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Absence of Eigenvalues for Quasi-Periodic Lattice Operators with Liouville Frequencies

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Dedicated to Barry Simon on the occasion of his 70th birthday

We show that a lattice Schrödinger operator $\Delta + v$ acting in $\mathbb{C}^{\mathbb{Z}^d}$ does not have l^2 eigenfunctions if its potential $v(\cdot)$ admits fast local approximation by periodic functions. A special case of this result states that if $v(x) = V(\alpha_1 x_1, \ldots, \alpha_d x_d)$, where $V(\cdot)$ is a $(1, \ldots, 1)$-periodic function on $\mathbb{R}^d$ satisfying the Hölder condition and $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is a vector admitting fast rational approximation, then the operator $\Delta + v$ has no eigenfunctions in $l^2(\mathbb{Z}^d)$. The one-dimensional case of this statement has been known since 1970s, and the question whether its multidimensional generalization was possible remained open since then.

1 Introduction

The goal of this paper is to show that the lattice Schrödinger operator $H = \Delta + v$ acting in $\mathbb{C}^{\mathbb{Z}^d}$ as follows:

$$(Hu)(x) = \sum_{s \in \mathbb{Z}^d:\ \|s\|_1 = 1} u(x+s) + v(x)u(x), \quad x \in \mathbb{Z}^d, \tag{1}$$

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does not have l^2 eigenfunctions if its potential $v\colon \mathbb{Z}^d \to \mathbb{C}$ can be approximated with high accuracy by periodic functions on a suitable increasing sequence of finite sets. This result can be applied, in particular, to operators with potentials of the form

$$v(x) = V(\alpha_1 x_1, \dots, \alpha_d x_d), \quad x \in \mathbb{Z}^d, \tag{2}$$

where $V\colon \mathbb{R}^d \to \mathbb{C}$ is a $(1, \dots, 1)$-periodic function and α_i's are irrational numbers. One of the main results of this paper (Theorem 4.1) states: given a function V satisfying the Hölder condition, there is an explicit $\vartheta > 0$ with the property that if real numbers $\alpha_1, \dots, \alpha_d$ satisfy the inequality $|||\nu_1\alpha_1||| + \cdots + |||\nu_d\alpha_d||| < \vartheta^{\nu_1 \cdots \nu_d}$ ($|||\cdot|||$ is the distance from a real number to the nearest integer) for all d-tuples $\nu = (\nu_1, \dots, \nu_d)$ from some infinite set $\mathcal{T} \subset \mathbb{N}^d$ such that $\lim_{\mathcal{T} \ni \nu \to \infty} \min\{\nu_1, \dots, \nu_d\} = \infty$, then the operator $\Delta + v$ with $v(\cdot)$ given by (2) has no eigenfunctions in $l^2(\mathbb{Z}^d)$.

The one-dimensional case of this result, for real-valued potentials, was essentially established in [2] (that paper dealt with the equation $-y'' + v(x)y = \lambda y$ on $\mathbb{R}$, but the adaptation to the discrete case was immediate). Since then, the question whether a multidimensional generalization of that one-dimensional fact was possible remained open.

An intermediate step in this direction was made in a recent paper [3], where, under conditions similar to those of Theorem 4.1, the absence of l^1 eigenfunctions was established (see also Krüger's preprint [4], where a slightly stronger condition was imposed on α_i's).

Both Theorem 4.1, giving the final answer to the above question, and its l^1 counterpart contrast the result of Bourgain [1], according to which for a fixed real analytic function V on $\mathbb{T}^d$ satisfying a mild non-degeneracy condition, the operator $\Delta + \lambda v$ with $v(\cdot)$ given by (2) exhibits Anderson localization (that is, the set of all exponentially decaying eigenfunctions is complete in $l^2(\mathbb{Z}^d)$) for all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{T}^d \setminus \Omega_\lambda$, where $\operatorname{mes} \Omega_\lambda \to 0$ as $\lambda \to \infty$.

The main method of this paper is a strengthening of the method used in [3], thus there are many similarities with [3]. We provide detailed arguments in order to make the present paper self-contained.

The remaining part of the paper is organized as follows. In Section 2 we establish inequalities of the form $|u(q)|^2 \le C \sum_{x \in K} |u(q+x)|^2$, where $u(\cdot)$ is an arbitrary solution of a periodic linear homogeneous lattice equation and K is a certain finite subset of the group of periods not containing 0. These inequalities generalize the inequality $|y(0)| \le 2 \max_{r=\pm1,\pm2} |y(rT)|$ [2] that holds for any solution $y(\cdot)$ of the equation $y(j-1)+y$

$(j+1) + v(j)y(j) = \lambda y(j)$ if $v(j+T) \equiv v(j)$, $j \in \mathbb{Z}$. In Section 3 we use the results of Section 2 to prove the absence of l^2 eigenfunctions for a Schrödinger operator whose potential admits fast periodic approximation on a suitable increasing sequence of finite sets. In Section 4 we consider a special case: that of a quasi-periodic potential with Liouville frequencies.

In the rest of the paper we use the following notation. Given a set $X \subset \mathbb{Z}^d$, we denote the set $X \setminus \{0\}$ by X^*. The cardinality of a finite set X is denoted by $|X|$. The dimension d of the lattice $\mathbb{Z}^d$ is assumed to be ≥ 2 (except for Lemma 2.1 and Theorem 2.1, where d may be equal to one).

2 Periodic Operators

Lemma 2.1. Let Γ be a subgroup of $\mathbb{Z}^d$ and H a Γ-periodic linear operator in $\mathbb{C}^{\mathbb{Z}^d}$ (Γ-periodicity means that, letting $(T^\gamma u)(x) = u(x+\gamma)$, we have $T^\gamma H = HT^\gamma$ for all $\gamma \in \Gamma$.) Suppose $F \subset \Gamma$ and $Y \subset \mathbb{Z}^d$ are such finite sets and $\lambda \in \mathbb{C}$ is such a number that
(a) if $u(\cdot)$ is a solution of the equation

$$Hu = \lambda u \tag{3}$$

and $u|_Y = 0$, then $u|_F = 0$;
(b) $|F| > |Y|$.
Then for any solution $u(\cdot)$ of (3)

$$|u(0)|^2 \leq \left(\frac{|F|}{|Y|} - 1\right)^{-1} \sum_{x \in (F-F)^*} |u(x)|^2. \tag{4}$$

□

Proof. Let N denote the vector space of all solutions of (3), and let $M = N|_Y$. It follows from (a) that for each $x \in F$ the value of a solution $u \in N$ at x is uniquely determined by $u|_Y$; therefore, $u(x)$ is a linear functional on M.

Set $k = |F|$, $n = |Y|$, and $m = \dim M$. Since $m \leq n$, the assumption (b) implies that $k > m$. We will need the following statement.

Lemma 2.2. Let Q_i, $i = 1, \ldots, k$, be positive semidefinite Hermitian forms on an m-dimensional complex vector space L. If $k > m$, then there is $l \in \{1, \ldots, k\}$ such that

$$Q_l(x) \leq \left(\frac{k}{m} - 1\right)^{-1} \sum_{\substack{1 \leq j \leq k \\ j \neq l}} Q_j(x) \ \text{ for all } \ x \in L. \tag{5}$$

□

Proof of Lemma 2.2. Let

$$Q(x) = \sum_{1 \le j \le k} Q_j(x), \quad x \in L.$$

We may assume that the Hermitian form $Q(\cdot)$ is positive definite—otherwise, we can replace L with the quotient vector space $\widetilde{L} = L/Z$, where $Z = \{x \in L \colon Q(x) = 0\}$; this will only reduce the constant factor in (5).

The Hermitian form Q makes L a complex Euclidean space; let $e_1, \dots, e_m$ be its orthonormal basis, and A_j ($j = 1, \dots, k$) be the matrix of the form Q_j relative to that basis. The Hermitian matrices A_j are positive semidefinite (notation: $A_j \ge 0$), their sum being the identity matrix I. Hence, the sum of their traces is m, and there is $l \in \{1, \dots, k\}$ such that $\operatorname{tr} A_l \le m/k$. Therefore, the largest eigenvalue of A_l does not exceed m/k so that $A_l \le (m/k)I$; in other words, $A_l \le (m/k) \sum_{1 \le j \le k} A_j$, or

$$\left(1 - \frac{m}{k}\right) A_l \le \frac{m}{k} \sum_{\substack{1 \le j \le k \\ j \ne l}} A_j,$$

which is equivalent to (5). ■

End of proof of Lemma 2.1. For each $x \in F$, $|u(x)|^2$ is a positive semidefinite Hermitian form on the m-dimensional space M. We have $k = |F| > m$, and by Lemma 2.2 there is $a \in F$ such that for all solutions $u(\cdot)$ of the equation (3) we have

$$|u(a)|^2 \le \left(\frac{k}{m} - 1\right)^{-1} \sum_{x \in F \setminus \{a\}} |u(x)|^2. \tag{6}$$

Since the operator H is Γ-periodic, the vector space N of solutions of (3) is invariant under translations by elements of Γ. Therefore, (6) implies that for all solutions $u(\cdot)$ of (3)

$$|u(0)|^2 \le \left(\frac{k}{m} - 1\right)^{-1} \sum_{x \in (F - a) \setminus \{0\}} |u(x)|^2,$$

from which (4) follows. ■

Theorem 2.1. Let Γ be a subgroup of $\mathbb{Z}^d$, and H a Γ-periodic linear operator in $C^{\mathbb{Z}^d}$. Let $F \subset \Gamma$ and $Y \subset \mathbb{Z}^d$ be two finite sets with the following properties:

(a) for any $q \in \mathbb{Z}^d$ and any solution $u(\cdot)$ of the equation (3) such that $u|_{Y+q} = 0$, we also have $u|_{F+q} = 0$;
(b) $|F| > |Y|$.
Then for any solution $u(\cdot)$ of (3) and any $q \in \mathbb{Z}^d$

$$|u(q)|^2 \le \left(\frac{|F|}{|Y|} - 1\right)^{-1} \sum_{x \in (F-F)^*} |u(q+x)|^2. \tag{7}$$

□

Proof. For $q \in \mathbb{Z}^d$, let $H^q = T^q H T^{-q}$. The operator H^q and the sets F and Y satisfy the conditions of Lemma 2.1. Putting $u_q = T^q u$, where u is a solution of $Hu = \lambda u$, we have $H^q u_q = \lambda u_q$ and, by Lemma 2.1,

$$|u_q(0)|^2 \le \left(\frac{|F|}{|Y|} - 1\right)^{-1} \sum_{x \in (F-F)^*} |u_q(x)|^2,$$

which is equivalent to (7). ■

Let $H = \Delta + v$ be the lattice Schrödinger operator (1) in $l^2(\mathbb{Z}^d)$, $d \ge 2$, with a Γ-periodic potential $v(\cdot)$. We will assume now that Γ is a subgroup of $\mathbb{Z}^d$ generated by d linearly independent vectors $f_1, \dots, f_d$, where $f_j = \left(f_j^{(i)}\right)_{i=1}^d \in \mathbb{Z}^d$, $j = 1, \dots, d$:

$$\Gamma = \left\{ \sum_{j=1}^d m_j f_j : \ m_j \in \mathbb{Z},\ j = 1, \dots, d \right\}.$$

Denote by A_Γ the fundamental region of the lattice Γ in $\mathbb{R}^d$:

$$A_\Gamma = \left\{ \sum_{j=1}^d \theta_j f_j : \ 0 \le \theta_j < 1,\ j = 1. \dots, d \right\}$$

and by V_Γ its volume:

$$V_\Gamma = \mathrm{Vol}(A_\Gamma) = \left| \det \left[f_j^{(i)} \right]_{i,j=1}^d \right|.$$

Theorem 2.2. For any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that if $u \colon \mathbb{Z}^d \to \mathbb{C}$ is a solution of the equation

$$(\Delta + v)u = \lambda u \tag{8}$$

with a Γ-periodic potential v, then for any point $q \in \mathbb{Z}^d$

$$|u(q)|^2 \le C_\varepsilon \sum_{x\in\Gamma^*:\ \|x\|_\infty \le (2d+\varepsilon)V_\Gamma} |u(q+x)|^2. \tag{9}$$

□

Proof. Fix an integer $n \ge 3$ and consider the following subset of $\mathbb{Z}^d$:

$$C_0 = \{0, 1, \ldots, n-1\}^d.$$

For $z \in \mathbb{Z}^d$, let $C_z = C_0 + z$ and

$$F_z = \Gamma \cap C_z.$$

The lattice Γ has "density"

$$\lim_{\mathbb{N}\ni r\to\infty} \frac{|\{x \in \Gamma:\ \|x\|_\infty \le r\}|}{(2r+1)^d} = \frac{1}{V_\Gamma},$$

and since $\mathbb{Z}^d = \bigsqcup_{z\in n\mathbb{Z}^d}(C_0 + z)$, we have $\sup_{z\in n\mathbb{Z}^d}|F_z| \ge |C_0|/V_\Gamma = n^d/V_\Gamma$. The integer $|F_z|$ takes only finitely many values, so there is $z \in n\mathbb{Z}^d$ such that

$$|F_z| \ge \frac{n^d}{V_\Gamma}. \tag{10}$$

Let, furthermore, $Y_0 = C_0 \setminus S_0$, where $S_0 = \{1, \ldots, n-2\}^{d-1} \times \{2, \ldots, n-1\}$, and

$$Y_z = Y_0 + z.$$

Due to the structure of the operator $\Delta + v$, any solution u of the equation (8) that vanishes on the set Y_0 also vanishes on C_0. Given $q \in \mathbb{Z}^d$, the same is true for the sets $Y_z + q = Y_0 + (z+q)$ and $C_z + q = C_0 + (z+q)$. And since $F_z + q \subset C_z + q$, it follows that for any solution $u(\cdot)$ of the equation (8) such that $u|_{Y_z+q} = 0$ we also have $u|_{F_z+q} = 0$.

We want to use Theorem 2.1 with $F = F_z$ and $Y = Y_z$. Its condition (a) is, therefore, fulfilled, and to apply the theorem we need that condition (b) be fulfilled as well, that is, the ratio $|Y_z|/|F_z|$ should be < 1.

In view of the equality $|Y_z| = |Y_0| = n^d - (n-2)^d$ and inequality (10),

$$\frac{|Y_z|}{|F_z|} \le V_\Gamma\left(1 - \left(1 - \frac{2}{n}\right)^d\right) < \frac{2dV_\Gamma}{n}. \tag{11}$$

So far the integer n ($n \geq 3$) has been arbitrary. We set now

$$n = n_\Gamma = \lfloor (2d+\varepsilon) V_\Gamma \rfloor \tag{12}$$

($\lfloor a \rfloor$ is the largest integer $\leq a$). The second inequality in (11), together with (12), implies that the quantity $V_\Gamma\big(1-\big(1-\frac{2}{n_\Gamma}\big)^d\big)$, which depends only on V_Γ, is < 1 for all values of V_Γ. Moreover, its upper limit, as $V_\Gamma \to \infty$, is < 1. Consequently, there is $\delta_\varepsilon > 0$ such that, setting $n = n_\Gamma$, we have $|Y_z|/|F_z| \leq (1+\delta_\varepsilon)^{-1}$ for any Γ and an arbitrary z satisfying (10).

By Theorem 2.1,

$$|u(q)|^2 \leq \left(\frac{|F_z|}{|Y_z|} - 1\right)^{-1} \sum_{x \in (F_z - F_z)^*} |u(q+x)|^2 \leq \delta_\varepsilon^{-1} \sum_{x \in \Gamma^*:\ \|x\|_\infty < n_\Gamma} |u(q+x)|^2. \tag{13}$$

The second inequality uses the fact that

$$F_z - F_z = (\Gamma \cap (C_0 + z)) - (\Gamma \cap (C_0 + z)) \subset \Gamma \cap ((C_0+z) - (C_0+z)) = \{x \in \Gamma:\ \|x\|_\infty < n_\Gamma\}.$$

Inequalities (13) imply that (9) holds with $C_\varepsilon = \delta_\varepsilon^{-1}$. ■

Corollary 2.1. If $u(\cdot)$ is a solution of $(\Delta + v)u = \lambda u$, where the function $v(\cdot)$ on $\mathbb{Z}^d$ is $(\tau_1, \ldots, \tau_d)$-periodic ($\tau_1, \ldots, \tau_d \in \mathbb{N}$), then for any $\varepsilon > 0$ and any $q \in \mathbb{Z}^d$

$$|u(q)|^2 \leq C_\varepsilon \sum_{x \in \Gamma_\tau^*:\ \|x\|_\infty \leq (2d+\varepsilon)\tau_1 \ldots \tau_d} |u(q+x)|^2,$$

where C_ε is the same as in Theorem 2.2 and

$$\Gamma_\tau = \{(j_1\tau_1, \ldots, j_d\tau_d) : j_1, \ldots, j_d \in \mathbb{Z}\}. \tag{14}$$

□

3 Operators Approximable by Periodic Ones

Theorem 3.1. Let $H = \Delta + v$, where $v(\cdot)$ is a bounded complex-valued function on $\mathbb{Z}^d$, and let θ be a constant satisfying the inequalities

$$0 < \theta < (2\|v\|_\infty + 4d - 1)^{-1}. \tag{15}$$

Suppose for some fixed $\varepsilon > 0$ and each $\tau = (\tau_1, \dots, \tau_d)$ in an infinite set $\mathcal{T} \subset \mathbb{N}^d$ such that

$$\lim_{\mathcal{T} \ni \tau \to \infty} \tau_i = \infty, \quad i = 1, \dots, d, \tag{16}$$

there is a $(\tau_1, \dots, \tau_d)$-periodic function $v_\tau(\cdot)$ satisfying the inequality

$$\max_{\|x\|_\infty \le (2d+\varepsilon)\tau_1 \dots \tau_d} |v_\tau(x) - v(x)| < \theta^{\,2d\tau_1 \dots \tau_d}. \tag{17}$$

Then the equation $Hu = \lambda u$ with any $\lambda \in \mathbb{C}$ does not have nontrivial l^2 solutions. □

Proof. Fix $D \in \mathbb{R}$ such that

$$2\|v\|_\infty + 4d - 1 < D < \theta^{-1} \tag{18}$$

(this is possible due to (15)). Then reduce, if necessary, the given positive ε so that

$$D^{2d+\varepsilon} < \theta^{-2d} \tag{19}$$

(inequality (17) will still hold for all $\tau \in \mathcal{T}$).

For any $\tau \in \mathcal{T}$, let

$$m_\tau = \lfloor (2d + \varepsilon)\tau_1 \dots \tau_d \rfloor.$$

Inequality (17) can be rewritten in the form

$$\rho_\tau \equiv \max_{x \in Q_\tau} |v_\tau(x) - v(x)| < \theta^{\,2d\tau_1 \dots \tau_d}, \tag{20}$$

where

$$Q_\tau = \{x \in \mathbb{Z}^d : \|x\|_\infty \le m_\tau\}.$$

Suppose $u \colon \mathbb{Z}^d \to \mathbb{C}$ is a solution of the equation $(\Delta + v)u = \lambda u$ such that

$$\|u\|_2 \le 1. \tag{21}$$

Pick any $\tau \in \mathcal{T}$ so there is a τ-periodic function $v_\tau(\cdot)$ satisfying (20). Define a subset Z_τ of Q_τ as follows:

$$Z_\tau = \{x \in Q_\tau : x_d \in \{-1, 0\} \text{ or } |x_i| = m_\tau \text{ for some } i \in \{1, \dots, d-1\}\}.$$

Denote by Q_τ^{o} the "interior"of the cube Q_τ:

$$Q_\tau^{\mathrm{o}} = \{x \in \mathbb{Z}^d : \ \|x\|_\infty \le m_\tau - 1\}$$

and by $u_\tau(\cdot)$ the unique function on Q_τ such that

(i) $(\Delta u_\tau)(x) + v_\tau(x) u_\tau(x) = \lambda u_\tau(x)$ for all $x \in Q_\tau^{\mathrm{o}}$;

(ii) $u_\tau|_{Z_\tau} = u|_{Z_\tau}$.

The function

$$w_\tau(x) = u_\tau(x) - u(x), \quad x \in Q_\tau,$$

satisfies the equations $w_\tau|_{Z_\tau} = 0$ and

$$(\Delta w_\tau)(x) + (v(x) - \lambda) w_\tau(x) + r_\tau(x) u(x) + r_\tau(x) w_\tau(x) = 0, \quad x \in Q_\tau^{\mathrm{o}},$$

where

$$r_\tau(x) = v_\tau(x) - v(x).$$

By representing any $x \in \mathbb{Z}^d$ in the form $x = (j, k)$, where $j \in \mathbb{Z}^{d-1}$ and $k \in \mathbb{Z}$, we transform the previous equation into

$$\begin{aligned} w_\tau(j, k+1) + w_\tau(j, k-1) + \sum_{s \in \mathbb{Z}^{d-1} : \, \|s\|_1 = 1} w_\tau(j+s, k) + (v(j,k) - \lambda) w_\tau(j,k) \\ + r_\tau(j,k) u(j,k) + r_\tau(j,k) w_\tau(j,k) = 0, \quad (j,k) \in Q_\tau^{\mathrm{o}}. \end{aligned} \tag{22}$$

Setting

$$\sigma_\tau(k) = \left(\sum_{x \in Q_\tau : \, x_d = k} |w_\tau(x)|^2 \right)^{1/2} \equiv \left(\sum_{j \in \mathbb{Z}^{d-1} : \, \|j\|_\infty \le m_\tau - 1} |w_\tau(j,k)|^2 \right)^{1/2}, \quad -m_\tau \le k \le m_\tau,$$

we obtain from (22), using the Minkowski inequality:

$$\sigma_\tau(k \pm 1) \le \sigma_\tau(k \mp 1) + B_\tau \sigma_\tau(k) + \rho_\tau, \qquad -m_\tau + 1 \le k \le m_\tau - 1, \tag{23}$$

where

$$B_\tau = 2(d-1) + \|v\|_\infty + |\lambda| + \rho_\tau \tag{24}$$

(we use the facts that $|r_\tau(j,k)| \le \rho_\tau$ for all $(j,k) \in Q_\tau$ and $\sum_{j\in\mathbb{Z}^{d-1}} |u(j,k)|^2 \le 1$ for all $k \in \mathbb{Z}$, due to (20) and (21), respectively).

Note that $|\lambda|$ does not exceed the norm of the operator $H = \Delta + v$, which is $\le 2d + \|v\|_\infty$. Therefore,

$$B_\tau \le 4d + 2\|v\|_\infty - 2 + \rho_\tau, \qquad \tau \in \mathcal{T}.$$

In view of the first inequality in (18) and inequality (20), we have

$$B_\tau < D - 1 \tag{25}$$

for all but finitely many $\tau \in \mathcal{T}$. By removing this finite set from $\mathcal{T}$, we may assume that (25) holds for all $\tau \in \mathcal{T}$.

Inequalities (23) and (25) imply that

$$\sigma_\tau(k \pm 1) \le \sigma_\tau(k \mp 1) + (D-1)\sigma_\tau(k) + \rho_\tau, \qquad -m_\tau + 1 \le k \le m_\tau - 1.$$

It follows by induction (using the equalities $\sigma_\tau(-1) = \sigma_\tau(0) = 0$) that

$$\sigma_\tau(k) \le D^{|k|-1}\rho_\tau, \qquad -m_\tau \le k \le m_\tau.$$

Consequently,

$$\sum_{|k|\le m_\tau} \sigma_\tau^2(k) \le D^{2m_\tau}\rho_\tau^2,$$

or, equivalently,

$$\left(\sum_{x\in Q_\tau} |u_\tau(x) - u(x)|^2\right)^{1/2} \le D^{m_\tau}\rho_\tau, \tag{26}$$

which holds for all $\tau \in \mathcal{T}$.

The function $u_\tau(\cdot)$ is defined on the cube Q_τ and satisfies the equation $\Delta u_\tau(x) + v_\tau(x)u_\tau(x) = \lambda u_\tau(x)$ on Q_τ^{o}. According to the lemma below, this function has an extension to $\mathbb{Z}^d$ that satisfies the same equation for all $x \in \mathbb{Z}^d$.

We will denote the convex hull of a set $X \subset \mathbb{R}^d$ by $\mathrm{Conv}(X)$; a set $G \subset \mathbb{Z}^d$ will be called *z-convex* if $\mathrm{Conv}(G) \cap \mathbb{Z}^d = G$. Given $y \in \mathbb{Z}^d$, the set $\mathrm{St}(y) = \{x \in \mathbb{Z}^d\colon \|x - y\|_1 \le 1\}$

will be called *the star centered at* y. For $G \subset \mathbb{Z}^d$, we will denote by G° the set $\{y \in G\colon \mathrm{St}(y) \subset G\}$.

Lemma 3.1. Let G be a finite z-convex subset of $\mathbb{Z}^d$; suppose $v\colon \mathbb{Z}^d \to \mathbb{C}$ and $u\colon G \to \mathbb{C}$ are such functions that

$$\Delta u(x) + v(x)u(x) = \lambda u(x) \tag{27}$$

for all $x \in G^{\circ}$. Then $u(\cdot)$ can be extended to $\mathbb{Z}^d$ so that the resulting function satisfies (27) for all $x \in \mathbb{Z}^d$. □

Proof. Let us enumerate the points of G arbitrarily: $y_1, \ldots, y_N$, where $N = |G|$, then enumerate the points of the set $\mathbb{Z}^d \setminus G$: $y_{N+1}, y_{N+2}, \ldots$ in such a way that the set $G_n = \{y_1, \ldots, y_n\}$ is z-convex for any $n \geq N$. (One way to do so is to always choose y_n ($n > N$) at the smallest possible Euclidean distance from the set $\mathrm{Conv}(G_{n-1})$.) Then we determine the values $u(y_n)$, $n > N$, recursively as follows. If there is a star, say $\mathrm{St}(w)$, containing y_n and contained in $G_{n-1} \sqcup \{y_n\}$ (note that in this case $w \in G_{n-1}$ and there is only one such star, due to the z-convexity of G_{n-1}), then we define $u(y_n)$ so that equation (27) holds for $x = w$; otherwise, we define $u(y_n)$ arbitrarily. ■

Due to the lemma, we can consider $u_\tau(\cdot)$ as a function defined on $\mathbb{Z}^d$ and satisfying equation $(\Delta + v_\tau)u_\tau = \lambda u_\tau$ on the entire lattice $\mathbb{Z}^d$. The function $v_\tau(\cdot)$ is Γ_τ-periodic, where the lattice Γ_τ is defined by (14).

Let $\delta = \varepsilon/2$. Pick any $q \in \mathbb{Z}^d$. According to Corollary 2.1,

$$|u_\tau(q)| \leq C_\delta^{1/2} \left(\sum_{x \in q + P_\tau^*} |u_\tau(x)|^2 \right)^{1/2}, \tag{28}$$

where

$$P_\tau = \{x \in \Gamma_\tau : \ \|x\|_\infty \leq (2d + \delta)\, \tau_1 \ldots \tau_d\}.$$

Assuming that

$$\|q\|_\infty \leq \delta \cdot \tau_1 \ldots \tau_d$$

(which is true for all $\tau \in \mathcal{T}$ with large enough $\|\tau\|_\infty$), we have $q + P_\tau \subset Q_\tau$, so that inequalities (26) and (28), together with the Minkowski inequality, imply that

$$\begin{aligned}|u(q)| &\le |u_\tau(q)| + |u(q) - u_\tau(q)| \\ &\le C_\delta^{1/2}\left(\sum_{x\in q+P_\tau^*}|u(x)|^2\right)^{1/2} + \left(C_\delta^{1/2}+1\right)\left(\sum_{x\in Q_\tau}|u_\tau(x)-u(x)|^2\right)^{1/2} \\ &\le C_\delta^{1/2}\left(\sum_{x\in q+P_\tau^*}|u(x)|^2\right)^{1/2} + \left(C_\delta^{1/2}+1\right)D^{m_\tau}\rho_\tau \end{aligned} \tag{29}$$

for all $\tau \in \mathcal{T}$ with large enough $\|\tau\|_\infty$.

As $\mathcal{T} \ni \tau \to \infty$, the first summand on the right converges to 0 due to (21) and (16). The second summand does not exceed the quantity

$$\left(C_\delta^{1/2}+1\right)D^{(2d+\varepsilon)\tau_1\ldots\tau_d}\theta^{2d\tau_1\ldots\tau_d} = \left(C_\delta^{1/2}+1\right)\left(D^{2d+\varepsilon}\theta^{2d}\right)^{\tau_1\ldots\tau_d},$$

which, due to (19), goes to 0 as $\mathcal{T} \ni \tau \to \infty$. It follows that the right-hand side of (29) goes to 0 as well. Therefore, $u(q) = 0$. Since $q \in \mathbb{Z}^d$ was chosen arbitrarily, this completes the proof of Theorem 3.1. ■

4 Quasi-Periodic Operators

As was said in the Introduction, we denote the distance from a real number a to the nearest integer by $|||a|||$.

Theorem 4.1. Let $V\colon \mathbb{R}^d \to \mathbb{C}$ be a $(1,\ldots,1)$-periodic function satisfying the Hölder condition

$$|V(t_1,\ldots,t_d) - V(t_1',\ldots,t_d')| \le M\sum_{i=1}^d |t_i - t_i'|^\beta,$$

where $0 < \beta \le 1$. Suppose $\eta \in \mathbb{R}$ is such that

$$0 < \eta < (2\|V\|_\infty + 4d - 1)^{-1}. \tag{30}$$

If real numbers $\alpha_1,\ldots,\alpha_d$ satisfy the inequalities

$$|||\nu_1\alpha_1||| + \cdots + |||\nu_d\alpha_d||| < \eta^{(2d/\beta)\nu_1\ldots\nu_d} \tag{31}$$

for all d-tuples $(\nu_1, \dots, \nu_d)$ in an infinite set $\mathcal{T} \in \mathbb{N}^d$ such that

$$\lim_{\mathcal{T} \ni \nu \to \infty} \nu_i = \infty, \quad i = 1, \dots, d,$$

then the operator $\Delta + v$ with potential

$$v(x) = V(\alpha_1 x_1, \dots, \alpha_d x_d) \tag{32}$$

has no eigenfunctions in $l^2(\mathbb{Z}^d)$. □

Remark 4.1. The fact that the absence of eigenvalues for operators $\Delta + v$ with potentials (32) is topologically typical, which follows from Theorem 4.1, has been known for a long time: due to Simon's result [6, Theorem 1.1], the set of all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ for which that operator has no eigenvalues is a G_δ set; this set is dense as it contains all α's with rational components (since the corresponding potentials are periodic). However, the essence of Theorem 4.1 is that it provides an *explicit* rate of rational approximation of the frequency vector α that guarantees the absence of eigenvalues. □

Proof. Denote by $\mathcal{T}$ the infinite set of those $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$ for which (31) holds. Given $\nu \in \mathcal{T}$, there is $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{Z}^d$ such that $|\mu_i - \nu_i \alpha| = |||\nu_i \alpha|||$ for each $i = 1, \dots, d$. Let

$$\alpha_i^\nu = \frac{\mu_i}{\nu_i}, \quad i = 1, \dots, d,$$

and

$$v_\nu(x_1, \dots, x_d) = V(\alpha_1^\nu x_1, \dots, \alpha_d^\nu x_d).$$

The function $v_\nu(\cdot)$ is $(\nu_1, \dots, \nu_d)$-periodic. In order to apply Theorem 3.1, we need to estimate (for a fixed $\varepsilon > 0$) the number

$$\rho_\nu = \max_{\|x\|_\infty \le (2d+\varepsilon)\, \nu_1 \dots \nu_d} |v_\nu(x) - v(x)|.$$

If $\nu \in \mathcal{T}$ and $\|x\|_\infty \le (2d + \varepsilon)\, \nu_1 \dots \nu_d$, then

$$\begin{aligned} |v_\nu(x) - v(x)| &= \left| V(\alpha_1^\nu x_1, \dots, \alpha_d^\nu x_d) - V(\alpha_1 x_1, \dots, \alpha_d x_d) \right| \\ &\le M \sum_{i=1}^{d} \left| \frac{\mu_i}{\nu_i} x_i - \alpha_i x_i \right|^\beta \end{aligned}$$

$$\leq M \sum_{i=1}^{d} |x_i|^{\beta} \, |\mu_i - \nu_i \alpha_i|^{\beta}$$

$$\leq M \|x\|_{\infty}^{\beta} \sum_{i=1}^{d} \|\nu_i \alpha_i\|^{\beta}$$

$$< Md \left((2d + \varepsilon)\, \nu_1 \ldots \nu_d\right)^{\beta} \; \left(\eta^{\,\nu_1 \ldots \nu_d}\right)^{2d} \tag{33}$$

(the last inequality uses (31)).

Pick any $\theta \in \mathbb{R}$ such that

$$\eta < \theta < (2\|V\|_{\infty} + 4d - 1)^{-1}, \tag{34}$$

which is possible due to (30). It follows from (33) and the first inequality in (34) that $\rho_\nu < \theta^{\,2d\nu_1 \ldots \nu_d}$ for all but finitely many $\nu \in \mathcal{T}$. This, together with the second inequality in (34), shows that the conditions of Theorem 3.1 are fulfilled and, therefore, the equation $(\Delta + v)u = \lambda u$ does not have nontrivial l^2 solutions. ■

5 Concluding Remarks

Remark 5.1. Theorem 3.1 can be extended to the case where the potential $v(\cdot)$ is unbounded. □

Theorem 5.1. Suppose $v\colon \mathbb{Z}^d \to \mathbb{C}$ is such that

$$\sup_{x \in Z^d} |v(x)| = \infty$$

and there exist $\varepsilon > 0$, $\gamma > 0$ and an infinite set $\mathcal{T} \subset \mathbb{N}^d$ satisfying (16) such that for any $\tau \in \mathcal{T}$ there is a $(\tau_1, \ldots, \tau_d)$-periodic function $v_\tau\colon \mathbb{Z}^d \to \mathbb{C}$ with the property that

$$\rho_\tau < M_\tau^{-(2d+\gamma)\tau_1 \ldots \tau_d}, \tag{35}$$

where

$$\rho_\tau = \max_{x \in \mathbb{Z}^d :\, \|x\|_\infty \leq (2d+\varepsilon)\tau_1 \ldots \tau_d} |v_\tau(x) - v(x)|; \quad M_\tau = \max_{x \in \mathbb{Z}^d :\, \|x\|_\infty \leq (2d+\varepsilon)\tau_1 \ldots \tau_d} |v(x)|.$$

Then the equation $(\Delta + v)u = \lambda u$ with any $\lambda \in \mathbb{C}$ has no nontrivial solutions in $l^2(\mathbb{Z}^d)$. □

The proof follows the same lines as that of Theorem 3.1. The necessary changes are presented in the Appendix.

Remark 5.2. The main results of the paper pertain to the lattice Schrödinger operator; however, the method that we use can be adapted to other lattice operators, such as $\widetilde{\Delta}+v$, where $\widetilde{\Delta}$ is the diagonal Laplacian introduced in [5]:

$$(\widetilde{\Delta}u)(x) = \sum_{z\in\{-1,+1\}^d} u(x+z), \quad x \in \mathbb{Z}^d. \qquad \square$$

Appendix

Proof of Theorem 5.1. The beginning of the proof is similar to that of Theorem 3.1, with two changes. First, in the definition (24) of the coefficient B_τ used in (23) we replace $\|v\|_\infty$ with M_τ:

$$B_\tau = 2(d-1) + M_\tau + |\lambda| + \rho_\tau.$$

Second, instead of the constant D satisfying inequalities (18) we introduce

$$D_\tau = 2M_\tau. \tag{36}$$

Then (25) gets replaced with the inequality

$$B_\tau < D_\tau - 1,$$

which holds for all but finitely many $\tau \in \mathcal{T}$.

We then replace D with D_τ in all the subsequent inequalities up to (29), which now gives

$$|u(q)| \le C_\delta^{1/2}\left(\sum_{x\in q+P_\tau^*} |u(x)|^2\right)^{1/2} + \left(C_\delta^{1/2}+1\right) D_\tau^{m_\tau}\rho_\tau; \tag{37}$$

this inequality holds for all but finitely many $\tau \in \mathcal{T}$.

By the assumption, $u \in l^2(\mathbb{Z}^d)$; therefore, as $\mathcal{T} \ni \tau \to \infty$, the first summand on the right converges to 0. Furthermore, in view of (35) and (36),

$$D_\tau^{m_\tau}\rho_\tau < (2M_\tau)^{(2d+\varepsilon)\tau_1\dots\tau_d} M_\tau^{-(2d+\gamma)\tau_1\dots\tau_d} = \left(2^{2d+\varepsilon}M_\tau^{\varepsilon-\gamma}\right)^{\tau_1\dots\tau_d}. \tag{38}$$

The positive constant ε can be made arbitrarily small without violating (35), so we may assume from the very beginning that $0 < \varepsilon < \gamma$. Since $\|v\|_\infty = \infty$, we have $M_\tau \to \infty$ as $\mathcal{T} \ni \tau \to \infty$; therefore, the right-hand side of (38) converges to 0. Consequently, the right-hand side of (37) converges to 0 as well, which completes the proof. ■

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