# Robust solutions of Linear Programming problems contaminated with uncertain data[1]

**Aharon Ben-Tal** and **Arkadi Nemirovski**
morbt@ie.technion.ac.il nemirovs@ie.technion.ac.il
Faculty of Industrial Engineering and Management
Technion – Israel Institute of Technology

**Abstract**

Optimal solutions of Linear Programming problems may become severely infeasible if the nominal data is slightly perturbed. We demonstrate this phenomenon by studying 90 LPs from the well-known NETLIB collection. We then apply the Robust Optimization methodology (Ben-Tal and Nemirovski [1-3]; El Ghaoui et al. [5,6]) to produce "robust" solutions of the above LPs which are in a sense immuned against uncertainty. Surprisingly, for the NETLIB problems these robust solutions nearly lose nothing in optimality.

## 1 Introduction

To motivate the research summarized in this paper, let us start with an example – problem PILOT4 from the well-known NETLIB library. This is a Linear Programming problem with 1,000 variables and 410 constraints; one of the constraints (# 372) is

$$\begin{array}{rcl} a^T x & \equiv & -15.79081x_{826} - 8.598819x_{827} - 1.88789x_{828} - 1.362417x_{829} - 1.526049x_{830} \\ & & -0.031883x_{849} - 28.725555x_{850} - 10.792065x_{851} - 0.19004x_{852} - 2.757176x_{853} \\ & & -12.290832x_{854} + 717.562256x_{855} - 0.057865x_{856} - 3.785417x_{857} - 78.30661x_{858} \\ & & -122.163055x_{859} - 6.46609x_{860} - 0.48371x_{861} - 0.615264x_{862} - 1.353783x_{863} \\ & & -84.644257x_{864} - 122.459045x_{865} - 43.15593x_{866} - 1.712592x_{870} - 0.401597x_{871} \\ & & +x_{880} - 0.946049x_{898} - 0.946049x_{916} \\ & \geq & b \equiv 23.387405 \end{array} \tag{C}$$

The related *nonzero* coordinates in the optimal solution $x^*$ of the problem, as reported by CPLEX, are as follows:

$$\begin{array}{lll} x^*_{826} = 255.6112787181108 & x^*_{827} = 6240.488912232100 & x^*_{828} = 3624.613324098961 \\ x^*_{829} = 18.20205065283259 & x^*_{849} = 174397.0389573037 & x^*_{870} = 14250.00176680900 \\ x^*_{871} = 25910.00731692178 & x^*_{880} = 104958.3199274139 & \end{array}$$

Note that within machine precision the indicated optimal solution makes (C) an equality.

Observe that most of the coefficients in (C) are "ugly reals" like -15.79081 or -84.644257. We have many reasons to believe that coefficients of this type characterize certain technological devices/processes, and as such *they could hardly be known to high accuracy*. It is quite natural to assume that the "ugly coefficients" are in fact uncertain – they coincide with the "true" values of the corresponding data within accuracy of 3-4 digits, not more. The only exception is the coefficient 1 of $x_{880}$ – it perhaps reflects the structure of the problem and is therefore exact – "certain".

Assuming that the uncertain entries of $a$ are, say, 0.1%-accurate approximations of unknown entries of the "true" vector of coefficients $\widetilde{a}$, we looked what would be the effect of this uncertainty on the validity of the "true" constraint $\widetilde{a}^T x \geq b$ at $x^*$. Here is what we have found:

• The minimum, (over all vectors of coefficients $\widetilde{a}$ compatible with our "0.1%-uncertainty hypothesis"), value of $\widetilde{a}^T x^* - b$, is $< -104.9$; in other words, the violation of the constraint can be as large as 450% of the right hand side!

[1]Funded by the Germany-Israeli Foundation for R&D and the Israel Ministry of Science.

• Treating the above worst-case violation as "too pessimistic" (why should the true values of all uncertain coefficients differ from the values indicated in (C) in the "most dangerous" way?), consider a more realistic measure of violation. Specifically, assume that the true values of the uncertain coefficients in (C) are obtained from the "nominal values" (those shown in (C)) by random perturbations $a_j \mapsto \widetilde{a}_j = (1 + \xi_j)a_j$ with independent and, say, uniformly distributed on $[-0.001, 0.001]$ "relative perturbations" $\xi_j$. What will be a "typical" relative violation

$$V = \frac{b - \widetilde{a}^T x^*}{b} \times 100\%$$

of the "true" (now random) constraint $\widetilde{a}^T x \geq b$ at $x^*$? The answer is nearly as bad as for the worst scenario:

| Prob$\{V > 0\}$ | Prob$\{V > 150\%\}$ | Mean$(V)$ |
|---|---|---|
| 0.50 | 0.18 | 125% |

**Table 1.** Relative violation of constraint # 372 in `PILOT4`
(1,000-element sample of 0.1% perturbations of the uncertain data)

We see that *quite small (just 0.1%) perturbations of "obviously uncertain" data coefficients can make the "nominal" optimal solution $x^*$ heavily infeasible and thus – practically meaningless.* Our intention in this, mainly experimental, paper is to investigate how common is this phenomenon and how to struggle with it when it occurs. Specifically, we look through the list of `NETLIB` problems and for every one of them

— quantify the level of infeasibility of the nominal solution in face of small uncertainty;

— when this infeasibility is "too large", apply the Robust Optimization methodology (see [1-6]), thus generating another solution which in a sense is immuned against data perturbations, and, finally,

— look what is the price, in terms of the value of the objective function, of our "immunization".

It is important to emphasize that our approach has nearly nothing in common with Sensitivity Analysis – a traditional tool for investigating the stability of optimal solutions with respect to data perturbations:

- In Sensitivity Analysis, one is interested in *how much the optimal solution to a perturbed problem can differ from the one of the nominal problem.* In contrast to this, we want to know *by how much the optimal solution to the nominal problem can violate the constraints of the perturbed problem*, which is a completely different question.

- Sensitivity Analysis is a "post-mortem" tool – at best, it can quantify locally the stability of the nominal solution with respect to infinitesimal data perturbations, but it does not say how to improve this stability when necessary. The latter issue is exactly what is addressed by the Robust Optimization methodology.

The rest of the paper is organized as follows. In Section 2 we describe our methodology for quantifying the level to which data perturbations may affect the quality of a feasible solution to a LP program and present the associated results for the `NETLIB` problems. In Section 3 we explain how the Robust Optimization methodology can be used to immune solutions against data perturbations and discuss the results of this immunization for the `NETLIB` problems.

# 2 Quantifying the influence of data perturbations

## 2.1 Detecting uncertain data entries

Consider an LP problem in the form

$$\text{minimize } c^T x \text{ s.t.} \left\{ \begin{array}{rcll} Ex & = & e & (a) \\ Ax & \leq & b & (b) \\ \ell \leq x & \leq & u & (c) \end{array} \right. \tag{LP}$$

and let $\widehat{x}$ be a feasible solution to the problem. We intend to define a "reliability index" of $\widehat{x}$ with respect to perturbations of the uncertain data elements of (LP). In order to define this quantity, we first should decide what are the uncertain data elements and how they are affected by uncertainty. In the ideal case, this information should be provided by the user responsible for the model. Here, however, we intend to carry out a `NETLIB` case study, so that all we know about the problem is its *nominal data* – that given in the `mps` file specifying the problem. In accordance with what was explained in Introduction, we resolve the question of "what is uncertain" as follows:

> *"Uncertain data elements" in (LP) are the "ugly reals" appearing as entries in the matrix $A$ of inequality constraints (LP.b), specifically reals which cannot be represented as rational fractions $\frac{p}{q}$ with $1 \leq q \leq 100$.*

Note that we ignore possible uncertainty in the data of the objective, the equality and the box constraints, or in the right hand side of the inequality constraints. The motivation for that is as follows:

- If there was uncertainty in the data of an equality constraint, a good model-builder would not model the constraint as an equality, rather as a range constraint with the right hand side bounds close to one another;
- Possible uncertainty in the data of the box constraints and the right hand side vector of the inequality constraints in principle should be taken into account. However, it will become clear that with our approach this uncertainty hardly affects the results, so for the sake of simplicity we treat this data as certain;
- Possible uncertainty in the objective should and could in principle be treated similarly to that in the inequality constraints. In our `NETLIB` case study we have checked the influence of this uncertainty and found that it does not essentially affect the quality of the nominal solution; consequently, we assume the objective data certain.

## 2.2 The uncertainty and the Reliability index

The next issue is how the uncertainty affects the data. Here we intend to work with the simplest assumption:

> *The "true" value $\widetilde{a}_{ij}$ of an uncertain data entry is obtained from the nominal value $a_{ij}$ of the entry by random perturbation:*
>
> $$\widetilde{a}_{ij} = (1 + \epsilon \xi_{ij}) a_{ij},$$
>
> *where $\epsilon > 0$ is a given uncertainty level and $\xi_{ij}$ are random variables distributed symmetrically in the interval $[-1, 1]$. The random perturbations affecting the uncertain data entries of a particular inequality constraint are iid.*

With this assumption, the value of the left hand side of the "true" constraint

$$\zeta_x^i \equiv \sum_j \widetilde{a}_{ij} x_j - b_i \leq 0$$

at a candidate solution $x$ is a random variable with mean and the standard deviation given by

$$\mathrm{Mean}(\zeta_x^i) = \sum_j a_{ij} x_j - b_i; \quad \mathrm{StD}(\zeta_x^i) = \sigma_i D_i(x),\ D_\epsilon^i(x) = \epsilon \sqrt{\sum_{j \in J_i} a_{ij}^2 x_j^2},$$

where $J_i$ is the set of indices of uncertain data entries of the $i$-th constraint and $\sigma_i$ is the standard deviation of $\xi_{ij}$. Assuming that $\sigma_i$ is of order of 1, we see that "typical" values of the difference $\delta_x^i \equiv \zeta_x^i - \mathrm{Mean}(\zeta_x^i)$ are of the order of $D_\epsilon^i(x)$, values of opposite sign being equally possible. For instance, when $\xi_{ij}$ are distributed uniformly on $[-1,1]$, the probability of the event $\{\delta_x^i > 0.92 D_\epsilon^i(x)\}$ is more than 2%, while the probability of the event $\{\delta_x^i > 5.24 D_\epsilon^i(x)\}$ is $< 10^{-6}$, whatever the (symmetric) distribution of $\xi_{ij}$ in $[-1,1]$.

It follows that the "typical" violation of the true constraint in question at $x$ is of order of the quantity

$$\max\left[\mathrm{Mean}(\zeta_x^i) + D_\epsilon^i(x); 0\right].$$

Normalizing the violation by the absolute value of the right hand side $b_i$, we come to the *$\epsilon$-reliability index*

$$\mathrm{Rel}_\epsilon^i(x) = \frac{\max\left[\sum\limits_j a_{ij} x_j - b_i + 0.92\epsilon \sqrt{\sum\limits_{j \in J_i} a_{ij}^2 x_j^2}; 0\right]}{\max[1; |b_i|]} \times 100\%$$

*of $x$ with respect to i-th constraint*[2)]

Finally, we define the *$\epsilon$-reliability index of a candidate solution $x$ to* (LP) as the quantity

$$\mathrm{Rel}_\epsilon(x) = \max_i \mathrm{Rel}_\epsilon^i(x),$$

the maximum being taken over all inequality constraints of the problem.

### 2.3 The NETLIB case study: analyzing reliability of nominal solutions

Now we are ready to explain the methodology of our case study in its analysis part. We looked at 90 NETLIB problems and for every one of them we

1. solved the problem using CPLEX 6.2; let $x^*$ be the optimal solution as reported by the solver.

2. computed the reliability index of the nominal solution $x^*$ for the three uncertainty levels $\epsilon = 10^{-4}, 10^{-3}, 10^{-2}$.

   Given an uncertainty level $\epsilon$, we qualify an inequality constraint as *bad* at $x^*$, if the corresponding reliability index is greater than 5%. We qualify $x^*$ as a *bad* nominal solution to the problem, if at least one of the inequality constraints of the problem is bad at $x^*$.

[2)]To evaluate the "level of relevance" of the reliability index, note that for the constraint and the solution of problem PILOT4 considered in the Introduction the 0.001-reliability index is 260%, which is quite consistent with the simulation results in Table 1.

The results of the Analysis phase of our case study are as follows. From the total of 90 NETLIB problems we have processed,

• in 27 problems the nominal solution turned out to be bad at the largest ($\epsilon = 1\%$) level of uncertainty;

• 19 of these 27 problems are already bad at the 0.01%-level of uncertainty, and in 13 of these 19 problems, 0.01% perturbations of the uncertain data can make the nominal solution more than 50%-infeasible.

The details are given in Table 2.

| Problem | Size[a)] | $\epsilon = 0.01\%$ | | $\epsilon = 0.1\%$ | | $\epsilon = 1\%$ | |
|---|---|---|---|---|---|---|---|
| | | Nbad[b)] | Index[c)] | Nbad | Index | Nbad | Index |
| 80BAU3B | $2263 \times 9799$ | 37 | 84 | 177 | 842 | 364 | 8,420 |
| 25FV47 | $822 \times 1571$ | 14 | 16 | 28 | 162 | 35 | 1,620 |
| ADLITTLE | $57 \times 97$ | | | 2 | 6 | 7 | 58 |
| AFIRO | $28 \times 32$ | | | 1 | 5 | 2 | 50 |
| BNL2 | $2325 \times 3489$ | | | | | 24 | 34 |
| BRANDY | $221 \times 249$ | | | | | 1 | 5 |
| CAPRI | $272 \times 353$ | | | 10 | 39 | 14 | 390 |
| CYCLE | $1904 \times 2857$ | 2 | 110 | 5 | 1,100 | 6 | 11,000 |
| D2Q06C | $2172 \times 5167$ | 107 | 1,150 | 134 | 11,500 | 168 | 115,000 |
| E226 | $224 \times 282$ | | | | | 2 | 15 |
| FFFFF800 | $525 \times 854$ | | | | | 6 | 8 |
| FINNIS | $498 \times 614$ | 12 | 10 | 63 | 104 | 97 | 1,040 |
| GREENBEA | $2393 \times 5405$ | 13 | 116 | 30 | 1,160 | 37 | 11,600 |
| KB2 | $44 \times 41$ | 5 | 27 | 6 | 268 | 10 | 2,680 |
| MAROS | $847 \times 1443$ | 3 | 6 | 38 | 57 | 73 | 566 |
| NESM | $751 \times 2923$ | | | | | 37 | 20 |
| PEROLD | $626 \times 1376$ | 6 | 34 | 26 | 339 | 58 | 3,390 |
| PILOT | $1442 \times 3652$ | 16 | 50 | 185 | 498 | 379 | 4,980 |
| PILOT4 | $411 \times 1000$ | 42 | 210,000 | 63 | 2,100,000 | 75 | 21,000,000 |
| PILOT87 | $2031 \times 4883$ | 86 | 130 | 433 | 1,300 | 990 | 13,000 |
| PILOTJA | $941 \times 1988$ | 4 | 46 | 20 | 463 | 59 | 4,630 |
| PILOTNOV | $976 \times 2172$ | 4 | 69 | 13 | 694 | 47 | 6,940 |
| PILOTWE | $723 \times 2789$ | 61 | 12,200 | 69 | 122,000 | 69 | 1,220,000 |
| SCFXM1 | $331 \times 457$ | 1 | 95 | 3 | 946 | 11 | 9,460 |
| SCFXM2 | $661 \times 914$ | 2 | 95 | 6 | 946 | 21 | 9,460 |
| SCFXM3 | $991 \times 1371$ | 3 | 95 | 9 | 946 | 32 | 9,460 |
| SHARE1B | $118 \times 225$ | 1 | 257 | 1 | 2,570 | 1 | 25,700 |

**Table 2.** NETLIB problems with bad nominal solutions.

a) # of linear constraints (excluding the box ones) plus 1 and # of variables

b) # of constraints with $\mathrm{Rel}^i_\epsilon(x^*) > 5\%$

c) $\epsilon$-reliability index of the nominal solution $x^*$, %

The analysis stage of our Case Study leads to the following conclusion:

> *In real-world applications of Linear Programming one cannot ignore the possibility that a small uncertainty in the data* (intrinsic for most real-world LP programs) *can make the usual optimal solution of the problem completely meaningless from a practical viewpoint.*

Consequently,

*In applications of LP, there exists a real need of a technique capable of detecting cases when data uncertainty can heavily affect the quality of the nominal solution, and in these cases to generate a "reliable" solution, one which is immuned against uncertainty.*

# 3 Robust solutions to linear programs

## 3.1 The methodology

The methodology for generating robust ("uncertainty-immune") solutions to uncertain LPs we intend to implement originates in the *Robust Optimization* paradigm proposed and developed independently in [1-3] and [5-6]. Within the outlined framework, there are two ways to implement this methodology, depending on whether we treat the uncertainty affecting the data as "unknown-but-bounded", or as random.

**"Unknown-but-bounded" uncertainty.** Assume that we intend to make a solution immune against entry-wise uncertainty of given (relative) magnitude $\epsilon > 0$ affecting uncertain coefficients of (LP); specifically, we intend to build a solution $x$ with the following characteristics:

(i) $x$ is feasible for the nominal problem,

and

(ii) Suppose that in inequality constraint $i$ the true values $\widetilde{a}_{ij}$, $j \in J_i$, of uncertain data can range in the interval $[a_{ij} - \epsilon|a_{ij}|, a_{ij} + \epsilon|a_{ij}|]$. Whatever are the true values of uncertain coefficients from these intervals, $x$ must satisfy the $i$-th constraint with an error of at most $\delta \max[1, |b_i|]$, where $\delta$ is a given infeasibility tolerance:

$$\forall i \, \forall (\widetilde{a}_{ij} : |\widetilde{a}_{ij} - a_{ij}| \leq \epsilon |a_{ij}|): \quad \sum_{j \notin J_i} a_{ij} x_j + \sum_{j \in J_i} \widetilde{a}_{ij} x_j \leq b_i + \delta \max[1, |b_i|].$$

We shall call a solution satisfying (i) and (ii) *reliable*[3)].

It is clearly seen that $x$ is reliable if and only if $x$ is a feasible solution of the following optimization problem:

$$\begin{array}{lc}
 & c^T x \to \min \\
\text{s.t.} & \\
 & Ex = e \\
 & Ax \leq b \\
 & \sum\limits_j a_{ij} x_j + \epsilon \sum\limits_{j \in J_i} |a_{ij}||x_j| \leq b_i + \delta \max[1, |b_i|] \quad \forall i \\
 & \ell \leq x \leq u
\end{array} \qquad (*)$$

The optimal solution to the latter problem can be treated as the best, in terms of the objective, of the reliable solutions. It is easily seen that $(*)$ is equivalent to the Linear Programming

3) A rigorous name should, of course, be "$(\epsilon, \delta)$-reliable". In what follows, however, the values $\epsilon$ and $\delta$ will be clear from the context, which allows us to use the nickname "reliable". This is also the case with the notion of an "almost reliable" solution to be introduced later.

program

$$
\begin{array}{c}
c^T x \to \min \\
\text{s.t.} \\
Ex = e \\
Ax \le b \\
\sum_j a_{ij} x_i + \epsilon \sum_{j \in J_i} |a_{ij}| y_j \le b_i + \delta \max[1, |b_i|] \quad \forall i \\
-y_j \le x_j \le y_j \quad \forall j \\
\ell \le x \le u
\end{array}
\qquad (\mathrm{IRC}[\epsilon,\delta])
$$

Thus, one way to get a "robust optimal" solution to (LP) is to solve the $(\epsilon, \delta)$-*interval robust counterpart* ($\mathrm{IRC}[\epsilon, \delta]$) of our uncertain problem. Note that this scheme in fact goes back to L. Soyster [7].

**"Random symmetric uncertainty".** Now assume that the true values $\widetilde{a}_{ij}$ of uncertain data entries in $i$-th inequality constraint

$$\sum_j a_{ij} x_j \le b_i$$

of (LP) are obtained from the nominal values $a_{ij}$ of the entries by random perturbations:

$$\widetilde{a}_{ij} = (1 + \epsilon \xi_{ij}) a_{ij},$$

where $\xi_{ij} = 0$ for $j \notin J_i$ and the perturbations $\{\xi_{ij}\}_{j \in J_i}$ are independent random variables symmetrically distributed in the interval $[-1, 1]$.

In this situation, when speaking about robust solutions to (LP), it makes sense to pass from the deterministic requirement (ii) to its probabilistic version, specifically, to the requirement

(ii$'$) For every $i$, the probability of the event

$$\sum_j \widetilde{a}_{ij} x_j > b_i + \delta \max[1, |b_i|]$$

is at most $\kappa$, where $\delta > 0$ is a given feasibility tolerance and $\kappa > 0$ is a given "reliability level". We shall call a solution satisfying (i) and (ii') an *almost reliable* solution to (LP).

**Proposition 3.1** *Assume that $x$ can be extended to a feasible solution $(x, y, z)$ of the optimization problem*

$$
\begin{array}{c}
c^T x \to \min \\
\text{s.t.} \\
Ex = e, \\
Ax \le b, \\
\sum_j a_{ij} x_j + \epsilon \left[ \sum_{j \in J} |a_{ij}| y_{ij} + \Omega \sqrt{\sum_{j \in J_i} a_{ij}^2 z_{ij}^2} \right] \le b_i + \delta \max[1, |b_i|] \quad \forall i, \\
\ell \le x \le u, \\
-y_{ij} \le x_j - z_{ij} \le y_{ij} \quad \forall i, j.
\end{array}
\qquad (RC[\epsilon,\delta,\Omega])
$$

*where $\Omega > 0$ is a positive parameter. Then $x$ satisfies* (i) *and* (ii$'$) *with* $\kappa = \exp\{-\Omega^2/2\}$.

**Proof.** Let $(x, y, z)$ be feasible for (RC$[\epsilon, \delta, \Omega]$). Then

$$\begin{array}{rl}
& \Pr\left\{\sum\limits_j \widetilde{a}_{ij}x_j > \overbrace{b_i + \delta\max[1, |b_i|]}^{b_i^+}\right\} \\
= & \Pr\left\{\sum\limits_j a_{ij}x_j + \epsilon\sum\limits_{j\in J_i}\xi_{ij}|a_{ij}|(x_i - y_{ij}) + \epsilon\sum\limits_{j\in J_i}\xi_{ij}|a_{ij}|y_{ij} > b_i^+\right\} \\
\le & \Pr\left\{\sum\limits_j a_{ij}x_j + \epsilon\sum\limits_{j\in J_i}\xi_{ij}|a_{ij}|z_{ij} + \epsilon\sum\limits_{j\in J_i}|a_{ij}|y_{ij} > b_i^+\right\} \\
\le & \Pr\left\{\sum\limits_{j\in J_i}\xi_{ij}|a_{ij}|y_j > \Omega\sqrt{\sum\limits_{j\in J_i}a_{ij}^2y_{ij}^2}\right\},
\end{array}$$

and the concluding probability is $\le \exp\{-\Omega^2/2\}$ due to the following well-known fact:

> *Let $p_j$ be given reals and $\eta_j$ be symmetrically distributed in $[-1, 1]$ independent random variables. Then for every $\Omega > 0$ one has*
>
> $$\Pr\left\{\sum_j \eta_j p_j > \Omega\sqrt{\sum_j p_j^2}\right\} \le \exp\{-\Omega^2/2\}. \quad (1)$$
>
> For the sake of completeness, here is the proof of (1): By homogeneity arguments, it suffices to consider the case of $\sum\limits_j p_j^2 = 1$. In this case
>
> $$\begin{array}{rcl}
\Pr\left\{\sum\limits_j \eta_j p_j > \Omega\right\} & \underbrace{\le}_{(a)} & \exp\{-\Omega^2\}\mathbf{E}\left\{\exp\{\Omega\sum\limits_j\eta_jp_j\}\right\} \underbrace{=}_{(b)} \exp\{-\Omega^2\}\prod\limits_j \mathbf{E}\left\{\exp\{\Omega\eta_jp_j\}\right\} \\
& \underbrace{=}_{(c)} & \exp\{-\Omega^2\}\prod\limits_j\left[\sum\limits_{\ell=0}^{\infty}\frac{(\Omega p_j)^{2\ell}}{(2\ell)!}\right] \le \exp\{-\Omega^2\}\prod\limits_j\exp\{\Omega^2p_j^2/2\} \underbrace{=}_{(d)} \exp\{-\Omega^2/2\}
\end{array}$$
>
> with $(a)$ being the Tschebyshev inequality, $(b)$ coming from independence of $\eta_j$ for distinct $j$, $(c)$ given by symmetry of the distribution of $\eta_j \in [-1, 1]$ and $(d)$ given by $\sum\limits_j p_j^2 = 1$. ■

Note that (RC$[\epsilon, \delta, \Omega]$) is "less conservative" that (IRC$[\epsilon, \delta]$): if $\{x_j, y_j\}$ is a feasible solution of the latter problem, then $\{x_j, y_{ij} = y_j, z_{ij} = 0\}$ is a feasible solution to the former one. In fact, in the case of "large" sets $J_i$ (IRC) can be "much more restrictive" than (RC). Indeed, a *necessary and sufficient* condition for a vector $x$ to admit an extension to a feasible solution to (IRC) is that $x$ satisfies all the constraints of the nominal problem along with the inequalities

$$\sum_j a_{ij}x_j + \epsilon\alpha_i(x) \le b_i^+ \quad \forall i, \quad \alpha_i(x) = \sum_{j\in J_i}|a_{ij}||x_j|$$

while a *sufficient* condition for $x$ to admit an extension to a feasible solution of (RC) is to be feasible for (LP) and to satisfy the inequalities

$$\sum_j a_{ij}x_j + \epsilon\beta_i(x) \le b_i^+ \quad \forall i, \quad \beta_i(x) = \Omega\sqrt{\sum_{j\in J_i}a_{ij}^2x_j^2}.$$

Now, the ratio $\frac{\alpha_i(x)}{\beta_i(x)}$ can be as large as $\sqrt{\text{card}\,(J_i)}$.

A practical drawback of (RC) as compared to (IRC) is that the former problem, although convex and "well-structured", is more demanding computationally than the LP program (IRC).

### 3.2 The NETLIB case study: results of immunization

We have implemented the two aforementioned schemes – the (IRC)- and the (RC)-based ones – to get robust solutions to the bad NETLIB problems presented in Section 2. In the RC-scheme, the "safety parameter" $\Omega$ was set to 5.24, which corresponds to the reliability level $\kappa = 10^{-6}$ in (ii′), while the tolerance $\delta$ was set to 5%, the same as at the Analysis stage. The goal of the "immunization" stage of our case study was to understand what is the price of immunization in terms of the objective value. The results can be summarized as follows:

- Reliable solutions do exist, except for the four cases corresponding to the highest ($\epsilon = 1\%$) uncertainty level (see the right column in Table 3). Moreover, the price of immunization in terms of the objective value is surprisingly low: when $\epsilon \leq 0.1\%$, it never exceeds 1% and it is less than 0.1% in 13 of 23 cases. Thus, passing to the robust solutions, we gain a lot in the ability of a solution to withstand data uncertainty, while losing nearly nothing in optimality.

The detailed description of the results is given in Table 3.

| Problem | $\text{Opt}^{\text{nom}}$ | Objective at robust solution | | |
|---|---|---|---|---|
| | | $\epsilon = 0.01\%$ | $\epsilon = 0.1\%$ | $\epsilon = 1\%$ |
| 80BAU3B | 987224.2 | 987311.8 (+ 0.01%) | 989084.7 (+ 0.19%) | 1009229 (+ 2.23%) |
| 25FV47 | 5501.846 | 5501.862 (+ 0.00%) | 5502.191 (+ 0.01%) | 5505.653 (+ 0.07%) |
| ADLITTLE | 225495.0 | | 225594.2 (+ 0.04%) | 228061.3 (+ 1.14%) |
| AFIRO | -464.7531 | | -464.7500 (+ 0.00%) | -464.2613 (+ 0.11%) |
| BNL2 | 1811.237 | | 1811.237 (+ 0.00%) | 1811.338 (+ 0.01%) |
| BRANDY | 1518.511 | | | 1518.581 (+ 0.00%) |
| CAPRI | 1912.621 | | 1912.738 (+ 0.01%) | 1913.958 (+ 0.07%) |
| CYCLE | 1913.958 | 1913.958 (+ 0.00%) | 1913.958 (+ 0.00%) | 1913.958 (+ 0.00%) |
| D2Q06C | 122784.2 | 122793.1 (+ 0.01%) | 122893.8 (+ 0.09%) | Infeasible |
| E226 | -18.75193 | | | -18.75173 (+ 0.00%) |
| FFFFF800 | 555679.6 | | | 555715.2 (+ 0.01%) |
| FINNIS | 172791.1 | 172808.8 (+ 0.01%) | 173269.4 (+ 0.28%) | 178448.7 (+ 3.27%) |
| GREENBEA | -72555250 | -72526140 (+ 0.04%) | -72192920 (+ 0.50%) | -68869430 (+ 5.08%) |
| KB2 | -1749.900 | -1749.877 (+ 0.00%) | -1749.638 (+ 0.01%) | -1746.613 (+ 0.19%) |
| MAROS | -58063.74 | -58063.45 (+ 0.00%) | -58011.14 (+ 0.09%) | -57312.23 (+ 1.29%) |
| NESM | 14076040 | | | 14172030 (+ 0.68%) |
| PEROLD | -9380.755 | -9380.755 (+ 0.00%) | -9362.653 (+ 0.19%) | Infeasible |
| PILOT | -557.4875 | -557.4538 (+ 0.01%) | -555.3021 (+ 0.39%) | Infeasible |
| PILOT4 | -64195.51 | -64149.13 (+ 0.07%) | -63584.16 (+ 0.95%) | -58113.67 (+ 9.47%) |
| PILOT87 | 301.7109 | 301.7188 (+ 0.00%) | 302.2191 (+ 0.17%) | Infeasible |
| PILOTJA | -6113.136 | -6113.059 (+ 0.00%) | -6104.153 (+ 0.15%) | -5943.937 (+ 2.77%) |
| PILOTNOV | -4497.276 | -4496.421 (+ 0.02%) | -4488.072 (+ 0.20%) | -4405.665 (+ 2.04%) |
| PILOTWE | -2720108 | -2719502 (+ 0.02%) | -2713356 (+ 0.25%) | -2651786 (+ 2.51%) |
| SCFXM1 | 18416.76 | 18417.09 (+ 0.00%) | 18420.66 (+ 0.02%) | 18470.51 (+ 0.29%) |
| SCFXM2 | 36660.26 | 36660.82 (+ 0.00%) | 36666.86 (+ 0.02%) | 36764.43 (+ 0.28%) |
| SCFXM3 | 54901.25 | 54902.03 (+ 0.00%) | 54910.49 (+ 0.02%) | 55055.51 (+ 0.28%) |
| SHARE1B | -76589.32 | -76589.32 (+ 0.00%) | -76589.32 (+ 0.00%) | -76589.29 (+ 0.00%) |

**Table 3.** Objective values for nominal and robust solutions to bad NETLIB problems

Note that Table 3 represents only a single result per problem plus uncertainty level, in spite of the fact that we have implemented two immunization schemes. The reason is that with our setup, both (IRC[$\epsilon, \delta$]) and (RC[$\epsilon, \delta, \Omega$]) have "essentially the same" (coinciding with each other within relative inaccuracy $10^{-7}$) optimal values. A possible explanation of this phenomenon is that we require high ($\kappa = 10^{-6}$) reliability and thus use "large" $\Omega$ ($\Omega = 5.24$). With this $\Omega$, (RC) is indeed less conservative than (IRC) only when there are at least 30 uncertain data entries per constraint, which is not the case for the majority of bad NETLIB problems.

**How close is the nominal solution to reliable ones?** The outlined results demonstrate that as far as a reasonable (0.1%) level of uncertainty is concerned, the nominal solution, even if it itself "cannot withstand uncertainty", results in the "true" optimal value, in the sense that there exists another solution, capable of withstanding the uncertainty, with nearly the same value of the objective. An immediate question arises: can the nominal solution itself be made reliable by a small correction?

To answer this question, we have carried out the following experiment. Given a bad NETLIB problem (LP), we build a "local robust counterpart" (LRC) of the problem by adding to the constraints of (IRC) an additional requirement that the $x$-component of the solution of (IRC) must belong to a given "moderately small" neighbourhood $X$ of the nominal solution $x^*$ of (LP). If the optimal value in (LRC) is close to that of (IRC), we can say that the nominal solution is not that bad – a small correction suffices to make it reliable; in the opposite case, when the optimal value in (LRC) differs significantly from that of (IRC) (in particular, when (LRC) is infeasible), the nominal solution "is indeed bad".

To carry out the outlined experiment, we need a reasonable way to define $X$, i.e., to decide somehow what is "small" and what is "large", as far as changes in values of $x_j$ are concerned. Since we do not know the origin of the models we are dealing with, the only way to make such a decision is to look at the data themselves and to retrieve from them a "natural scale" for every one of the decision variables $x_j$. Note that the data plus the nominal solution give us a number of "clearly meaningful" values of $x_j$, namely

- the nominal optimal value $x_j^*$ of the variable,
- the lower bound $\ell_j$ on the variable, provided that it is not $-\infty$,
- the upper bound $u_j$ on the variable, provided that it is not $+\infty$.

From these values, we reconstruct a "natural scale" $d_j$ for $x_j$:

$$d_j = \max\left\{|x_j^*|, \widehat{\ell}_j, \widehat{u}_j\right\}$$
$$\left[\widehat{\ell}_j = \begin{cases} |\ell_j|, & \ell_j > -\infty \\ 0, & \text{otherwise} \end{cases}, \quad \widehat{u}_j = \begin{cases} |u_j|, & u_j < \infty \\ 0, & \text{otherwise} \end{cases}\right]$$

and define $\omega$-neighbourhood of $x^*$ as

$$X_\omega = \{x \mid |x_j - x_j^*| \leq \omega d_j, \ j = 1, ..., \dim x\}.$$

With this approach, the local robust counterpart of (LP) becomes the Linear Programming program

$$
\begin{array}{ll}
& c^T x \to \min \\
\text{s.t.} & \\
& Ex = e \\
& Ax \leq b \\
& \sum_j a_{ij} x_i + \epsilon \sum_{j \in J_i} |a_{ij}| y_j \leq b_i + \delta \max[1, |b_i|] \quad \forall i \\
& -y_j \leq x_j \leq y_j \quad \forall j \\
& |x_j - x_j^*| \leq \omega d_j \quad \forall j \\
& \ell \leq x \leq u
\end{array}
\tag{LRC[$\epsilon, \delta, \omega$]}
$$

In our experiments, we used $\delta = 0.05$ and $\omega = 0.05$. The results are presented in Table 4.

| | $\epsilon = 0.01\%$ | | $\epsilon = 0.1\%$ | | $\epsilon = 1\%$ | |
|---|---|---|---|---|---|---|
| Problem | $\Delta$[IRC][a] | $\Delta$[LRC][a] | $\Delta$[IRC] | $\Delta$[LRC] | $\Delta$[IRC] | $\Delta$[LRC] |
| 80BAU3B | 0.00 | 0.00 | 0.19 | $+\infty$[b] | 2.23 | $+\infty$ |
| 25FV47 | 0.00 | 0.00 | 0.01 | $+\infty$ | 0.07 | $+\infty$ |
| ADLITTLE | | | 0.04 | 0.04 | 1.14 | $+\infty$ |
| AFIRO | | | 0.00 | 0.00 | 0.16 | 0.16 |
| BNL2 | | | 0.00 | 0.00 | 0.01 | 0.01 |
| BRANDY | | | | | 0.00 | 0.01 |
| CAPRI | | | 0.01 | $+\infty$ | 0.07 | $+\infty$ |
| CYCLE | 0.00 | 0.00 | 0.00 | $+\infty$ | 0.00 | $+\infty$ |
| D2Q6C | 0.00 | $+\infty$ | 0.09 | $+\infty$ | $+\infty$ | $+\infty$ |
| E226 | | | | | 0.00 | 0.00 |
| FFFFF800 | | | | | 0.01 | 0.01 |
| FINNIS | 0.01 | 0.01 | 0.28 | $+\infty$ | 3.27 | $+\infty$ |
| GREENBEA | 0.04 | 0.29 | 0.50 | $+\infty$ | 5.08 | $+\infty$ |
| KB2 | 0.00 | 0.00 | 0.02 | 1.95 | 0.19 | $+\infty$ |
| MAROS | 0.00 | $+\infty$ | 0.09 | $+\infty$ | 1.29 | $+\infty$ |
| NESM | | | | | 0.68 | 0.68 |
| PEROLD | 0.00 | 0.00 | 0.19 | 0.20 | $+\infty$ | $+\infty$ |
| PILOT | 0.00 | $+\infty$ | 0.39 | $+\infty$ | $+\infty$ | $+\infty$ |
| PILOT4 | 0.07 | 0.07 | 0.95 | 0.95 | 9.47 | $+\infty$ |
| PILOT87 | 0.00 | 0.00 | 0.17 | 0.17 | $+\infty$ | $+\infty$ |
| PILOTJA | 0.00 | 0.00 | 0.15 | 0.15 | 2.77 | $+\infty$ |
| PILOTNOV | 0.02 | 0.02 | 0.21 | 0.21 | 2.04 | $+\infty$ |
| PILOTWE | 0.02 | 0.02 | 0.25 | 0.26 | 2.51 | $+\infty$ |
| SCFXM1 | 0.00 | 0.00 | 0.02 | $+\infty$ | 0.29 | $+\infty$ |
| SCFXM2 | 0.00 | 0.00 | 0.02 | $+\infty$ | 0.28 | $+\infty$ |
| SCFXM3 | 0.00 | 0.00 | 0.02 | $+\infty$ | 0.28 | $+\infty$ |
| SHARE1B | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

**Table 4.** (IRC) vs. (LRC)

[a] $\Delta[\text{xxx}] = \frac{\text{Opt(xxx)} - \text{Opt(LP)}}{|\text{Opt(LP)}|} \times 100\%$

[b] (xxx) is infeasible

We see that all bad nominal solutions, except two of them, are "nearly capable" of withstanding 0.01%-uncertainty: a reliable solution with essentially the same value of the objective can be found already in the 5%-neighbourhood of the nominal solution. The situation changes dramatically when the uncertainty level is increased to 0.1%: here, to get a reliable solution, one should "reshape essentially" the nominal one in 12 of 23 cases. The latter phenomenon becomes even stronger when the uncertainty level is further increased to 1%.

A somewhat surprising observation related to the data of Table 4 is that whenever a bad nominal solution can be converted to a reliable one by a "5%-correction", the value of the objective at the resulting solution is nearly as good as the one corresponding to the best reliable solution; the only exception is problem `KB2` at the uncertainty level 0.1%.

**Conclusions.** We see that

1. In many cases the feasibility of the usual optimal solution to a Linear Programming program can be heavily affected by quite small, from the practical viewpoint, perturbations of the data. At the same time, there exists a systematic and computationally reasonable way to construct reliable solutions, those capable to withstand data uncertainty of a given level.

2. When passing from the usual optimal solution to a reliable one, we do not necessarily lose a lot in optimality (in fact, for the `NETLIB` problems with 0.1%-perturbations in the data, the losses never exceed 1%).

3. In many cases, a reliable solution cannot be obtained by a moderately small correction of the nominal solution; in other words, the immunization methodology we have presented is "essential".

# References


[1] Ben-Tal, A., and Nemirovski, A. "Stable Truss Topology Design via Semidefinite Programming" – *SIAM Journal on Optimization* **v.** 7 (1997), 991-1016.

[2] Ben-Tal, A., and Nemirovski, A. "Robust solutions to uncertain linear programs" – *OR Letters* **v.** 25 (1999), 1-13.

[3] Ben-Tal, A., and Nemirovski, A. "Robust Convex Optimization" – *Mathematics of Operations Research* **23** (1998), 769–805.

[4] Ben-Tal, A., El-Ghaoui, L., and Nemirovski, A. "Robust semidefinite programming", in: R. Saigal, L. Vandenberghe, H. Wolkowicz, Eds. *Semidefinite Programming and Applications*, Kluwer Academic Publishers, 2000.

[5] El-Ghaoui, L., and Lebret, H. "Robust solutions to least-square problems with uncertain data matrices", *SIAM J. of Matrix Anal. and Appl.* **v.** 18 (1997), 1035-1064.

[6] El-Ghaoui, L., Oustry, F., and Lebret, H. "Robust solutions to uncertain semidefinite programs", *SIAM J. on Optimization* **v.** 9 (1998), 33-52.

[7] Soyster, A.L. "Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming" – *Oper. Res.* **21** (1973), 1154–1157.