\exp\left\{-\theta_\ell^+\sqrt{\alpha}\|g^\ell\|_2\right\} \\ &+\sum_{\ell=1}^{m\mu} \exp\left\{\theta_\ell^- \sum_{\nu=1}^{k} g_\nu^\ell x_\nu\right\}\left[\prod_{\nu=k+1}^{n} \cosh(\theta_\ell^- g_\nu^\ell)\right]\exp\left\{\theta_\ell^-\sqrt{\alpha}\|g^\ell\|_2\right\}. \end{aligned}$$

Now let $\xi$ be a random vector with independent coordinates taking values $\pm 1$ with probabilities $1/2$. Same as in item $2^0$.c) of the proof of Proposition 1, for every $k$, $0 \leq k \leq n$, and every collection $x_1, ..., x_k$ of $\pm 1$'s we have

$$\sum_{\ell=1}^{m\mu} \text{Prob}\left\{\left|\sum_{\nu=1}^{n} \xi_\nu g_\nu^\ell\right| > \sqrt{\alpha}\|g^\ell\|_2 \,\middle|\, \xi_1 = x_1, ..., \xi_k = x_k\right\} \leq F_k(x_1, ..., x_k).$$

At the same time, we have

$$F_0 \leq 1 \tag{21}$$

by origin of $\alpha$ and

$$F_k(x_1,...,x_k) = \frac{1}{2}\left[F_{k+1}(x_1,...,x_k,1) + F_{k+1}(x_1,...,x_k,-1)\right]. \tag{22}$$

From (22) it follows that given a collection $x_1, ..., x_k$ of $\pm 1$'s with $k < n$ such that $F_k(x_1, ..., x_k) \leq 1$ and computing two explicitly given quantities $F_{k+1}(x_1, ..., x_k, 1)$ and $F_{k+1}(x_1, ..., x_k, -1)$, we can extend $(x_1, ..., x_k)$ by setting $x_{k+1}$ either to 1 or to $-1$ in order to ensure $F_{k+1}(x_1, ..., x_{k+1}) \leq 1$. By (21), we may start our construction with $k = 0$, and after $n$ steps will end up with a collection $(x_1, ..., x_n)$ of $\pm 1$'s such that

$$\begin{aligned} 1 \geq\ & F_n(x_1,...,x_n) \\ =& \sum_{\ell=1}^{m\mu} \exp\left\{\theta_\ell^+ \sum_{\nu=1}^{n} g_\nu^\ell x_\nu\right\}\exp\left\{-\theta_\ell^+\sqrt{\alpha}\|g^\ell\|_2\right\} \\ &+\sum_{\ell=1}^{m\mu} \exp\left\{\theta_\ell^- \sum_{\nu=1}^{n} g_\nu^\ell x_\nu\right\}\exp\left\{\theta_\ell^-\sqrt{\alpha}\|g^\ell\|_2\right\}. \end{aligned}$$

In particular, for every $\ell$ we have

$$\begin{gathered} \exp\left\{\theta_\ell^+ \textstyle\sum_{\nu=1}^n g_\nu^\ell x_\nu\right\}\exp\left\{-\theta_\ell^+\sqrt{\alpha}\|g^\ell\|_2\right\} \leq 1 \\ \Downarrow \\ \theta_\ell^+ \textstyle\sum_{\nu=1}^n g_\nu^\ell x_\nu \leq \theta_\ell^+\sqrt{\alpha}\|g^\ell\|_2 \\ \Downarrow \\ \textstyle\sum_{\nu=1}^n g_\nu^\ell x_\nu \leq \sqrt{\alpha}\|g^\ell\|_2 \end{gathered}$$

and

$$\exp\left\{\theta_\ell^- \sum_{\nu=1}^n g_\nu^\ell x_\nu\right\} \exp\left\{\theta_\ell^- \sqrt{\alpha}\|g^\ell\|_2\right\} \le 1$$
$$\Downarrow$$
$$\theta_\ell^- \sum_{\nu=1}^n g_\nu^\ell x_\nu \le -\theta_\ell^- \sqrt{\alpha}\|g^\ell\|_2$$
$$\Downarrow$$
$$\sum_{\nu=1}^n g_\nu^\ell x_\nu \ge -\sqrt{\alpha}\|g^\ell\|_2$$

as required in (20).

## 5. Extensions

We are about to extend our results in two directions.

*"Combined" problems.* Consider the case when (P) is replaced with a more general problem

$$x^T Ax \to \max \mid x^T A_i x \le t_i, 1, \ i = 1, \dots, m; t = (t_1, \dots, t_m)^T \in T, \tag{P$^+$}$$

where $T$ is closed and bounded convex set contained in the nonnegative orthant, and $A_i$ are positive semidefinite matrices with positive definite sum. The natural convex relaxation of (P$^+$) is the problem

$$\mathrm{Tr}(AX) \to \max \mid \mathrm{Tr}(A_i X) \le t_i, \ i = 1, \dots, m; t \in T. \tag{C}$$

Applying the technique from Sects. 2, 4, we immediately conclude that

$$\mathrm{Opt}(\mathrm{P}^+) \ge \frac{\mathrm{Opt}(\mathrm{C})}{2\ln(2m\mu)}$$

and that every feasible solution of (C) with positive value $\gamma$ of the objective for every $\alpha \ge 2\ln(2m\mu)$ can be efficiently converted into a feasible solution of (P$^+$) with the value of the objective $\le \gamma/\alpha$.

*Inhomogeneous case.* Now consider the case when (P) is replaced with the problem

$$f(x) = x^T Px + 2b^T x \to \max \mid x^T P_i x \le 1, \ i = 1, \dots, m, \tag{P$'$}$$

$P_i \succeq 0$, $\sum_i P_i \succ 0$. The problem clearly is equivalent to

$$\begin{gathered} g(z) = z^T Az \to \max \mid z^T A_i z \le 1, \ i = 1, \dots, m+1, \\ z = \begin{pmatrix} x \\ \tau \end{pmatrix}, \ A = \begin{pmatrix} P & b \\ b^T & 0 \end{pmatrix}, \ A_i = \begin{pmatrix} P_i & 0 \\ 0 & 0 \end{pmatrix}, \ i = 1, \dots, m, \ A_{m+1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \end{gathered} \tag{$\widehat{\mathrm{P}}$}$$

note that $A_i \succeq 0$, $\sum\limits_i A_i \succ 0$. From the results of Sects. 2, 4 it follows that for the optimal value of the semidefinite relaxation (SDP) associated with $(\widehat{P})$ it holds

$$\mathrm{Opt}(\mathrm{P}') \ge \frac{\mathrm{Opt}(\mathrm{SDP})}{2\ln(2(m+1)\mu)}, \quad \mu = \min\left[m+1, \max_i \mathrm{Rank} P_i\right], \tag{23}$$

and that a feasible solution $X$ to (SDP) with a positive value of the objective $\gamma$, for every $\alpha \ge 2\ln(2(m+1)\mu)$, can be efficiently converted, in a deterministic fashion, into a feasible solution to (P$'$) with the value of the objective $\ge \gamma/\alpha$.

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