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Participation in auctions

Shmuel Gal [a], Michael Landsberger [b,*], Arkadi Nemirovski [c]

[a] *Department of Statistics, University of Haifa, Haifa, Israel*
[b] *Department of Economics, University of Haifa, Haifa, Israel*
[c] *Faculty of Industrial Engineering and Management, Technion—Israel Institute of Technology, Haifa, Israel*

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Abstract

Participation (preparing offers) in auction and procurement games—e.g., bidding for spectrum rights or for control of corporations—is very costly, which calls for analyzing participation (no less than how to play the games) as part of equilibrium. The result is a model with *bi-dimensional* distribution of types (as in [Jackson, M.O., 1999. Nonexistence of equilibria in auctions with both private and common values. Mimeo. http://www.stanford.edu/~jacksonm/nonexist.pdf]) and *endogenous* distribution of participants. Players with low production but high participation costs may decide not to participate, forcing the auctioneer to pay higher-than-necessary prices. To mitigate this phenomenon, auctioneers may consider partially reimbursing participants for the costs they incur in preparing offers. Restricting attention to symmetric games, we prove that

- The game we consider has a unique symmetric equilibrium, with and without the reimbursement mechanism.
- Partially reimbursing bidders for the cost of preparing bids pays off for virtually *all* joint distributions of types.
- A constant reimbursement rule, independent of participation cost, causes losses.

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* Corresponding author.
E-mail addresses: sgal@univ.haifa.ac.il (S. Gal), lmichael@econ.haifa.ac.il (M. Landsberger), nemirovs@ie.technion.ac.il (A. Nemirovski).

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1. Introduction

Participating in auctions and procurements is costly, often highly so. Bidding for contracts to develop a new product, to build an opera house or airport, or to acquire an institution like a bank is generally preceded by intensive—and expensive—preparations. Hence, the question of whether to participate in these projects at all may be more crucial than the standard question of how to bid, suggesting that such decisions should be modeled and be part of an equilibrium. Since participation costs vary across players, as do production costs, the resulting game is one with *bi-dimensional* types and *endogenous* distribution of participants.

There may be players with low production (marginal) costs who might make good offers, but who choose not to participate because of high participation costs. This motivates the question of whether such players might be attracted to participate by partially reimbursing them for the participation costs they incur—and whether the benefits of attracting them to participate would exceed this reimbursement cost.[1] We address this question in the paper.

We are not the first to acknowledge that bidders may face participation costs. Samuelson (1985), McAfee and McMillan (1987, 1988) and Harstad (1990) were among the first to do so. Their work was augmented by Levin and Smith (1994), Menezes and Monteiro (2000), and most recently Ye (2004), who considered participation decisions in an even more general setting. Stegeman (1996) further augmented the type of problems considered by allowing asymmetric bidders and by establishing a 'semirevelation' principle for mechanisms with participation costs. By allowing asymmetric equilibria and correlated valuations, Cremer et al. (2004) augmented results obtained by McAfee and McMillan (1988) for the case where the auctioneer must expand resources to inform players (sequentially) about the auction. Thus, the literature on participation has addressed a host of interesting problems. However, each of these papers has assumed that participation costs are the same across players. The core of our paper lies in the assumption, which we believe more accurately reflects reality, that marginal as well as fixed costs vary across players, resulting in a model with bi-dimensional types, and setting the stage for analyzing the profitability of reimbursement policies.[2]

Restricting attention to symmetric games and assuming a quite general class of joint distributions of types, we prove that:

- The game considered, which includes a reimbursement mechanism, has a unique symmetric equilibrium for virtually all joint distributions of types.
- Partially reimbursing participants creates a more competitive environment by attracting players with low production costs who otherwise would not take part because of high participation costs. At the same time, it deters players with high production costs and low participation costs, who cannot compete successfully in this more competitive environment. Consequently, the auctioneer receives better offers that compensate for the cost of reimburse-

[1] McAfee and McMillan (1987) noted that the US Department of Defense had a dual sourcing policy, which involved subsidizing an extra firm's costs of investment specific to the production of a particular weapons system in order to generate bidding competition against the incumbent producer. Also, it is common in many countries to partially reimburse architectural and engineering firms that participate in procurements but do not win.

[2] Since we prove that reimbursing participants independently of their participation costs causes losses, an efficient policy of reimbursement must depend on the costs incurred. Consequently, a reimbursement policy could not be considered in the models mentioned above.

ment. A numerical analysis of equilibrium with a simple reimbursement rule shows that the reimbursement procedure may boost the auctioneer's profits by as much as 10–14 percent.
- The auctioneer cannot gain by instituting a constant reimbursement rule (independent of participation costs). Although such a rule creates a more competitive environment, reimbursement costs are higher than the gains from obtaining good offers.

In Section 2, we present a "Toy" model that mimics the structure and results derived from the general model, but is easy to follow (because it assumes a simple joint distribution of types and only two players). In Section 3 we present the general model. Formulation of the reimbursement problem and an analysis of its effects can be found in Section 4. Proofs were relegated to Appendix A.

2. The "Toy" model

A principal offers a contract for the delivery of a certain object that he values as 1. Producers (players) are invited to submit bids for the contract, whereby the player who makes the lowest offer below 1 is granted the contract and, for convenience, is required to pay the second highest bid.[3] Preparing an offer (bid) is costly, making participation in the auction a strategic variable. Once a player has decided to participate, he incurs the participation cost K (the cost of, for example, assembling and financing a team of experts to develop the new product). At the second stage of the game, the winner must begin producing and so incurs the marginal production cost C. Both C and K are privately observed, but we assume that *after* bids have been submitted, participants may submit claims to be partially reimbursed for their participation costs—claims that must be supported by appropriate documentation, making K verifiable by the auctioneer. The pair (c, k) is a realization of a bi-dimensional random variable (C, K), where C and K are typically dependent. We assume that there are only two players.

2.1. Joint distribution of types

Assume that the joint distribution of types (C, K) is

$$f(c,k) = \begin{cases} c \text{ is uniform in } (0,1), \text{and } k = 0, & \text{with probability } p, \\ c = 0, \text{and } k = k_* = 1 - p/2 + \Delta, & \text{with probability } 1 - p \end{cases} \tag{1}$$

where $p \in [0, 1]$ and $\Delta > 0$. Hence, a player may belong to one of two types: σ_u, with production costs uniformly distributed on (0, 1) and participation costs of 0, or σ_0, with no production costs and positive, constant participation cost. If both participants belong to σ_0, the winner is chosen randomly. To make our example less stylized and to avoid dealing with atoms, which leads to the use of mixed strategy, we could allow production costs for types in σ_0 to be distributed on a smaller support, say $[0, \alpha < 1]$, but since such a change would not affect our conclusions, we let simplicity have the upper hand. Note that the joint distribution is designed so that σ_u-type players *always* participate, since they incur no participation costs. σ_0-type players are all alike, have high participation costs and no production costs; they do not participate unless reimbursed (above a certain level); k_* and p are set so as to assure this property. We consider a reimbursement rule

$$R(k) = r \text{ if } k = k_* \quad \text{and} \quad 0 \text{ otherwise.} \tag{2}$$

[3] We show in Section 3 that Revenue Equivalence holds.

2.2. Equilibrium

Theorem 2.1. *In equilibrium, the reimbursement policy stipulated in* (2) *induces the following participation strategies*:

(i) *if* $r \leqslant r_{\min} \equiv \Delta$, *only* σ_u*-type players participate.*
(ii) *if* $r_{\min} \leqslant r \leqslant r_{\max} \equiv 1 - p + \Delta$, σ_0 *-type players participate with probability* $\theta = \frac{r-\Delta}{1-p}$.
(iii) *if* $r_{\max} \leqslant r < k_*$, *both type players participate with probability* 1.

The theorem illustrates how a higher reimbursement rate intensifies the participation of zero-production-cost, σ_0-type players.

Proof.[4] (i) To prove that if $r < r_{\min} \equiv \Delta$, σ_0-type players do not participate, we need show only that the expected gain from bidding of σ_0-type players who defect from the strategy of non-participation does not cover participation costs. To this effect, note that the gain from bidding of such a player is

$$(1-p)\cdot 1 + p\cdot\frac{1}{2} = 1 - p\cdot\frac{1}{2}. \tag{3}$$

Thus, the expected gain of σ_0-type players who deviate from the non-participation strategy is

$$E(\pi_2) = \left(1 - p\cdot\frac{1}{2}\right) - \left(1 - p\cdot\frac{1}{2} + \Delta - r\right) = r - \Delta < 0.$$

(ii) It follows immediately that if $r_{\min} < r < r_{\max}$ and σ_0-type players do not participate, it pays for a single σ_0 player to deviate from the non-participation strategy. On the other hand, if all σ_0-type players participate, the gain from bidding does not cover participation costs. Therefore, assume that σ_0-type players resort to a mixed strategy θ. Then the expected gain from bidding of a σ_0-type player is

$$(1-p)(1-\theta) + \frac{p}{2}. \tag{4}$$

This result is obtained as in (3), except that in this situation we must account for the fact that σ_0-type players participate with probability θ. Consequently, the expected gain from participation is

$$E(\pi_2) = (1-p)(1-\theta) + \frac{p}{2} - k_* + r. \tag{5}$$

Since in equilibrium $E(\pi_2) = 0$, we obtain $\theta = \frac{r-\Delta}{1-p}$, as claimed.

(iii) If $r_{\max} \leqslant r < k_*$ and $\theta = 1$, the expected gain from bidding of a σ_0-type participant is $\frac{p}{2}$, and the expected gain from participation is

$$E(\pi_2) = -k_* + r + \frac{p}{2} > 0, \tag{6}$$

since $k_* > r \geqslant r_{\max} \equiv 1 - p + \Delta$ and $k_* = 1 - \frac{p}{2} + \Delta$. Note that if reimbursement is set so that $r < 1 - p + \Delta$, the strategy $\theta = 1$ is not an equilibrium. □

We show below how the increased participation of σ_0-type players forces σ_u-type players to make better (lower) offers and, consequently, settle for lower gains from bidding, $g(c)$.

[4] A rigorous proof of the theorem establishing uniqueness can be found in Appendix A.2.

- When $r \leqslant r_{\min}$, σ_0-type players do not participate, $\theta = 0$, and the expected gain from participation (of σ_u-type players) is

$$g_{(\theta=0)}(c) = \frac{p}{2}(1-c)^2 + (1-p)(1-c). \tag{7}$$

- Higher reimbursement, $r_{\max} \geqslant r \geqslant r_{\min}$, attracts σ_0-type players into participation and lowers the expected gain from bidding of σ_u-type players;

$$g_{(0<\theta<1)}(c) = \frac{p}{2}(1-c)^2 + \big[(1-p)(1-\theta)\big](1-c) < g_{(\theta=0)}(c). \tag{8}$$

- Finally, when $r \geqslant r_{\max}$, σ_0-type players fully participate ($\theta = 1$), and $g(c)$ stabilizes at its lowest level

$$g_{(\theta=1)}(c) = \frac{p}{2}(1-c)^2 < g_{(0<\theta<1)}(c). \tag{9}$$

2.3. *Net gains from competition*

To prove that the reimbursement policy pays off, we must show that the lower offers received by the principal more than offset the cost of reimbursement—that is, they boost the principal's expected profits. The latter is given by $(1 \cdot P^d - J(r))$, where P^d stands for the probability that the project is delivered (at least one player has chosen to participate), and $J(r)$ is the total expected cost (payment to the winning bidder and reimbursement). We adhere to the convention that players who decided not to participate, deliver the project and are paid 1. This convention assures that when there are no participants the net gain of the auctioneer is 0, which conforms with the model, and it allows us to replace the expression for the net expected gain, $(1 \cdot P^d - J(r))$, by $(1 - J(r))$. Obviously, the probability of the event 'no participants' is part of equilibrium and $J(r)$ is calculated by using the above-mentioned convention. Consequently, instead of maximizing net expected gain, $(1 - J(r))$, we can simply minimize expected cost, $J(r)$.

We are now ready to prove that if $k_* < 1$ and partial reimbursement is instituted, the auctioneer's total costs, $J(r)$, decrease. To this end, we need show only that $J'(r_{\min}) < 0$, where $J(r)$ is composed of the expected reimbursement, $R(r)$, plus the expected payment to the winner, $W(r)$

$$J(r) = R(r) + W(r). \tag{10}$$

We begin with $R(r)$, making a distinction between two events: only one, or both players must be reimbursed. The first event happens with probability $2\theta(1-p)(1-\theta(1-p))$, and in this case the auctioneer pays r. The second event takes place with probability $(\theta(1-p))^2$, and the auctioneer pays $2r$. Thus,

$$R(r) = 2r\big[\big(\theta(1-p)\big)^2 + \theta(1-p)\big(1-\theta(1-p)\big)\big] = 2r(r-\Delta). \tag{11}$$

The derivation of the expected payment to the winner, $W(r)$, is calculated as a conditional expectation in four events:

- At least one of the players does not participate, which happens with probability $1 - (p + \theta(1-p))^2$; the payment to winner is 1.
- The other three events assume that both players participate, but a distinction must be made between three situation:
  - Both participants belong to σ_u, which happens with probability p^2: the expected payment to the winner is $\frac{2}{3}$, which is the expectation of the second-order statistic from a uniform distribution on (0, 1). Hence, the expected payment to the winner is $\frac{2}{3} \cdot p^2$.

- One player belongs to σ_u and the other to σ_0. The probability of this event is $2p\theta(1-p)$, and the expected payment to the winner (σ_0-type) is $\frac{1}{2}$.
- Both players belong to σ_0; the probability of both participating is $(\theta(1-p))^2$), and the payment to the winner is 0.

Combining all these, we obtain (remember that $\theta = \frac{r-\Delta}{1-p}$)

$$W(r) = 2(k_* - r) - (k_* - r)^2 - \frac{p^2}{12}. \tag{12}$$

Using (10), (11) and (12) we obtain

$$J(r) = 2r(r-\Delta) + 2(k_* - r) - (k_* - r)^2 - \frac{p^2}{12}. \tag{13}$$

Differentiating $J(r)$ with respect to r we obtain

$$J'(r) = 2(r - \Delta - 1 + k_*) \tag{14}$$

and since $r_{\min} = \Delta$, $J'(r_{\min}) = 2(k_* - 1)$. Thus, $J'(r_{\min}) < 0$ if and only if $k_* < 1$, which proves that

Theorem 2.2. *If $k_* < 1$, partial reimbursement pays off.*

Solving the equilibrium for a numerical example shows that cost savings come close to 14 percent; see Gal et al. (2006). Moreover, it is shown that upon instituting the reimbursement rule the expected payment to the winner decreases monotonically in θ; at $\theta = 1$ it is reduced by almost 35 percent.

3. Uniqueness (and existence) of equilibrium: the general case

We assume N, a fixed number, *potential* bidders (players) who privately observe their bi-dimensional types defined by production cost, C, and the cost of preparing the bid, K (participation cost). Types are drawn independently from a probability distribution $F_{C,K}(c,k)$ defined on $\Omega = \{(c,k) \mid 0 \leqslant c \leqslant 1,\ k \geqslant 0\}$. We augment the "Toy" model by assuming any number of players and a general, bi-dimensional, joint distribution of types. To make the presentation more concise, we derive our results by assuming a given reimbursement rule, $R(k)$; equilibrium without reimbursement follows immediately by letting $R(k) = 0$.

Assumption 3.1. Given the reimbursement rule $R(\cdot)$, the participation cost function

$$\psi(k) \equiv k - R(k) \tag{15}$$

is non-negative, strictly increasing and continuously differentiable on the non-negative ray.

Assumption 3.1 is innocuous; all it says is that the reimbursement rule is such that after being reimbursed, a player with higher participation costs will incur higher costs. We believe that such a property can be obtained endogenously in any reasonable setting.

Assumption 3.2. The distribution $F(c,k)$ possesses continuously differentiable density $p(c,k)$.

Assumption 3.3.

$$\int_{\phi(0)}^{\infty} p(1,k)\,\mathrm{d}k > 0,$$

where the ϕ function is defined in Eq. (16).

Assumption 3.3 stipulates that the set of players with the highest possible production cost is not empty. To prove that partial reimbursement lowers the total expected cost, it is not necessary to make this assumption, but doing so makes the proof easier.

A type (c,k) who decides to participate bears the participation cost $\psi(k)$. A player participates if participation costs do not exceed the (conjectured) expected profits from bidding, $g(c)$. Of course, once a decision about participation has been made, k is history and the expected gain from bidding $g(c)$ depends only on the *endogenous* marginal distribution of production cost C of players who have decided to participate. Hence, a type (c, k) participates if and only if $\psi(k) \leqslant g(c)$, or

$$k \leqslant \psi^{-1}\big(g(c)\big) \equiv \phi\big(g(c)\big). \tag{16}$$

An equilibrium consists of a pair of functions (P, g) where P is the *endogenously* determined distribution of participating bidders' production costs, when every bidder believes that his or her expected payoff conditional on her participation is determined by the function g, and g determines a participant's expected payoff with the provision that (i) a participant's production cost is distributed according to P and (ii) the participant with the lowest production cost wins. More precisely,

Definition 3.1. An equilibrium is a pair of functions (P, g) such that

$$\begin{aligned} P(x) &= \text{Prob(a participant's production cost} \leqslant x) \\ &= \text{Prob}\big\{(c,k) : (k - R(k)) \leqslant g(c);\ c \leqslant x\big\}, \end{aligned} \tag{17}$$

where

$$g(c) = E\left[\int_c^1 \big(1 - P(x)\big)^{b-1}\,\mathrm{d}x\right] \tag{18}$$

and b is the endogenously determined random number of participants.

To obtain an explicit expression for the function $g(\cdot)$ we proceed as follows: As in the "Toy" model we adhere to the convention that non-participants are considered as players with production cost 1, who deliver the project and are paid 1. Of course, the probability of this event is determined endogenously as part of equilibrium. This convention allows to replace (18) by

$$g(c) = \int_c^1 \big(1 - P(x)\big)^{N-1}\,\mathrm{d}x \tag{19}$$

and induces the one-dimensional random variable

$$\widehat{C}(R) = \begin{cases} 1 & \text{with probability } Q(R), \\ 0 \leqslant c < 1 & \text{with density } P(c, \phi(g(c))) = \int_0^{\phi(g(c))} \mathrm{d}F(c,k) \end{cases} \tag{20}$$

that governs bidding behavior, where

$$Q(R) = \int_0^1 \int_{\phi(g(c))}^{\infty} \mathrm{d}F(c,k). \tag{21}$$

The inner integral in (21) "collects" the densities of non-participating types given c, and then the integration over all possible c values gives the desired probability mass of non-participants. $P(c, \phi(g(c)))$ in (20) is the density of participants with participation cost $0 \leqslant k \leqslant \phi(g(c))$ that have a positive probability of winning. Although the random variable $\widehat{C}(R)$ depends on R, we suppress, for convenience, the notation indicating the dependence. Whenever needed, this dependence is accounted for. Finally, the following lemma gives the probability distribution function of $\widehat{C}$.

Lemma 3.1. *The one-dimensional marginal distribution function of production costs of participating types, $\widehat{C}$, evaluated at some point $c < 1$ is*

$$\int_0^c P\bigl(t, \phi\bigl(g(t)\bigr)\bigr)\,\mathrm{d}t. \tag{22}$$

At $c = 1$ there is a probability mass of non-participants, $Q(R)$, given by (21).

The distribution function given by the lemma, complicated as it is, stands for $P(\cdot)$ in the definition of equilibrium. Note that the distribution stipulated in the lemma is one-dimensional; hence, the complication is the price paid in order to avoid handling bi-dimensional distributions. The probability mass $Q(R)$ in our model plays a role similar to that of the probability mass at the reserve price in standard auctions without participation costs. There is, however, one important difference; in these auction models the probability mass at the reserve price is given exogenously, and therefore the distribution function of participants follows immediately. In our model this probability mass is determined *endogenously* by (21), where $g(c)$ is part of equilibrium.

With one-dimensional types the expected profit from bidding of a type c under a distribution function $G_{\hat{C}}(\cdot)$ with a support, say [0, 1], is given by[5]

$$\Pi(c) = \int_c^1 \bigl(1 - G_{\hat{C}}(t)\bigr)^{N-1}\,\mathrm{d}t. \tag{23}$$

Using the distribution function defined in Lemma 3.1 we can use (23) to obtain an explicit, if not simple, expression for the equilibrium function $g(c)$. Hence, $g(c)$ supports an equilibrium if and only if it satisfies the integral equation

$$g(c) = \int_c^1 \left[1 - \int_0^u P\bigl(t, \phi\bigl(g(t)\bigr)\bigr)\,\mathrm{d}t\right]^{N-1} \mathrm{d}u, \quad 0 \leqslant c \leqslant 1 \tag{24}$$

[5] The one-dimensional distribution derived in Lemma 3.1 is continuous and has no atoms except at the upper end of the support. Since types are drawn independently, Revenue Equivalence holds. Note that the expected profit of a bidder in our setting is a mirror image of the standard case since the winner is the bidder who submitted the lowest bid. Hence, (23) replaces $\Pi(c) = \int_0^c [G_{\hat{C}}(t)]^{N-1}\,\mathrm{d}t$.

where P in the integrand of the inner integral is the distribution function of $\widehat{C}$ as defined in (20).

Theorem 3.1. *Under Assumptions* 3.1–3.3*, Eq.* (24) *has a unique smooth solution which is the unique solution of the mixed boundary problem*

$$\begin{aligned} g''(c) &= (N-1)\big[-g'(c)\big]^{(N-2)/(N-1)} P\big(c, \phi\big(g(c)\big)\big), \\ g(1) &= 0, \\ g'(0) &= -1. \end{aligned} \tag{25}$$

To prove the theorem it is sufficient to prove (i) and (ii) stated below.

(i) If the integral equation stated in (24) has a continuous and nonnegative solution on [0, 1], then it is twice continuously differentiable and its derivative is negative on the segment. Moreover, the solution solves the boundary problem stated in (25).

(ii) The boundary problem stated in (25) has exactly one continuous and nonnegative solution on [0, 1], and this solution satisfies (24). The formal, technical, proof can be found in Appendix A.1. Here are a few non-technical hints: The proof of (i) is immediate. The proof of (ii), which gives uniqueness, poses a difficulty since the standard existence and uniqueness theorems in ordinary differential equations identify the first equation in (25) as the Cauchy problem, which can be solved. This, however, does not include the two mixed boundary conditions. To account for these, we show that there exist *uniquely* defined initial conditions for $g(0)$ that *enforce* the required boundary conditions for $g(1)$ stipulated in (25), so that the existence and uniqueness of a solution to (25) follows from the standard existence and uniqueness results for the Cauchy problem. This is a non-trivial mathematical procedure that requires careful treatment, which is one reason for the complexity of the problem and the proof. Lizzeri and Persico (2000) used a similar method in their analysis of equilibria in a large class of asymmetric auctions with interdependent values.

Theorem 3.1 assures that, with and without reimbursement, in equilibrium, there exists a unique expected-gain-from-bidding function, $g(\cdot)$, and a unique participation function, $\phi(g(c))$; of course, if no reimbursement is applied, $g(\cdot)$ is also the participation function. Namely, a player of type (c, k) participates iff $k \leqslant g(c)$. Consequently, we have a unique one-dimensional distribution function of participants, defined in Lemma 3.1. Since the proof of (ii) in Theorem 3.1 is by construction, it opens the door to the construction of an algorithm that solves the integral equation in (24), allowing us to obtain the expected-gain-from-bidding function, $g(\cdot)$, in explicit form, and to perform numerical analyses documented in Gal et al. (2006).

As an illustration, we solve a case where there is no reimbursement, $R(k) \equiv 0$, the joint distribution of types is uniform in the unit square $\{0 \leqslant c, k \leqslant 1\}$, and the number of players—potential bidders—is $N = 8$. In this case, $g(c)$ is the expected gain from bidding and the participation function; see Fig. 1.

We measure c on the horizontal axis and k on the vertical. Hence, the participating types are below the $g(c)$ function. To obtain graphically the density of the marginal distribution of $\widehat{C}$ for participating types at some c, we need to collect the densities of $F_{C,K}(c, k)$ along the vertical distance from the horizontal axis to the curve $g(c)$ in the graph. This is done in Eq. (20). Not surprisingly, the density of participants decreases with c.

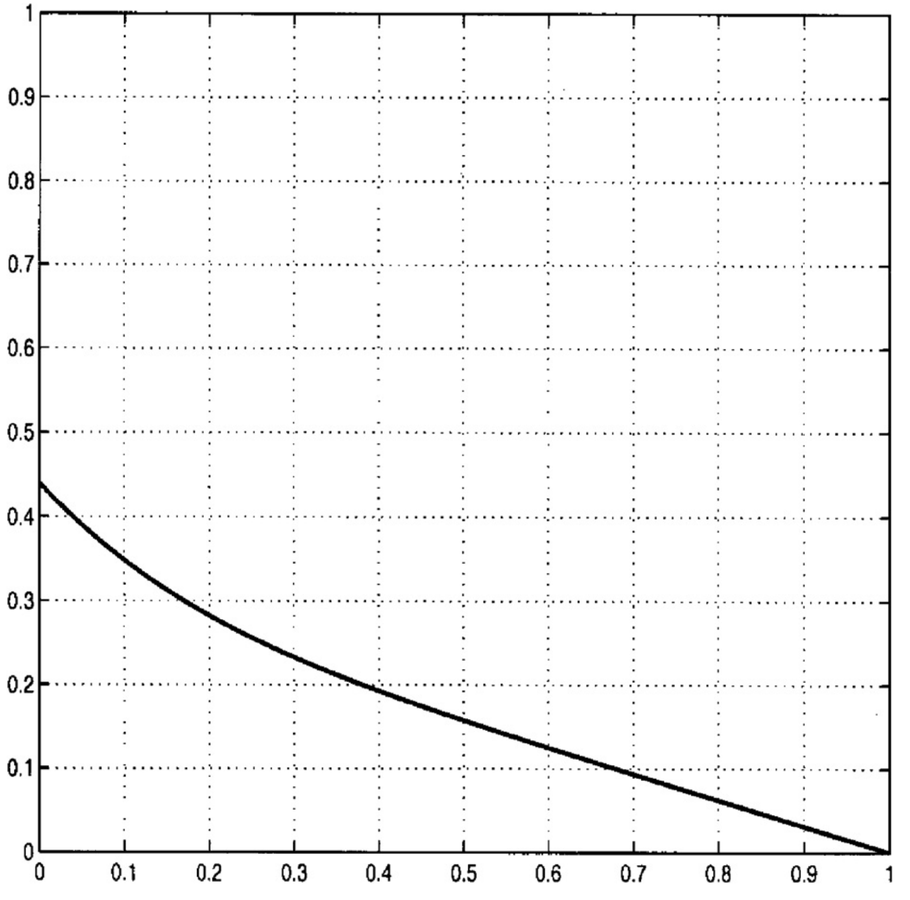

Fig. 1. The $g(c)$ function.

4. Reimbursement and gains

Having established the uniqueness and existence of equilibrium with and without reimbursement, we prove that partially reimbursing participants for their participation costs raises the auctioneer's expected profit under *all* joint distributions of types.

Assume the following reimbursement rule:

$$R_\epsilon(k) = r\big[k - k^*(\epsilon)\big]_+, \quad k^*(\epsilon) = g_0(0) - \epsilon, \tag{26}$$

where $[a]_+ = \max[a, 0]$ and $g_0(0)$ is the equilibrium expected profit from bidding under no reimbursement, evaluated at $c = 0$.[6]

[6] Note that ϕ induced by (26) does not satisfy exactly Assumption 3.1 since under (26) ϕ is nonnegative and Lipschitz continuous, but not continuously differentiable. This, however, can be ignored since $R_\epsilon(\cdot)$ given by (26) can be approximated uniformly by a smooth function with the derivative varying in $[0, r]$, and it can be easily shown that the arguments to follow work for a good enough approximation of this type.

Theorem 4.1. *Assume that F satisfies Assumptions* 3.1–3.3 *and has a positive density at* $(0, g_0(0))$. *Then for every fixed r, $0 < r < 1$, the reimbursement rule stated in* (26) *strictly raises the auctioneer's expected profits, provided that $\epsilon > 0$ is small enough.*

The essence of the (quite technical) proof of the theorem (presented in Appendix A.4) is that for small ϵ, reimbursement to the participants is of the order of ϵ^3, while the savings arising from the reduced payment to the winning bidder are of the order of ϵ^2. As they stand, these arguments are quite intuitive, and seem necessary, even a-priori, if reimbursement is going to pay off. However, given our complicated setting a rigorous proof of these naive statements turns out to be quite complex, since several technical points require delicate mathematical treatment.

The expected costs of the auctioneer is composed of (i) the expected reimbursement

$$\bar{R} = N \cdot \int_0^1 \int_0^{\phi(g(c))} R(k)\, \mathrm{d}F(c,k), \tag{27}$$

and (ii) the expected payment to the winning bidder, which is the second order statistic (in an ascending order) of N i.i.d. random variables distributed as in Lemma 3.1; we denote it by W. In Appendix A.3 we prove that it is given by

$$W = Ng(0) - (N-1) \int_0^1 \left[-g'(c)\right]^{N/(N-1)} \mathrm{d}c \tag{28}$$

where $g'(c)$ is the derivative of $g(\cdot)$ (see Eq. (A.1) in Appendix A.1). Combining (27) and (28), we obtain the total expected cost

$$J[R] = N \int_0^1 \int_0^{\phi(g(c))} R(k) \cdot \mathrm{d}F(c,k) + Ng(0) - (N-1) \int_0^1 \left[-g'(c)\right]^{N/(N-1)} \mathrm{d}c. \tag{29}$$

To help impart an intuitive understanding of the interactions set in motion by reimbursement and to reinforce and augment the results obtained in the "Toy" model, we solve a more general numerical example and present it graphically; see Fig. 2. To begin, we choose a joint uniform distribution in the unit square $\{0 \leqslant c, k \leqslant 1\}$ and $N = 8$. Applying the algorithm derived in the proof of (ii) of Theorem 3.1, we calculate the equilibrium expected profit function when there is no reimbursement, $g_0(c)$, by solving the integral equation in (24) for $R(k) \equiv 0$. The curve is denoted by (a) in Fig. 2 (in all figures we measure c on the horizontal axis and k on the vertical). Hence, the participating types are below the curve.

The reimbursement rule is

$$R(k) = r \cdot [k - 0.220]_+, \quad r = 0.5; \tag{30}$$

here and below $a_+ = \max[a, 0]$. Under (30), types with $k > 0.220$ are eligible and are reimbursed by $r = 0.5$ times the excess of their participation cost above 0.220; see the horizontal solid line in Fig. 2. Applying this reimbursement rule, we solve again the integral equation (24) for $g(c)$. This gives us the participation function $\phi(g_r(c))$ (see (16)), where $g_r(c)$ is the equilibrium expected profit from bidding *after* the institution of reimbursement; the two functions are denoted in Fig. 2 as 'c' for ϕ and 'b' for $g_r(c)$. Note that $\phi(g_r(c))$ and $g_r(c)$ coincide when $k \leqslant 0.220$, since types with $k \leqslant 0.220$ are not eligible for reimbursement.

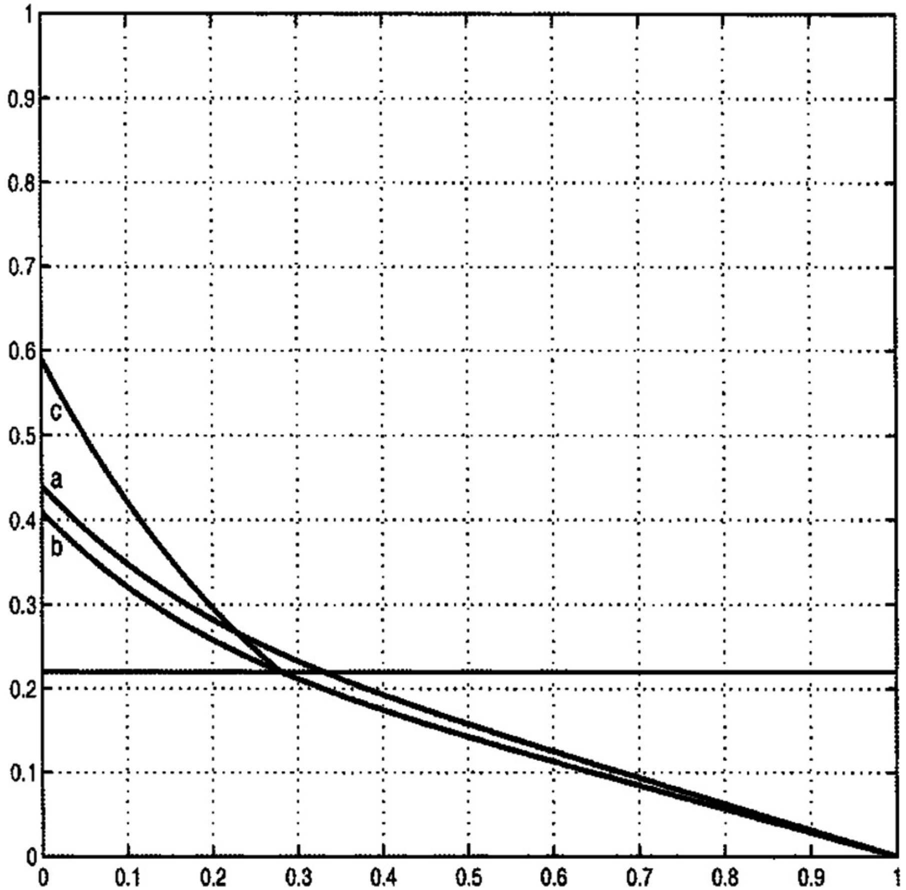

Fig. 2. Equilibria with and without reimbursement under a uniform distribution, a: $g_0(c)$, no reimbursement; b: $g_r(c)$, with reimbursement; c: $\phi(g_r(c))$, with reimbursement. The horizontal line is the lower boundary of the domain where the types are reimbursed for participation.

The (almost) triangular area between 'a' and 'c' to the left of the crossing represents types that are attracted to participate by the promise of reimbursement. The (almost) triangular area to the right of the crossing point represents types that would participate if there were no reimbursement, but do not participate under the reimbursement policy. Since reimbursement attracts into participation types with low production costs, the more competitive environment lowers the expected gains from bidding, which is why g_r is below g_0. Consequently, players with high production costs drop out and are replaced by participants with low production costs.

The reader can see now that by applying a more general joint distribution than we did in the "Toy" model, we can arrive at a deeper insight. For example, in the "Toy" model the entrance of the low-cost σ_0-type players forced the high-cost incumbents, σ_u-type, to make better offers, but since their participation costs were 0, they kept participating. Under the more general joint distribution, some incumbents make better offers under the reimbursement policy (as reflected by lower gains from bidding, curve 'b' in comparison with curve 'a' in Fig. 2), but, as noted above, some players with high production cost drop out.

Since verifying K may be difficult in some cases, it is interesting to investigate how much savings can be obtained if we apply a constant reimbursement rule. The following theorem provides an answer.

Theorem 4.2. *For any number $N \geqslant 2$ of players and any distribution F satisfying Assumptions* 3.1 *and* 3.2*, a constant reimbursement rule $R(k) \equiv r$ induces an expected cost function, $J(r)$, that is non-decreasing in $r \geqslant 0$.*

The proof was relegated to Appendix A.5. This result is somewhat surprising, since it would seem possible to construct a joint distribution under which at least small cost savings could be obtained. In particular, consider the toy model. There are players who always participate since they bear no participation costs, but their production costs are high and therefore they make high (bad) offers; the proportion of these players is p. And there is a group $(1 - p)$ of players who do not participate because they have to incur participation costs; these players have production cost of zero and therefore would make very good (low) offers if they participated. Under these conditions it is reasonable to think that attracting the non participants by paying a small flat reimbursement to all players should pay off. We can make p very small so the "unnecessary" payment made to the σ_u-type players is small in contrast with the large group of low cost players who move in. And yet Theorem 4.2 defies this intuition, suggesting that something important is missing.

Note that to obtain $J(r)$ in the "Toy" model while paying r to each participant, all that is needed is to add $2p \cdot r$ to $J(r)$ in (13). Differentiating the resulting expression with respect to r we obtain

$$\begin{aligned} J'(r) &= 2(r - \Delta - 1 + k_\star) + 2p \\ &= 2(k_\star - 1) + 2p, \quad \text{when evaluated at } r_{\min} = \Delta, \\ &\quad \text{which can be further simplified to produce} \\ &= 2(k_\star - 1 + p/2) + p = 2\Delta + p > 0, \end{aligned} \tag{31}$$

in accord with the result stated in Theorem 4.2.

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Appendix A

A.1. Proof of Theorem 3.1

(i) Let g be a nonnegative continuous solution to (24). Then, it follows that g is continuously differentiable and

$$g'(c) = -\left[1 - \int_0^c P\big(t, \phi\big(g(t)\big)\big)\,\mathrm{d}t\right]^{N-1}, \quad 0 \leqslant c \leqslant 1. \tag{A.1}$$

From (24) we have $g(1) = 0$, and by Assumption 3.3 $\int_0^c P(t, \phi(g(t)))\,\mathrm{d}t < 1$ for all $c \in [0, 1]$; hence, $g'(c)$ is strictly negative on $[0, 1]$, and from (A.1) we see that g' is continuously differentiable on $[0, 1]$. Differentiating (A.1), we get

$$\begin{aligned} g''(c) &= (N-1)\left[1 - \int_0^c P\big(t, \phi\big(g(t)\big)\big)\,\mathrm{d}t\right]^{N-2} P\big(c, \phi\big(g(c)\big)\big) \\ &= (N-1)\big[-g'(c)\big]^{(N-2)/(N-1)} P\big(c, \phi\big(g(c)\big)\big) \\ &\quad \text{[we have used (A.1)].} \end{aligned} \tag{A.2}$$

Thus, g satisfies the differential equation in (25). It remains to verify that g satisfies the boundary conditions stated in (25), which follows since $g(1) = 0$ by (24) and $g'(0) = -1$ by (A.1). □

(ii) First, let us replace in (25) the function $g(c)$ by $\gamma(s) = g(1 - s)$; then, all we need is to prove that the problem

$$\begin{aligned} &\text{(a)} \quad \gamma''(s) = (N-1)\big[\gamma'(s)\big]^{(N-2)/(N-1)} P\big(1 - s, \phi\big(\gamma(s)\big)\big), \\ &\text{(b)} \quad \gamma(0) = 0, \\ &\text{(c)} \quad \gamma'(1) = 1 \end{aligned} \tag{A.3}$$

has a solution which is nonnegative on $[0, 1]$ and possesses a positive derivative at 0, and that such a solution is unique. Note that this claim requires a delicate proof since the standard existence and uniqueness theorems in ordinary differential equations deal with the Cauchy problem (initial conditions for (A.3(a)) rather than the boundary conditions in (A.3)). Our plan is to demonstrate that there are *uniquely defined* initial conditions for (A.3(a)) which enforce the required boundary conditions (A.3(b),(c)), so that the existence and the uniqueness of a solution to (A.3) will finally follow from the standard existence and uniqueness results for the Cauchy problem. As a byproduct, we shall see how to specify numerically the required initial conditions for (A.3(a)), which will yield an algorithm for computing $g(\cdot)$.

Consider the domain

$$G = \big\{(s, y_0, y_1) \in \mathbf{R}^3 \,\big|\, 0 \leqslant s \leqslant 1,\ y_1 > 0\big\}$$

along with the Cauchy problem

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s} y_0(s) = y_1(s), \\ &\frac{\mathrm{d}}{\mathrm{d}s} y_1(s) = (N-1)\big[y_1(s)\big]^{(N-2)/(N-1)} Q\big(s, y_0(s)\big), \\ &y_0(0) = 0, \\ &y_1(0) = \theta > 0 \end{aligned} \tag{A.4}$$

where $Q(s, y_0)$ is a continuously differentiable bounded function in $\{0 \leqslant s \leqslant 1, -\infty < y_0 < \infty\}$ which coincides with $P(1 - s, \phi(y_0))$ for $y_0 \geqslant 0$. By standard considerations, for every $\theta > 0$ the Cauchy problem (A.4) has a unique solution defined on certain part $[0, d(\theta))$ of the segment $[0, 1]$ and such that the limit, $\lim_{s \to d(\theta) - 0} y(s) \equiv y^* \equiv (y_0^*, y_1^*)$ exists, and the point $(d(\theta), y^*)$

belongs to the boundary of G. We claim that $d(\theta) = 1$, $y_1^* > 0$, so that (A.4) has a unique solution on $[0, 1]$, and the graph of this solution belongs to G.

By standard arguments from the theory of differential equations, to prove our claim is the same as to demonstrate that $y_1^* > 0$. Assume to the contrary, that $y_1^* = 0$ (note that $y_1^* \geqslant 0$ by origin of this quantity). Let us start with the observation that $y_0(c)$ is nonnegative on $[0, d(\theta))$. Indeed, $y_0(c)$ is strictly positive on $(0, \epsilon)$ for some positive ϵ, due to the fact that $y_0'(0) = y_1 > 0$ and $y_0(0) = 0$. Therefore, assuming that $y_0(s)$ is negative for some $s \in [0, d(\theta))$, we would conclude that there exists $s^* \in (0, d(\theta))$ such that $y_0(s)$ is positive on $(0, s^*)$, while $y_0(0) = y_0(s^*) = 0$. But since y_0 is positive on $(0, s^*)$ and $y_1(s) = y_0'(s)$ is positive on this interval, (A.4) says that $y_0(s)$ is convex on $[0, s^*]$; since $y_0(0) = 0$ and $y_0'(0) = \theta > 0$, this convex function is positive and strictly increasing in $(0, s^*)$ and cannot, therefore, vanish at s^*.

Thus, $y_0(s)$ is positive on $(0, d(\theta))$. But then (A.4) says that $y_1(s)$ is strictly increasing on $[0, d(\theta))$; since $\theta > 0$, we conclude that $y_1^* = \lim_{s \to d(\theta)-0} y_1(s) > 0$, as claimed. We have proved the following

Lemma A.1. *Whenever $\theta > 0$, the Cauchy problem* (A.4) *has a unique solution $y^\theta(s) = (y_0^\theta(s), y_1^\theta(s))$ defined on* $[0, 1]$, *and for* $0 \leqslant s \leqslant 1$,

$$\begin{aligned} & y_0^\theta(s) \geqslant 0; \\ & \frac{\mathrm{d}}{\mathrm{d}s} y_0(s) \equiv y_1^\theta(s) > 0, \\ & \frac{\mathrm{d}^2}{\mathrm{d}s^2} y_0^\theta(s) \equiv \frac{\mathrm{d}}{\mathrm{d}s} y_1^\theta(s) \geqslant 0. \end{aligned} \tag{A.5}$$

Note also that from the fact that the graph of $y^\theta(\cdot)$ belongs to G and from the smoothness assumptions on the density of F, it follows that $y^\theta(s)$ and $\frac{\mathrm{d}}{\mathrm{d}s} y^\theta(s)$ are continuously differentiable functions of $s \in [0, 1]$, $\theta > 0$.

We proceed by showing that a solution to the Cauchy problem (A.4) that corresponds to certain uniquely defined θ, constitutes a unique solution to (A.3). To this effect, we need the following lemma:

Lemma A.2. *Both $y_0^\theta(s)$ and $y_1^\theta(s)$ are strictly increasing in $\theta > 0$ for every $s \in (0, 1]$, and* $\frac{\partial}{\partial \theta} y_1^\theta(1) > 0$.

Proof. As we have just mentioned, the derivative of y^θ with respect to θ exists; by standard arguments from the theory of differential equations, this derivative $\delta^\theta(s) = (\delta_0^\theta(s), \delta_1^\theta(s))$ solves the Cauchy problem

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}s} \delta_0^\theta(s) = \delta_1^\theta(s); \\ & \frac{\mathrm{d}}{\mathrm{d}s} \delta_1^\theta(s) = (N-2)\big[y_1^\theta(s)\big]^{-1/(N-1)} \delta_1^\theta(s) Q\big(s, y_0^\theta(s)\big) \\ & \qquad\qquad + (N-1)\big[y_1^\theta(s)\big]^{(N-2)/(N-1)} q\big(s, y_0^\theta(s)\big) \delta_0^\theta(s), \\ & q(s, y) = \frac{\partial}{\partial y} Q(s, y); \\ & \delta_0^\theta(0) = 0; \\ & \delta_1^\theta(0) = 1. \end{aligned} \tag{A.6}$$

To prove Lemma A.2 is exactly the same as to demonstrate that both components of the solution to this problem are positive on $(0, 1]$. Let us first show that $\delta_0^\theta(s)$ is positive for $s > 0$. Indeed, $\delta_0^\theta(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}s}\delta_0^\theta(0) = 1$, so that there exists $\epsilon > 0$ such that $\delta_0^\theta(s)$ is positive on $(0, \epsilon)$. Assuming that δ_0^θ is nonpositive somewhere on $(0, 1]$, we would conclude that there exists $s^* \in (0, 1]$ such that $\delta_0^\theta(s)$ is positive on $(0, s^*)$ and is zero at $s = s^*$. Now, due to its origin $q(s, y)$ is nonnegative whenever $y \geqslant 0$; consequently, the second relation in (A.6) says that $\delta_1^\theta(s) = \frac{\mathrm{d}}{\mathrm{d}s}\delta_0^\theta(s)$ is nonnegative on $[0, s^*]$. We see that $\delta_0^\theta(\cdot)$ is nondecreasing on $[0, s^*]$, which contradicts the fact that $\delta_0^\theta(\cdot)$, being positive on $(0, s^*)$, vanishes at both $s = 0$ and $s = s^*$.

Thus, $\delta_0^\theta(s)$ is positive on $(0, 1]$. This fact combined with nonnegativity of $q(s, y)$, positivity of $\delta_1^\theta(0)$ and the second relation in (A.6) immediately implies that $\delta_1^\theta(s)$ is a nondecreasing positive function on $[0, 1]$. □

Now we can prove that (A.3) indeed possesses a unique nonnegative solution with positive derivative at 0. This is exactly the same as to verify that there exists a unique $\theta > 0$ satisfying the relation

$$y_1^\theta(1) = 1; \tag{A.7}$$

the solution to (A.3) is nothing but the component $y_0^\theta(s)$ of the solution, corresponding to the above θ, of the Cauchy problem (A.4).

Thus, all we need to prove is that (A.7) possesses a positive root and this root is unique. Uniqueness is given by Lemma A.2—the left-hand side of our equation is strictly increasing in $\theta > 0$. In order to prove the existence of the root, it suffices to show that the value of the left-hand side in (A.7) is < 1 for small enough positive θ, and is > 1 for large enough positive θ (recall that the left-hand side in (A.7) is continuous, and even continuously differentiable in $\theta > 0$). Now, from (A.5) we know that $y_1^\theta(s)$ is nondecreasing in s and consequently $y_1^\theta(1) \geqslant y_1^\theta(0) = \theta$; thus, the left-hand side in (A.7) is indeed > 1 whenever $\theta > 1$. It remains to prove that the left-hand side in (A.7) is < 1 for all small enough positive θ. To see this, let us set

$$L(s) = Q\big(s, y_0^1(s)\big);$$

note that L is nonnegative, and in view of Assumptions 3.2 and 3.3 combined with the fact that $y_0^1(0) = 0$ we have

$$\int_0^1 L(s)\,\mathrm{d}s = \int_0^1 P\big(1 - s, \phi\big(y_0^1(s)\big)\big)\,\mathrm{d}s < 1. \tag{A.8}$$

Now, for $\theta \in (0, 1)$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\big[y_1^\theta(s)\big]^{1/(N-1)} &= \left[\frac{\mathrm{d}}{\mathrm{d}s}y_1^\theta(s)\right]\big[(N-1)\big[y_1^\theta(s)\big]^{(N-2)/(N-1)}\big]^{-1} \\
&= Q\big(s, y_0^\theta(s)\big) \\
&\quad \text{[see the second relation in (A.4)]} \\
&\leqslant L(s) \\
&\quad \text{[since } 0 \leqslant y_0^\theta(s) \leqslant y_0^1(s) \text{ by Lemmas A.1, A.2];}
\end{aligned}$$

hence

$$\left[y_1^\theta(1)\right]^{1/(N-1)} \leqslant \int_0^1 L(s)\,\mathrm{d}s + \left[y_1^\theta(0)\right]^{1/(N-1)} = \int_0^1 L(s)\,\mathrm{d}s + \theta^{1/(N-1)}.$$

This relation combined with (A.8) implies $y_1^\theta(1) < 1$ for all small enough positive θ, as claimed.

We have proved that the boundary problem (25) indeed possesses a unique solution of the required type, and that in such a solution g' is negative (see Lemma A.1). To complete the proof, it remains to verify that such a solution solves (24) as well, which is immediate:

$$\begin{aligned}
&g''(c) = (N-1)\left[-g'(c)\right]^{(N-2)/(N-1)} P\big(c, \phi\big(g(c)\big)\big),\\
&g(1) = 0,\\
&g'(0) = -1\\
&\Rightarrow\\
&\frac{\mathrm{d}}{\mathrm{d}c}\left[-g'(c)\right]^{1/(N-1)} = -P\big(c, \phi\big(g(c)\big)\big),\\
&-g'(0) = 1,\\
&g(1) = 0\\
&\Rightarrow\\
&g'(c) = -\left[1 - \int_0^c P\big(u, \phi\big(g(u)\big)\big)\,\mathrm{d}u\right]^{N-1},\\
&g(1) = 0\\
&\Rightarrow\\
&g(c) = \int_c^1 \left[1 - \int_0^u P\big(t, \phi\big(g(t)\big)\big)\,\mathrm{d}t\right]^{N-1} \mathrm{d}u.
\end{aligned}$$

Remark A.1. Note that our procedure implies an algorithm for constructing the equilibrium participation rule numerically. Indeed, given θ and integrating numerically the Cauchy problem (A.4), we can compute the left-hand side of (A.7) at this θ; applying further bisection in $\theta \in [0, 1]$, we can easily approximate to a high accuracy the root of (A.7). Integrating (A.4) for the resulting θ, we end up with a tight approximation of the equilibrium participation rule $g(\cdot)$.

A.2. Proof of Theorem 2.1

Note that the distribution function induced by the random variables in the "Toy" model does not satisfy Assumptions 3.1 and 3.2 made in Theorem 3.1. To use the results established in Theorem 3.1 in the case of the "Toy" model in order to assure uniqueness and provide a rigorous proof of other results considered there, we show (see Theorem A.1) that we can establish a very close approximation of the distribution used in the "Toy" model, let it be called F_0, by distributions F_δ satisfying Assumptions 3.1–3.2 and weakly converging to F_0 as $\delta \to +0$, which allows to use the results established in Theorem 3.1.

Assume, that F_δ is as follows:

- with probability p, (c,k) belongs to the narrow strip S_δ $\{0 \leqslant c \leqslant 1, 0 \leqslant k \leqslant \delta\}$, and the corresponding conditional density is smooth and does not depend on c;
- with probability $1-p$, (c,k) takes values in a small rectangle

$$D_\delta = \{(c,k) \mid 0 \leqslant c \leqslant \delta, |k-k_*| \leqslant \delta\}.$$

The resulting conditional density is continuously differentiable, and the distributions F_δ, $\delta > 0$, along with our linear reimbursement policy do satisfy Assumptions 3.1–3.3.

Let $p_\delta(c,k)$ be the density of F_δ, and let $g_r^\delta(c)$ be the solution to (25) corresponding to $F = F_\delta$ and $R(k) = rk$. Let us set

$$h_r^\delta(c) = \phi(g_r^\delta(c)) = g_r^\delta(c)/(1-r), \qquad \pi_r^\delta = -(g_r^\delta)'(1);$$

then, (25) can be rewritten as

$$\begin{aligned} &\text{(a)} \quad (g_r^\delta)''(c) = P_\delta(c, h_r^\delta(c)),\\ &\text{(b)} \quad g_r(1) = 0,\\ &\text{(c)} \quad \pi_r^\delta + \int_0^1 P_\delta(c, h_r^\delta(c))\,\mathrm{d}c = 1, \end{aligned} \tag{A.9}$$

where, $P_\delta(c,k) = \int_0^k p_\delta(c,v)\,\mathrm{d}v$.

The following Theorem stipulates that as $\delta \to +0$, the functions $g_r^\delta(\cdot)$ converge to explicitly given functions $g_r(\cdot)$.

Theorem A.1. *Assume that $\Delta > 0$. Then, as $\delta \to +0$, for every $r \in [0,1)$ the functions $g_r^\delta(c)$ converge to functions $g_r(c)$ as follows*:

$$\begin{aligned} &0 \leqslant r \leqslant r_{\min} \equiv \frac{\Delta}{k_*}:\\ &g_r(c) = \frac{p}{2}(1-c)^2 + (1-p)(1-c);\\ &r_{\min} \leqslant r \leqslant r_{\max} \equiv \frac{1-p+\Delta}{k_*}:\\ &g_r(c) = \frac{p}{2}(1-c)^2 + \Big[(1-r)k_* - \frac{p}{2}\Big](1-c);\\ &r_{\max} \leqslant r < 1:\\ &g_r(c) = \frac{p}{2}(1-c)^2. \end{aligned} \tag{A.10}$$

The convergence is uniform in $c \in [0,1]$ and uniform along with the first order derivative on every segment $c \in [\epsilon, 1]$ with $\epsilon > 0$.

Proof of Theorem A.1. (a) Let $\epsilon^\delta(r)$ be the F_δ-mass of the part of the rectangle S_δ which is above the graph of h_r^δ. We claim that

$$\epsilon^\delta(r) \to 0 \quad \text{uniformly in } r \in [0,1) \text{ as } \delta \to +0. \tag{A.11}$$

Assume to the contrary, that there exist sequences $\delta_i \to +0$ and $r_i \in [0,1)$ such that $\epsilon^{\delta_i}(r_i) \geqslant \epsilon > 0$. Then,

$$\pi_{r_i}^{\delta_i} \to 0, \quad i \to \infty. \tag{A.12}$$

Indeed, if the quantities $\pi_{r_i}^{\delta_i}$, i.e., the minus slopes at $c = 1$ of the *convex* functions $g_{r_i}^{\delta_i}$, were bounded away from 0, then the F_δ-mass of the part of S_δ above the graph of $g_{r_i}^{\delta_i}$ (and therefore the mass above the graph of $h_{r_i}^{\delta_i} \geqslant g_{r_i}^{\delta_i}$) would be small for small δ_i, while we have assumed that it is not the case: $\epsilon^{\delta_i}(r_i) \geqslant \epsilon > 0$.

Since

$$\pi_{r_i}^{\delta_i} + \int_0^1 P_\delta\big(c, h_{r_i}^{\delta_i}(c)\big)\,\mathrm{d}c = 1$$

(see (A.9)), it follows from (A.12) that

$$\lim_{i\to\infty} \int_0^1 P_\delta\big(c, h_{r_i}^{\delta_i}(c)\big)\,\mathrm{d}c \to 1.$$

The latter clearly implies $\epsilon^{\delta_i}(r_i) \to 0$, and we come to a contradiction.

As an immediate corollary of (A.11), we get the relation

$$\liminf_{\delta\to+0} \int_0^1 P_\delta\big(c, h_r^{\delta}(c)\big)\,\mathrm{d}c \geqslant p \quad \forall r \in [0, 1). \tag{A.13}$$

(b) From (a), (A.9) and the structure of F_δ one can easily derive the following

Lemma A.3. *Let $r \in [0, 1)$, and let $\delta_i \to +0$. Assume that the sequence $\pi^{\delta_i}(r)$ has a limit π as $i \to \infty$. Then $\pi \in [0, 1 - p]$, and the functions $g_r^{\delta_i}(\cdot)$ converge uniformly on $[0, 1]$ and uniformly along with the first order derivative on $[\epsilon, 1]$ for every $\epsilon > 0$, to the function*

$$g^{(\pi)}(c) = \frac{p}{2}(1-c)^2 + \pi(1-c).$$

(c) Let us fix $r \in [0, 1)$, and let π be a limiting point of π_r^δ as $\delta \to +0$:

$$\pi = \lim_{i\to\infty} \pi_r^{\delta_i}$$

for a sequence $\delta_i \to +0$. In view of Lemma A.3, the functions $g_r^{\delta_i}(\cdot)$ converge uniformly on $[0, 1]$ to $g^{(\pi)}$, whence the functions $h_r^{\delta_i}(\cdot)$ converge uniformly on $[0, 1]$ to $h^{(\pi)} = g^{(\pi)}/(1 - r)$. Note that by (c) in (A.9 we have

$$\pi + \lim_{i\to\infty} \int_0^1 P_{\delta_i}\big(c, h_r^{\delta_i}(c)\big)\,\mathrm{d}c = 1. \tag{A.14}$$

Now, from (A.13) it is clear that the limit in the left-hand side of (A.14) is always $\geqslant p$. From the structure of F_δ it follows that this limit is equal to p when $h^{(\pi)}(0) < k_*$ and is equal to 1 when $h^{(\pi)}(0) > k_*$. Thus, we see that π should satisfy at least one of the following 3 relations:

$$\begin{aligned} &(\mathrm{A}) \quad h^{(\pi)}(0) < k_* \quad \text{and} \quad \pi + p = 1,\\ &(\mathrm{B}) \quad h^{(\pi)}(0) = k_* \quad \text{and} \quad \pi + p \leqslant 1,\\ &(\mathrm{C}) \quad h^{(\pi)}(0) > k_* \quad \text{and} \quad \pi = 0. \end{aligned} \tag{A.15}$$

It is easily seen that (A.15) implies, first, that π is uniquely defined by r (which, in view of Lemma A.3, means that $g_r^\delta(\cdot)$ indeed converges as $\delta \to 0$ in the sense required by Theorem A.1), and, second, that the functions $g_r(\cdot) = \lim_{\delta\to+0} g_r^\delta(\cdot)$ are exactly as claimed in the theorem. Specifically, from (A)–(C) it follows that $\pi \leqslant 1-p$; this observation immediately implies that if $r \leqslant r_{\min}$, then the only possible are (A) and {(B) and $\pi = 1-p$}, so that we always have $\pi = 1-p$ and g_r is as in the first relation in (A.10). Similarly, if $r \geqslant r_{\max}$, then from (A)–(C) it follows that the only possible are (C) and {(B) and $\pi = 0$}, so that we always have $\pi = 0$, and g_r is as in the third relation in (A.10). Finally, if $r_{\min} < r < r_{\max}$, then the only possible case is (B), which immediately leads to the second relation in (A.10). □

The theorem demonstrates that representing the singular distribution $F = F_0$ as a limit of a natural family of "nice" distributions F_δ, we get "approximate equilibria" g_r^δ which have limits as $\delta \to +0$. These limits, *by definition*, are the equilibria corresponding to the distribution F. As it is easily seen, the convergence established in the Theorem implies that for every linear reimbursement policy $R(k) = rk$ the quantities $R_\delta(r)$, $J_\delta(r)$ and $W_\delta(r)$ associated with distributions F_δ converge, as $\delta \to 0$, to certain limits, $R(r)$, $J(r)$, $W(r)$, that *by definition*, are, respectively, the expected cost of reimbursement, the expected total cost of the auctioneer and the expected payments to the winner, induced by the distribution F and the piecewise linear reimbursement rule.

A.3. Derivation of $W(r)$

Let $\widehat{C}_i$, $i = 1, \ldots, N$, be independent random variables distributed according to Lemma 3.1. Let us approximate the random variables $\widehat{C}_i$, $i = 1, \ldots, N$, by independent continuously distributed random variables with density $h_\epsilon(c)$, being equal to

$$h(c) = P\big(c, \phi\big(g(c)\big)\big)$$

for $0 \leqslant c \leqslant 1$ and vanishing to the right of $1+\epsilon$. We get

$$W = \lim_{\epsilon\to 0} W_\epsilon,$$

$$W_\epsilon = N(N-1)\int_G c_2\left[\prod_1^N h_\epsilon(c_i)\right]\mathrm{d}c,$$

$$G = \big\{c = (c_1, \ldots, c_N) \geqslant 0\colon\ c_1 \leqslant c_2 \leqslant \min[c_3, \ldots, c_N]\big\}$$

$$\Rightarrow \quad W_\epsilon = N(N-1)\int_0^\infty c_2 h_\epsilon(c_2)\left[\int_0^{c_2} h_\epsilon(u)\,\mathrm{d}u\right]\left[\int_{c_2}^\infty h_\epsilon(v)\,\mathrm{d}v\right]^{N-2}\mathrm{d}c_2$$

$$= -N(N-1)\int_0^1 s\big[1 - H_\epsilon(s)\big]H_\epsilon^{N-2}(s)H_\epsilon'(s)\,\mathrm{d}s$$

$$\left[H_\epsilon(s) = \int_s^\infty h_\epsilon(u)\,\mathrm{d}u\right]$$

$$= -\int_0^\infty s\left[N\frac{\mathrm{d}}{\mathrm{d}s}H_\epsilon^{N-1}(s) - (N-1)\frac{\mathrm{d}}{\mathrm{d}s}H_\epsilon^N(s)\right]\mathrm{d}s$$

$$= \int_0^\infty \left[NH_\epsilon^{N-1}(s) - (N-1)H_\epsilon^N(s)\right]\mathrm{d}s.$$

Hence

$$W = \int_0^1 \left[NH^{N-1}(s) - (N-1)H^N(s)\right]\mathrm{d}s,$$

$$H(s) = \int_s^1 P\big(c, \phi\big(g(c)\big)\big)\,\mathrm{d}c + \left[1 - \int_0^1 P\big(c, \phi\big(g(c)\big)\big)\,\mathrm{d}c\right]$$

$$= 1 - \int_0^s P\big(c, \phi\big(g(c)\big)\big)\,\mathrm{d}c.$$

Taking into account that (see (25))

$$P\big(c, \phi\big(g(c)\big)\big) = g''(c)\big[(N-1)\big[-g'(c)\big]^{(N-2)/(N-1)}\big]^{-1} = -\frac{d}{dc}\big[\big[-g'(c)\big]^{1/(N-1)}\big],$$

we get

$$\int_0^s P\big(c, \phi\big(g(c)\big)\big)\,\mathrm{d}c = \big[-g'(0)\big]^{1/(N-1)} - \big[-g'(s)\big]^{1/(N-1)} = 1 - \big[-g'(s)\big]^{1/(N-1)};$$

hence

$$H(s) = \big[-g'(s)\big]^{1/(N-1)}$$

and finally,

$$W = \int_0^1 \big[N\big[-g'(s)\big] - (N-1)\big[-g'(s)\big]^{N/(N-1)}\big]\,\mathrm{d}s$$

$$= N\big[g(0) - g(1)\big] - (N-1)\int_0^1 \big[-g'(s)\big]^{N/(N-1)}\,\mathrm{d}s$$

$$= Ng(0) - (N-1)\int_0^1 \big[-g'(s)\big]^{N/(N-1)}\,\mathrm{d}s$$

$$[\text{since } g(1) = 0 \text{ by (25)}], \tag{A.16}$$

as stated in (28). □

A.4. Proof of Theorem 4.1

In what follows we write $a(\epsilon)=\Omega(b(\epsilon))$, $a(\epsilon)\leqslant\Omega(b(\epsilon))$, $a(\epsilon)\geqslant\Omega(b(\epsilon))$ if there exist constants $l, L>0$ such that, respectively,

$$\begin{aligned} &lb(\epsilon)\leqslant a(\epsilon)\leqslant Lb(\epsilon),\\ &0\leqslant a(\epsilon)\leqslant Lb(\epsilon),\\ &0\leqslant lb(\epsilon)\leqslant a(\epsilon) \end{aligned} \tag{A.17}$$

for all small enough values of $\epsilon\geqslant 0$. We use the usual notation $O(b(\epsilon))$ for a function $a(\epsilon)$ such that $|a(\epsilon)|\leqslant\Omega(b(\epsilon))$.

To make the proof more friendly to the reader we split it into two parts. In the first part, we derive the theorem from two lemmas, which in turn are proved in the (more technical) second part of the proof.

Let us denote by $g_\epsilon(\cdot)$ the equilibrium corresponding to the reimbursement policy $R_\epsilon(\cdot)$ (see (26)), and let $J(\epsilon)$ be auctioneer's total cost associated with this reimbursement. Note that $g_0(\cdot)$ is exactly the same as the equilibrium participation rule $g(\cdot)$ for the case of no reimbursement, since the reimbursement $R_0(k)$ vanishes when $k\leqslant g(0)$, so that *on the graph of* $g(\cdot)$ the right hand sides of differential equations (25) specifying $g(\cdot)$ and $g_0(\cdot)$, respectively, are equal to each other. It is convenient to conduct the analysis in terms of the functions $f(t)=g(1-t)$.

1⁰. The equilibrium equation (25) reads

$$\begin{aligned} &f_\epsilon''(t)=\Pi\big(1-t,\phi_\epsilon\big(f_\epsilon(t)\big)\big)\big[f_\epsilon'(t)\big]^{(N-2)/(N-1)},\\ &f_\epsilon(0)=0,\\ &f_\epsilon'(1)=1. \end{aligned} \tag{A.18}$$

where, $f_\epsilon(t)=g_\epsilon(1-t)$, $\Pi(c,k)=(N-1)P(c,k)$ and,

$$\phi_\epsilon(s)=\begin{cases} s & \text{if } s\leqslant k^*(\epsilon),\\ \frac{s-rk_*(\epsilon)}{1-r} & \text{if } s\geqslant k^*(\epsilon). \end{cases} \tag{A.19}$$

The expression for auctioneer's cost, (29), under an ϵ reimbursement, is

$$\begin{aligned} J(\epsilon)&=\underbrace{N\int\limits_{0\leqslant k\leqslant\phi_\epsilon(f_\epsilon(t))} R_\epsilon(k)p(1-t,k)\,\mathrm{d}k\,\mathrm{d}t}_{A(\epsilon)}+Nf_\epsilon(1)-(N-1)\int_0^1\big(f_\epsilon'(t)\big)^{N/(N-1)}\,\mathrm{d}t\\ &=A(\epsilon)+\int_0^1\big[Nf_\epsilon'(t)-(N-1)\big[f_\epsilon'(t)\big]^{N/(N-1)}\big]\,\mathrm{d}t\\ &=A(\epsilon)+\int_0^1 G\big(f_\epsilon'(t)\big)\,\mathrm{d}t,\\ G(t)&=Nt-(N-1)t^{N/(N-1)}. \end{aligned} \tag{A.20}$$

Note that $A(\epsilon)$ stands for reimbursement payments and $\int_0^1 G(f'_\epsilon(t))\,\mathrm{d}t$ is the payment to the winning bidder.

Let

$$\Delta(\epsilon) = J(\epsilon) - J(0) = A(\epsilon) + \big(B(\epsilon) - B(0)\big) \tag{A.21}$$

where

$$B(0) - B(\epsilon) = \int_0^1 \big[G\big(f'_0(t)\big) - G\big(f'_\epsilon(t)\big)\big]\,\mathrm{d}t, \tag{A.22}$$

is the decrease in payment to the winning bidder, and

$$A(\epsilon) = N \int_{\Xi(\epsilon)} R_\epsilon(k) p(1-t,k)\,\mathrm{d}t\,\mathrm{d}k, \quad \Xi(\epsilon) = \big\{(t,k)\colon\ f_\epsilon(t) \geqslant k \geqslant k^*(\epsilon)\big\}. \tag{A.23}$$

We must prove that $\Delta(\epsilon)$ is strictly negative for ϵ small enough and positive. This is an immediate consequence of the following pair of lemmas:

Lemma A.4. *Reimbursement payments are at most of order of* ϵ^3:

$$A(\epsilon) \leqslant O\big(\epsilon^3\big). \tag{A.24}$$

Lemma A.5. *The decrease in expected payment to the winning bidder,* $B(0) - B(\epsilon)$*, is at least of order of* ϵ^2:

$$B(0) - B(\epsilon) \geqslant \Omega\big(\epsilon^2\big). \tag{A.25}$$

What remains is to prove Lemmas A.4 and A.5. This is the ultimate goal of the reasoning to follow.

$\mathbf{2^0}$. Let

$$\delta(\epsilon) = f'_0(0) - f'_\epsilon(0).$$

It can be shown that the equilibrium participation rule corresponding to a nontrivial reimbursement (the one resulting in $\phi(s) \geqslant s$ for all $s \geqslant 0$) is pointwise $\leqslant$ the equilibrium participation rule corresponding to the "no reimbursement" policy. Consequently,

$$f_\epsilon(t) \leqslant f_0(t), \quad 0 \leqslant t \leqslant 1; \quad \delta(\epsilon) \geqslant 0. \tag{A.26}$$

$\mathbf{3^0}$. Let $T(\epsilon)$ be the solution to the equation

$$f_0(t) = k^*(\epsilon) \quad \big[\equiv f_0(0) - \epsilon\big]. \tag{A.27}$$

Since f_0 is twice continuously differentiable and $f'_0(1) = 1$, we have

$$1 - T(\epsilon) = \epsilon + O\big(\epsilon^2\big). \tag{A.28}$$

$\mathbf{4^0}$. We need the following technical lemma.

Lemma A.6. *One has*

$$\text{(i)}\quad \delta(\epsilon) \leqslant O(\epsilon^2);$$
$$\text{(ii)}\quad \left|f_\epsilon^{(\ell)}(t) - f_0^{(\ell)}(t)\right| \leqslant O(\delta(\epsilon)), \quad 0 \leqslant t \leqslant T(\epsilon),\ \ell = 0, 1, 2;$$
$$\text{(iii)}\quad \left|f_\epsilon^{(\ell)}(t) - f_0^{(\ell)}(t)\right| \leqslant O(\delta(\epsilon) + \epsilon^{3-\ell}), \quad T(\epsilon) \leqslant t \leqslant 1,\ \ell = 0, 1, 2, \tag{A.29}$$

where, as usual, $f^{(\ell)}(t)$ is ℓth derivative of function f.

Proof. $\mathbf{a^0}$. By (A.26), on the segment $0 \leqslant t \leqslant T(\epsilon)$, i.e., before $f_0(t)$ "enters" the domain where the reimbursement $R_\epsilon(\cdot)$ is nontrivial, the functions $f_0(\cdot)$ and $f_\epsilon(\cdot)$ are solutions to the same differential equation of the second order with Lipschitz continuous right-hand side, the initial conditions for the solutions differing by $\leqslant O(\delta(\epsilon))$, which implies (A.29.ii).

$\mathbf{b^0}$. We need the following fact:

Lemma A.7. *The function $f_0'(t) - f_\epsilon'(t)$ is nonnegative and nondecreasing on* $[0, T(\epsilon)]$.

Proof of Lemma A.7. There is nothing to prove if $\delta(\epsilon) = 0$ (see (A.29.ii)). Now assume that $\delta(\epsilon) > 0$. Let us first demonstrate that $f_\epsilon'(t) \leqslant f_0'(t)$, $0 \leqslant t \leqslant T(\epsilon)$. Assuming the opposite, there exists $\bar{t} \in (0, T(\epsilon)]$ such that $f_\epsilon'(t) < f_0'(t)$ for $0 \leqslant t < \bar{t}$, while $f_\epsilon'(\bar{t}) = f_0'(\bar{t})$. Consequently, there exists $t' \in (0, \bar{t})$ such that

$$\begin{aligned} 0 &= f_0''(t') - f_\epsilon''(t') \\ &= \Pi(1 - t', f_0(t'))[f_0'(t')]^{(N-2)/(N-1)} - \Pi(1 - t', f_\epsilon(t'))[f_\epsilon'(t')]^{(N-2)/(N-1)}, \end{aligned} \tag{A.30}$$

which is impossible, since $f_\epsilon(t') < f_0(t')$ (note that $f_0(0) = f_\epsilon(0)$ and $0 \leqslant f_\epsilon'(t') < f_0'(t')$). Since $0 \leqslant f_\epsilon'(t) \leqslant f_0'(t)$, $0 \leqslant f_\epsilon(t) \leqslant f_0(t)$ for $0 \leqslant t \leqslant T(\epsilon)$ and due to

$$f''(t) = \Pi(1 - t, f(t))[f'(t)]^{(N-2)/(N-1)}, \quad 0 \leqslant t \leqslant T(\epsilon) \tag{A.31}$$

for both $f = f_0$ and $f = f_\epsilon$, we see that $f_\epsilon''(t) \leqslant f_0''(t)$, $0 \leqslant t \leqslant T(\epsilon)$. It remains to recall that $f_0(0) = f_\epsilon(0) = 0$. □

$\mathbf{c^0}$. In view of Lemma A.7 we have

$$f_0'(T(\epsilon)) - f_\epsilon'(T(\epsilon)) \geqslant \delta(\epsilon) \quad [\equiv f_0'(0) - f_\epsilon'(0)]. \tag{A.32}$$

The latter observation combined with (A.29.ii) implies that

$$f_0'(T(\epsilon)) - f_\epsilon'(T(\epsilon)) = \Omega(\delta(\epsilon)). \tag{A.33}$$

Now, on the segment $T(\epsilon) \leqslant t \leqslant 1$, in view of the corresponding equilibrium equations (A.18) and (A.29.ii) one has

$$\begin{aligned} &\left|f_\epsilon''(t) - f_0''(t)\right| \\ &= \left|\Pi(1 - t, \phi_\epsilon(f_\epsilon(t)))[f_\epsilon'(t)]^{(N-2)/(N-1)} - \Pi(1 - t, f_0(t))[f_0'(t)]^{(N-2)/(N-1)}\right| \\ &\leqslant \left|[\Pi(1 - t, \phi_\epsilon(f_\epsilon(t))) - \Pi(1 - t, f_0(t))][f_\epsilon'(t)]^{(N-2)/(N-1)}\right| \\ &\quad + \left|\Pi(1 - t, f_0(t))[[f_\epsilon'(t)]^{(N-2)/(N-1)} - [f_0'(t)]^{(N-2)/(N-1)}]\right| \end{aligned}$$

$$\begin{aligned}
&\leqslant O(1)\big[\big|\phi_\epsilon(f_\epsilon(t)) - f_0(t)\big| + \big|f'_\epsilon(t) - f'_0(t)\big|\big] \\
&\leqslant O(1)\big[\big|\phi_\epsilon\big(f_\epsilon(t)\big) - \phi_\epsilon\big(f_0(t)\big)\big| + \underbrace{\big|\phi_\epsilon\big(f_0(t)\big) - f_0(t)\big|}_{=\frac{r[f_0(t)-f_0(T(\epsilon))]}{1-r}\leqslant O(\epsilon)} + \big|f'_\epsilon(t) - f'_0(t)\big|\big] \\
&\leqslant O(1)\big|f_\epsilon(t) - f_0(t)\big| + \big|f'_\epsilon(t) - f'_0(t)\big| + O(\epsilon),
\end{aligned} \tag{A.34}$$

which combined with (A.29.ii) and (A.28) implies (A.29.iii).

$\mathbf{d^0}$. By (A.29.iii) applied with $\ell = 2$ and in view of $f'_0(1) = 1 = f'_\epsilon(1)$ we have

$$T(\epsilon) \leqslant t \leqslant 1 \quad \Rightarrow \quad \big|f'_\epsilon(t) - f'_0(t)\big| \leqslant O\big(\delta(\epsilon) + \epsilon\big)\big(1 - T(\epsilon)\big), \tag{A.35}$$

so that by (A.28) it holds

$$T(\epsilon) \leqslant t \leqslant 1 \quad \Rightarrow \quad \big|f'_\epsilon(t) - f'_0(t)\big| \leqslant O\big(\delta(\epsilon)\epsilon + \epsilon^2\big); \tag{A.36}$$

the latter relation (as applied with $t = T(\epsilon)$) combined with (A.33) implies (A.29.i). The proof of Lemma A.6 is completed. □

$\mathbf{5^0}$. Let $\pi = (N-1)p(0, f_0(0))$. Since $p(\cdot,\cdot)$ is continuously differentiable and in view of (A.29) we have

$$\left.\begin{array}{c} T(\epsilon) \leqslant t \leqslant 1 \\ \min[f_0(t), \phi_\epsilon(f_\epsilon(t))] \leqslant k \leqslant \max[f_0(t), \phi_\epsilon(f_\epsilon(t))] \end{array}\right\}
\Rightarrow \quad \big|(N-1)p(1-t,k) - \pi\big| \leqslant O(\epsilon). \tag{A.37}$$

For $T(\epsilon) \leqslant t \leqslant 1$ we have

$$\begin{aligned}
f''_0(t) &= \Pi\big(1-t, f_0(t)\big)\big[f'_0(t)\big]^{(N-2)/(N-1)}, \\
f''_\epsilon(t) &= \Pi\big(1-t, \phi_\epsilon\big(f_\epsilon(t)\big)\big)\big[f'_\epsilon(t)\big]^{(N-2)/(N-1)},
\end{aligned} \tag{A.38}$$

whence by (A.37), (A.29) and (A.28) for the indicated t one has

$$\begin{aligned}
&f''_\epsilon(t) - f''_0(t) \\
&\quad = \Pi\big(1-t, \phi_\epsilon\big(f_\epsilon(t)\big)\big)\big[\big[f'_\epsilon(t)\big]^{(N-2)/(N-1)} - \big[f'_0(t)\big]^{(N-2)/(N-1)}\big] \\
&\qquad + \underbrace{\big[\Pi\big(1-t, \phi_\epsilon\big(f_\epsilon(t)\big)\big) - \Pi\big(1-t, f_0(t)\big)\big]}_{\substack{=\pi[\phi_\epsilon(f_\epsilon(t)) - f_0(t)] + a_\epsilon(t), \\ |a_\epsilon(t)| \leqslant O(\epsilon^3)}} \\
&\qquad + \big[\Pi\big(1-t, \phi_\epsilon\big(f_\epsilon(t)\big)\big) - \Pi\big(1-t, f_0(t)\big)\big]\big[\big[f'_0(t)\big]^{(N-2)/(N-1)} - 1\big] \\
&\quad = \pi\big[\phi_\epsilon\big(f_\epsilon(t)\big) - f_0(t)\big] + d_\epsilon(t), \\
&|d_\epsilon(t)| \leqslant O(1)\big[\delta(\epsilon) + \epsilon^2 + \big|a_\epsilon(t)\big| + \epsilon\pi\big|\phi_\epsilon(f_\epsilon(t)) - f_0(t)\big| + \epsilon\big|a_\epsilon(t)\big|\big] \\
&\qquad = \pi\big[\phi_\epsilon\big(f_\epsilon(t)\big) - f_0(t)\big] + O(1)\big[\epsilon^2 + \epsilon\pi\big|\phi_\epsilon(f_\epsilon(t)) - f_0(t)\big|\big].
\end{aligned} \tag{A.39}$$

Further, by (A.29) we have for $T(\epsilon) \leqslant t \leqslant 1$:

$$\begin{aligned}
&\pi\big[\phi_\epsilon\big(f_\epsilon(t)\big) - f_0(t)\big] = \pi\big[\phi_\epsilon\big(f_0(t)\big) - f_0(t)\big] + e_\epsilon(t), \\
&\qquad \big|e_\epsilon(t)\big| \equiv \pi\big|\phi_\epsilon\big(f_0(t)\big) - \phi_\epsilon\big(f_\epsilon(t)\big)\big| \leqslant O\big(\epsilon^2\big),
\end{aligned} \tag{A.40}$$

and

$$\phi_\epsilon\big(f_0(t)\big) - f_0(t) = \frac{f_0(t) - rk_*(\epsilon)}{1-r} - f_0(t) = \frac{r[f_0(t) - k^*(\epsilon)]}{1-r}$$
$$= \frac{r[f_0(t) - f_0(T(\epsilon))]}{1-r} = \frac{r(t - T(\epsilon))}{1-r} + v_\epsilon(t),$$
$$\big|v_\epsilon(t)\big| \leqslant O(\epsilon^2) \tag{A.41}$$

(we have used the fact that $|f_0'(t) - 1| \leqslant O(1)(1-t)$ due to $f_0'(1) = 1$ and that $1 - T(\epsilon) \leqslant O(\epsilon)$ by (A.28)). Combining the latter relation with (A.39), (A.40), we finally get

$$T(\epsilon) \leqslant t \leqslant 1$$
$$\Rightarrow \quad \begin{cases} f_\epsilon''(t) - f_0''(t) = \kappa(t - T(\epsilon)) + \gamma_\epsilon(t), & \text{(i)} \\ \Pi(1-t, \phi_\epsilon(f_\epsilon(t))) - \Pi(1-t, f_0(t)) = \kappa[t - T(\epsilon)] + \tilde{\gamma}_\epsilon(t), & \text{(ii)} \end{cases}$$
$$\kappa = \frac{\pi r}{1-r},$$
$$\big|\gamma_\epsilon(t)\big| + \big|\tilde{\gamma}_\epsilon(t)\big| \leqslant O\big(\epsilon^2\big), \tag{A.42}$$

6^0. We need the following

Lemma A.8. *One has*

$$\delta(\epsilon) = \frac{\alpha(\epsilon)}{\beta'(1)} + \bar{o}(\epsilon^2), \quad \alpha(\epsilon) = \frac{\kappa\epsilon^2}{2} > 0. \tag{A.43}$$

Here $\kappa > 0$ is given by (A.42) *and, as usual, $\bar{o}(\epsilon^\ell)$ denotes a function $a(\epsilon)$ such that $a(\epsilon)/\epsilon^\ell \to 0$ as $\epsilon \to +0$.*

Proof. From (A.42) and (A.28) it follows that

$$0 = f_\epsilon'(1) - f_0'(1) = f_\epsilon'\big(T(\epsilon)\big) - f_0'\big(T(\epsilon)\big) + \frac{\kappa(1 - T(\epsilon))^2}{2} + O\big(\epsilon^3\big), \tag{A.44}$$

i.e., that

$$f_\epsilon'\big(T(\epsilon)\big) - f_0'\big(T(\epsilon)\big) = -\frac{\kappa\epsilon^2}{2} + O\big(\epsilon^3\big). \tag{A.45}$$

On the other hand, on the segment $[0, T(\epsilon)]$ both $f_\epsilon(\cdot)$, $f_0(\cdot)$ are solutions to the same differential equation

$$f''(t) = \Pi\big(1-t, f(t)\big)\big[f'(t)\big]^{(N-2)/(N-1)}, \tag{A.46}$$

the initial conditions for these solutions being $(f(0) = 0, f'(0) = f_0'(0) - \delta(\epsilon) > 0)$ and $(f(0) = 0, f'(0) = f_0'(0) > 0)$, respectively. It follows that if

$$\Phi(t) = (N-1)p\big(1-t, f_0(t)\big)\big[f_0'(t)\big]^{(N-2)/(N-1)},$$
$$\Psi(t) = \frac{N-2}{N-1}\Pi\big(1-t, f_0(t)\big)\big[f_0'(t)\big]^{-1/(N-1)}, \tag{A.47}$$

and $\beta(t)$ is the solution to the system

$$\begin{aligned} &\beta''(t) = \Phi(t)\beta(t) + \Psi(t)\beta'(t),\\ &\beta(0) = 0,\\ &\beta'(0) = 1, \end{aligned} \tag{A.48}$$

then

$$\begin{aligned} &0 \leqslant t \leqslant T(\epsilon) \quad \Rightarrow \quad f_\epsilon^{(\ell)}(t) = f_0^{(\ell)}(t) - \delta(\epsilon)\beta^{(\ell)}(t) + \theta_\epsilon^{(\ell)}(t), \quad \ell = 0, 1, 2,\\ &|\theta_\epsilon^{(\ell)}(t)| \leqslant o(1)\delta(\epsilon) \underbrace{=}_{\text{(A.29.i)}} o(\epsilon^2), \quad \ell = 0, 1, 2. \end{aligned} \tag{A.49}$$

From (A.49) it follows that

$$f'_\epsilon\big(T(\epsilon)\big) - f'_0\big(T(\epsilon)\big) = -\delta(\epsilon)\beta'\big(T(\epsilon)\big) + \bar{o}\big(\epsilon^2\big) = -\delta(\epsilon)\beta'(1) + \bar{o}\big(\epsilon^2\big) \tag{A.50}$$

(we have used (A.28)). Comparing (A.45) and (A.50), we come to (A.43). □

7⁰. Now we are ready to prove Lemma A.4. Indeed, by (A.26), (A.29) and the definition of $T(\epsilon)$ the domain of integration $\Xi(\epsilon)$ in the expression defining $A(\epsilon)$ is contained in the rectangle $\{T(\epsilon) \leqslant t \leqslant 1,\ f_0(T(\epsilon)) \leqslant k \leqslant f_0(T(\epsilon)) + u(\epsilon)\}$ with $u(\epsilon) \leqslant O(\epsilon)$. The area of this rectangle does not exceed $O(\epsilon^2)$, the integrand $R_\epsilon(k)$ in the rectangle does not exceed $O(\epsilon)$, and (A.24) follows.

8⁰. Now let us prove Lemma A.5. The function $G(s)$ clearly is nondecreasing on $[0, 1]$, and its derivative is positive on $[0, 1)$. Taking into account Lemma A.7, we have

$$\begin{aligned} B(\epsilon) - B(0) &= \int_0^{T(\epsilon)} \big[G\big(f'_\epsilon(t)\big) - G\big(f'_0(t)\big)\big]\,\mathrm{d}t + \underbrace{\int_{T(\epsilon)}^{1} \big[G\big(f'_\epsilon(t)\big) - G\big(f'_0(t)\big)\big]\,\mathrm{d}t}_{\leqslant O(\epsilon^3) \text{ by (A.28),(A.29),(A.36)}}\\ &\leqslant \int_0^{T(\epsilon)} \big[G\big(f'_0(t) - \delta(\epsilon)\big) - G\big(f'_0(t)\big)\big]\,\mathrm{d}t + O(\epsilon^3)\\ &\leqslant -\Omega\big(\delta(\epsilon)\big) + O\big(\epsilon^3\big). \end{aligned} \tag{A.51}$$

Combining this observation with (A.43), we arrive at (A.25). Lemma A.5 is proved, and this completes the proof of Theorem 4.1. □

A.5. *Proof of Theorem 4.2*

The proof is that $J'(r) \geqslant 0$ for all $r \geqslant 0$. Let us fix a nonnegative r and denote by $g_r(c)$ the equilibrium participation rule associated with the reimbursement rule $R(\cdot) \equiv r$. Let further

$$\begin{aligned} &\gamma(s) = \gamma_r(s) \equiv g_r(1-s),\\ &\delta(s) = \frac{\partial}{\partial r}\gamma_r(s),\\ &\Delta(s) = \delta(s) + 1. \end{aligned}$$

Note that (29) in the case of $R(k) \equiv r$ results in

$$J(r) = Nr \int_0^{\phi(g(c))} \mathrm{d}F(c,k) + Ng(0) - (N-1)\int_0^1 \left[-g'(c)\right]^{N/(N-1)} \mathrm{d}c. \tag{A.52}$$

From (A.52) combined with $\gamma_r(0) \equiv 0$ (see (25)),

$$\begin{aligned}
J'(r) &= N\int_0^1 P\big(1-s, \gamma(s)+r\big)\,\mathrm{d}s + Nr\int_0^1 p\big(1-s,\gamma(s)+r\big)\big[\delta(s)+1\big]\,\mathrm{d}s \\
&\quad + \int_0^1 \big[N\delta'(s) - N\big[\gamma'(s)\big]^{1/(N-1)}\delta'(s)\big]\,\mathrm{d}s \\
&= Nr\int_0^1 p\big(1-s,\gamma(s)+r\big)\Delta(s)\,\mathrm{d}s + N\int_0^1 \gamma''(s)\big[(N-1)\big[\gamma'(s)\big]^{(N-2)/(N-1)}\big]^{-1}\,\mathrm{d}s \\
&\quad + N\int_0^1 \big[1-\big[\gamma'(s)\big]^{1/(N-1)}\big]\Delta'(s)\,\mathrm{d}s \\
&\qquad \text{[we have used (A.3)]} \\
&= \left\{Nr\int_0^1 p(1-s,\gamma(s)+r)\Delta(s)\,\mathrm{d}s\right\}_1 \\
&\quad + N\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\big[\gamma'(s)\big]^{1/(N-1)}\,\mathrm{d}s + N\int_0^1\big[1-\big[\gamma'(s)\big]^{1/(N-1)}\big]\Delta'(s)\,\mathrm{d}s \\
&= \{\cdot\}_1 + N\left\{1-\big[\gamma'(0)\big]^{1/(N-1)} + \int_0^1 \big[1-\big[\gamma'(s)\big]^{1/(N-1)}\big]\Delta'(s)\,\mathrm{d}s\right\}_2 \\
&\qquad \text{[recall that } \gamma'(1)=1 \text{ by (A.3)].}
\end{aligned} \tag{A.53}$$

It can be shown that

$$\begin{aligned}
\delta''(s) &= (N-2)\big[\gamma'(s)\big]^{-1/(N-1)}\delta'(s)P\big(1-s,\gamma(s)+r\big) \\
&\quad + (N-1)\big[\gamma'(s)\big]^{(N-2)/(N-1)} p\big(1-s,\gamma(s)+r\big)\big[\delta(s)+1\big], \\
\delta(0) &= 0, \\
\delta'(1) &= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Delta''(s) &= \mu(s)\Delta'(s) + \nu(s)\Delta(s), \\
&\qquad [\mu(\cdot), \nu(\cdot) \geqslant 0] \\
\Delta(0) &= 1, \\
\Delta'(1) &= 0.
\end{aligned} \tag{A.54}$$

We claim that $\Delta(s)$ is nonnegative and $\Delta'(s)$ is nonpositive on [0, 1]. Let us start from the first claim. Assume, on the contrary, that $\Delta(s) < 0$ for some $s \in (0, 1)$. Since $\Delta(0) = 1 > 0$, then there would exist a point $s^* \in (0, 1)$ such that both $\Delta(s)$ and $\Delta'(s)$ are negative at $s = s^*$. But then the differential equation in (A.54) would imply that Δ is negative, and Δ' is negative and nonincreasing to the right of s^*, which contradicts the boundary condition $\Delta'(1) = 0$. Thus, we know that $\Delta(s)$ is nonnegative on [0, 1]. When assuming that $\Delta'(s^*) > 0$ for some s^*, we conclude from the differential equation in (A.54) that $\Delta'(s)$ is increasing and positive to the right of s^*, which again contradicts to the equality $\Delta'(1) = 0$. Thus, $\Delta'(s) \leqslant 0$ on [0, 1]. Now, since Δ is nonnegative, the term $\{\cdot\}_1$ in (A.53) is nonnegative. To see that the term $\{\cdot\}_2$ in this expression is also nonnegative, note that since $\Delta'(s) \leqslant 0$, we have

$$\begin{aligned}\int_0^1 \bigl[1 - \bigl[\gamma'(s)\bigr]^{1/(N-1)}\bigr]\Delta'(s)\,\mathrm{d}s &\geqslant \max_{0\leqslant s\leqslant 1}\bigl[1 - \bigl[\gamma'(s)\bigr]^{1/(N-1)}\bigr]\int_0^1 \Delta'(s)\,\mathrm{d}s \\ &= \bigl[1 - \bigl[\gamma'(0)\bigr]^{1/(N-1)}\bigr]\bigl[\Delta(1) - \Delta(0)\bigr] \\ &\quad \text{[recall that } \gamma'(s) \text{ is nondecreasing and positive on [0, 1]]} \\ &= \bigl[1 - \bigl[\gamma'(0)\bigr]^{1/(N-1)}\bigr]\bigl[\Delta(1) - 1\bigr].\end{aligned}$$

Hence,

$$\{\cdot\}_2 \geqslant \bigl[1 - \bigl[\gamma'(0)\bigr]^{1/(N-1)}\bigr]\Delta(1) \geqslant 0.$$

Thus, $J'(r) \geqslant 0$. □

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