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A. Ben-Tal · A. Goryashko · E. Guslitzer · A. Nemirovski

Adjustable robust solutions of uncertain linear programs*

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Abstract. We consider linear programs with uncertain parameters, lying in some prescribed uncertainty set, where part of the variables must be determined before the realization of the uncertain parameters ("non-adjustable variables"), while the other part are variables that can be chosen after the realization ("adjustable variables"). We extend the Robust Optimization methodology ([1, 3–6, 9, 13, 14]) to this situation by introducing the Adjustable Robust Counterpart (ARC) associated with an LP of the above structure. Often the ARC is significantly less conservative than the usual Robust Counterpart (RC), however, in most cases the ARC is computationally intractable (NP-hard). This difficulty is addressed by restricting the adjustable variables to be *affine* functions of the uncertain data. The ensuing Affinely Adjustable Robust Counterpart (AARC) problem is then shown to be, in certain important cases, equivalent to a tractable optimization problem (typically an LP or a Semidefinite problem), and in other cases, having a tight approximation which is tractable. The AARC approach is illustrated by applying it to a multi-stage inventory management problem.

Key words. Uncertain linear programs – robust optimization – conic optimization – semidefinite programming – NP-hard continuous optimization problems – adjustable robust counterpart – affinely-adjustable robust counterpart

1. Introduction

Uncertain linear programming problems. Real-world optimization problems, and in particular Linear Programming problems, often possess the following attributes:

- The data are not known exactly and can "drift" around their nominal values, varying in some given *uncertainty set*.
- In many cases small data perturbations can heavily affect the feasibility/optimality properties of the nominal optimal solution, yet the constraints must remain feasible for all "reasonable" realizations (i.e., those in the uncertainty set) of the data.
- The dimensions of the data and the decision vectors are large, and therefore efficient solution methods are required to solve the underlying large scale optimization problems.

A methodology aimed at dealing with uncertain optimization problems under the above "decision environment" was recently developed under the name *Robust Optimization* (RO), see [1, 3–6, 9, 13, 14] and references therein. In this paper we consider

A. Ben-Tal, A. Goryashko, A. Nemirovski: Minerva Optimization Center, Faculty of Industrial Engineering and Management, Technion, Haifa, Israel

E. Guslitzer: Graduate Business School, Stanford University, Stanford, USA

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uncertain linear programs, and extend the scope of the RO in a significant way by introducing the Adjustable RO methodology.

Robust optimization methodology. An *uncertain Linear Programming problem* is defined as a family

$$\left\{\min_x \left\{c^T x : Ax \leq b\right\}\right\}_{\zeta\equiv[A,b,c]\in\mathcal{Z}} \tag{1}$$

of usual Linear Programming problems ("instances") with $m \times n$ matrices A and the *data* $\zeta \equiv [A, b, c]$ varying in a given *uncertainty set* $\mathcal{Z} \subset \mathbf{R}^n \times \mathbf{R}^{m\times n} \times \mathbf{R}^m$ (a nonempty compact convex set). The Robust Optimization methodology associates with such an uncertain LP its *Robust Counterpart* (RC)

$$\min_x \left\{ \sup_{\zeta\equiv[A,b,c]\in\mathcal{Z}} (c^T x) : A^T x - b \leq 0 \quad \forall\zeta \equiv [A, b, c] \in \mathcal{Z} \right\} \tag{2}$$

and treats feasible/optimal solutions of the latter problem as uncertainty-immunized feasible/optimal solutions of the original uncertain LP. Indeed, an uncertainty-immunized solution of (2) satisfies all realizations of the constraints associated with $\zeta \equiv [A, b, c] \in \mathcal{Z}$, while the optimal uncertainty-immunized solution optimizes, under this restriction, the *guaranteed* value of the (uncertain) objective.

Adjustable and non-adjustable variables. The Robust Optimization approach corresponds to the case when *all* the variables represent decisions that must be made before the actual realization of the uncertain data becomes known. There are cases in reality when this is indeed the case, so that the Robust Optimization approach seems to be an adequate way to model the decision-making process. At the same time, in the majority of optimization problems of real-world origin *only part* of the decision variables x_i are actual "here and now" decisions. Some variables are auxiliary, such as slack or surplus variables, or variables introduced in order to convert a model into a Linear Programming form by eliminating piecewise-linear functions like $|x_i|$ or $\max\{x_i, 0\}$. These variables *do not correspond to actual decisions and can tune themselves to varying data*. Another kind of variables represent "wait and see" decisions, those that can be made when part of the uncertain data become known. Thus it is reasonable to assume that the "wait and see" variables can adjust themselves to a corresponding part of the data.

Example 1.1. Consider a factory that produces $p(t)$ units of product to satisfy demand d_t, on each one of the days $t = 1, \ldots, T$. We know that

$$(*)\text{the data}(d_1, \ldots, d_T)\text{takes values in a set } \mathcal{Z},$$

and the actual value of d_t becomes known only at the end of day t, while the decision on how much to produce on day t must be made at the beginning of that day.

When choosing $p(1)$, we indeed know nothing on the actual demand, except for the fact that it obeys the model $(*)$, thus $p(1)$ represents a "here and now" decision. In contrast to this, when implementing every one of the subsequent decisions $p(t), t \geq 2$ we already know the actual demands of the days $I_t = \{1, \ldots, t-1\}$. Thus, it is reasonable to assume that $p(t)$ is a function of $d_r : r \in I_t$, i.e., to let the "wait and see" variables depend on part of the uncertain data.

We have just distinguished between variables that cannot be adjusted to the data ("here and now" decisions) and the variables that can adjust themselves to all the data or to a part of it (auxiliary variables and "wait and see" decisions). In general, every variable x_i may have its own information basis, i.e., can depend on a prescribed portion ζ_i of the true data ζ, as it is the case in Example 1.1. Here, for simplicity of notation, we focus only on the "black and white" case where part of the variables cannot tune themselves to the true value of the data at all, while the remaining variables are allowed to depend on *all* the data. It can easily be seen that the main results of this paper can be straightforwardly extended from this "black and white" case to the general one where every variable has its own information basis. In what follows, we call all the variables that may depend on the realizations of the data *adjustable*, while other variables are called *non-adjustable*. Consequently, the vector x of variables in (1) will be partitioned as $x = (u^T, v^T)^T$, where the sub-vector u represents the non-adjustable and v the adjustable variables.

Adjustable robust counterpart. Distinguishing between the adjustable and the non-adjustable variables we rewrite (1) equivalently as

$$\min_{(s,u),v} \left\{ s : c^T \begin{pmatrix} u \\ v \end{pmatrix} \leq s,\ Uu + Vv \leq b \right\}_{[U,V,b,c]\in\mathcal{Z}}$$

and treat (u, s) as the non-adjustable part of the solution. The above representation of (1) is "normalized" in the sense that its objective is independent of both the uncertain data and the adjustable part v of the variables. In the sequel we assume, w.l.o.g., that the uncertain Linear Programming problem under consideration is normalized and thus write this problem as

$$\mathcal{LP}_{\mathcal{Z}} = \left\{ \min_{u,v} c^T u : Uu + Vv \leq b \right\}_{\zeta=[U,V,b]\in\mathcal{Z}}. \tag{3}$$

Borrowing from the terminology of "Two-stage stochastic programming under uncertainty" ([10],[17]), the matrix V is called *recourse matrix*. When V is not uncertain, we call the corresponding uncertain LP

$$\left\{ \min_{u,v} c^T u : Uu + Vv \leq b \right\}_{\zeta=[U,b]\in\mathcal{Z}}. \tag{4}$$

a *fixed recourse* one.

Definition. *We define the Adjustable Robust Counterpart (ARC) of the uncertain Linear Programming problem $\mathcal{LP}_{\mathcal{Z}}$ as*

$$\text{(ARC)} : \min_u \left\{ c^T u : \forall(\zeta = [U, V, b] \in \mathcal{Z})\, \exists v : Uu + Vv \leq b \right\}. \tag{5}$$

In contrast, the usual Robust Counterpart of $\mathcal{LP}_{\mathcal{Z}}$ is:

$$\text{(RC)} : \min_u \left\{ c^T u : \exists v\, \forall(\zeta = [U, V, b] \in \mathcal{Z}) : Uu + Vv \leq b \right\}, \tag{6}$$

It can easily be seen that the ARC is more flexible than the RC, i.e., it has a larger robust feasible set, enabling a better optimal value while still satisfying all possible realizations of the constraints. The difference between the ARC and the RC can be very significant, as demonstrated in the following example.

Example 1.2. Consider an uncertain Adjustable Linear Programming problem with a single equality constraint: $\alpha u + \beta v = 1$, where the uncertain data (α, β) can take values in the uncertainty set $\mathcal{Z} = \{(\alpha, \beta) | \alpha \in [\frac{1}{2}, 1], \beta \in [\frac{1}{2}, 1]\}$. Then the feasible set of the RC of this problem $\{u | \exists v \forall (\alpha, \beta) \in \mathcal{Z} : \alpha u + \beta v = 1\} = \emptyset$. This happens because in particular for $\alpha = 1$ we get $\forall \beta \in [\frac{1}{2}, 1] : u + \beta v = 1 \Rightarrow u = 1, v = 0$ is the unique solution. And then $\forall \alpha \in [\frac{1}{2}, 1] : \alpha \cdot 1 + \beta \cdot 0 = 1$ does not hold. At the same time, the feasible set of the ARC $\{u | \forall (\alpha, \beta) \in \mathcal{Z} \, \exists v : \alpha u + \beta v = 1\} = \mathbf{R}$, since for any fixed $\bar{u}$ the constraint can be satisfied by taking $v = \frac{1 - \alpha \bar{u}}{\beta}$.

The goal of this paper is to investigate the concept of the Adjustable Robust Counterpart of an uncertain LP problem. The main body of the paper is organized as follows. We start by identifying conditions under which the ARC of an uncertain LP is equivalent to the RC of the problem (Section 2). The conditions turn out to be quite demanding, thus the rest of the paper considers the much more general situations when these conditions are not fulfilled, in which case ARC can be significantly less conservative than the RC. The next issue to be addressed is the tractability of the ARC (Section 2). It turns out that while the RC of an uncertain LP typically is a computationally tractable problem (see [4]), this is not the case with ARC. This unfortunate fact motivates the notion of *Affinely Adjustable Robust Counterpart* (AARC) of an uncertain LP, where we restrict the adjustable variables to be *affine* functions of the corresponding data. This notion is introduced and motivated in Section 3, where we demonstrate that the AARC of an uncertain LP is efficiently (polynomially) solvable in the fixed recourse case. The case of uncertain recourse matrix is considered in Section 4, where we demonstrate that the AARC of an uncertain LP, although itself not necessarily tractable, always admits tight tractable approximation by a Semidefinite program. In Section 5 we apply the AARC methodology to a multi-stage uncertain inventory system.

2. Adjustable robust counterpart of an uncertain LP problem

The very reason for defining ARC is to enable more flexibility in cases when RC is unjustifiably conservative. Nevertheless, there are some cases when the ARC and the RC of uncertain Adjustable Linear Programming problem are equivalent. These include the cases where the uncertainty affecting every one of the constraints is independent of the uncertainty affecting all other constraints.

Theorem 2.1. *Let $\mathcal{LP}_{\mathcal{Z}}$ (3) satisfy the following two assumptions:*

1. *The uncertainty is constraint-wise, i.e., there is a partition $\zeta = (\zeta^1, \ldots, \zeta^m)$ of vector ζ in non-overlapping sub-vectors such that*
   - *For every $i = 1, ..., m$, and for all u, v the quantity $(U(\zeta)u + V(\zeta)v - b(\zeta))_i$ depends on ζ^i only:*

$$(U(\zeta)u + V(\zeta)v - b(\zeta))_i = (U(\zeta^i)u + V(\zeta^i)v - b(\zeta^i))_i;$$

- *There exist nonempty convex compact sets* $\mathcal{Z}_i \subset \mathbf{R}^{\dim \zeta^i}$ *such that* $\mathcal{Z} = \mathcal{Z}_1 \times \ldots \times \mathcal{Z}_m = \{\zeta = (\zeta^1, ..., \zeta^m) : \zeta^i \in \mathcal{Z}_i,\ i = 1, \ldots, m\}$.

2. *Whenever* u *is feasible for* (ARC)*, there exists a compact set* V_u *such that for every* $\zeta \equiv [U, V, b] \in \mathcal{Z}$ *the relation*

$$U(\zeta)u + V(\zeta)v \leq b(\zeta)$$

implies that $v \in V_u$.

Then the RC *of* $\mathcal{LP}_\mathcal{Z}$ *is equivalent to its* ARC.

Proof. See Appendix.

The conditions under which ARC is equivalent to RC are quite stringent. Even in simple situations when two or more constraints can depend on the same uncertain parameter, the ARC can significantly improve the solution obtained by the RC. As an example consider the following uncertain LP:

$$\min_{u,v} \{-u : (1 - 2\xi)u + v \geq 0,\ \xi u - v \geq 0,\ u \leq 1\}_{0 \leq \xi \leq 1}$$

Note that here the uncertainty ξ influences both the first and the second constraints. It can easily be seen that the optimal value of the RC of this problem is

$$\min_u \{-u \mid \exists\, v\, \forall(\xi \in [0, 1]) : (1 - 2\xi)u + v \geq 0,\ \xi u - v \geq 0,\ u \leq 1\} = 0$$

achieved at the unique solution $u = 0$, $v = 0$. The optimal value of the ARC is

$$\min_u \{-u \mid \forall(\xi \in [0, 1])\, \exists\, v : (1 - 2\xi)u + v \geq 0,\ \xi u - v \geq 0,\ u \leq 1\} = -1,$$

where for any $\bar{u} \leq 1$ we can take $v = \xi \bar{u}$ to obtain feasibility.

Tractability of the ARC. Unfortunately, it turns out that there is a "price to pay" for the flexibility of the ARC. Specifically, the usual RC of $\mathcal{LP}_\mathcal{Z}$ (6), with computationally tractable uncertainty set $\mathcal{Z}$, can be solved efficiently (see [4]). Usually this is not the case with the Adjustable Robust Counterpart of $\mathcal{LP}_\mathcal{Z}$ (5). One simple case when ARC *is* tractable (in fact is an LP) is the following:

Theorem 2.2 (see [12])**.** *Assume that the uncertainty set* $\mathcal{Z}$ *is given as the convex hull of a finite set :*

$$\mathcal{Z} = \mathrm{Conv}\,\{[U_1, V_1, b_1], \ldots, [U_N, V_N, b_N]\}\,. \tag{A}$$

Then in the case $V_1 = ... = V_N$ *of fixed recourse the* ARC *is given by the usual* LP

$$\min_{u, v_1, \ldots, v_N} \left\{c^T u : U_\ell u + V v_\ell \leq b_\ell,\ \ell = 1, \ldots, N\right\},$$

and as such is computationally tractable.

Even in the fixed recourse case with a general-type polytope $\mathcal{Z}$ given by a list of linear inequalities, the ARC can be NP-hard. The same is true when $\mathcal{Z}$ is given by (A), but V_i do depend on i (for proofs of these claims, see [12]).

From the practical viewpoint, the Robust Optimization approach is useful only when the resulting robust counterpart of the original uncertain problem is a computationally tractable program, which is not the typical case for the Adjustable RC approach. A possible remedy is to find a computationally tractable *approximation* of the ARC. This is the issue we address next.

3. Affinely adjustable robust counterpart

When passing from general uncertain problem $\mathcal{LP}_{\mathcal{Z}}$ to its Adjustable Robust Counterpart, we allow the adjustable variables v to tune themselves to the true data ζ. What we are interested in are the non-adjustable variables u which can be extended, by appropriately tuned adjustable variables, to a feasible solution of the instance of $\mathcal{LP}_{\mathcal{Z}}$, *whatever be the instance*. Now let us impose a restriction on *how* the adjustable variables can be tuned to the data. The simplest restriction of this type is to require of the adjustable variables to be *affine* functions of the data. The main motivation behind this restriction is that, as we shall see in a while, it results in a computationally tractable robust counterpart. The *affine* dependency restriction we have chosen here is very much in the spirit of using *linear* feedbacks in controlled dynamical systems.

Example 2.1. Consider the controlled dynamical system:

$$\begin{array}{l} s_{t+1} = A(t)s_t + B(t)u_t + D(t)d_t \\ y_t = C(t)s_t \end{array}, \ t = 0, 1, 2, ... \tag{7}$$

where:

- s_t is the state of the plant,
- y_t is the output,
- u_t is the control,
- $A(t), B(t), C(t), D(t)$ are the certain (time-varying) data of the system, and d_t is the uncertain exogenous input.

A typical control law for (7) is given by a *linear feedback* $u_t = K(t)y_t$. With such a control, the dynamics of the system becomes $s_{t+1} = [A(t)+B(t)K(t)C(t)]s_t + D(t)d_t$. Assuming that the only uncertainty affecting the system is represented by the exogenous input d_t, one can observe that (for a given feedback and initial state) the states s_t (and thus – the controls u_t) become affine functions of d_r, $r < t$. Consequently, treating u_t as the adjustable decision variables, we can say that *control via a linear feedback, in a linear dynamical system affected by uncertain exogenous input, is a particular case of decision-making with adjustable variables specified as affine functions of the uncertainty*.

Coming back to the general problem $\mathcal{LP}_{\mathcal{Z}}$, we assume that *for u given, v is forced to be an affine function of the data:*

$$v = w + W\zeta.$$

With this approach, the possibility to extend non-adjustable variables u to feasible solutions of instances of $\mathcal{LP}_{\mathcal{Z}}$ is equivalent to the fact that u can be extended, by properly chosen vector w and matrix W, to a solution of the infinite system of inequalities

$$Uu + V(w + W\zeta) \leq b \quad \forall(\zeta = [U, V, b] \in \mathcal{Z}),$$

in variables u, w, W, and the *Affinely Adjustable Robust Counterpart (AARC) of $\mathcal{LP}_{\mathcal{Z}}$* (3) is defined as the optimization program

$$\min_{u,w,W} \left\{ c^T u : Uu + V(w + W\zeta) \leq b \quad \forall(\zeta = [U, V, b] \in \mathcal{Z}) \right\}. \tag{8}$$

Note that (8) is "in-between" the usual RC of $\mathcal{LP}_{\mathcal{Z}}$ and the Adjustable RC of the problem; to get the RC, one should set to zero the variable W in (8). Since (8) seems simpler than the general ARC of $\mathcal{LP}_{\mathcal{Z}}$, there is a hope that it is computationally tractable in cases where the ARC is intractable. We are about to demonstrate that

1. In the case of *fixed recourse*, the AARC is computationally tractable, provided that that the uncertainty set $\mathcal{Z}$ itself is *computationally tractable* (Theorem 3.1). The latter concept means that for any vector z there is a tractable computational scheme (so-called "separation oracle") that either asserts correctly that $z \in \mathcal{Z}$ or, otherwise, generates a separator – a nonzero vector s such that $s^T z \geq \max_{y \in \mathcal{Z}} s^T y$ (see [11] for a comprehensive treatment of these notions). Moreover, if the uncertainty set is given by system of linear/second-order cone/semidefinite constraints, the AARC can be reformulated as an explicit Linear/Second-Order Cone/Semidefinite program, respectively (Theorems 3.2, 3.3);
2. In the case of uncertainty-affected recourse and the uncertainty set being the intersection of concentric ellipsoids, the AARC admits tight computationally tractable *approximation* which is an explicit Semidefinite program; this approximation is exact when the uncertainty set is an ellipsoid (Theorem 4.1).

Theorem 3.1. *Consider the fixed recourse* $\mathcal{LP}_{\mathcal{Z}}$ (4). *If* $\mathcal{Z}$ *is a computationally tractable set, then the* AARC (8) *of this* $\mathcal{LP}_{\mathcal{Z}}$ *is computationally tractable as well.*

Proof. It can easily be seen that since the recourse matrix V is fixed, the AARC (8) can be rewritten as:

$$\min_{x=[u,w,W]} \left\{ c^T u : A(\zeta)x \leq b(\zeta), \ \forall \zeta \in \mathcal{Z} \right\}, \tag{9}$$

with properly chosen $A(\zeta)$, $b(\zeta)$ affinely depending on ζ and with $x = [u, w, W]$. An equivalent representation of (9) is :

$$\begin{aligned} &\min_x \ \left\{ c^T x : x \in G \right\} \\ &\quad G = \{x \mid \forall \zeta \in \mathcal{Z} : \ -A(\zeta)x + b(\zeta) \geq 0\} \end{aligned} \tag{10}$$

The latter problem is the usual Robust Counterpart of an uncertain Linear Programming problem with the data affinely parameterized by $\zeta \in \mathcal{Z}$, so that the results on computational tractability of (10) (and thus – of (8)) are readily given by [4]. □

Theorem 3.1 claims that when the uncertainty set $\mathcal{Z}$ in a fixed recourse $\mathcal{LP}_{\mathcal{Z}}$ is computationally tractable, then so is the AARC of the problem. We are about to demonstrate that when $\mathcal{Z}$ is not only computationally tractable, but is also "well-structured" (specifically, given by a list of linear inequalities or, much more generally, by a list of Linear Matrix Inequalities), then the corresponding AARC is also "well structured" (and thus can be solved, even in the large-scale case, by powerful optimization techniques known for Linear and Semidefinite Programming). To derive these results it is more convenient to pass to an equivalent form of AARC.

Parameterized uncertainty and equivalent form of AARC. It is convenient to assume the uncertainty set $\mathcal{Z}$ to be affinely parameterized by a "vector of perturbations" ξ varying in a nonempty convex compact *perturbation set* $\mathcal{X} \subset \mathbf{R}^L$, so that

$$\mathcal{Z} = \{[U, V, b] = [U^0, V^0, b^0] + \sum_{\ell=1}^{L} \xi_\ell [U^\ell, V^\ell, b^\ell] : \xi \in \mathcal{X}\}, \tag{11}$$

Observe that this assumption does not restrict generality (since we always can set $U^0 = 0$, $V^0 = 0$, $b^0 = 0$, $\mathcal{X} = \mathcal{Z}$ and define ξ as the vector of coordinates of $[U, V, b]$ with respect to a certain basis $\{[U^\ell, V^\ell, b^\ell]\}$ in the space of triples $[U, V, b]$). Without loss of generality we may assume that the *parameterization mapping*

$$\xi \mapsto [U^0, V^0, b^0] + \sum_{\ell=1}^{L} \xi_\ell [U^\ell, V^\ell, b^\ell] \tag{12}$$

is an embedding; since $\mathcal{Z}$ is a nonempty convex compact set and the parameterization is an embedding, the set $\mathcal{X}$ is indeed a nonempty convex compact.

Note that since the parameterization (11) of $\mathcal{Z}$ is an affine embedding, the restriction on the adjustable variables to be affine functions of ζ is equivalent to the restriction on these variables to be affine functions of ξ. Thus, when building the AARC, we lose nothing by assuming that the adjustable variables are affine functions of ξ:

$$v = v(\xi) = v^0 + \sum_\ell \xi_\ell v^\ell.$$

In terms of the original non-adjustable variables u and the just introduced non-adjustable variables $v^0, v^1, \dots, v^L$, the AARC of the uncertain LP (3) becomes

$$\min_{u, v^0, v^1, \dots, v^L} \left\{ c^T u : \begin{array}{l} \left[U^0 + \sum \xi_\ell U^\ell\right] u + \left[V^0 + \sum \xi_\ell V^\ell\right]\left[v^0 + \sum \xi_\ell v^\ell\right] \\ \leq \left[b^0 + \sum \xi_\ell b^\ell\right], \forall \xi \in \mathcal{X} \end{array} \right\}. \tag{13}$$

We are currently dealing with the case when the recourse matrix V is fixed. Thus, (13) becomes:

$$\min_{u, v^0, v^1, \dots, v^L} \left\{ c^T u : \begin{array}{l} \left[U^0 + \sum \xi_\ell U^\ell\right] u + V\left[v^0 + \sum \xi_\ell v^\ell\right] \\ \leq \left[b^0 + \sum \xi_\ell b^\ell\right], \forall \xi \in \mathcal{X} \end{array} \right\}. \tag{14}$$

The AARC of a fixed recourse LP with a cone-represented uncertainty set. Consider the case of a perturbation set given by a "conic representation"

$$\mathcal{X} = \{\xi \mid \exists \omega : A\xi + B\omega \geq_{\mathcal{K}} d\} \subset \mathbf{R}^L; \tag{15}$$

here A, B, d are the data of the representation, $\mathcal{K}$ is a closed, pointed convex cone with a nonempty interior and $a \geq_{\mathcal{K}} b$ means that $a - b \in \mathcal{K}$. Let us set

$$\begin{array}{l} \chi = (u, v^0, v^1, \dots, v^L), \\ a_\ell^i \equiv a_\ell^i(\chi) \equiv (-U^\ell u - V v^\ell + b^\ell)_i, \ \ell = 0, 1, \dots, L, \ i = 1, ..., m. \end{array}$$

Theorem 3.2. *Assume that the perturbation set is represented by* (15) *and the representation is strictly feasible, i.e., there exist* $\bar{\xi}, \bar{\omega}$ *such that*

$$A\bar{\xi} + B\bar{\omega} - d \in \mathrm{int}\mathcal{K}.$$

Then the AARC *of the fixed recourse* LP (4) *is equivalent to the optimization program*

$$\begin{array}{rcl}
\min\limits_{u,v^0,v^1,\ldots,v^L,\lambda^1,\ldots,\lambda^m} & c^T u & \\
 & s.t. & \\
A^T\lambda^i - a^i(u, v^0, v^1, \ldots, v^L) & = & 0,\ i = 1, \ldots, m; \\
B^T\lambda^i & = & 0,\ i = 1, \ldots, m; \\
d^T\lambda^i + a_0^i(u, v^0, v^1, \ldots, v^L) & \geq & 0,\ i = 1, \ldots, m; \\
\lambda^i & \geq_{\mathcal{K}_*} & 0,\ i = 1, \ldots, m,
\end{array} \tag{16}$$

where $\mathcal{K}_*$ *is the cone dual to* $\mathcal{K}$.

In particular, in the case when $\mathcal{K}$ *is a direct product of Lorentz cones or a semidefinite cone, the* AARC *of the fixed recourse* LP (4) *is, respectively, an explicit Conic Quadratic or Semidefinite Programming program. The sizes of the latter problems are polynomial in those of the description of the perturbation set* (15) *and the parameterization mapping* (11).

Proof. The objective in (14) is the same as in (16). Thus, all we should verify is that a collection $\chi = (u, v^0, v^1, \ldots, v^L)$ is feasible for (14) if and only if it can be extended to a feasible solution of (16).

Observe that $\chi = (u, v^0, v^1, \ldots, v^L)$ is feasible for (14) if and only if for every $i = 1, \ldots, m$ relation

$$a_0^i(\chi) + \sum_{\ell=1}^{L} \xi_\ell a_\ell^i(\chi) \geq 0,\ \forall \xi \in \mathcal{X}. \tag{17}$$

holds, or, which is the same, if and only if for all $i = 1, ..., m$ the optimal values in the conic programming programs

$$\begin{array}{l}
\mathrm{Opt}_i \equiv \min\limits_{\xi,\omega} a_0^i(\chi) + \sum a_\ell^i(\chi)\xi_\ell \\
\qquad\quad \text{s.t. } A\xi + B\omega \geq_{\mathcal{K}} d;
\end{array} \tag{$CP_i[\chi]$}$$

in variables ξ, ω are nonnegative.

Since (15) is strictly feasible, problem $(CP_i[\chi])$ is strictly feasible. Therefore by the Conic Duality Theorem, $\mathrm{Opt}_i \geq 0$ if and only if the corresponding conic dual problem, which is

$$\max_{\lambda} \left\{ a_0^i(\chi) + d^T\lambda : A^T\lambda = a^i(\chi) \equiv (a_1^i(\chi), ..., a_L^i(\chi))^T,\ B^T\lambda = 0,\ \lambda \in \mathcal{K}_* \right\}. \tag{$D_i[\chi]$}$$

has a nonnegative optimal value, i.e., if and only if

$$\exists \lambda : A^T\lambda = a^i(\chi) \equiv (a_1^i(\chi), \ldots, a_L^i(\chi))^T,\ B^T\lambda = 0,\ \lambda \in \mathcal{K}_*,\ a_0^i(\chi) + d^T\lambda \geq 0.$$

Recalling that χ is feasible for (14) if and only if $\mathrm{Opt}_i \geq 0,\ i = 1, \ldots, m$, we conclude that χ is feasible for (14) if and only if χ can be extended by properly chosen λ^i, $i = 1, \ldots, m$, to a feasible solution to (16), as claimed. □

In the case when $\mathcal{K}$ is a nonnegative orthant, i.e., $\mathcal{Z}$ is a polyhedral set given by

$$\mathcal{X} = \{\xi \mid \exists \omega : A\xi + B\omega \geq d\} \subset \mathbf{R}^L, \tag{18}$$

the proof of Theorem 3.2 clearly remains valid without imposing the assumption of strict feasibility of (18), and we arrive at the following result.

Theorem 3.3. *In the case of polyhedral perturbation set* (18)*, the* AARC *of the fixed recourse* LP (4) *is equivalent to an explicit* LP *program. The sizes of this* LP *are polynomial in the sizes of the description of the perturbation set and the parameterization mapping* (12).

4. Approximating the AARC in the case of uncertainty-affected recourse matrix

In the previous Section we have seen that the AARC of a fixed recourse uncertain LP with tractable uncertainty is computationally tractable. Unfortunately, when the recourse matrix V is affected by uncertainty, the AARC can become computationally intractable. We demonstrate this by the following example.

Example 3.1. Consider the following uncertain LP:

$$\left\{\min_{u,v}\left\{u \,\middle|\, u - \zeta^T v \geq 0, v \geq Q\zeta, -v \geq -Q\zeta\ \right\}\right\}_{\zeta\in\triangle}$$

where Q is an $n \times n$ matrix and $\triangle = \{\xi \geq 0 : \sum\limits_i \xi_i = 1\}$ is the standard simplex. The AARC of this uncertain problem is:

$$\min_{u,w,W}\left\{u \left| \begin{array}{ll} u - \zeta^T(w + W\zeta) \geq 0 & \\ (w + W\zeta) \geq Q\zeta & \forall \zeta \in \triangle \\ -(w + W\zeta) \geq -Q\zeta & \end{array} \right.\right\},$$

or, which is the same:

$$\min_u\{u | u \geq \zeta^T Q\zeta\ \forall \zeta \in \triangle\}.$$

The optimal value in the AARC is $\max_{\zeta\in\triangle}\{\zeta^T Q\zeta\}$. Thus, if the AARC in question would be computationally tractable, so would be the problem of maximizing a quadratic form over the standard simplex. But the latter problem is equivalent to the problem of checking copositivity, a problem known to be NP-hard (see, e.g., [15, 16]).

We have just seen that when the recourse matrix V is not fixed, the AARC of $\mathcal{LP}_\mathcal{Z}$ can become computationally intractable. The goal of this section is to utilize recent results of [6] (obtained there for the robust counterparts of general-type uncertain conic quadratic constraints affected by ellipsoidal uncertainty) in order to demonstrate that in many important cases this intractable AARC admits a *tight computationally tractable approximation*.

It is more convenient to use the "perturbation-based" model (11) of the uncertainty set and the description of the AARC as it appears in (13). Then the feasible set of the AARC is given by a system of m scalar constraints:

$$\begin{aligned}\forall \xi \in \mathcal{X} : &\left[U_i^0 + \sum \xi_\ell U_i^\ell\right] u + \left[V_i^0 + \sum \xi_\ell V_i^\ell\right]\left[v^0 + \sum \xi_\ell v^\ell\right] - \left[b_i^0 + \sum \xi_\ell b_i^\ell\right] \\ &= \left[U_i^0 u + V_i^0 v^0 - b_i^0\right] + \sum \xi_\ell \left[U_i^\ell u + V_i^0 v^\ell + V_i^\ell v^0 - b_i^\ell\right] \\ &\quad + \left[\sum \xi_k \xi_\ell V_i^k v^\ell\right] \le 0. \end{aligned} \tag{19}$$

Setting $x \equiv [u, v^0, v^1, \ldots, v^L]$, we define the functions

- $\alpha_i(x) \equiv -\left[U_i^0 u + V_i^0 v^0 - b_i^0\right]$
- $\beta_i^\ell(x) \equiv -\frac{\left[U_i^\ell u + V_i^0 v^\ell + V_i^\ell v^0 - b_i^\ell\right]}{2}, \ell = 1, \ldots, L$
- $\Gamma_i^{(\ell,k)}(x) \equiv -\frac{V_i^k v^\ell + V_i^\ell v^k}{2}, \ell, k = 1, \ldots, L,$

$i = 1, \ldots, m$, and observe that these are affine functions of x. We can now represent (19) as:

$$\forall \xi \in \mathcal{X} : \alpha_i(x) + 2\xi^T \beta_i(x) + \xi^T \Gamma_i(x)\xi \ge 0, \tag{20}$$

where $\Gamma_i(x)$, by construction, is a symmetric matrix. Assume that we deal with an $\cap$-*ellipsoidal uncertainty*, specifically, that the perturbation vector ξ takes values in the set

$$\mathcal{X} = \mathcal{X}_\rho \equiv \left\{\xi \,\middle|\, \xi^T S_k \xi \le \rho^2,\ k = 1, \ldots, K\right\},$$

with $\rho > 0,\ S_k \succeq 0,\ \sum S_k \succ 0$, i.e., the perturbation set is explicitly given as an intersection of a finite number of concentric ellipsoids and elliptic cylinders. Note that the $\cap$-ellipsoidal uncertainty allows for a wide variety of convex perturbation sets symmetric with respect to the origin. For example,

- if $K = 1$, we get a perturbation set which is an ellipsoid centered at the origin;
- if $K = \dim \xi$ and $\xi^T S_k \xi = a_k^{-2}\xi_k^2$, $k = 1, \ldots, K$, we get a perturbation set which is a box $\{|\xi_k| \le \rho a_k,\ k = 1, \ldots, K\}$ centered at the origin;
- if S_k is a dyadic matrix $g_k g_k^T$, we get a perturbation set which is a polytope symmetric with respect to the origin: $\{\xi : |g_k^T \xi| \le \rho,\ k = 1, \ldots, K\}$.

An equivalent representation of $\mathcal{X}$ is:

$$\mathcal{X} = \left\{\xi \,\middle|\, \xi^T \left(\rho^{-2} S_k\right) \xi \le 1,\ k = 1, \ldots, K\right\}.^{1)} \tag{21}$$

[1] For $\rho > 0$ still holds: $(\rho^{-2} S_k) \succeq 0,\ \sum(\rho^{-2} S_k) \succ 0$.

Now let us make use of the following simple observation:

Lemma 4.1. *For every x, the implication*

$$\begin{array}{c}\forall t,\ \xi:\ t^2 \le 1,\ \xi^T(\rho^{-2}S_k)\xi \le 1,\ k=1,\dots,K\\ \Downarrow\\ -2\,\xi^T\beta_i(x)t - \xi^T\,\Gamma_i(x)\xi \le \alpha_i(x).\end{array} \qquad \mathrm{A}(i)$$

is valid if and only if x is feasible for i-th constraint (20) *of the* AARC*:*

$$\forall \xi \in \mathcal{X} : \alpha_i(x) + 2\,\xi^T\beta_i(x) + \xi^T\,\Gamma_i(x)\xi \ge 0. \qquad \mathrm{B}(i)$$

Proof. Suppose that x is such that A(i) is true, and let $\xi \in \mathcal{X}$. By (21) the latter means that the pair $(\xi, t=1)$ satisfies the premise in A(i); thus, if A(i) holds true, then $(\xi, t=1)$ satisfies the conclusion in A(i) as well, and this conclusion for $t=1$ is exactly the inequality in B(i). We see that if x is such that A(i) holds true, then B(i) holds true as well. Vice versa, let x be such that B(i) holds true. Since the set $\mathcal{X}$ is given by (21) (and thus is symmetric), it follows that

$$\forall(\xi : \xi^T\rho^{-2}S_k\xi \le 1,\ k=1,\dots,K) : \pm 2\,\xi^T\beta_i(x) - \xi^T\,\Gamma_i(x)\xi \le \alpha_i(x),$$

so that the inequality in the conclusion of A(i) is valid for all (ξ, t) such that $\xi^T\rho^{-2}S_k\xi \le 1$, $k=1,\dots,K$, and $t=\pm 1$. Since the left hand side of this inequality is linear in t, the inequality in fact is valid for all (ξ, t) satisfying the premise of A(i), so that A(i) holds true. □

Now let $\mu \ge 0, \lambda_k \ge 0, k=1,\dots,K$. For every pair (ξ, t) satisfying the premise in A(i) we have $t^2 \le 1, \xi^T\rho^{-2}S_k\xi \le 1, k=1,\dots,K$, therefore the relation

$$\mu t^2 + \xi^T\left(\sum_{k=1}^{K}\rho^{-2}\lambda_k S_k\right)\xi \le \mu + \sum_{k=1}^{K}\lambda_k \qquad (*)$$

is a consequence of the inequalities in the premise of A(i). It follows that if for a given x there exist $\mu \ge 0, \lambda_k \ge 0, k=1,\dots,K$ such that $\mu + \sum_{k=1}^{K}\lambda_k \le \alpha_i(x)$ and at the same time for all $t,\ \xi$

$$-2\,\xi^T\beta_i(x)t - \xi^T\,\Gamma_i(x)\xi \le \mu t^2 + \xi^T\left(\sum_{k=1}^{K}\rho^{-2}\lambda_k S_k\right)\xi,$$

then, in particular, for $t,\ \xi$ satisfying the premise of A(i) one has:

$$-2\,\xi^T\beta_i(x)t - \xi^T\,\Gamma_i(x)\xi \le \mu t^2 + \xi^T\left(\sum_{k=1}^{K}\rho^{-2}\lambda_k S_k\right)\xi \le \mu + \sum_{k=1}^{K}\lambda_k \le \alpha_i(x),$$

i.e., the conclusion of A(i) is satisfied. Thus, in the case in question A(i) (and consequently B(i)) holds true. In other words, the condition

$$\begin{array}{c}\exists \mu, \lambda_1,\dots,\lambda_K \ge 0 : \left(\begin{array}{c|c}\Gamma_i(x) + \rho^{-2}\sum_{k=1}^{K}\lambda_k S_k & \beta_i(x)\\ \hline \beta_i^T(x) & \mu\end{array}\right) \succeq 0,\\ \mu + \sum_{k=1}^{K}\lambda_k \le \alpha_i(x)\end{array} \qquad (22)$$

is a sufficient condition for the validity of B(i). Recalling that x is feasible for AARC if and only if the corresponding relations B(i) hold true for $i = 1, \dots, m$, and eliminating μ, we arrive at the following result:

Theorem 4.1. *The explicit Semidefinite program*

$$\begin{array}{ll} \min\limits_{\lambda^1,\dots,\lambda^m,\, x=[u,v^0,v^1,\dots,v^K]} & c^T u \\ \text{s.t.} & \left(\begin{array}{c|c} \Gamma_i(x) + \rho^{-2}\sum_{k=1}^K \lambda_k^i S_k & \beta_i(x) \\ \hline \beta_i^T(x) & \alpha_i(x) - \sum_{k=1}^K \lambda_k^i \end{array}\right) \succeq 0,\ i = 1, \dots m \\ & \lambda^i \geq 0,\ i = 1, \dots, m \end{array} \tag{23}$$

is a "conservative approximation" to the AARC (13)*, i.e., if x can be extended, by some $\lambda^1, \dots, \lambda^m$, to a feasible solution of* (23)*, then x is feasible for* (13)*.*

We are about to prove that

- when $K = 1$, i.e., when the perturbation set $\mathcal{X}$ is an ellipsoid centered at the origin ("simple ellipsoidal uncertainty"), problem (23) is *exactly equivalent* to the AARC, and
- in the general case $K > 1$, problem (23) is a tight, in certain precise sense, approximation of the AARC.

The case of simple ellipsoidal uncertainty. We start with the following modification of Lemma 4.1:

Lemma 4.2. *Let $K = 1$. For every x, the implication*

$$\begin{array}{c} \forall\, \xi, t:\ \xi^T(\rho^{-2}S_1)\xi \leq t^2 \\ \Downarrow \\ \alpha_i(x)t^2 + 2\,\xi^T\beta_i(x)t + \xi^T\,\Gamma_i(x)\xi \geq 0 \end{array} \tag{C(i)}$$

holds true if and only if x is feasible for i'th constraint (20) *of the* AARC*:*

$$\forall \xi \in \mathcal{X}: \alpha_i(x) + 2\,\xi^T\beta_i(x) + \xi^T\,\Gamma_i(x)\xi \geq 0. \tag{B(i)}$$

Proof. Indeed, in view of Lemma 4.1, in order to prove Lemma 4.2 it suffices to verify that for a given x, C(i) takes place if and only if A(i) takes place. Let A(i) be valid; then if ξ and $t \neq 0$ satisfy the premise of C(i), i.e., $\xi^T(\rho^{-2}S_1)\xi \leq t^2$, then $\xi/|t|,\ t/|t|$ satisfy the premise of A(i), and the conclusion of A(i)

$$-\frac{2\xi^T\beta_i(x)t}{t^2} - \frac{\xi^T\Gamma_i(x)\xi}{t^2} \leq \alpha_i(x)$$

is exactly the desired conclusion of C(i). Since we are in the situation when $S_1 = \sum\limits_{k=1}^{K} S_k \succ 0$, the only pair (ξ, t) with $t = 0$ satisfying the premise in C(i) is the trivial pair ($\xi = 0, t = 0$), and for this pair the conclusion in C(i) is evident. Thus, when A(i) is valid, so is C(i).

Now let us prove that the validity of C(i) implies the validity of A(i). Assume that C(i) is valid, and let (ξ, t) satisfy the premise in A(i), i.e., $\xi^T \rho^{-2} S_1 \xi \leq 1$, $t^2 \leq 1$. The pairs $(\xi, 1)$ and $(\xi, -1)$ satisfy the premise in C(i); since C(i) is valid, the conclusion of C(i) is satisfied for both these pairs, i.e.,

$$\alpha_i(x) \pm 2\,\xi^T \beta_i(x) + \xi^T\, \Gamma_i(x)\xi \geq 0.$$

Since $|t| \leq 1$, we conclude that

$$\alpha_i(x) + 2\,\xi^T \beta_i(x)t + \xi^T\, \Gamma_i(x)\xi \geq 0$$

as well, i.e., (ξ, t) satisfy the conclusion in A(i), so that A(i) is valid. □

Now let us recall the following fundamental fact (see, e.g., [2, 8]):

Lemma 4.3 **($\mathcal{S}$ - Lemma).** *Let A, B be symmetric matrices of the same size, and let the homogeneous quadratic inequality*

$$y^T A y \geq 0 \tag{24}$$

be strictly feasible (i.e., $\bar{y}^T A \bar{y} > 0$ for some $\bar{y}$). A homogeneous quadratic inequality

$$y^T B y \geq 0$$

is a consequence of (24) *if and only if there exists nonnegative λ such that*

$$B \succeq \lambda A.$$

Let us apply the $\mathcal{S}$-Lemma to the following pair of quadratic forms of $y = \begin{pmatrix} \xi \\ t \end{pmatrix}$:

$$\begin{aligned} y^T A y &= t^2 - \xi^T \rho^{-2} S_1 \xi & \text{where } A &= \begin{pmatrix} -\rho^{-2} S_1 & 0 \\ 0 & 1 \end{pmatrix} \\ y^T B y &= \alpha_i(x)t^2 + 2\,\xi^T \beta_i(x)t + \xi^T\, \Gamma_i(x)\xi & \text{where } B &= \begin{pmatrix} \Gamma_i(x) & \beta_i(x) \\ \beta_i^T(x) & \alpha_i(x) \end{pmatrix}. \end{aligned}$$

Clearly, the quadratic inequality $y^T A y \geq 0$ is strictly feasible. Applying the $\mathcal{S}$ - Lemma we conclude that the implication C(i) takes place if and only if

$$\exists \lambda \geq 0 : \quad B - \lambda A \equiv \begin{pmatrix} \Gamma_i(x) + \lambda \rho^{-2} S_1 & \beta_i(x) \\ \beta_i^T(x) & \alpha_i(x) - \lambda \end{pmatrix} \succeq 0. \tag{25}$$

Since C(i) is equivalent to B(i), and x is feasible for the AARC if and only if the predicates B(i) holds for $i = 1, \ldots, m$, we conclude that x is feasible for the AARC if and only if (25) holds for every $i = 1, \ldots, m$, or, which is the same, if and only if x can be extended to a feasible solution of (23). We arrive at the following result:

Theorem 4.2. *In the case of simple ellipsoidal uncertainty $K = 1$, the explicit semidefinite program* (23) *is equivalent to the* AARC.

The case of $K > 1$. In order to derive a result on the quality of the approximation of (13), given by Theorem 4.1, we need the following "approximate" $\mathcal{S}$-Lemma proved in [6]:

Lemma 4.4 (Approximate S-lemma). *Let $R, R_0, R_1, \ldots, R_K$ be symmetric $n \times n$ matrices such that $R_1, \ldots, R_K \succeq 0$, and assume that*

$$\exists \nu_0, \nu_1, \ldots, \nu_K \geq 0 \text{ such that } \sum_{k=0}^{K} \nu_k R_k \succ 0.$$

Consider the following quadratically constrained quadratic program:

$$QCQ = \max_{y \in \mathbf{R}^n} \left\{ y^T R y : y^T R_0 y \leq r_0, y^T R_k y \leq 1, k = 1, \ldots, K \right\} \tag{26}$$

and the semidefinite optimization problem:

$$SDP = \min_{\mu_0, \mu_1, \ldots, \mu_K} \left\{ r_0 \mu_0 + \sum_{k=1}^{K} \mu_k : \sum_{k=0}^{K} \mu_k R_k \succeq R, \mu \geq 0 \right\}. \tag{27}$$

Then

(i) *If problem* (26) *is feasible, then problem* (27) *is bounded below and* $SDP \geq QCQ$. *Moreover there exist* $y_* \in \mathbf{R}^n$ *such that*

$$\begin{aligned} (a) \quad & y_*^T R y_* = SDP \\ (b) \quad & y_*^T R_0 y_* \leq r_0 \\ (c) \quad & y_*^T R_k y_* \leq \Omega^2, \; k = 1, \ldots, K \end{aligned} \tag{28}$$

where

$$\Omega = \sqrt{2 \ln \left(6 \sum_{k=1}^{K} \mathrm{Rank}(R_k) \right)},$$

if R_0 *is a dyadic matrix, and*

$$\Omega = \sqrt{2 \ln \left(16 n^2 \sum_{k=1}^{K} \mathrm{Rank}(R_k) \right)} \tag{29}$$

otherwise.

(ii) *If* $r_0 > 0$, *then* (26) *is feasible, problem* (27) *is solvable and*

$$0 \leq QCQ \leq SDP \leq \Omega^2 QCQ. \tag{30}$$

Now assume that certain x *cannot* be extended to a feasible solution of (23), so that for certain $i \leq m$ the condition (22) is *not* valid. Given x and i, let us specify the entities appearing in the Approximate $\mathcal{S}$-Lemma as follows:

- $y = \begin{pmatrix} \xi \\ t \end{pmatrix}$;
- $y^T R y = -2\xi^T \beta_i(x) t - \xi^T \Gamma_i(x) \xi$;

- $y^T R_0 y = t^2, r_0 = 1$;
- $y^T R_k y = \xi^T \rho^{-2} S_k \xi, k = 1, \dots, K$.

It is immediately seen that with this setup

- all the conditions of the Approximate $\mathcal{S}$-Lemma are satisfied (with $r_0 > 0$ and R_0 being dyadic);
- the validity of the condition (22) is equivalent to the validity of the inequality $SDP \leq \alpha_i(x)$. Since we are in the situation when the former condition is not valid, we conclude that $SDP > \alpha_i(x)$.

By the conclusion of the Approximate $\mathcal{S}$-Lemma, there exists $y_* = \begin{pmatrix} \xi_* \\ t_* \end{pmatrix}$ such that (28) holds true, i.e., such that

$$\begin{array}{ll} (a) & \alpha_i(x) < SDP = y_*^T R y_* \equiv -2\xi_*^T \beta_i(x) t_* - \xi_*^T \Gamma_i(x) \xi_*; \\ (b) & y_*^T R_0 y_* \equiv t_*^2 \leq 1; \\ (c) & 1 \geq y_*^T \Omega^{-2} R_k y_* \equiv \xi_* (\Omega\rho)^{-2} S_k \xi_*, \ k = 1, \dots, K, \end{array} \tag{31}$$

where

$$\Omega = \sqrt{2 \ln \left(6 \sum_{k=1}^{K} \mathrm{Rank}(S_k) \right)}. \tag{32}$$

Relations (31) say that *when the perturbation level ρ is replaced with $\Omega\rho$, the implication* A(i) *fails to be true*, so that *with this new perturbation level*, B(i) does not hold, and therefore x is *not* feasible for the AARC. We have arrived at the following statement which quantifies the "tightness" of the approximation to AARC we have built:

Theorem 4.3. *The projection on the x-space of the feasible set of the approximate* AARC (23) *is contained in the feasible set of the true* AARC*, the perturbation level being ρ, and contains the feasible set of the* AARC*, the perturbation level being $\Omega\rho$. In particular, the optimal value in* (23) *is in-between the optimal values of the* AARC'*s corresponding to the perturbation levels ρ and $\Omega\rho$.*

Note that in reality the quantity Ω given by (32) is a moderate constant: it is ≤ 6, provided that the total rank of the matrices S_k participating in the description of the perturbation set is less than 65,000,000.

Remark 4.1. We have considered the case when the adjustable variables are restricted to be affine functions of the *entire* perturbation vector ξ. It is immediately seen that all constructions and results of Sections 3, 4 remain valid in the case when every one of the adjustable variables v_i is allowed to be an affine function of a "prescribed portion" $P_i\xi$ of the perturbation vector, where P_i are given matrices.

5. Example: an inventory model

In this section we illustrate the use of the AARC approach by considering an inventory management problem.

The model. Consider a single product inventory system, which is comprised of a warehouse and I factories. The planning horizon is T periods. At a period t:

- d_t is the demand for the product. All the demand must be satisfied;
- $v(t)$ is the amount of the product in the warehouse at the beginning of the period ($v(1)$ is given);
- $p_i(t)$ is the i-th order of the period – the amount of the product to be produced during the period by factory i and used to satisfy the demand of the period (and, perhaps, to replenish the warehouse);
- $P_i(t)$ is the maximal production capacity of factory i;
- $c_i(t)$ is the cost of producing a unit of the product at a factory i.

Other parameters of the problem are:

- $V_{\min}$ – the minimal allowed level of inventory at the warehouse;
- $V_{\max}$ – the maximal storage capacity of the warehouse;
- Q_i – the maximal cumulative production capacity of i'th factory throughout the planning horizon.

The goal is to minimize the total production cost over all factories and the entire planning period. When all the data are certain, the problem can be modelled by the following linear program:

$$\begin{aligned} \min_{p_i(t),v(t),F} \quad & F \\ \text{s.t.} \quad & \sum_{t=1}^{T}\sum_{i=1}^{I} c_i(t)p_i(t) \le F \\ & 0 \le p_i(t) \le P_i(t),\ i=1,\dots,I,\ t=1,\dots,T \\ & \sum_{t=1}^{T} p_i(t) \le Q(i),\ i=1,\dots,I \\ & v(t+1) = v(t) + \sum_{i=1}^{I} p_i(t) - d_t,\ t=1,\dots,T \\ & V_{\min} \le v(t) \le V_{\max},\ t=2,\dots,T+1. \end{aligned} \tag{33}$$

Eliminating v-variables, we get an inequality constrained problem:

$$\begin{aligned} \min_{p_i(t),F} \quad & F \\ \text{s.t.} \quad & \sum_{t=1}^{T}\sum_{i=1}^{I} c_i(t)p_i(t) \le F \\ & 0 \le p_i(t) \le P_i(t),\ i=1,\dots,I,\ t=1,\dots,T \\ & \sum_{t=1}^{T} p_i(t) \le Q(i),\ i=1,\dots,I \\ & V_{\min} \le v(1) + \sum_{s=1}^{t}\sum_{i=1}^{I} p_i(s) - \sum_{s=1}^{t} d_s \le V_{\max},\ t=1,\dots,T. \end{aligned} \tag{34}$$

Assume that the decision on supplies $p_i(t)$ is made at the beginning of period t, and that we are allowed to make these decisions on the basis of demands d_r observed at periods $r \in I_t$, where I_t is a given subset of $\{1, \dots, t\}$. Further, assume that we should specify our supply policies before the planning period starts ("at period 0"), and that when specifying these policies, we do not know exactly the future demands; all we know is that

$$d_t \in [d_t^* - \theta d_t^*, d_t^* + \theta d_t^*],\ t = 1, \dots, T, \tag{35}$$

with given positive θ and positive nominal demand d_t^*. We have now an uncertain LP, where the uncertain data are the actual demands d_t, the decision variables are the supplies $p_i(t)$, and these decision variables are allowed to depend on the data $\{d_\tau : \tau \in I_t\}$ which become known when $p_i(t)$ should be specified. Applying the AARC methodology, we restrict our decision-making policy with *affine decision rules*

$$p_i(t) = \pi_{i,t}^0 + \sum_{r \in I_t} \pi_{i,t}^r d_r, \tag{36}$$

where the coefficients $\pi_{i,t}^r$ are our new non-adjustable variables. With this approach, (34) becomes the following uncertain Linear Programming problem in variables $\pi_{i,t}^s$, F:

$$\begin{aligned}
\min_{\pi, F}\ & F \\
\text{s.t. } & \sum_{t=1}^{T} \sum_{i=1}^{I} c_i(t) \left(\pi_{i,t}^0 + \sum_{r \in I_t} \pi_{i,t}^r d_r \right) \le F \\
& 0 \le \pi_{i,t}^0 + \sum_{r \in I_t} \pi_{i,t}^r d_r \le P_i(t),\ i = 1, \dots, I,\ t = 1, \dots, T \\
& \sum_{t=1}^{T} \left(\pi_{i,t}^0 + \sum_{r \in I_t} \pi_{i,t}^r d_r \right) \le Q(i),\ i = 1, \dots, I \\
& V_{\min} \le v(1) + \sum_{s=1}^{t} \left(\sum_{i=1}^{I} \pi_{i,s}^0 + \sum_{r \in I_s} \pi_{i,s}^r d_r \right) - \sum_{s=1}^{t} d_s \le V_{\max}, \\
& \quad t = 1, \dots, T \\
& \forall \{d_t \in [d_t^* - \theta d_t^*, d_t^* + \theta d_t^*],\ t = 1, \dots, T\},
\end{aligned} \tag{37}$$

or, which is the same,

$$\begin{aligned}
\min_{\pi, F}\ & F \\
\text{s.t. } & \sum_{t=1}^{T} \sum_{i=1}^{I} c_i(t) \pi_{i,t}^0 + \sum_{r=1}^{T} \left(\sum_{i=1}^{I} \sum_{t: r \in I_t} c_i(t) \pi_{i,t}^r \right) d_r - F \le 0 \\
& \pi_{i,t}^0 + \sum_{r \in I_t}^{t} \pi_{i,t}^r d_r \le P_i(t),\ i = 1, \dots, I,\ t = 1, \dots, T \\
& \pi_{i,t}^0 + \sum_{r \in I_t} \pi_{i,t}^r d_r \ge 0,\ i = 1, \dots, I,\ t = 1, \dots, T
\end{aligned}$$

$$\begin{array}{l}\sum_{t=1}^{T}\pi_{i,t}^{0}+\sum_{r=1}^{T}\left(\sum_{t:r\in I_t}\pi_{i,t}^{r}\right)d_r \le Q_i,\ i=1,\dots,I\\ \sum_{s=1}^{t}\sum_{i=1}^{I}\pi_{i,s}^{0}+\sum_{r=1}^{t}\left(\sum_{i=1}^{I}\sum_{s\le t,r\in I_s}\pi_{i,s}^{r}-1\right)d_r \le V_{\max}-v(1)\\ \quad t=1,\dots,T\\ -\sum_{s=1}^{t}\sum_{i=1}^{I}\pi_{i,s}^{0}-\sum_{r=1}^{t}\left(\sum_{i=1}^{I}\sum_{s\le t,r\in I_s}\pi_{i,s}^{z}-1\right)d_z \le v(1)-V_{\min}\\ \quad t=1,\dots,T\\ \forall\{d_t\in[d_t^*-\theta d_t^*,d_t^*+\theta d_t^*],\ t=1,\dots,T\}.\end{array} \tag{38}$$

Now, using the following equivalences

$$\begin{array}{c}\sum_{t=1}^{T}d_t x_t \le y,\ \forall d_t\in[d_t^*(1-\theta),d_t^*(1+\theta)]\\ \Updownarrow\\ \sum_{t:x_t<0}d_t^*(1-\theta)x_t+\sum_{t:x_t>0}d_t^*(1+\theta)x_t\le y\\ \Updownarrow\\ \sum_{t=1}^{T}d_t^*x_t+\theta\sum_{t=1}^{T}d_t^*|x_t|\le y,\end{array}$$

and defining additional variables

$$\alpha_r\equiv\sum_{t:r\in I_t}c_i(t)\pi_{i,t}^{r};\quad \delta_i^r\equiv\sum_{t:r\in I_t}\pi_{i,t}^{r};\quad \xi_t^r\equiv\sum_{i=1}^{I}\sum_{s\le t,r\in I_s}\pi_{i,s}^{r}-1,$$

we can straightforwardly convert the AARC (38) into an equivalent LP (cf. Theorem 3.3):

$$\begin{array}{ll}\min\limits_{\pi,F,\alpha,\beta,\gamma,\delta,\zeta,\xi,\eta} & F\\ & \sum_{i=1}^{I}\sum_{t:r\in I_t}c_i(t)\pi_{i,t}^{r}=\alpha_r,\ -\beta_r\le\alpha_r\le\beta_r,\ 1\le r\le T,\\ & \sum_{t=1}^{T}\sum_{i=1}^{I}c_i(t)\pi_{i,t}^{0}+\sum_{r=1}^{T}\alpha_r d_r^*+\theta\sum_{r=1}^{T}\beta_r d_r^*\le F;\\ & -\gamma_{i,t}^{r}\le\pi_{i,t}^{r}\le\gamma_{i,t}^{r},\ r\in I_t,\ \pi_{i,t}^{0}+\sum_{r\in I_t}\pi_{i,t}^{r}d_r^*\\ & \quad +\theta\sum_{r\in I_t}\gamma_{i,t}^{r}d_r^*\le P_i(t),\ 1\le i\le I,1\le t\le T;\\ & \pi_{i,t}^{0}+\sum_{r\in I_t}\pi_{i,t}^{r}d_r^*-\theta\sum_{r\in I_t}\gamma_{i,t}^{r}d_r^*\ge 0,\\ & \sum_{t:r\in I_t}\pi_{i,t}^{r}=\delta_i^r,\ -\zeta_i^r\le\delta_i^r\le\zeta_i^r,\ 1\le i\le I,1\le r\le T,\end{array} \tag{39}$$

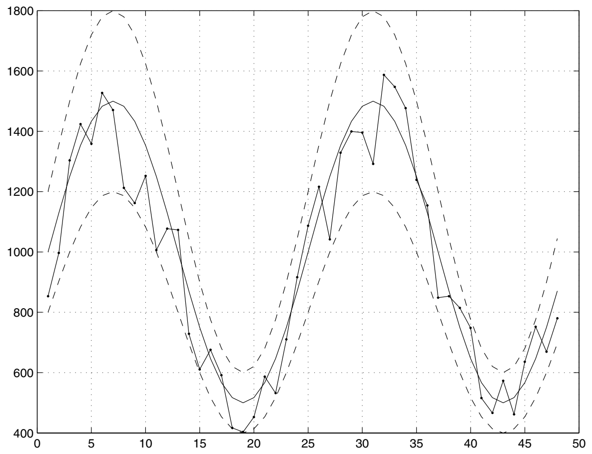
$$\sum_{t=1}^{T} \pi_{i,t}^{0} + \sum_{r=1}^{T} \delta_i^r d_r^* + \theta \sum_{r=1}^{T} \zeta_i^r d_r^* \leq Q_i,\ 1 \leq i \leq I;$$
$$\sum_{i=1}^{I} \sum_{s \leq t, r \in I_s} \pi_{i,s}^{r} - \xi_t^r = 1,\ -\eta_t^r \leq \xi_t^r \leq \eta_t^r,\ 1 \leq r \leq t \leq T,$$
$$\sum_{s=1}^{t} \sum_{i=1}^{I} \pi_{i,s}^{0} + \sum_{r=1}^{t} \xi_t^r d_r^* + \theta \sum_{r=1}^{t} \eta_t^r d_r^* \leq V_{\max} - v(1),\ \ 1 \leq t \leq T,$$
$$\sum_{s=1}^{t} \sum_{i=1}^{I} \pi_{i,s}^{0} + \sum_{r=1}^{t} \xi_t^r d_r^* - \theta \sum_{r=1}^{t} \eta_t^r d_r^* \geq v(1) - V_{\min},\ 1 \leq t \leq T.$$

An illustrative example. There are $I = 3$ factories producing a seasonal product, and one warehouse. The decisions concerning production are made every two weeks, and we are planning production for 48 weeks, thus the time horizon is $T = 24$ periods. The nominal demand d^* is seasonal, reaching its maximum in winter, specifically,

$$d_t^* = 1000 \left(1 + \frac{1}{2} \sin\left(\frac{\pi\ (t-1)}{12}\right)\right), \quad t = 1, \ldots, 24.$$

We assume that the uncertainty level θ is 20%, i.e., $d_t \in [0.8 d_t^*, 1.2 d_t^*]$, as shown on Fig. 1.

The production costs per unit of the product depend on the factory and on time and follow the same seasonal pattern as the demand, i.e., rise in winter and fall in summer.



- Nominal demand (solid)
- "demand tube" – nominal demand ±20% (dashed)
- a sample realization of actual demand (dotted)

Fig. 1. Demand.

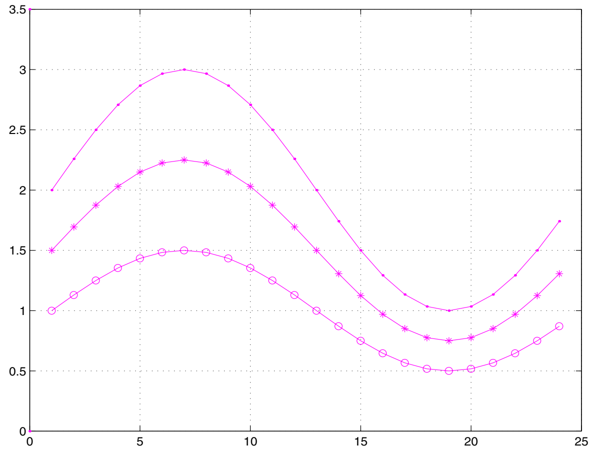


Fig. 2. Production costs for the 3 factories.

The production cost for a factory i at a period t is given by:

$$c_i(t) = \alpha_i \left(1 - \frac{1}{2}\sin\left(\frac{\pi\,(t-1)}{12}\right)\right), \quad t = 1, \dots, 24.$$
$$\alpha_1 = 1$$
$$\alpha_2 = 1.5$$
$$\alpha_3 = 2$$

The maximal production capacity of each one of the factories at each two-weeks period is $P_i(t) = 567$ units, and the integral production capacity of each one of the factories for a year is $Q_i = 13600$. The inventory at the warehouse should not be less then 500 units, and cannot exceed 2000 units.

With this data, the AARC (39) of the uncertain inventory problem is an LP, the dimensions of which vary, depending on the "information basis" (see below), from 919 variables and 1413 constraints (empty information basis) to 2719 variables and 3213 constraints (on-line information basis). In the experiments to be reported, these LP's were solved by the commercial code MOSEKOPT (see www.mosek.com).

The experiments. In every one of the experiments, the corresponding management policy was tested against a given number (100) of simulations; in every one of the simulations, the actual demand d_t of period t was drawn at random, according to the uniform distribution on the segment $[(1-\theta)d_t^*, (1+\theta)d_t^*]$ where θ was the "uncertainty level" characteristic for the experiment. The demands of distinct periods were independent of each other.

We have conducted two series of experiments:

1. The aim of the first series of experiments was to check the influence of the demand uncertainty θ on the total production costs corresponding to the *robustly adjustable* management policy – the policy (36) yielded by the optimal solution to the AARC (39). We compared this cost to the "ideal" one, i.e., the cost we would have paid in

the case when all the demands were known to us in advance and we were using the corresponding optimal management policy as given by the optimal solution of (33).

2. The aim of the second series of experiments was to check the influence of the "information basis" allowed for the management policy, on the resulting management cost. Specifically, in our model as described in the previous section, when making decisions $p_i(t)$ at time period t, we can make these decisions depending on the demands of periods $r \in I_t$, where I_t is a given subset of the segment $\{1, 2, \dots, t\}$. The larger are these subsets, the more flexible can be our decisions, and hopefully the less are the corresponding management costs. In order to quantify this phenomenon, we considered 4 "information bases" of the decisions:
   (a) $I_t = \{1, \dots, t\}$ (the richest "on-line" information basis);
   (b) $I_t = \{1, \dots, t-1\}$ (this *standard information basis* seems to be the most natural "information basis": past is known, present and future are unknown);
   (c) $I_t = \{1, \dots, t-4\}$ (the information about the demand is received with a four-day delay);
   (d) $I_t = \emptyset$ (i.e., no adjusting of future decisions to actual demands at all. This "information basis" corresponds exactly to the management policy yielded by the usual RC of our uncertain LP.).

The results of our experiments are as follows:

1. **The influence of the uncertainty level on the management cost.** Here we tested the robustly adjustable management policy *with the standard information basis* against different levels of uncertainty, specifically, the levels of 20%, 10%, 5% and 2.5%. For every uncertainty level, we have computed the average (over 100 simulations) management costs when using the corresponding robustly adaptive management policy. We saved the simulated demand trajectories and then used these trajectories to compute the ideal management costs. The results are summarized in Table 1. As expected, the less is the uncertainty, the closer are our management costs to the ideal ones. What is surprising, is the low "price of robustness": even at the 20% uncertainty level, the average management cost for the robustly adjustable policy was just by 3.4% worse than the corresponding ideal cost; the similar quantity for 2.5%-uncertainty in the demand was just 0.3%.
2. **The influence of the information basis.** The influence of the information basis on the performance of the robustly adjustable management policy is displayed in Table 2. These experiments were carried out at the uncertainty level of 20%. We see that the poorer is the information basis of our management policy, the worse

Table 1. Management costs vs. uncertainty level.

	AARC		Ideal case		
Uncertainty	Mean	Std	Mean	Std	price of robustness
2.5%	33974	190	33878	194	0.3%
5%	34063	432	33864	454	0.6%
10%	34471	595	34009	621	1.6%
20%	35121	1458	33958	1541	3.4%

Table 2. The influence of the information basis on the management costs.

information basis for decision $p_i(t)$ is demand in periods	Management cost	
	Mean	Std
$1, \dots, t$	34583	1475
$1, \dots, t-1$	35121	1458
$1, \dots, t-4$	Infeasible	
$\emptyset$	Infeasible	

are the results yielded by this policy. In particular, with 20% level of uncertainty, there does not exist a robust *non-adjustable* management policy: the usual RC of our uncertain LP is infeasible. In other words, in our illustrating example, passing from a priori decisions yielded by RC to "adjustable" decisions yielded by AARC is indeed crucial.

An interesting question is what is the uncertainty level which still allows for a priori decisions. It turns out that the RC is infeasible even at the 5% uncertainty level. Only at the uncertainty level as small as 2.5% the RC becomes feasible and yields the following management costs:

	RC		Ideal cost		
Uncertainty	Mean	Std	Mean	Std	price of robustness
2.5%	35287	0	33842	172	4.3%

Note that even at this unrealistically small uncertainty level the price of robustness for the policy yielded by the RC is by 4.3% larger than the ideal cost (while for the robustly adjustable management this difference is just 0.3%, see Table 1).

The preliminary numerical results we have presented are highly encouraging and clearly demonstrate the advantage of the AARC-based approach to LP-based multi-stage decision making under dynamical uncertainty.

Comparison with Dynamic Programming. An Inventory problem we have considered is a typical example of sequential decision-making under dynamical uncertainty, where the information basis for the decision x_t made at time t is the part of the uncertainty revealed at instant t. This example allows for an instructive comparison of the AARC-based approach with Dynamic Programming, which is the traditional technique for sequential decision-making under dynamical uncertainty. Restricting ourselves with the case where the decision-making problem can be modelled as a Linear Programming problem with the data affected by dynamical uncertainty, we could say that (minimax-oriented) *Dynamic Programming is a specific technique for solving the ARC of this uncertain LP.* Therefore *when applicable,* Dynamic Programming has a significant advantage as compared to the above AARC-based approach, since it does *not* impose on the adjustable variables an "ad hoc" restriction (motivated solely by the desire to end up with a tractable problem) to be affine functions of the uncertain data. At the same time, the above "if applicable" is highly restrictive: the computational effort in Dynamical Programming explodes exponentially with the dimension of the state space of the decision-making process in question. For example, the simple Inventory problem we have considered has 4-dimensional state space (the current amount of product in the warehouse plus remaining total capacities of the three factories), which is already computationally too demanding for accurate implementation of Dynamic Programming. In our opinion, the main advantage of the AARC-based dynamical decision-making as compared with Dynamic Programming (as well as with Multi-Stage Stochastic Programming) comes from the "built-in" computational tractability of our approach, which prevents the "curse of dimensionality" and allows to process routinely fairly complicated models with high-dimensional state spaces and many stages.

Appendix

Proof of Theorem 2.1. It is clear that the set of the feasible solutions to the RC (6) is contained in the feasible set of the ARC (5), so that all we need to show is that if a given u is infeasible for the RC, then it is infeasible for the ARC as well.

Let u be infeasible for the RC and let V_u be the corresponding set from the premise of the Theorem. Then

$$\forall(v \in V_u)\exists(\zeta_v \in \mathcal{Z}, i_v \in \{1,\dots,m\}) : [U(\zeta_v)u + V(\zeta_v)v - b(\zeta_v)]_{i_v} > 0. \tag{40}$$

It follows that for every $v \in V_u$ there exist $\zeta_v \in \mathcal{Z}$, $i_v \in \{1,\dots,m\}$ and $\epsilon_v > 0$ such that

$$[U(\zeta_v)u + V(\zeta_v)v - b(\zeta_v)]_{i_v} > \epsilon_v. \tag{41}$$

The sets $B_v \equiv \left\{\widetilde{v} \in V_u : [U(\zeta_v)u + V(\zeta_v)\widetilde{v} - b(\zeta_v)]_{i_v} > \epsilon_v\right\}$ constitute an open cover of V_u; since V_u is compact, we can extract from this covering a finite sub-cover, i.e.,

$$\exists(v_1,\dots,v_N \in V_u) : \bigcup_{k=1}^{N} B_{v_k} = V_u. \tag{42}$$

Therefore

$$\forall(v \in V_u)\exists(k \in \{1,\dots,N\}) : \left(v \in B_{v_k} \Leftrightarrow [U(\zeta_{v_k})u + V(\zeta_{v_k})v - b(\zeta_{v_k})]_{i_{v_k}} > \epsilon_{v_k}\right). \tag{43}$$

Setting $B_k = B_{v_k},\ i_k = i_{v_k},\ \zeta_k = \zeta_{v_k},\ \epsilon_k = \epsilon_{v_k}, \epsilon = \min_k \epsilon_k$, (43) implies that

$$\forall(v \in V_u)\exists(k \in \{1,\dots,N\}) : [U(\zeta_k)u + V(\zeta_k)v - b(\zeta_k)]_{i_k} > \epsilon. \tag{44}$$

As a result of (44) the system of inequalities

$$[U(\zeta_k)u + V(\zeta_k)v - b(\zeta_k)]_i - \epsilon < 0,\ i = 1,\dots,m,\ k = 1,\dots,N \tag{45}$$

in variables v has no solution in V_u. By the Karlin-Bohnenblust Theorem [7] it follows that there exists collection of weights $\{\lambda_{i,k} \geq 0\}$, $\sum_{i,k}\lambda_{i,k} = 1$, such that the corresponding combination of the left hand sides of the inequalities (45) is nonnegative everywhere on V_u, so that

$$
\begin{aligned}
\forall(v \in V_u) : \epsilon
&\le \sum_{i,k} \lambda_{i,k}[U(\zeta_k)u + V(\zeta_k)v - b(\zeta_k)]_i \\
&= \sum_i \left(\sum_k \lambda_{i,k}\, [U(\zeta_k)u + V(\zeta_k)v - b(\zeta_k)]_i \right) \\
&= \sum_{i:\sum_k \lambda_{i,k}>0} \left(\sum_k \lambda_{i,k}\, [U(\zeta_k)u + V(\zeta_k)v - b(\zeta_k)]_i \right) \\
&= \sum_{i:\sum_k \lambda_{i,k}>0} \left(\underbrace{\left(\sum_k \lambda_{i,k} \right)}_{\mu_i} \left(\sum_k \frac{\lambda_{i,k}}{\left(\sum_k \lambda_{i,k}\right)} [U(\zeta_k)u + V(\zeta_k)v - b(\zeta_k)]_i \right) \right) \\
&= \sum_{i:\mu_i>0} \left(\mu_i \left[U\left(\sum_k \frac{\lambda_{i,k}}{\mu_i}\zeta_k \right) u + V\left(\sum_k \frac{\lambda_{i,k}}{\mu_i}\zeta_k \right) v - b\left(\sum_k \frac{\lambda_{i,k}}{\mu_i}\zeta_k \right) \right]_i \right)
\end{aligned} \tag{46}
$$

Due to the structure of $\mathcal{Z}$ as the product of convex sets $\mathcal{Z}_1, \ldots, \mathcal{Z}_m$, the data $\widetilde{\zeta} = (\widetilde{\zeta}^1, \ldots, \widetilde{\zeta}^m)$ given by

$$
\widetilde{\zeta}^i = \begin{cases} \left[\sum_k \frac{\lambda_{i,k}}{\mu_i}\zeta_k \right], & \mu_i > 0 \\ \text{an arbitrary point from } \mathcal{Z}_i, & \mu_i = 0 \end{cases}
$$

is in $\mathcal{Z}$ (i'th block of $\widetilde{\zeta}$ is a convex combination of elements from $\mathcal{Z}_i$). Since the uncertainty is constraint-wise, we have

$$
\begin{aligned}
&\left[U\left(\sum_k \frac{\lambda_{i,k}}{\mu_i}\zeta_k \right) u + V\left(\sum_k \frac{\lambda_{i,k}}{\mu_i}\zeta_k \right) v - b\left(\sum_k \frac{\lambda_{i,k}}{\mu_i}\zeta_k \right) \right]_i \\
&\quad = \left[U(\widetilde{\zeta}^i)u + V(\widetilde{\zeta}^i)v - b(\widetilde{\zeta}^i) \right]_i = \left[U(\widetilde{\zeta})u + V(\widetilde{\zeta})v - b(\widetilde{\zeta}) \right]_i .
\end{aligned} \tag{47}
$$

Combining (46) and (47), we get

$$
\forall(v \in V_u) : \epsilon \le \sum_{i:\sum_k \lambda_{i,k}>0} \left(\sum_k \lambda_{i,k} \right) \left[U(\widetilde{\zeta})u + V(\widetilde{\zeta})v - b(\widetilde{\zeta}) \right]_i . \tag{48}
$$

Thus, the data $\widetilde{\zeta} \in \mathcal{Z}$ we have built are such that for every $v \in V_u$ at least one of the quantities $\left[U(\widetilde{\zeta})u + V(\widetilde{\zeta})v - b(\widetilde{\zeta}) \right]_i$, $i = 1, \ldots, m$, is positive, so that no $v \in V_u$ can complete u to a feasible solution to the instance of our uncertain LP given by the data $\widetilde{\zeta}$. Recalling the definition of V_u, we conclude that u is infeasible for the ARC, as required. □

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