Lectures on Robust Convex Optimization

Arkadi Nemirovski

School of Industrial and Systems Engineering
Georgia Institute of Technology

Optimization & Applications Seminar Fall 2009

Organized Jointly by ETH Zurich and University of Zurich

Subject. The data of optimization problems of real world origin typically is *uncertain* - not known exactly when the problem is solved. With the traditional approach, "small" (fractions of percents) data uncertainty is merely ignored, and the problem is solved as if the nominal data — our *guesses* for the actual data — were identical to the actual data. However, experiments demonstrate that already pretty small perturbations of uncertain data can make the nominal (i.e., corresponding to the nominal data) optimal solution heavily infeasible and thus practically meaningless. For example, in 13 of 90 LP programs from the NETLIB library, 0.01% random perturbations of uncertain data lead to more than 50% violations of some of the constraints as evaluated at the nominal optimal solutions. Thus, in applications there is an actual need in a methodology which produces *robust*, "immunized against uncertainty" solutions. Essentially, the only traditional methodology of this type is offered by Stochastic Programming, where one assigns data perturbations a probability distribution and replaces the original constraints with their "chance versions", imposing on a candidate solution the requirement to satisfy the constraints with probability $\geq 1-\epsilon$, $\epsilon \ll 1$ being a given tolerance. In many cases, however, there is no natural way to assign the data perturbations with a probability distribution; besides this, Chance Constrained Stochastic Programming typically is computationally intractable – aside of a small number of special cases, chance constrained versions of simple – just linear – constraints are nonconvex and difficult to verify, which makes optimization under these constraints highly problematic.

Robust Optimization can be viewed as a complementary to Stochastic Programming approach to handling optimization problems with uncertain data. Here one uses "uncertain-but-bounded" model of data perturbations, allowing the uncertain data to run through a given *uncertainty set*, and imposes on a candidate solution the requirement to be *robust feasible* – to satisfy the constraints whatever be a realization of the data from this set. Assuming that the objective is certain (i.e., is not affected by data perturbations; in fact, this assumption does not restrict generality), one then looks for the *robust optimal* solution — a robust feasible solution with as small value of the objective as possible. With this approach, one associates with the original uncertain problem its *Robust Counterpart* – the problem of building the robust optimal solution. Originating from Soyster (1973) and completely ignored for over two decades after its birth, the Robust Optimization was "reborn" circa 1997 and during the last decade became one of the most rapidly developing areas in Optimization. The mini-course in question is aimed at overview of basic concepts and recent developments in this area.

The contents. Our course will be focused on the basic theory of Robust Optimization, specifically, on

- Motivation and detailed presentation of the Robust Optimization paradigm, including in-depth investigation of the outlined notion of the Robust Counterpart of an uncertain optimization problem and its recent extensions (Adjustable and Globalized Robust Counterparts);

- *Computational tractability* of Robust Counterparts. In order to be a working tool rather than wishful thinking, the RC (which by itself is a specific optimization problem) should be efficiently solvable. This, as a minimum, requires efficient solvability of every certain instance of the uncertain problem in question; to meet this requirement, we restrict ourselves in our course with uncertain *conic* optimization, specifically, with uncertain Linear, Conic Quadratic and Semidefinite Optimization problems. Note, however, that tractability of instances is necessary, but by far not sufficient for the RC to be tractable. Indeed, the RC of an uncertain conic problem is a *semi-infinite* program: every conic constraint of the original problem gives rise to *infinitely many* "commonly structured" conic constraints parameterized by the data running through the uncertainty set. It turns out that the tractability of the RC of an uncertain conic problem depends on interplay between the geometries of the underlying cones and uncertainty sets. It will be shown that "uncertain Linear Optimization is tractable" — the RC of an uncertain LO is tractable whenever the uncertainty set is so; this is the major good news about Robust Optimization and its major advantage as compared to Stochastic Programming. In contrast to this, the tractability of the RC of an uncertain Conic Quadratic or Semidefinite problem is a "rare commodity;" the related goal of the course is to overview a number of important cases where the RCs of uncertain Conic Quadratic/Semidefinite problems are tractable or admit "tight", in certain precise sense, tractable approximations.

- Links with Chance Constrained Linear/Conic Quadratic/Semidefinite Optimization. As it was already mentioned, chance constrained versions of randomly perturbed optimization problems, even as simple as Linear Programming ones, usually are computationally intractable. It turns out that Robust Optimization offers an attractive way to build "safe," in certain natural sense, tractable approximations of chance constrained LO/CQO/SDO problems. As a result, information on the stochastic properties of data perturbations, when available, allows to build meaningful (and often highly nontrivial) uncertainty sets.

Prerequisites. Participants are expected to possess basic mathematical culture and to know the most elementary facts from Linear Algebra, Convex Optimization and Probabilities; all more specific and more advanced facts we intend to use (Conic Duality, Semidefinite Relaxation, Concentration Inequalities,...) will be explained in the course.

Textbook. The course is more than covered by the book A. Ben-Tal, L. El Ghaoui, A. Nemirovski, *Robust Optimization*, Princeton University Press, 2009 (freely available at
http://sites.google.com/site/robustoptimization
We hope to provide the participants, in an on-line fashion, with self-contained Lecture Notes.

Contents

Lecture 1

Robust Linear Optimization: Motivation, Concepts, Tractability

In this lecture, we introduce the concept of the uncertain Linear Optimization problem and its Robust Counterpart, and study the computational issues associated with the emerging optimization problems.

1.1 Data Uncertainty in Linear Optimization

1.1.1 Linear Optimization Problem, its data and structure

The Linear Optimization (LO) *problem* is

$$\min_x \left\{ c^T x + d : Ax \leq b \right\};\tag{1.1.1}$$

here $x \in \mathbb{R}^n$ is the vector of *decision variables*, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ form the *objective*, A is an $m \times n$ *constraint matrix*, and $b \in \mathbb{R}^m$ is the *right hand side vector*.

Clearly, the constant term d in the objective, while affecting the optimal value, does not affect the optimal solution, this is why it is traditionally skipped. When treating the LO problems with *uncertain data* there are good reasons not to neglect this constant term.

When speaking about optimization (or whatever other) problems, we usually distinguish between problems's *structure* and problems's *data*. When asked "what is the data of the LO problem (1.1.1)," everybody will give the same answer: "the data of the problem are the collection (c, d, A, b)." . As about the structure of (1.1.1), it, given the form in which we write the problem down, is specified by the number m of constraints and the number n of variables.

Usually not all constraints of an LO program, as it arises in applications, are of the form $a^T x \leq$ const; there can be linear "$\geq$" inequalities and linear equalities as well. Clearly, the constraints of the latter two types can be represented equivalently by linear "$\leq$" inequalities, and we will assume henceforth that these are the only constraints in the problem.

1.1.2 Data uncertainty: sources

Typically, the data of real world LOs is not known exactly when the problem is being solved. The most common reasons for data uncertainty are as follows:

- Some of data entries (future demands, returns, etc.) do not exist when the problem is solved and hence are replaced with their forecasts. These data entries are thus subject to *prediction errors*;

- Some of the data (parameters of technological devices/processes, contents associated with raw materials, etc.) cannot be measured exactly, and their true values drift around the measured "nominal" values; these data are subject to *measurement errors*;

- Some of the decision variables (intensities with which we intend to use various technological processes, parameters of physical devices we are designing, etc.) cannot be implemented exactly as computed. The resulting *implementation errors* are equivalent to appropriate artificial data uncertainties.

  Indeed, the contribution of a particular decision variable x_j to the left hand side of constraint i is the product $a_{ij}x_j$. A typical implementation error can be modeled as $x_j \mapsto (1+\xi_j)x_j + \eta_j$, where ξ_j is the multiplicative, and η_j is the additive component of the error. The effect of this error is *as if* there were no implementation error at all, but the coefficient a_{ij} got the multiplicative perturbation: $a_{ij} \mapsto a_{ij}(1+\xi_j)$, and the right hand side b_i of the constraint got the additive perturbation $b_i \mapsto b_i - \eta_j a_{ij}$.

1.1.3 Data uncertainty: dangers

In the traditional LO methodology, a small data uncertainty (say, 0.1% or less) is just ignored; the problem is solved *as if* the given ("nominal") data were exact, and the resulting *nominal* optimal solution is what is recommended for use, in hope that small data uncertainties will not affect significantly the feasibility and optimality properties of this solution, or that small adjustments of the nominal solution will be sufficient to make it feasible. In fact these hopes are not necessarily justified, and sometimes even small data uncertainty deserves significant attention. We are about to present two instructive examples of this type.

Motivating example I: Synthesis of Antenna Arrays.

Consider a monochromatic transmitting antenna placed at the origin. Physics says that

1. The directional distribution of energy sent by the antenna can be described in terms of *antenna's diagram* which is a complex-valued function $D(\delta)$ of a 3D direction δ. The directional distribution of energy sent by the antenna is proportional to $|D(\delta)|^2$.

2. When the antenna is comprised of several antenna elements with diagrams $D_1(\delta)$,..., $D_k(\delta)$, the diagram of the antenna is just the sum of the diagrams of the elements.

In a typical Antenna Design problem, we are given several antenna elements with diagrams $D_1(\delta)$,...,$D_n(\delta)$ and are allowed to multiply these diagrams by complex *weights* x_i (which in reality corresponds to modifying the output powers and shifting the phases of the elements). As a result, we can obtain, as a diagram of the array, any function of the form

$$D(\delta) = \sum_{j=1}^{n} x_j D_j(\delta),$$

and our goal is to find the weights x_j which result in a diagram as close as possible, in a prescribed sense, to a given "target diagram" $D_*(\delta)$.

Example 1.1 Antenna Design Consider a planar antenna comprised of a central circle and 9 concentric rings of the same area as the circle (figure 1.1.a) in the XY-plane ("Earth's surface"). Let the wavelength be $\lambda = 50$cm, and the outer radius of the outer ring be 1 m (twice the wavelength).

One can easily see that the diagram of a ring $\{a \le r \le b\}$ in the plane XY (r is the distance from a point to the origin) as a function of a 3-dimensional direction δ depends on the altitude (the angle θ between the direction and the plane) only. The resulting function of θ turns out to be *real-valued*, and its analytic expression is

$$D_{a,b}(\theta) = \frac{1}{2}\int\limits_a^b \left[\int\limits_0^{2\pi} r\cos\left(2\pi r\lambda^{-1}\cos(\theta)\cos(\phi)\right)d\phi\right]dr.$$

Fig. 1.1.b represents the diagrams of our 10 rings for $\lambda = 50$cm.

Assume that our goal is to design an array with a real-valued diagram which should be axial symmetric with respect to the Z-axis and should be "concentrated" in the cone $\pi/2 \ge \theta \ge \pi/2 - \pi/12$. In other words, our target diagram is a real-valued function $D_*(\theta)$ of the altitude θ with $D_*(\theta) = 0$ for $0 \le \theta \le \pi/2 - \pi/12$ and $D_*(\theta)$ somehow approaching 1 as θ approaches $\pi/2$. The target diagram $D_*(\theta)$ used in this example is given in figure 1.1.c (blue).

Let us measure the discrepancy between a synthesized diagram and the target one by the Tschebyshev distance, taken along the equidistant 240-point grid of altitudes, i.e., by the quantity

$$\tau = \max_{i=1,\dots,240}\left|D_*(\theta_i) - \sum_{j=1}^{10} x_j \underbrace{D_{r_{j-1},r_j}(\theta_i)}_{D_j(\theta_i)}\right|, \quad \theta_i = \frac{i\pi}{480}.$$

Our design problem is simplified considerably by the fact that the diagrams of our "building blocks" and the target diagram are real-valued; thus, we need no complex numbers, and the problem we should finally solve is

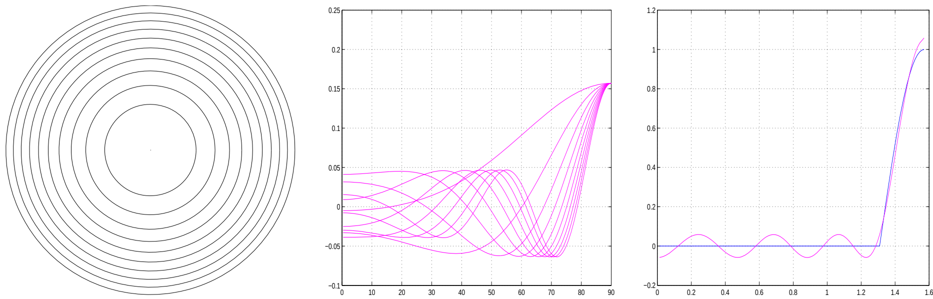
$$\min_{\tau\in\mathbb{R},x\in\mathbb{R}^{10}}\left\{\tau : -\tau \le D_*(\theta_i) - \sum_{j=1}^{10} x_j D_j(\theta_i) \le \tau,\ i = 1,\dots,240\right\}. \tag{1.1.2}$$

This is a simple LP program; its optimal solution x^* results in the diagram depicted at figure 1.1.c (magenta). The uniform distance between the actual and the target diagrams is ≈ 0.0589 (recall that the target diagram varies from 0 to 1).

Now recall that our design variables are characteristics of certain physical devices. In reality, of course, we cannot tune the devices to have precisely the optimal characteristics x_j^*; the best we may hope for is that the actual characteristics $x_j^{\rm fct}$ will coincide with the desired values x_j^* within a small margin ρ, say, $\rho = 0.1\%$ (this is a fairly high accuracy for a physical device):

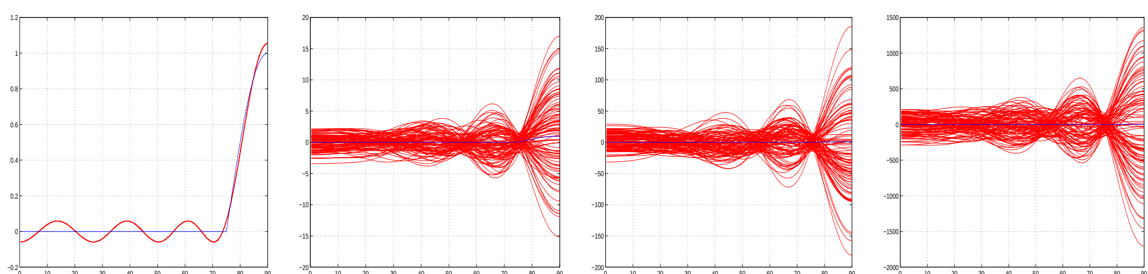
$$x_j^{\rm fct} = (1+\xi_j)x_j^*,\ |\xi_j| \le \rho = 0.001.$$

It is natural to assume that the *actuation errors* ξ_j are random with the mean value equal to 0; it is perhaps not a great sin to assume that these errors are independent of each other. Note that as it was already explained, the consequences of our actuation errors are *as if* there were no actuation errors at all, but the coefficients $D_j(\theta_i)$ of variables x_j in (1.1.2) were subject to perturbations $D_j(\theta_i) \mapsto (1+\xi_j)D_j(\theta_i)$.



(a) (b) (c)

Figure 1.1: Synthesis of antennae array.
(a): 10 array elements of equal areas in the XY-plane; the outer radius of the largest ring is 1m, the wavelength is 50cm.
(b): "building blocks" — the diagrams of the rings as functions of the altitude angle θ.
(c): the target diagram (blue) and the synthesized diagram (magenta).



$\rho = 0$ $\rho = 0.0001$ $\rho = 0.001$ $\rho = 0.01$

Figure 1.2: "Dream and reality," nominal optimal design: samples of 100 actual diagrams (red) for different uncertainty levels. Blue: the target diagram

	Dream	Reality								
	$\rho = 0$	$\rho = 0.0001$			$\rho = 0.001$			$\rho = 0.01$		
	value	min	mean	max	min	mean	max	min	mean	max
$\|\cdot\|_\infty$-distance to target	0.059	1.280	5.671	14.04	11.42	56.84	176.6	39.25	506.5	1484
energy concentration	85.1%	0.5%	16.4%	51.0%	0.1%	16.5%	48.3%	0.5%	14.9%	47.1%

Table 1.1: Quality of nominal antenna design: dream and reality. Data over 100 samples of actuation errors per each uncertainty level ρ.

Since the actual weights differ from their desired values x_j^*, the actual (random) diagram of our array of antennae will differ from the "nominal" one we see on figure 1.1.c. How large could be the difference? Looking at figure 1.2, we see that the difference can be dramatic. The diagrams corresponding to $\rho > 0$ are not even the worst case: given ρ, we just have taken as $\{\xi_j\}_{j=1}^{10}$ 100 samples of 10 independent numbers distributed uniformly in $[-\rho, \rho]$ and have plotted the diagrams corresponding to $x_j = (1 + \xi_j)x_j^*$. Pay attention not only to the shape, but also to the scale (table 1.1): the target diagram varies from 0 to 1, and the nominal diagram (the one corresponding to the exact optimal x_j) differs from the target by no more than by 0.0589 (this is the optimal value in the "nominal" problem (1.1.2)). The data in table 1.1 show that when $\rho = 0.001$, the typical $\|\cdot\|_\infty$ distance between the actual diagram and the target one is by 3 (!) orders of magnitude larger. Another meaningful way, also presented in table 1.1, to understand what is the quality of our design is via *energy concentration* – the fraction of the total emitted energy which "goes up," that is, is emitted along the spatial angle of directions forming angle at most $\pi/12$ with the Z-axis. For the nominal design, *the dream* (i.e., with no actuation errors) energy concentration is as high as 85% – quite respectable, given that the spatial angle in question forms just 3.41% of the entire hemisphere. This high concentration, however, exists only in our imagination, since actuation errors of magnitude ρ as low as 0.01% reduce the average energy concentration (which, same as the diagram itself, now becomes random) to just 16%; the lower 10% quantile of this random quantity is as small as 2.2% – 1.5 times less than the fraction (3.4%) which the "going up" directions form among all directions. The bottom line is that "un reality" our nominal optimal design is completely meaningless.

Motivating example II: NETLIB Case Study

NETLIB includes about 100 not very large LOs, mostly of real-world origin, used as the standard benchmark for LO solvers. In the study to be described, we used this collection in order to understand how "stable" are the feasibility properties of the standard – "nominal" – optimal solutions with respect to small uncertainty in the data. To motivate the methodology of this "case study", here is the constraint # 372 of the problem PILOT4 from NETLIB:

$$\begin{array}{rl} a^T x \equiv & -15.79081x_{826} - 8.598819x_{827} - 1.88789x_{828} - 1.362417x_{829} - 1.526049x_{830} \\ & -0.031883x_{849} - 28.725555x_{850} - 10.792065x_{851} - 0.19004x_{852} - 2.757176x_{853} \\ & -12.290832x_{854} + 717.562256x_{855} - 0.057865x_{856} - 3.785417x_{857} - 78.30661x_{858} \\ & -122.163055x_{859} - 6.46609x_{860} - 0.48371x_{861} - 0.615264x_{862} - 1.353783x_{863} \\ & -84.644257x_{864} - 122.459045x_{865} - 43.15593x_{866} - 1.712592x_{870} - 0.401597x_{871} \\ & +x_{880} - 0.946049x_{898} - 0.946049x_{916} \\ \geq & b \equiv 23.387405 \end{array} \quad \text{(C)}$$

The related *nonzero* coordinates in the optimal solution x^* of the problem, as reported by CPLEX (one of the best commercial LP solvers), are as follows:

$$\begin{array}{lll} x^*_{826} = 255.6112787181108 & x^*_{827} = 6240.488912232100 & x^*_{828} = 3624.613324098961 \\ x^*_{829} = 18.20205065283259 & x^*_{849} = 174397.0389573037 & x^*_{870} = 14250.00176680900 \\ x^*_{871} = 25910.00731692178 & x^*_{880} = 104958.3199274139 & \end{array}$$

The indicated optimal solution makes (C) an equality within machine precision.

Observe that most of the coefficients in (C) are "ugly reals" like -15.79081 or -84.644257. We have all reasons to believe that coefficients of this type characterize certain technological devices/processes, and as such *they could hardly be known to high accuracy*. It is quite natural to assume that the "ugly coefficients" are in fact uncertain – they coincide with the "true" values of the corresponding data within accuracy of 3-4 digits, not more. The only exception is the coefficient `1` of `x880` – it perhaps reflects the structure of the underlying model and is therefore exact – "certain".

Assuming that the uncertain entries of a are, say, 0.1%-accurate approximations of unknown entries of the "true" vector of coefficients $\widetilde{a}$, we looked what would be the effect of this uncertainty on the validity of the "true" constraint $\widetilde{a}^T x \geq b$ at x^*. Here is what we have found:

• The minimum (over all vectors of coefficients $\widetilde{a}$ compatible with our "0.1%-uncertainty hypothesis") value of $\widetilde{a}^T x^* - b$, is < -104.9; in other words, the violation of the constraint can be as large as 450% of the right hand side!

• Treating the above worst-case violation as "too pessimistic" (why should the true values of all uncertain coefficients differ from the values indicated in (C) in the "most dangerous" way?), consider a more realistic measure of violation. Specifically, assume that the true values of the uncertain coefficients in (C) are obtained from the "nominal values" (those shown in (C)) by random perturbations $a_j \mapsto \widetilde{a}_j = (1 + \xi_j)a_j$ with independent and, say, uniformly distributed on $[-0.001, 0.001]$ "relative perturbations" ξ_j. What will be a "typical" relative violation

$$V = \frac{\max[b - \widetilde{a}^T x^*, 0]}{b} \times 100\%$$

of the "true" (now random) constraint $\widetilde{a}^T x \geq b$ at x^*? The answer is nearly as bad as for the worst scenario:

$\text{Prob}\{V > 0\}$	$\text{Prob}\{V > 150\%\}$	$\text{Mean}(V)$
0.50	0.18	125%

Table 2.1. Relative violation of constraint # 372 in `PILOT4`
(1,000-element sample of 0.1% perturbations of the uncertain data)

We see that *quite small (just 0.1%) perturbations of "clearly uncertain" data coefficients can make the "nominal" optimal solution x^* heavily infeasible and thus – practically meaningless.*

A "case study" reported in [8] shows that the phenomenon we have just described is not an exception – in 13 of 90 *NETLIB* Linear Programming problems considered in this study, already 0.01%-perturbations of "ugly" coefficients result in violations of some constraints as evaluated at the nominal optimal solutions by more than 50%. In 6 of these 13 problems the magnitude of constraint violations was over 100%, and in `PILOT4` — "the champion" — it was as large as 210,000%, that is, 7 orders of magnitude larger than the relative perturbations in the data.

The conclusion is as follows:

> *In applications of LO, there exists a real need of a technique capable of detecting cases when data uncertainty can heavily affect the quality of the nominal solution,*

and in these cases to generate a "reliable" solution, one that is immunized against uncertainty.

We are about to introduce the *Robust Counterpart* approach to uncertain LO problems aimed at coping with data uncertainty.

1.2 Uncertain Linear Problems and their Robust Counterparts

1.2.1 Uncertain LO problem

Definition 1.1 *An uncertain Linear Optimization problem is a collection*

$$\left\{\min_x \left\{c^T x + d : Ax \le b\right\}\right\}_{(c,d,A,b)\in\mathcal{U}} \qquad (LO_{\mathcal{U}})$$

of LO problems (instances) $\min_x \left\{c^T x + d : Ax \le b\right\}$ *of common structure* (i.e., with common numbers m of constraints and n of variables) *with the data varying in a given* <u>*uncertainty set*</u> $\mathcal{U} \subset \mathbb{R}^{(m+1)\times(n+1)}$.

We always assume that the uncertainty set is parameterized, in an affine fashion, by *perturbation vector* ζ varying in a given *perturbation set* $\mathcal{Z}$:

$$\mathcal{U} = \left\{ \left[\begin{array}{c|c} c^T & d \\ \hline A & b \end{array}\right] = \underbrace{\left[\begin{array}{c|c} c_0^T & d_0 \\ \hline A_0 & b_0 \end{array}\right]}_{\substack{\text{nominal} \\ \text{data } D_0}} + \sum_{\ell=1}^{L} \zeta_\ell \underbrace{\left[\begin{array}{c|c} c_\ell^T & d_\ell \\ \hline A_\ell & b_\ell \end{array}\right]}_{\substack{\text{basic} \\ \text{shifts } D_\ell}} : \zeta \in \mathcal{Z} \subset \mathbb{R}^L \right\}. \qquad (1.2.1)$$

For example, when speaking about `PILOT4`, we, for the sake of simplicity, tacitly assumed uncertainty only in the constraint matrix, specifically, as follows: every coefficient a_{ij} is allowed to vary, independently of all other coefficients, in the interval $[a^{\rm n}_{ij} - \rho_{ij}|a^{\rm n}_{ij}|, a^{\rm n}_{ij} + \rho_{ij}|a^{\rm n}_{ij}|]$, where $a^{\rm n}_{ij}$ is the nominal value of the coefficient — the one in the data file of the problem as presented in NETLIB, and ρ_{ij} is the perturbation level, which in our experiment was set to 0.001 for all "ugly" coefficients $a^{\rm n}_{ij}$ and was set to 0 for "nice" coefficients, like the coefficient 1 at $\mathtt{x}_{880}$. Geometrically, the corresponding perturbation set is just a box

$$\zeta \in \mathcal{Z} = \{\zeta = \{\zeta_{ij} \in [-1,1]\}_{i,j:a^{\rm n}_{ij} \text{ is ugly}}\},$$

and the parameterization of the a_{ij}-data by the perturbation vector is

$$a_{ij} = \begin{cases} a^{\rm n}_{ij}(1+\zeta_{ij}), & a^{\rm n}_{ij} \text{ is ugly} \\ a^{\rm n}_{ij}, & \text{otherwise} \end{cases}$$

Remark 1.1 *If the perturbation set* $\mathcal{Z}$ *in* (1.2.1) *itself is represented as the image of another set* $\widehat{\mathcal{Z}}$ *under affine mapping* $\xi \mapsto \zeta = p + P\xi$*, then we can pass from perturbations* ζ *to perturbations*

ξ:

$$
\begin{array}{rcl}
\mathcal{U} & = & \left\{\left[\begin{array}{c|c} c^T & d \\ \hline A & b \end{array}\right] = D_0 + \sum\limits_{\ell=1}^{L} \zeta_\ell D_\ell : \zeta \in \mathcal{Z}\right\} \\
& = & \left\{\left[\begin{array}{c|c} c^T & d \\ \hline A & b \end{array}\right] = D_0 + \sum\limits_{\ell=1}^{L} [p_\ell + \sum\limits_{k=1}^{K} P_{\ell k}\xi_k] D_\ell : \xi \in \widehat{\mathcal{Z}}\right\} \\
& = & \left\{\left[\begin{array}{c|c} c^T & d \\ \hline A & b \end{array}\right] = \underbrace{\left[D_0 + \sum\limits_{\ell=1}^{L} p_\ell D_\ell\right]}_{\widehat{D}_0} + \sum\limits_{k=1}^{K} \xi_k \underbrace{\left[\sum\limits_{\ell=1}^{L} P_{\ell k} D_\ell\right]}_{\widehat{D}_k} : \xi \in \widehat{\mathcal{Z}}\right\}.
\end{array}
$$

It follows that when speaking about perturbation sets with simple geometry (parallelotopes, ellipsoids, etc.), we can normalize these sets to be "standard." For example, a parallelotope is by definition an affine image of a unit box $\{\xi \in \mathbb{R}^k : -1 \leq \xi_j \leq 1, j = 1, ..., k\}$*, which gives us the possibility to work with the unit box instead of a general parallelotope. Similarly, an ellipsoid is by definition the image of a unit Euclidean ball* $\{\xi \in \mathbb{R}^k : \|x\|_2^2 \equiv x^T x \leq 1\}$ *under affine mapping, so that we can work with the standard ball instead of the ellipsoid, etc. We will use this normalization whenever possible.*

1.2.2 Robust Counterpart of Uncertain LO

Note that a *family* of optimization problems like ($\mathrm{LO}_\mathcal{U}$), in contrast to a single optimization problem, is not associated by itself with the concepts of feasible/optimal solution and optimal value. How to define these concepts depends on the underlying "decision environment." Here we focus on an environment with the following characteristics:

A.1. All decision variables in ($\mathrm{LO}_\mathcal{U}$) represent "here and now" decisions; they should be assigned specific numerical values as a result of solving the problem *before* the actual data "reveals itself."

A.2. The decision maker is fully responsible for consequences of the decisions to be made when, and only when, the actual data is within the prespecified uncertainty set $\mathcal{U}$ given by (1.2.1).

A.3. The constraints in ($\mathrm{LO}_\mathcal{U}$) are "hard" — we cannot tolerate violations of constraints, even small ones, when the data is in $\mathcal{U}$.

Note that A.1 – A.3 are *assumptions* on our decision environment (in fact, the strongest ones within the methodology we are presenting); while being meaningful, these assumptions in no sense are automatically valid; In the mean time, we shall consider relaxed versions of these assumptions and consequences of these relaxations.

Assumptions A.1 — A.3 determine, essentially in a unique fashion, what are the meaningful, "immunized against uncertainty," feasible solutions to the uncertain problem ($\mathrm{LO}_\mathcal{U}$). By A.1, these should be fixed vectors; by A.2 and A.3, they should be *robust feasible*, that is, they should satisfy all the constraints, whatever the realization of the data from the uncertainty set. We have arrived at the following definition.

Definition 1.2 *A vector* $x \in \mathbb{R}^n$ *is a* <u>*robust feasible*</u> *solution to* ($\mathrm{LO}_\mathcal{U}$)*, if it satisfies all realizations of the constraints from the uncertainty set, that is,*

$$Ax \leq b \quad \forall (c, d, A, b) \in \mathcal{U}. \tag{1.2.2}$$

As for the objective value to be associated with a robust feasible) solution, assumptions A.1 — A.3 do not prescribe it in a unique fashion. However, "the spirit" of these worst-case-oriented assumptions leads naturally to the following definition:

Definition 1.3 *Given a candidate solution x, the robust value $\widehat{c}(x)$ of the objective in* ($\mathrm{LO}_{\mathcal{U}}$) *at x is the largest value of the "true" objective $c^T x + d$ over all realizations of the data from the uncertainty set:*

$$\widehat{c}(x) = \sup_{(c,d,A,b)\in\mathcal{U}} [c^T x + d]. \tag{1.2.3}$$

After we agree what are meaningful candidate solutions to the uncertain problem ($\mathrm{LO}_{\mathcal{U}}$) and how to quantify their quality, we can seek the best robust value of the objective among all robust feasible solutions to the problem. This brings us to the central concept of our methodology, *Robust Counterpart* of an uncertain optimization problem, which is defined as follows:

Definition 1.4 *The Robust Counterpart of the uncertain LO problem* ($\mathrm{LO}_{\mathcal{U}}$) *is the optimization problem*

$$\min_x \left\{ \widehat{c}(x) = \sup_{(c,d,A,b)\in\mathcal{U}} [c^T x + d] : Ax \le b \ \forall (c,d,A,b) \in \mathcal{U} \right\} \tag{1.2.4}$$

of minimizing the robust value of the objective over all robust feasible solutions to the uncertain problem.

An optimal solution to the Robust Counterpart is called a robust optimal solution to ($\mathrm{LO}_{\mathcal{U}}$), *and the optimal value of the Robust Counterpart is called the robust optimal value of* ($\mathrm{LO}_{\mathcal{U}}$).

In a nutshell, the robust optimal solution is simply "the best uncertainty-immunized" solution we can associate with our uncertain problem.

1.2.3 More on Robust Counterparts

We start with several useful observations.

A. The Robust Counterpart (1.2.4) of $\mathrm{LO}_{\mathcal{U}}$ can be rewritten equivalently as the problem

$$\min_{x,t} \left\{ t : \begin{array}{rcl} c^T x - t & \le & -d \\ Ax & \le & b \end{array} \right\} \forall (c,d,A,b) \in \mathcal{U} \right\}. \tag{1.2.5}$$

Note that we can arrive at this problem in another fashion: we first introduce the extra variable t and rewrite instances of our uncertain problem ($\mathrm{LO}_{\mathcal{U}}$) equivalently as

$$\min_{x,t} \left\{ t : \begin{array}{rcl} c^T x - t & \le & -d \\ Ax & \le & b \end{array} \right\},$$

thus arriving at an equivalent to ($\mathrm{LO}_{\mathcal{U}}$) uncertain problem in variables x, t with the objective t that is not affected by uncertainty at all. The RC of the reformulated problem is exactly (1.2.5). We see that

> *An uncertain LO problem can always be reformulated as an uncertain LO problem with certain objective. The Robust Counterpart of the reformulated problem has the same objective as this problem and is equivalent to the RC of the original uncertain problem.*

As a consequence, *we lose nothing when restricting ourselves with uncertain LO programs with certain objectives* and we shall frequently use this option in the future.

> We see now why the constant term d in the objective of (1.1.1) should not be neglected, or, more exactly, should not be neglected if it is uncertain. When d is certain, we can account for it by the shift $t \mapsto t - d$ in the slack variable t which affects only the optimal value, but not the optimal solution to the Robust Counterpart (1.2.5). When d is uncertain, there is no "universal" way to eliminate d without affecting the optimal solution to the Robust Counterpart (where d plays the same role as the right hand sides of the original constraints).

B. Assuming that (LO$_{\mathcal{U}}$) is with certain objective, the Robust Counterpart of the problem is

$$\min_x \left\{c^T x + d : Ax \leq b, \forall (A,b) \in \mathcal{U}\right\} \tag{1.2.6}$$

(note that the uncertainty set is now a set in the space of the constraint data $[A,b]$). We see that

The Robust Counterpart of an uncertain LO problem with a certain objective is a purely "constraint-wise" construction: to get RC, we act as follows:

- *preserve the original certain objective as it is, and*
- *replace every one of the original constraints*

$$(Ax)_i \leq b_i \Leftrightarrow a_i^T x \leq b_i \tag{C$_i$}$$

(a_i^T is i-th row in A) with its Robust Counterpart

$$a_i^T x \leq b_i \ \forall [a_i; b_i] \in \mathcal{U}_i, \tag{RC(C$_i$)}$$

where $\mathcal{U}_i$ is the projection of $\mathcal{U}$ on the space of data of i-th constraint:

$$\mathcal{U}_i = \{[a_i; b_i] : [A,b] \in \mathcal{U}\}.$$

In particular,

The RC of an uncertain LO problem with a certain objective remains intact when the original uncertainty set $\mathcal{U}$ is extended to the direct product

$$\widehat{\mathcal{U}} = \mathcal{U}_1 \times ... \times \mathcal{U}_m$$

of its projections onto the spaces of data of respective constraints.

Example 1.2 The RC of the system of uncertain constraints

$$\{x_1 \geq \zeta_1, \, x_2 \geq \zeta_2\} \tag{1.2.7}$$

with $\zeta \in \mathcal{U} := \{\zeta_1 + \zeta_2 \leq 1, \zeta_1, \zeta_2 \geq 0\}$ is the infinite system of constraints

$$x_1 \geq \zeta_1, \, x_1 \geq \zeta_2 \ \ \forall \zeta \in \mathcal{U};$$

on variables x_1, x_2. The latter system is clearly equivalent to the pair of constraints

$$x_1 \geq \max_{\zeta \in \mathcal{U}} \zeta_1 = 1, \ x_2 \geq \max_{\zeta \in \mathcal{U}} \zeta_2 = 1. \tag{1.2.8}$$

The projections of $\mathcal{U}$ to the spaces of data of the two uncertain constraints (1.2.7) are the segments $\mathcal{U}_1 = \{\zeta_1 : 0 \leq \zeta_1 \leq 1\}$, $\mathcal{U}_2 = \{\zeta_2 : 0 \leq \zeta_2 \leq 1\}$, and the RC of (1.2.7) w.r.t. the uncertainty set $\widehat{\mathcal{U}} = \mathcal{U}_1 \times \mathcal{U}_2 = \{\zeta \in \mathbb{R}^2 : 0 \leq \zeta_1, \zeta_2 \leq 1\}$ clearly is (1.2.8).

The conclusion we have arrived at seems to be counter-intuitive: it says that it is immaterial whether the perturbations of data in different constraints are or are not linked to each other, while intuition says that such a link should be important. We shall see later (lecture 5) that this intuition is valid when a more advanced concept of *Adjustable* Robust Counterpart is considered.

C. If x is a robust feasible solution of (C_i), then x remains robust feasible when we extend the uncertainty set $\mathcal{U}_i$ to its convex hull $\mathrm{Conv}(\mathcal{U}_i)$. Indeed, if $[\bar{a}_i;\bar{b}_i] \in \mathrm{Conv}(\mathcal{U}_i)$, then

$$[\bar{a}_i;\bar{b}_i] = \sum_{j=1}^{J} \lambda_j [a_i^j; b_i^j],$$

with appropriately chosen $[a_i^j; b_i^j] \in \mathcal{U}_i$, $\lambda_j \geq 0$ such that $\sum\limits_j \lambda_j = 1$. We now have

$$\bar{a}_i^T x = \sum_{j=1}^{J} \lambda_j [a_i^j]^T x \leq \sum_j \lambda_j b_i^j = \bar{b}_i,$$

where the inequality is given by the fact that x is feasible for $\mathrm{RC}(\mathrm{C}_i)$ and $[a_i^j; b_i^j] \in \mathcal{U}_i$. We see that $\bar{a}_i^T x \leq \bar{b}_i$ for all $[\bar{a}_i;\bar{b}_i] \in \mathrm{Conv}(\mathcal{U}_i)$, QED.

By similar reasons, the set of robust feasible solutions to (C_i) remains intact when we extend $\mathcal{U}_i$ to the closure of this set. Combining these observations with **B**, we arrive at the following conclusion:

The Robust Counterpart of an uncertain LO problem with a certain objective remains intact when we extend the sets $\mathcal{U}_i$ of uncertain data of respective constraints to their closed convex hulls, and extend $\mathcal{U}$ to the direct product of the resulting sets.

In other words, we lose nothing when assuming from the very beginning that the sets $\mathcal{U}_i$ of uncertain data of the constraints are closed and convex, and $\mathcal{U}$ is the direct product of these sets.

In terms of the parameterization (1.2.1) of the uncertainty sets, the latter conclusion means that

When speaking about the Robust Counterpart of an uncertain LO problem with a certain objective, we lose nothing when assuming that the set $\mathcal{U}_i$ of uncertain data of i-th constraint is given as

$$\mathcal{U}_i = \left\{ [a_i; b_i] = [a_i^0; b_i^0] + \sum_{\ell=1}^{L_i} \zeta_\ell [a_i^\ell; b_i^\ell] : \zeta \in \mathcal{Z}_i \right\}, \qquad (1.2.9)$$

with a closed and convex perturbation set $\mathcal{Z}_i$.

D. An important modeling issue. In the usual — with certain data — Linear Optimization, constraints can be modeled in various equivalent forms. For example, we can write:

$$\begin{array}{ll} (a) & a_1x_1 + a_2x_2 \leq a_3 \\ (b) & a_4x_1 + a_5x_2 = a_6 \\ (c) & x_1 \geq 0, x_2 \geq 0 \end{array} \qquad (1.2.10)$$

or, equivalently,

$$\begin{array}{ll} (a) & a_1x_1 + a_2x_2 \leq a_3 \\ (b.1) & a_4x_1 + a_5x_2 \leq a_6 \\ (b.2) & -a_5x_1 - a_5x_2 \leq -a_6 \\ (c) & x_1 \geq 0, x_2 \geq 0. \end{array} \tag{1.2.11}$$

Or, equivalently, by adding a slack variable s,

$$\begin{array}{ll} (a) & a_1x_1 + a_2x_2 + s = a_3 \\ (b) & a_4x_1 + a_5x_2 = a_6 \\ (c) & x_1 \geq 0, x_2 \geq 0, s \geq 0. \end{array} \tag{1.2.12}$$

However, when (part of) the data $a_1, ..., a_6$ become *uncertain*, not all of these equivalences remain valid: the RCs of our now uncertainty-affected systems of constraints are not equivalent to each other. Indeed, denoting the uncertainty set by $\mathcal{U}$, the RCs read, respectively,

$$\left.\begin{array}{ll} (a) & a_1x_1 + a_2x_2 \leq a_3 \\ (b) & a_4x_1 + a_5x_2 = a_6 \\ (c) & x_1 \geq 0, x_2 \geq 0 \end{array}\right\} \forall a = [a_1; ...; a_6] \in \mathcal{U}. \tag{1.2.13}$$

$$\left.\begin{array}{ll} (a) & a_1x_1 + a_2x_2 \leq a_3 \\ (b.1) & a_4x_1 + a_5x_2 \leq a_6 \\ (b.2) & -a_5x_1 - a_5x_2 \leq -a_6 \\ (c) & x_1 \geq 0, x_2 \geq 0 \end{array}\right\} \forall a = [a_1; ...; a_6] \in \mathcal{U}. \tag{1.2.14}$$

$$\left.\begin{array}{ll} (a) & a_1x_1 + a_2x_2 + s = a_3 \\ (b) & a_4x_1 + a_5x_2 = a_6 \\ (c) & x_1 \geq 0, x_2 \geq 0, s \geq 0 \end{array}\right\} \forall a = [a_1; ...; a_6] \in \mathcal{U}. \tag{1.2.15}$$

It is immediately seen that while the first and the second RCs are equivalent to each other,[1] they are *not* equivalent to the third RC. The latter RC is more conservative than the first two, meaning that whenever (x_1, x_2) can be extended, by a properly chosen s, to a feasible solution of (1.2.15), (x_1, x_2) is feasible for (1.2.13)≡(1.2.14) (this is evident), but not necessarily vice versa. In fact, the gap between (1.2.15) and (1.2.13)≡(1.2.14) can be quite large. To illustrate the latter claim, consider the case where the uncertainty set is

$$\mathcal{U} = \{a = a_\zeta := [1 + \zeta; 2 + \zeta; 4 - \zeta; 4 + \zeta; 5 - \zeta; 9] : -\rho \leq \zeta \leq \rho\},$$

where ζ is the data perturbation. In this situation, $x_1 = 1$, $x_2 = 1$ is a feasible solution to (1.2.13)≡(1.2.14), provided that the uncertainty level ρ is $\leq 1/3$:

$$(1 + \zeta) \cdot 1 + (2 + \zeta) \cdot 1 \leq 4 - \zeta \, \forall (\zeta : |\zeta| \leq \rho \leq 1/3) \ \&(4 + \zeta) \cdot 1 + (5 - \zeta) \cdot 1 = 9 \ \forall \zeta.$$

At the same time, when $\rho > 0$, our solution $(x_1 = 1, x_2 = 1)$ cannot be extended to a feasible solution of (1.2.15), since the latter system of constraints is infeasible and remains so even after eliminating the equality (1.2.15.b).

Indeed, in order for x_1, x_2, s to satisfy (1.2.15.a) for all $a \in \mathcal{U}$, we should have

$$x_1 + 2x_2 + s + \zeta[x_1 + x_2] = 4 - \zeta \ \forall (\zeta : |\zeta| \leq \rho);$$

when $\rho > 0$, we therefore should have $x_1 + x_2 = -1$, which contradicts (1.2.15.c)

[1] Clearly, this always is the case when an equality constraint, certain or uncertain alike, is replaced with a pair of opposite inequalities.

The origin of the outlined phenomenon is clear. Evidently the inequality $a_1x_1+a_2x_2 \leq a_3$, where all a_i and x_i are fixed reals, holds true if and only if we can "certify" the inequality by pointing out a real $s \geq 0$ such that $a_1x_1 + a_2x_2 + s = a_3$. When the data a_1, a_2, a_3 become uncertain, the restriction on (x_1, x_2) to be robust feasible for the uncertain inequality $a_1x_1 + a_2x_2 \leq a_3$ for all $a \in \mathcal{U}$ reads, "in terms of certificate," as

$$\forall a \in \mathcal{U}\, \exists s \geq 0 : a_1x_1 + a_2x_2 + s = a_3,$$

that is, the certificate s should be allowed to depend on the true data. In contrast to this, in (1.2.15) we require from both the decision variables x *and the slack variable ("the certificate")* s to be independent of the true data, which is by far too conservative.

What can be learned from the above examples is that when modeling an uncertain LO problem one should avoid whenever possible converting inequality constraints into equality ones, unless all the data in the constraints in question are certain. Aside from avoiding slack variables,[2] this means that restrictions like "total expenditure cannot exceed the budget," or "supply should be at least the demand," which in LO problems with certain data can harmlessly be modeled by equalities, in the case of uncertain data should be modeled by inequalities. This is in full accordance with common sense saying, e.g., that when the demand is uncertain and its satisfaction is a must, it would be unwise to forbid surplus in supply. Sometimes a good for the RO methodology modeling requires eliminating "state variables" — those which are readily given by variables representing actual decisions — via the corresponding "state equations." For example, time dynamics of an inventory is given in the simplest case by the state equations

$$\begin{array}{l} x_0 = c \\ x_{t+1} = x_t + q_t - d_t,\ t = 0, 1, ..., T, \end{array}$$

where x_t is the inventory level at time t, d_t is the (uncertain) demand in period $[t, t+1)$, and variables q_t represent actual decisions – replenishment orders at instants $t = 0, 1, ..., T$. A wise approach to the RO processing of such an inventory problem would be to eliminate the state variables x_t by setting

$$x_t = c + \sum_{\tau=1}^{t-1} q_\tau,\ t = 0, 1, 2, ..., T+1,$$

and to get rid of the state equations. As a result, typical restrictions on state variables (like "x_t should stay within given bounds" or "total holding cost should not exceed a given bound") will become uncertainty-affected inequality constraints on the actual decisions q_t, and we can process the resulting inequality-constrained uncertain LO problem via its RC.[3]

1.2.4 What is Ahead

After introducing the concept of the Robust Counterpart of an uncertain LO problem, we confront two major questions:

1. What is the "computational status" of the RC? When is it possible to process the RC efficiently?

2. How to come-up with meaningful uncertainty sets?

[2]Note that slack variables do not represent actual decisions; thus, their presence in an LO model contradicts assumption A.1, and thus can lead to too conservative, or even infeasible, RCs.

[3]For more advanced robust modeling of uncertainty-affected multi-stage inventory, see lecture 5.

The first of these questions, to be addressed in depth in section 1.3, is a "structural" one: what should be the structure of the uncertainty set in order to make the RC computationally tractable? Note that the RC as given by (1.2.5) or (1.2.6) is a *semi-infinite* LO program, that is, an optimization program with simple linear objective and *infinitely many* linear constraints. In principle, such a problem can be "computationally intractable" — NP-hard.

Example 1.3 Consider an uncertain "essentially linear" constraint

$$\{\|Px-p\|_1 \leq 1\}_{[P;p]\in\mathcal{U}}\,, \tag{1.2.16}$$

where $\|z\|_1 = \sum_j |z_j|$, and assume that the matrix P is certain, while the vector p is uncertain and is parameterized by perturbations from the unit box:

$$p \in \{p = B\zeta : \|\zeta\|_\infty \leq 1\}\,,$$

where $\|\zeta\|_\infty = \max_\ell |\zeta_\ell|$ and B is a given positive semidefinite matrix. To check whether $x=0$ is robust feasible is exactly the same as to verify whether $\|B\zeta\|_1 \leq 1$ whenever $\|\zeta\|_\infty \leq 1$; or, due to the evident relation $\|u\|_1 = \max\limits_{\|\eta\|_\infty\leq 1} \eta^T u$, the same as to check whether $\max\limits_{\eta,\zeta}\left\{\eta^T B\zeta : \|\eta\|_\infty \leq 1, \|\zeta\|_\infty \leq 1\right\} \leq 1$. The maximum of the bilinear form $\eta^T B\zeta$ with positive semidefinite B over η,ζ varying in a convex symmetric neighborhood of the origin is always achieved when $\eta=\zeta$ (you may check this by using the polarization identity $\eta^T B\zeta = \frac{1}{4}(\eta+\zeta)^T B(\eta+\zeta) - \frac{1}{4}(\eta-\zeta)^T B(\eta-\zeta)$). Thus, to check whether $x=0$ is robust feasible for (1.2.16) is the same as to check whether the maximum of a given nonnegative quadratic form $\zeta^T B\zeta$ over the unit box is ≤ 1. The latter problem is known to be NP-hard,[4] and therefore so is the problem of checking robust feasibility for (1.2.16).

The second of the above is a modeling question, and as such, goes beyond the scope of purely theoretical considerations. However, theory, as we shall see in section 2.1, contributes significantly to this modeling issue.

1.3 Tractability of Robust Counterparts

In this section, we investigate the "computational status" of the RC of uncertain LO problem. The situation here turns out to be as good as it could be: we shall see, essentially, that *the RC of the uncertain LO problem with uncertainty set $\mathcal{U}$ is computationally tractable whenever the convex uncertainty set $\mathcal{U}$ itself is computationally tractable.* The latter means that we know in advance the affine hull of $\mathcal{U}$, a point from the relative interior of $\mathcal{U}$, and we have access to an efficient *membership oracle* that, given on input a point u, reports whether $u \in \mathcal{U}$. This can be reformulated as a precise mathematical statement; however, we will prove a slightly restricted version of this statement that does not require long excursions into complexity theory.

1.3.1 The Strategy

Our strategy will be as follows. First, we restrict ourselves to uncertain LO problems with a certain objective — we remember from item **A** in Section 1.2.3 that we lose nothing by this restriction. Second, all we need is a "computationally tractable" representation of the RC of a *single* uncertain linear constraint, that is, an equivalent representation of the RC by an explicit

[4] In fact, it is NP-hard to compute the maximum of a nonnegative quadratic form over the unit box with inaccuracy less than 4% [55].

(and "short") system of efficiently verifiable convex inequalities. Given such representations for the RCs of every one of the constraints of our uncertain problem and putting them together (cf. item **B** in Section 1.2.3), we reformulate the RC of the problem as the problem of minimizing the original linear objective under a finite (and short) system of explicit convex constraints, and thus — as a computationally tractable problem.

To proceed, we should explain first what does it mean to represent a constraint by a system of convex inequalities. Everyone understands that the system of 4 constraints on 2 variables,

$$x_1 + x_2 \leq 1, x_1 - x_2 \leq 1, -x_1 + x_2 \leq 1, -x_1 - x_2 \leq 1, \tag{1.3.1}$$

represents the nonlinear inequality

$$|x_1| + |x_2| \leq 1 \tag{1.3.2}$$

in the sense that both (1.3.2) and (1.3.1) define the same feasible set. Well, what about the claim that the system of 5 linear inequalities

$$-u_1 \leq x_1 \leq u_1, -u_2 \leq x_2 \leq u_2, u_1 + u_2 \leq 1 \tag{1.3.3}$$

represents the same set as (1.3.2)? Here again everyone will agree with the claim, although we cannot justify the claim in the former fashion, since the feasible sets of (1.3.2) and (1.3.3) live in different spaces and therefore cannot be equal to each other!

What actually is meant when speaking about "equivalent representations of problems/constraints" in Optimization can be formalized as follows:

Definition 1.5 *A set $X^+ \subset \mathbb{R}^n_x \times \mathbb{R}^k_u$ is said to represent a set $X \subset \mathbb{R}^n_x$, if the projection of X^+ onto the space of x-variables is exactly X, i.e., $x \in X$ if and only if there exists $u \in \mathbb{R}^k_u$ such that $(x, u) \in X^+$:*

$$X = \left\{ x : \exists u : (x, u) \in X^+ \right\}.$$

A system of constraints $\mathcal{S}^+$ in variables $x \in \mathbb{R}^n_x$, $u \in \mathbb{R}^k_u$ is said to represent a system of constraints $\mathcal{S}$ in variables $x \in \mathbb{R}^n_x$, if the feasible set of the former system represents the feasible set of the latter one.

With this definition, it is clear that the system (1.3.3) indeed represents the constraint (1.3.2), and, more generally, that the system of $2n + 1$ linear inequalities

$$-u_j \leq x_j \leq u_j, \; j = 1, ..., n, \sum_j u_j \leq 1$$

in variables x, u represents the constraint

$$\sum_j |x_j| \leq 1.$$

To understand how powerful this representation is, note that to represent the same constraint in the style of (1.3.1), that is, without extra variables, it would take as much as 2^n linear inequalities.

Coming back to the general case, assume that we are given an optimization problem

$$\min_x \left\{ f(x) \text{ s.t. } x \text{ satisfies } \mathcal{S}_i, \; i = 1, ..., m \right\}, \tag{P}$$

where $\mathcal{S}_i$ are systems of constraints in variables x, and that we have in our disposal systems $\mathcal{S}_i^+$ of constraints in variables x, v^i which represent the systems $\mathcal{S}_i$. Clearly, the problem

$$\min_{x,v^1,...,v^m} \left\{ f(x) \text{ s.t. } (x, v^i) \text{ satisfies } \mathcal{S}_i^+,\, i = 1, ..., m \right\} \tag{P$^+$}$$

is equivalent to (P): the x component of every feasible solution to (P$^+$) is feasible for (P) with the same value of the objective, and the optimal values in the problems are equal to each other, so that the x component of an ϵ-optimal (in terms of the objective) feasible solution to (P$^+$) is an ϵ-optimal feasible solution to (P). We shall say that (P$^+$) represents equivalently the original problem (P). What is important here, is that a representation can possess desired properties that are absent in the original problem. For example, an appropriate representation can convert the problem of the form $\min_x \{\|Px - p\|_1 : Ax \le b\}$ with n variables, m linear constraints, and k-dimensional vector p, into an LO problem with $n + k$ variables and $m + 2k + 1$ linear inequality constraints, etc. Our goal now is to build a representation capable of expressing equivalently a semi-infinite linear constraint (specifically, the robust counterpart of an uncertain linear inequality) as a finite system of explicit convex constraints, with the ultimate goal to use these representations in order to convert the RC of an uncertain LO problem into an explicit (and as such, computationally tractable) convex program.

The outlined strategy allows us to focus on a *single* uncertainty-affected linear inequality — a family

$$\left\{ a^T x \le b \right\}_{[a;b] \in \mathcal{U}}, \tag{1.3.4}$$

of linear inequalities with the data varying in the uncertainty set

$$\mathcal{U} = \left\{ [a; b] = [a^0; b^0] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell; b^\ell] : \zeta \in \mathcal{Z} \right\} \tag{1.3.5}$$

— and on "tractable representation" of the RC

$$a^T x \le b \quad \forall \left([a; b] = [a^0; b^0] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell; b^\ell] : \zeta \in \mathcal{Z} \right) \tag{1.3.6}$$

of this uncertain inequality.

By reasons indicated in item **C** of Section 1.2.3, we assume from now on that the associated perturbation set $\mathcal{Z}$ is convex.

1.3.2 Tractable Representation of (1.3.6): Simple Cases

We start with the cases where the desired representation can be found by "bare hands," specifically, the cases of *interval* and *simple ellipsoidal* uncertainty.

Example 1.4 Consider the case of *interval uncertainty*, where $\mathcal{Z}$ in (1.3.6) is a box. W.l.o.g. we can normalize the situation by assuming that

$$\mathcal{Z} = \text{Box}_1 \equiv \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \le 1\}.$$

In this case, (1.3.6) reads

$$\begin{array}{lll}
 & [a^0]^T x + \sum\limits_{\ell=1}^{L} \zeta_\ell [a^\ell]^T x \le b^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell b^\ell & \forall (\zeta : \|\zeta\|_\infty \le 1) \\
\Leftrightarrow & \sum\limits_{\ell=1}^{L} \zeta_\ell [[a^\ell]^T x - b^\ell] \le b^0 - [a^0]^T x & \forall (\zeta : |\zeta_\ell| \le 1,\, \ell = 1, ..., L) \\
\Leftrightarrow & \max\limits_{-1 \le \zeta_\ell \le 1} \left[\sum\limits_{\ell=1}^{L} \zeta_\ell [[a^\ell]^T x - b^\ell] \right] \le b^0 - [a^0]^T x &
\end{array}$$

The concluding maximum in the chain is clearly $\sum\limits_{\ell=1}^{L} |[a^\ell]^T x - b^\ell|$, and we arrive at the representation of (1.3.6) by the explicit convex constraint

$$[a^0]^T x + \sum_{\ell=1}^{L} |[a^\ell]^T x - b^\ell| \leq b^0, \tag{1.3.7}$$

which in turn admits a representation by a system of linear inequalities:

$$\left\{ \begin{array}{l} -u_\ell \leq [a^\ell]^T x - b^\ell \leq u_\ell,\ \ell = 1, ..., L, \\ [a^0]^T x + \sum\limits_{\ell=1}^{L} u_\ell \leq b^0. \end{array} \right. \tag{1.3.8}$$

Example 1.5 Consider the case of *ellipsoidal uncertainty* where $\mathcal{Z}$ in (1.3.6) is an ellipsoid. W.l.o.g. we can normalize the situation by assuming that $\mathcal{Z}$ is merely the ball of radius Ω centered at the origin:

$$\mathcal{Z} = \mathrm{Ball}_\Omega = \{\zeta \in \mathbb{R}^L : \|\zeta\|_2 \leq \Omega\}.$$

In this case, (1.3.6) reads

$$\begin{array}{rl} & [a^0]^T x + \sum\limits_{\ell=1}^{L} \zeta_\ell [a^\ell]^T x \leq b^0 + \sum\limits_{\ell=1}^{L} \zeta_\ell b^\ell \qquad \forall (\zeta : \|\zeta\|_2 \leq \Omega) \\ \Leftrightarrow & \max\limits_{\|\zeta\|_2 \leq \Omega} \left[\sum\limits_{\ell=1}^{L} \zeta_\ell [[a^\ell]^T x - b^\ell] \right] \leq b^0 - [a^0]^T x \\ \Leftrightarrow & \Omega \sqrt{\sum\limits_{\ell=1}^{L} ([a^\ell]^T x - b^\ell)^2} \leq b^0 - [a^0]^T x, \end{array}$$

and we arrive at the representation of (1.3.6) by the explicit convex constraint ("conic quadratic inequality")

$$[a^0]^T x + \Omega \sqrt{\sum_{\ell=1}^{L} ([a^\ell]^T x - b^\ell)^2} \leq b^0. \tag{1.3.9}$$

1.3.3 Tractable Representation of (1.3.6): General Case

Now consider a rather general case when the perturbation set $\mathcal{Z}$ in (1.3.6) is given by a *conic representation* (cf. section A.2.4 in Appendix):

$$\mathcal{Z} = \left\{ \zeta \in \mathbb{R}^L : \exists u \in \mathbb{R}^K : P\zeta + Qu + p \in \mathbf{K} \right\}, \tag{1.3.10}$$

where $\mathbf{K}$ is a closed convex pointed cone in $\mathbb{R}^N$ with a nonempty interior, P, Q are given matrices and p is a given vector.

In the case when $\mathbf{K}$ is *not* a polyhedral cone, assume that this representation is strictly feasible:

$$\exists (\bar{\zeta}, \bar{u}) : P\bar{\zeta} + Q\bar{u} + p \in \mathrm{int} K. \tag{1.3.11}$$

In fact, in the sequel we would lose nothing by further restricting K to be a *canonical cone* – (finite) direct product of "simple" cones $K^1, ..., K^S$:

$$K = K^1 \times ... \times K^S, \tag{1.3.12}$$

where every K^s is

- either the nonnegative orthant $\mathbb{R}^n_+ = \{x = [x_1; ...; x_n] \in \mathbb{R}^n : x_i \geq 0\,\forall i\}$,
- or the Lorentz cone $\mathbf{L}^n = \{x = [x_1; ...; x_n] \in \mathbb{R}^n : x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2}\}$,
- or the semidefinite cone $\mathbf{S}^n_+$. This cone "lives" in the space $\mathbf{S}^n$ of real symmetric $n \times n$ matrices equipped with the Frobenius inner product $\langle A, B\rangle = \mathrm{Tr}(AB) = \mathrm{Tr}(AB^T) = \sum_{i,j} A_{ij}B_{ij}$; the cone itself is comprised of all positive semidefinite symmetric $n \times n$ matrices.

As a matter of fact,

- the family $\mathcal{F}$ of all convex sets admitting conic representations involving canonical cones is extremely nice – it is closed w.r.t. all basic operations preserving convexity, like taking finite intersections, arithmetic sums, images and inverse images under affine mappings, etc. Moreover, conic representation of the result of such an operation is readily given by conic representation of the operands; see section A.2.4 for the corresponding "calculus." As a result, handling convex sets from the family in question is fully algorithmic and computationally efficient;
- the family $\mathcal{F}$ is extremely wide: as a matter of fact, for all practical purposes one can think of $\mathcal{F}$ as of the family of *all* computationally tractable convex sets arising in applications.

Theorem 1.1 *Let the perturbation set $\mathcal{Z}$ be given by* (1.3.10)*, and in the case of non-polyhedral* $\mathbf{K}$*, let also* (1.3.11) *take place. Then the semi-infinite constraint* (1.3.6) *can be represented by the following system of conic inequalities in variables $x \in \mathbb{R}^n, y \in \mathbb{R}^N$:*

$$\begin{array}{l} p^T y + [a^0]^T x \leq b^0, \\ Q^T y = 0, \\ (P^T y)_\ell + [a^\ell]^T x = b^\ell, \ell = 1, ..., L, \\ y \in \mathbf{K}_*, \end{array} \tag{1.3.13}$$

where $\mathbf{K}_* = \{y : y^T z \geq 0\,\forall z \in \mathbf{K}\}$ is the cone dual to $\mathbf{K}$.

Proof. We have

$$\begin{array}{cl} & x \text{ is feasible for } (1.3.6) \\ \Leftrightarrow & \sup\limits_{\zeta\in\mathcal{Z}} \Big\{ \underbrace{[a^0]^T x - b^0}_{d[x]} + \sum\limits_{\ell=1}^{L} \zeta_\ell \underbrace{\left[[a^\ell]^T x - b^\ell\right]}_{c_\ell[x]} \Big\} \leq 0 \\ \Leftrightarrow & \sup\limits_{\zeta\in\mathcal{Z}} \left\{c^T[x]\zeta + d[x]\right\} \leq 0 \\ \Leftrightarrow & \sup\limits_{\zeta\in\mathcal{Z}} c^T[x]\zeta \leq -d[x] \\ \Leftrightarrow & \max\limits_{\zeta,v} \left\{c^T[x]\zeta : P\zeta + Qv + p \in \mathbf{K}\right\} \leq -d[x]. \end{array}$$

The concluding relation says that x is feasible for (1.3.6) if and only if the optimal value in the conic program

$$\max_{\zeta,v} \left\{c^T[x]\zeta : P\zeta + Qv + p \in \mathbf{K}\right\} \tag{CP}$$

is $\leq -d[x]$. Assume, first, that (1.3.11) takes place. Then (CP) is strictly feasible, and therefore, applying the Conic Duality Theorem (Theorem A.1), the optimal value in (CP) is $\leq -d[x]$ if and only if the optimal value in the conic dual to the (CP) problem

$$\min_y \left\{p^T y : Q^T y = 0, P^T y = -c[x], y \in \mathbf{K}_*\right\}, \tag{CD}$$

is attained and is $\leq -d[x]$. Now assume that $\mathbf{K}$ is a polyhedral cone. In this case the usual LO Duality Theorem, (which does not require the validity of (1.3.11)), yields exactly the same conclusion: the optimal value in (CP) is $\leq -d[x]$ if and only if the optimal value in (CD) is achieved and is $\leq -d[x]$. In other words, under the premise of the Theorem, x is feasible for (1.3.6) if and only if (CD) has a feasible solution y with $p^T y \leq -d[x]$. $\square$

Observing that nonnegative orthants, Lorentz and Semidefinite cones are self-dual, and thus their finite direct products, i.e., canonical cones, are self-dual as well,[5] we derive from Theorem 1.1 the following corollary:

Corollary 1.1 *Let the nonempty perturbation set in* (1.3.6) *be:*

(i) *polyhedral, i.e., given by* (1.3.10) *with a nonnegative orthant* $\mathbb{R}^N_+$ *in the role of* $\mathbf{K}$*, or*

(ii) *conic quadratic representable, i.e., given by* (1.3.10) *with a direct product* $\mathbf{L}^{k_1} \times ... \times \mathbf{L}^{k_m}$ *of Lorentz cones* $\mathbf{L}^k = \{x \in \mathbb{R}^k : x_k \geq \sqrt{x_1^2 + ... + x_{k-1}^2}\}$ *in the role of* $\mathbf{K}$*, or*

(iii) *semidefinite representable, i.e., given by* (1.3.10) *with the positive semidefinite cone* $\mathbf{S}^k_+$ *in the role of* $\mathbf{K}$*.*

In the cases of (ii), (iii) *assume in addition that* (1.3.11) *holds true. Then the Robust Counterpart* (1.3.6) *of the uncertain linear inequality* (1.3.4) — (1.3.5) *with the perturbation set* $\mathcal{Z}$ *admits equivalent reformulation as an explicit system of*

— *linear inequalities, in the case of* (i),

— *conic quadratic inequalities, in the case of* (ii),

— *linear matrix inequalities, in the case of* (iii).

In all cases, the size of the reformulation is polynomial in the number of variables in (1.3.6) *and the size of the conic description of* $\mathcal{Z}$*, while the data of the reformulation is readily given by the data describing, via* (1.3.10)*, the perturbation set* $\mathcal{Z}$*.*

Remark 1.2 A. Usually, the cone $\mathbf{K}$ participating in (1.3.10) is the direct product of simpler cones $\mathbf{K}^1, ..., \mathbf{K}^S$, so that representation (1.3.10) takes the form

$$\mathcal{Z} = \{\zeta : \exists u^1, ..., u^S : P_s\zeta + Q_s u^s + p_s \in \mathbf{K}^s,\ s = 1, ..., S\}. \tag{1.3.14}$$

In this case, (1.3.13) becomes the system of conic constraints in variables $x, y^1, ..., y^S$ as follows:

$$\begin{array}{l} \sum\limits_{s=1}^{S} p_s^T y^s + [a^0]^T x \leq b^0, \\ Q_s^T y^s = 0,\ s = 1, ..., S, \\ \sum\limits_{s=1}^{S} (P_s^T y^s)_\ell + [a^\ell]^T x = b^\ell, \ell = 1, ..., L, \\ y^s \in \mathbf{K}^s_*, s = 1, ..., S, \end{array} \tag{1.3.15}$$

where K^s_* is the cone dual to K^s.

B. Uncertainty sets given by LMIs seem "exotic"; however, they can arise under quite realistic circumstances, see section 1.5.

[5]Since the cone dual to a direct product of cones K^s clearly is the direct product of cones K^s_* dual to K^s.

Examples

We are about to apply Theorem 1.1 to build tractable reformulations of the semi-infinite inequality (1.3.6) in two particular cases. While at a first glance no natural "uncertainty models" lead to the "strange" perturbation sets we are about to consider, it will become clear later that these sets are of significant importance — they allow one to model *random* uncertainty.

Example 1.6 *$\mathcal{Z}$ is the intersection of concentric co-axial box and ellipsoid, specifically,*

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : -1 \le \zeta_\ell \le 1, \ell \le L, \sqrt{\sum_{\ell=1}^{L} \zeta_\ell^2/\sigma_\ell^2} \le \Omega\}, \tag{1.3.16}$$

where $\sigma_\ell > 0$ and $\Omega > 0$ are given parameters.

Here representation (1.3.14) becomes

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : P_1\zeta + p_1 \in \mathbf{K}^1, P_2\zeta + p_2 \in \mathbf{K}^2\},$$

where

- $P_1\zeta \equiv [\zeta; 0]$, $p_1 = [0_{L\times 1}; 1]$ and $\mathbf{K}^1 = \{(z,t) \in \mathbb{R}^L \times \mathbb{R} : t \ge \|z\|_\infty\}$, whence $\mathbf{K}^1_* = \{(z,t) \in \mathbb{R}^L \times \mathbb{R} : t \ge \|z\|_1\}$;
- $P_2\zeta = [\Sigma^{-1}\zeta; 0]$ with $\Sigma = \mathrm{Diag}\{\sigma_1, ..., \sigma_L\}$, $p_2 = [0_{L\times 1}; \Omega]$ and $\mathbf{K}^2$ is the Lorentz cone of the dimension $L+1$ (whence $\mathbf{K}^2_* = \mathbf{K}^2$)

Setting $y^1 = [\eta_1; \tau_1]$, $y^2 = [\eta_2; \tau_2]$ with one-dimensional τ_1, τ_2 and L-dimensional η_1, η_2, (1.3.15) becomes the following system of constraints in variables τ, η, x:

$$\begin{array}{llcl}
(a) & \tau_1 + \Omega\tau_2 + [a^0]^T x & \le & b^0, \\
(b) & (\eta_1 + \Sigma^{-1}\eta_2)_\ell & = & b^\ell - [a^\ell]^T x,\ \ell = 1, ..., L, \\
(c) & \|\eta_1\|_1 & \le & \tau_1 \quad [\Leftrightarrow [\eta_1; \tau_1] \in \mathbf{K}^1_*], \\
(d) & \|\eta_2\|_2 & \le & \tau_2 \quad [\Leftrightarrow [\eta_2; \tau_2] \in \mathbf{K}^2_*].
\end{array}$$

We can eliminate from this system the variables τ_1, τ_2 — for every feasible solution to the system, we have $\tau_1 \ge \bar\tau_1 \equiv \|\eta_1\|_1$, $\tau_2 \ge \bar\tau_2 \equiv \|\eta_2\|_2$, and the solution obtained when replacing τ_1, τ_2 with $\bar\tau_1$, $\bar\tau_2$ still is feasible. The reduced system in variables x, $z = \eta_1$, $w = \Sigma^{-1}\eta_2$ reads

$$\begin{array}{rcl}
\sum\limits_{\ell=1}^{L} |z_\ell| + \Omega\sqrt{\sum\limits_\ell \sigma_\ell^2 w_\ell^2} + [a^0]^T x & \le & b^0, \\
z_\ell + w_\ell & = & b^\ell - [a^\ell]^T x,\ \ell = 1, ..., L,
\end{array} \tag{1.3.17}$$

which is also a representation of (1.3.6), (1.3.16).

Example 1.7 [*"budgeted uncertainty"*] Consider the case where $\mathcal{Z}$ is the intersection of $\|\cdot\|_\infty$- and $\|\cdot\|_1$-balls, specifically,

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \le 1,\ \|\zeta\|_1 \le \gamma\}, \tag{1.3.18}$$

where γ, $1 \le \gamma \le L$, is a given "uncertainty budget."

Here representation (1.3.14) becomes

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : P_1\zeta + p_1 \in \mathbf{K}^1, P_2\zeta + p_2 \in \mathbf{K}^2\},$$

where

- $P_1\zeta \equiv [\zeta; 0]$, $p_1 = [0_{L\times 1}; 1]$ and $\mathbf{K}^1 = \{[z;t] \in \mathbb{R}^L \times \mathbb{R} : t \ge \|z\|_\infty\}$, whence $\mathbf{K}^1_* = \{[z;t] \in \mathbb{R}^L \times \mathbb{R} : t \ge \|z\|_1\}$;
- $P_2\zeta = [\zeta; 0]$, $p_2 = [0_{L\times 1}; \gamma]$ and $\mathbf{K}^2 = \mathbf{K}^1_* = \{[z;t] \in \mathbb{R}^L \times \mathbb{R} : t \ge \|z\|_1\}$, whence $\mathbf{K}^2_* = \mathbf{K}^1$.

Setting $y^1 = [z; \tau_1]$, $y^2 = [w; \tau_2]$ with one-dimensional τ and L-dimensional z, w, system (1.3.15) becomes the following system of constraints in variables τ_1, τ_2, z, w, x:

$$\begin{array}{llcl}
(a) & \tau_1 + \gamma\tau_2 + [a^0]^T x & \leq & b^0, \\
(b) & (z+w)_\ell & = & b^\ell - [a^\ell]^T x,\ \ell = 1, ..., L, \\
(c) & \|z\|_1 & \leq & \tau_1 \quad [\Leftrightarrow [\eta_1; \tau_1] \in \mathbf{K}^1_*], \\
(d) & \|w\|_\infty & \leq & \tau_2 \quad [\Leftrightarrow [\eta_2; \tau_2] \in \mathbf{K}^2_*].
\end{array}$$

Same as in Example 1.6, we can eliminate the τ-variables, arriving at a representation of (1.3.6), (1.3.18) by the following system of constraints in variables x, z, w:

$$\begin{array}{rcl}
\sum_{\ell=1}^L |z_\ell| + \gamma \max_\ell |w_\ell| + [a^0]^T x & \leq & b^0, \\
z_\ell + w_\ell & = & b^\ell - [a^\ell]^T x,\ \ell = 1, ..., L,
\end{array} \tag{1.3.19}$$

which can be further converted into the system of linear inequalities in z, w and additional variables.

1.4 How it Works: Motivating Examples Revisited

In this section, we outline the results of Robust Optimization methodology as applied to our "motivating examples."

1.4.1 Robust Synthesis of Antenna Arrays

In the situation of the Antenna Design problem (1.1.2), the "physical" uncertainty comes from the actuation errors $x_j \mapsto (1 + \xi_j)x_j$; as we have already explained, these errors can be modeled equivalently by the perturbations $D_j(\theta_i) \mapsto D_{ij} = (1 + \xi_j)D_j(\theta_i)$ in the coefficients of x_j. Assuming that the errors ξ_j are bounded by a given *uncertainty level* ρ, and that this is the only a priori information on the actuation errors, we end up with the uncertain LO problem

$$\left\{ \min_{x,\tau} \left\{ \tau : -\tau \leq \sum_{j=1}^{J=10} D_{ij} x_j - D_*(\theta_i) \leq \tau, 1 \leq i \leq I = 240 \right\} : |D_{ij} - D_j(\theta_i)| \leq \rho |D_j(\theta_i)| \right\}.$$

The Robust Counterpart of the problem is the semi-infinite LO program

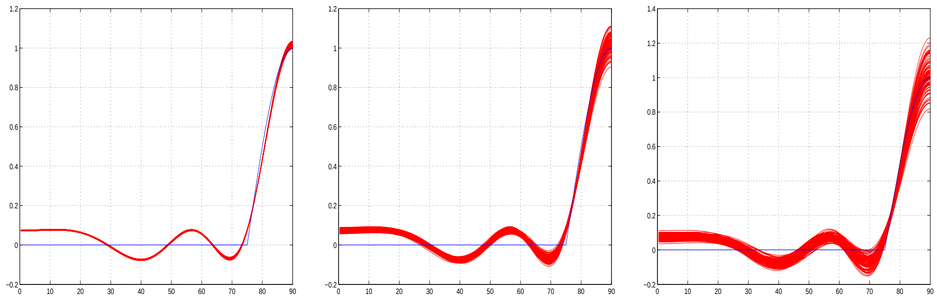
$$\min_{x,\tau} \left\{ \tau : -\tau \leq \sum_j D_{ij} x_j \leq \tau, 1 \leq i \leq I\ \forall D_{ij} \in [\underline{G}_{ij}, \overline{G}_{ij}] \right\} \tag{1.4.1}$$

with $\underline{G}_{ij} = G_j(\theta_i) - \rho|G_j(\theta_i)|$, $\overline{G}_{ij} = G_j(\theta_i) + \rho|G_j(\theta_i)|$. The generic form of this semi-infinite LO is

$$\min_y \left\{ c^T y : Ay \leq b \forall [A, b] : [\underline{A}, \underline{b}] \leq [A, b] \leq [\overline{A}, \overline{b}] \right\} \tag{1.4.2}$$

where $\leq$ for matrices is understood entrywise and $[\underline{A}, \underline{b}] \leq [\overline{A}, \overline{b}]$ are two given matrices. This is a very special case of polyhedral uncertainty set, so that our theory says that the RC is equivalent to an explicit LO program. In fact we can point out (one of) LO reformulation of the Robust Counterpart without reference to any theory: it is immediately seen that (1.4.2) is equivalent to the LO program

$$\min_{y,z} \left\{ c^T y : \underline{A} z + \overline{A}(y + z) \leq \underline{b}, z \geq 0, y + z \geq 0 \right\}. \tag{1.4.3}$$



$\rho = 0.01$ $\rho = 0.05$ $\rho = 0.1$

Figure 1.3: "Dream and reality," robust optimal design: samples of 100 actual diagrams (red) for different uncertainty levels. Blue: the target diagram.

	Reality								
	$\rho = 0.01$			$\rho = 0.05$			$\rho = 0.1$		
	min	mean	max	min	mean	max	min	mean	max
$\|\cdot\|_\infty$-distance to target	0.075	0.078	0.081	0.077	0.088	0.114	0.082	0.113	0.216
energy concentration	70.3%	72.3%	73.8%	63.6%	71.6%6	79.3%	52.2%	70.8%	87.5%

Table 1.2: Quality of robust antenna design. Data over 100 samples of actuation errors per each uncertainty level ρ.
For comparison: for nominal design, with the uncertainty level as small as $\rho = 0.001$, the average $\|\cdot\|_\infty$-distance of the actual diagram to target is as large as 56.8, and the expected energy concentration is as low as 16.5%.

Solving (1.4.1) for the uncertainty level $\rho = 0.01$, we end up with the robust optimal value 0.0815, which, while being by 39% worse than the nominal optimal value 0.0589 (which, as we have seen, exists only in our imagination and says nothing about the actual performance of the nominal optimal design), still is reasonably small. Note that the robust optimal value, in sharp contrast with the nominally optimal one, does say something meaningful about the actual performance of the underlying *robust* design. In our experiments, we have tested the robust optimal design associated with the uncertainty level $\rho = 0.01$ versus actuation errors of this and larger magnitudes. The results are presented on figure 1.3 and in table 1.2. Comparing these figure and table with their "nominal design" counterparts, we see that the robust design is incomparably better than the nominal one.

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The corresponding uncertainty model ("ugly coefficients a_{ij} in the constraint matrix independently of each other vary in the segments $[a^{\rm n}_{ij} - \rho|a^{\rm n}_{ij}|, a^{\rm n}_{ij} + \rho|a^{\rm n}_{ij}|]$, $\rho > 0$ being the uncertainty level) clearly yields the RCs of the generic form (1.4.2). As explained above, these RCs can be straightforwardly converted to explicit LO programs which are of nearly the same sizes and sparsity as the instances of the uncertain LPs in question. It turns out that at the uncertainty level 0.1% ($\rho = 0.001$), all these RCs are feasible, that is, we can immunize the solutions against this uncertainty. Surprisingly, this immunization is "nearly costless" – the robust optimal values of all 90 NETLIB LOs considered in [8] remain within 1% margin of the nominal optimal values. For further details, including what happens at larger uncertainty levels, see [8].

1.5 Non-Affine Perturbations

In the first reading this section can be skipped.

So far we have assumed that the uncertain data of an uncertain LO problem are *affinely* parameterized by a perturbation vector ζ varying in a closed convex set $\mathcal{Z}$. We have seen that this assumption, combined with the assumption that $\mathcal{Z}$ is computationally tractable, implies tractability of the RC. What happens when the perturbations enter the uncertain data in a nonlinear fashion? Assume w.l.o.g. that every entry a in the uncertain data is of the form

$$a = \sum_{k=1}^{K} c_k^a f_k(\zeta),$$

where c_k^a are given coefficients (depending on the data entry in question) and $f_1(\zeta), ..., f_K(\zeta)$ are certain basic functions, perhaps non-affine, defined on the perturbation set $\mathcal{Z}$. Assuming w.l.o.g. that the objective is certain, we still can define the RC of our uncertain problem as the problem of minimizing the original objective over the set of robust feasible solutions, those which remain feasible for all values of the data coming from $\zeta \in \mathcal{Z}$, but what about the tractability of this RC? An immediate observation is that the case of nonlinearly perturbed data can be immediately reduced to the one where the data are affinely perturbed. To this end, it suffices to pass from the original perturbation vector ζ to the new vector

$$\widehat{\zeta}[\zeta] = [\zeta_1; ...; \zeta_L; f_1(\zeta); ...; f_K(\zeta)].$$

As a result, the uncertain data become *affine* functions of the new perturbation vector $\widehat{\zeta}$ which now runs through the image $\widetilde{\mathcal{Z}} = \widehat{\zeta}[\mathcal{Z}]$ of the original uncertainty set $\mathcal{Z}$ under the mapping $\zeta \mapsto \widehat{\zeta}[\zeta]$. As we know, in the case of affine data perturbations the RC remains intact when replacing a given perturbation set with its closed convex hull. Thus, we can think about our uncertain LO problem as an affinely perturbed problem where the perturbation vector is $\widehat{\zeta}$, and this vector runs through the closed convex set $\widehat{\mathcal{Z}} = \mathrm{cl}\,\mathrm{Conv}(\widehat{\zeta}[\mathcal{Z}])$. We see that formally speaking, the case of general-type perturbations can be reduced to the one of affine perturbations. This, unfortunately, does not mean that non-affine perturbations do not cause difficulties. Indeed, in order to end up with a computationally tractable RC, we need more than affinity of perturbations and convexity of the perturbation set — we need this set to be computationally tractable. And the set $\widehat{\mathcal{Z}} = \mathrm{cl}\,\mathrm{Conv}(\widehat{\zeta}[\mathcal{Z}])$ may fail to satisfy this requirement even when both $\mathcal{Z}$ and the *nonlinear* mapping $\zeta \mapsto \widehat{\zeta}[\zeta]$ are simple, e.g., when $\mathcal{Z}$ is a box and $\widehat{\zeta} = [\zeta; \{\zeta_\ell \zeta_r\}_{\ell,r=1}^L]$, (i.e., when the uncertain data are quadratically perturbed by the original perturbations ζ).

We are about to present two generic cases where the difficulty just outlined does not occur (for justification and more examples, see section 5.3.2).

Ellipsoidal perturbation set $\mathcal{Z}$, quadratic perturbations. Here $\mathcal{Z}$ is an ellipsoid, and the basic functions f_k are the constant, the coordinates of ζ and the pairwise products of these coordinates. This means that the uncertain data entries are quadratic functions of the perturbations. W.l.o.g. we can assume that the ellipsoid $\mathcal{Z}$ is centered at the origin: $\mathcal{Z} = \{\zeta : \|Q\zeta\|_2 \leq 1\}$, where Ker$Q = \{0\}$. In this case, representing $\widehat{\zeta}[\zeta]$ as the matrix $\left[\begin{array}{c|c} & \zeta^T \\ \hline \zeta & \zeta\zeta^T \end{array}\right]$, we have the following semidefinite representation of $\widehat{\mathcal{Z}} = \mathrm{cl}\,\mathrm{Conv}(\widehat{\zeta}[\mathcal{Z}])$:

$$\widehat{\mathcal{Z}} = \left\{ \left[\begin{array}{c|c} & w^T \\ \hline w & W \end{array}\right] : \left[\begin{array}{c|c} 1 & w^T \\ \hline w & W \end{array}\right] \succeq 0, \mathrm{Tr}(QWQ^T) \leq 1 \right\}$$

(for proof, see Lemma 5.4).

Separable polynomial perturbations. Here the structure of perturbations is as follows: ζ runs through the box $\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1\}$, and the uncertain data entries are of the form

$$a = p_1^a(\zeta_1) + ... + p_L^a(\zeta_L),$$

where $p_\ell^a(s)$ are given algebraic polynomials of degrees not exceeding d; in other words, the basic functions can be split into L groups, the functions of ℓ-th group being $1 = \zeta_\ell^0, \zeta_\ell, \zeta_\ell^2, ..., \zeta_\ell^d$. Consequently, the function $\widehat{\zeta}[\zeta]$ is given by

$$\widehat{\zeta}[\zeta] = [[1; \zeta_1; \zeta_1^2; ...; \zeta_1^d]; ...; [1; \zeta_L; \zeta_L^2; ...; \zeta_L^d]].$$

Setting $P = \{\widehat{s} = [1; s; s^2; ...; s^d] : -1 \leq s \leq 1\}$, we conclude that $\widetilde{\mathcal{Z}} = \widehat{\zeta}[\mathcal{Z}]$ can be identified with the set $P^L = \underbrace{P \times ... \times P}_{L}$, so that $\widehat{\mathcal{Z}}$ is nothing but the set $\underbrace{\mathcal{P} \times ... \times \mathcal{P}}_{L}$, where $\mathcal{P} = \mathrm{Conv}(P)$. It remains to note that the set $\mathcal{P}$ admits an explicit semidefinite representation, see Lemma 5.2.

1.6 Exercises

Exercise 1.1 Prove the fact stated in the beginning of section 1.4.1:

(!) The RC of an uncertain LO problem with certain objective and simple interval uncertainty in the constraints — the uncertain problem

$$\mathcal{P} = \left\{ \min_x \left\{ c^T x : Ax \leq b \right\}, [\underline{A}, \underline{b}] \leq [A, b] \leq [\overline{A}, \overline{b}] \right\}$$

is equivalent to the explicit LO program

$$\min_{u,v} \left\{ c^T x : \overline{A}u - \underline{A}v \leq \underline{b}, u \geq 0, v \geq 0, u - v = x \right\} \tag{1.6.1}$$

Exercise 1.2 Represent the RCs of every one of the uncertain linear constraints given below:

$$\begin{array}{lr} a^T x \leq b, [a; b] \in \mathcal{U} = \{[a; b] = [a^{\mathrm{n}}; b^{\mathrm{n}}] + P\zeta : \|\zeta\|_p \leq \rho\} & \\ & [p \in [1, \infty]] \quad (a) \\ a^T x \leq b, [a; b] \in \mathcal{U} = \{[a; b] = [a^{\mathrm{n}}; b^{\mathrm{n}}] + P\zeta : \|\zeta\|_p \leq \rho, \zeta \geq 0\} & \\ & [p \in [1, \infty]] \quad (b) \\ a^T x \leq b, [a; b] \in \mathcal{U} = \{[a; b] = [a^{\mathrm{n}}; b^{\mathrm{n}}] + P\zeta : \|\zeta\|_p \leq \rho\} & \\ & [p \in (0, 1)] \quad (c) \end{array}$$

as explicit convex constraints.

Exercise 1.3 Represent in tractable form the RC of uncertain linear constraint

$$a^T x \leq b$$

with $\cap$-ellipsoidal uncertainty set

$$\mathcal{U} = \{[a, b] = [a^{\rm n}; b^{\rm n}] + P\zeta : \zeta^T Q_j \zeta \leq \rho^2, \, 1 \leq j \leq J\},$$

where $Q_j \succeq 0$ and $\sum_j Q_j \succ 0$.

The goal of subsequent exercises is to find out whether there is a "gap" between feasibility/optimality properties of *instances* of an uncertain LO problem

$$\mathcal{P} = \left\{ \min_x \left\{ c^T x : Ax \leq b \right\} : [A, b] \in \mathcal{U} \right\}$$

and similar properties of its RC

$$\text{Opt} = \min_x \left\{ c^T x : Ax \leq b \ \forall [A, b] \in \mathcal{U} \right\}. \tag{RC}$$

Specifically, we want to answer the following questions:

- Is it possible that every instance of $\mathcal{P}$ is feasible, while (RC) is not so?
- Is it possible that (RC) is feasible, but its optimal value is worse than those of all instances?
- Under which natural conditions feasibility of (RC) is equivalent to feasibility of all instances, and the robust optimal value is the maximum of optimal values of instances.

Exercise 1.4 Consider two uncertain LO problems

$$\begin{array}{r}
\mathcal{P}_1 = \left\{ \min_x \left\{ -x_1 - x_2 : 0 \leq x_1 \leq b_1, 0 \leq x_2 \leq b_2, x_1 + x_2 \geq p \right\} : b \in \mathcal{U}_1 \right\}, \\
\mathcal{U}_1 = \{ b : 1 \geq b_1 \geq 1/3, 1 \geq b_2 \geq 1/3, b_1 + b_2 \geq 1 \}, \\
\mathcal{P}_2 = \left\{ \min_x \left\{ -x_1 - x_2 : 0 \leq x_1 \leq b_1, 0 \leq x_2 \leq b_2, x_1 + x_2 \geq p \right\} : b \in \mathcal{U}_2 \right\}, \\
\mathcal{U}_2 = \{ b : 1 \geq b_1 \geq 1/3, 1 \geq b_2 \geq 1/3 \}.
\end{array}$$

(p is a parameter).

1. Build the RC's of the problems.
2. Set $p = 3/4$. Is there a gap between feasibility properties of the instances of $\mathcal{P}_1$ and those of the RC of $\mathcal{P}_1$? Is there a similar gap in the case of $\mathcal{P}_2$?
3. Set $p = 2/3$. Is there a gap between the largest of the optimal values of instances of $\mathcal{P}_1$ and the optimal value of the RC? Is there a similar gap in the case of $\mathcal{P}_2$?

The results of Exercise 1.4 demonstrate that there could be a huge gap between feasibility/optimality properties of the RC and those of instances. We are about to demonstrate that this phenomenon does *not* occur in the case of a "constraint-wise" uncertainty.

Definition 1.6 *Consider an uncertain LO problem with certain objective*

$$\mathcal{P} = \left\{\min_x \left\{c^T x : Ax \le b\right\} : [A,b] \in \mathcal{U}\right\}$$

with convex compact uncertainty set $\mathcal{U}$*, and let* $\mathcal{U}_i$ *be the projection of* $\mathcal{U}$ *on the set of data of* i*-th constraint:*

$$\mathcal{U}_i = \left\{[a_i^T, b_i] : \exists [A,b] \in \mathcal{U} \text{ such that } [a_i^T, b_i] \text{ is } i\text{-th row in } [A,b]\right\}.$$

Clearly, $\mathcal{U} \subset \mathcal{U}^+ = \prod_i \mathcal{U}_i$*. We call uncertainty constraint-wise, if* $\mathcal{U} = \mathcal{U}^+$*, and call the uncertain problem, obtained from* $\mathcal{P}$ *by extending the original uncertainty set* $\mathcal{U}$ *the constraint-wise envelope of* $\mathcal{P}$*.*

Note that by claim on p. 10, when passing from uncertain LO problem to its constraint-wise envelope, the RC remains intact.

Exercise 1.5

1. Consider the uncertain problems $\mathcal{P}_1$ and $\mathcal{P}_2$ from Exercise 1.4. Which one of them, if any, has constraint-wise uncertainty? Which one of them, if any, is the constraint-wise envelope of the other problem?

2. * Let $\mathcal{P}$ be an uncertain LO program with constraint-wise uncertainty such that the feasible sets of all instances belong to a given in advance convex compact set X (e.g., all instances share common system of certain box constraints). Prove that in this case there is no gap between feasibility/optimality properties of instances and those of the RC: the RC if feasible if and only if all instances are so, and in this case the optimal value of the RC is equal to the maximum, over the instances, of the optimal values of the instances.

Lecture 2

Robust Linear Optimization and Chance Constraints

2.1 How to Specify an Uncertainty Set

The question posed in the title of this section goes beyond general-type theoretical considerations — this is mainly a modeling issue that should be resolved on the basis of application-driven considerations. There is however a special case where this question makes sense and can, to some extent, be answered — this is the case where our goal is not to build an uncertainty model "from scratch," but rather to *translate* an already existing uncertainty model, namely, a stochastic one, to the language of "uncertain-but-bounded" perturbation sets and the associated robust counterparts. By exactly the same reasons as in the previous section, we can restrict our considerations to the case of a *single* uncertainty-affected linear inequality – a family

$$\left\{a^T x \le b\right\}_{[a;b]\in\mathcal{U}}, \tag{1.3.4}$$

of linear inequalities with the data varying in the uncertainty set

$$\mathcal{U} = \left\{[a;b] = [a^0;b^0] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell;b^\ell] : \zeta \in \mathcal{Z}\right\}. \tag{1.3.5}$$

of this uncertain inequality.

Probabilistic vs. "uncertain-but-bounded" perturbations. When building the RC

$$a^T x \le b \quad \forall \left([a;b] = [a^0;b^0] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell;b^\ell] : \zeta \in \mathcal{Z}\right) \tag{1.3.6}$$

of uncertain linear inequality (1.3.4), we worked with the so called "uncertain-but-bounded" data model (1.3.5) — one where all we know about the possible values of the data $[a;b]$ is their domain $\mathcal{U}$ defined in terms of a given affine parameterization of the data by perturbation vector ζ varying in a given perturbation set $\mathcal{Z}$. It should be stressed that we did not assume that the perturbations are of a stochastic nature and therefore used the only approach meaningful under the circumstances, namely, we looked for solutions that remain feasible whatever the data perturbation from $\mathcal{Z}$. This approach has its advantages:

1. More often than not there are no reasons to assign the perturbations a stochastic nature.

   Indeed, stochasticity makes sense only when one repeats a certain action many times, or executes many similar actions in parallel; here it might be reasonable to think of frequencies of successes, etc. Probabilistic considerations become, methodologically, much more problematic when applied to a unique action, with no second attempt possible.

2. Even when the unknown data can be thought of as stochastic, it might be difficult, especially in the large-scale case, to specify reliably data distribution. Indeed, the mere fact that the data are stochastic does not help unless we possess at least a partial knowledge of the underlying distribution.

   Of course, the uncertain-but-bounded models of uncertainty also require a priori knowledge, namely, to know what is the uncertainty set (a probabilistically oriented person could think about this set as the *support* of data distribution, that is, the smallest closed set in the space of the data such that the probability for the data to take a value outside of this set is zero). Note, however, that it is much easier to point out the support of the relevant distribution than the distribution itself.

With the uncertain-but-bounded model of uncertainty, we can make clear predictions like "with such and such behavior, we definitely will survive, provided that the unknown parameters will differ from their nominal values by no more than 15%, although we may die when the variation will be as large as 15.1%." In case we do believe that 15.1% variations are also worthy to worry about, we have an option to increase the perturbation set to take care of 30% perturbations in the data. With luck, we will be able to find a robust feasible solution for the increased perturbation set. This is a typical engineering approach — after the required thickness of a bar supporting certain load is found, a civil engineer will increase it by factor like 1.2 or 1.5 "to be on the safe side" — to account for model inaccuracies, material imperfections, etc. With a stochastic uncertainty model, this "being on the safe side" is impossible — increasing the probability of certain events, one must decrease simultaneously the probability of certain other events, since the "total probability budget" is once and for ever fixed. While all these arguments demonstrate that there are situations in reality when the uncertain-but-bounded model of data perturbations possesses significant methodological advantages over the stochastic models of uncertainty, there are, of course, applications (like communications, weather forecasts, mass production, and, to some extent, finance) where one can rely on probabilistic models of uncertainty. Whenever this is the case, the much less informative uncertain-but-bounded model and associated worst-case-oriented decisions can be too conservative and thus impractical. The bottom line is that *while the stochastic models of data uncertainty are by far not the only meaningful ones, they definitely deserve attention*. Our goal in this lecture is to *develop techniques that are capable to utilize, to some extent, knowledge of the stochastic nature of data perturbations when building uncertainty-immunized solutions*. This goal will be achieved via a specific "translation" of stochastic models of uncertain data to the language of uncertain-but-bounded perturbations and the associated robust counterparts. Before developing the approach in full detail, we will explain why we choose such an implicit way to treat stochastic uncertainty models instead of treating them directly.

2.2 Chance Constraints and their Safe Tractable Approximations

The most direct way to treat stochastic data uncertainty in the context of uncertain Linear Optimization is offered by an old concept (going back to 50s [38]) of *chance constraints*. Consider an uncertain linear inequality

$$a^T x \leq b,\ \ [a;b] = [a^0;b^0] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell;b^\ell] \tag{2.2.1}$$

(cf. (1.3.4), (1.3.5)) and assume that the perturbation vector ζ is random with, say, completely known probability distribution P. Ideally, we would like to work with candidate solutions x that make the constraint valid with probability 1. This "ideal goal," however, means coming back to the uncertain-but-bounded model of perturbations; indeed, it is easily seen that a given x satisfies (2.2.1) for almost all realizations of ζ if and only if x is robust feasible w.r.t. the perturbation set that is the closed convex hull of the support of P. The only meaningful way to utilize the stochasticity of perturbations is to require a candidate solution x to satisfy the constraint for "nearly all" realizations of ζ, specifically, to satisfy the constraint with probability at least $1-\epsilon$, where $\epsilon \in (0,1)$ is a prespecified small tolerance. This approach associates with the randomly perturbed constraint (2.2.1) the *chance constraint*

$$p(x) := \mathrm{Prob}_{\zeta\sim P}\left\{\zeta : [a^0]^T x + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell]^T x > b^0 + \sum_{\ell=1}^{L} \zeta_\ell b^\ell \right\} \leq \epsilon, \tag{2.2.2}$$

where $\mathrm{Prob}_{\zeta\sim P}$ is the probability associated with the distribution P. Note that (2.2.2) is a usual certain constraint. Replacing all uncertainty-affected constraints in an uncertain LO problem with their chance constrained versions and minimizing the objective function, (which we, w.l.o.g., may assume to be certain) under these constraints, we end up with the *chance constrained* version of $(\mathrm{LO}_{\mathcal{U}})$, which is a deterministic optimization problem.

Strictly speaking, from purely modeling viewpoint the just outlined scheme of passing from an uncertain LO problem with stochastic uncertainty

$$\mathcal{P} = \left\{ \min_x \left\{ c^T x : A_\zeta x \leq b_\zeta \right\} : [A_\zeta, b_\zeta] = [A^0, b^0] + \sum_{\ell=1}^{L} \zeta_\ell [A^\ell, b^\ell], \zeta \sim P \right\}$$

(we have assumed w.l.o.g. that the objective is certain) to its chance constrained version

$$\min_x \left\{ c^T x : \mathrm{Prob}_{\zeta\sim P}\left\{ (A_\zeta x)_i \leq (b_\zeta)_i \right\} \geq 1-\epsilon,\ 1 \leq i \leq m := \dim b_\zeta \right\} \tag{2.2.3}$$

is neither the only natural, nor the most natural one. When the constraints are hard, a candidate solution is meaningless when it violates *at least one* of the constraints, so that a natural chance constrained version of $\mathcal{P}$ would be

$$\min_x \left\{ c^T x : \mathrm{Prob}_{\zeta\sim P}\left\{ (A_\zeta x)_i \leq (b_\zeta)_i\ \forall i \right\} \geq 1-\epsilon \right\}. \tag{2.2.4}$$

The latter problem is *not* the same as (2.2.3): while it is true that a feasible solution to (2.2.3) is feasible for (2.2.4) as well, the inverse statement usually is not true. All we can say in general about a feasible solution x to (2.2.4) is that x is feasible for a *relaxed* version of (2.2.3), specifically, the one where ϵ is increased to $k\epsilon$, $k \leq m$ being the number of scalar linear constraints in $\mathcal{P}$ which indeed are affected by uncertainty.

In spite of this drawback, the chance constrained model (2.2.3) is *the* major entity of interest in Chance Constrained LO. The underlying rationale is as follows:

- As we shall see in a while, aside of a handful of very special particular cases, numerical processing of the constraint-wise chance constrained problem is a highly challenging task; these challenges are amplified drastically when passing from this problem to the "true" chance constrained problem (2.2.4).
- As we have already seen, one can approximate the "true" chance constrained problem (2.2.4) with problem (2.2.3); to this end, it suffices to associate the latter problem with a smaller tolerance than the one required in (2.2.4), namely, to reset ϵ to ϵ/k. With this approximation (which is much simpler than (2.2.4)), we "stay at the safe side" – what is feasible for approximation, clearly is feasible for the true problem (2.2.4) as well. At the same time, in many cases, especially when ϵ is small, it turns out that the conservatism built into the approximation nearly does not affect the optimal value we can achieve.

In this lecture, we focus solely on the chance constrained model (2.2.3), or, which is the same, on the chance constrained version (2.2.2) of a single scalar linear inequality affected by stochastic uncertainty; some results on handling chance constrained problems of the form (2.2.4) will be presented in lecture 3 (section 3.6.4).

While passing from a randomly perturbed scalar linear constraint (2.2.1) to its chance constrained version (2.2.2) seems to be quite natural, this approach suffers from a severe drawback — *typically, it results in a severely computationally intractable problem.* The reason is twofold:

1. Usually, it is difficult to evaluate with high accuracy the probability in the left hand side of (2.2.2), even in the case when P is simple.

   For example, it is known [61] that computing the left hand side in (2.2.2) is NP-hard already when ζ_ℓ are independent and uniformly distributed in $[-1,1]$. This means that unless P=NP, there is no algorithm that, given on input a rational x, rational data $\{[a^\ell;b^\ell]\}_{\ell=0}^L$ and rational $\delta\in(0,1)$, allows to evaluate $p(x)$ within accuracy δ in time polynomial in the bit size of the input. Unless ζ takes values in a finite set of moderate cardinality, the only known general method to evaluate $p(x)$ is based on Monte-Carlo simulations; this method, however, requires samples with cardinality of order of $1/\delta$, where δ is the required accuracy of evaluation. Since the meaningful values of this accuracy are $\leq\epsilon$, we conclude that in reality the Monte-Carlo approach can hardly be used when ϵ is like 0.0001 or less.

2. More often than not the feasible set of (2.2.2) is non-convex, which makes optimization under chance constraints a highly problematic task.

   Note that while the first difficulty becomes an actual obstacle only when ϵ is small enough, the second difficulty makes chance constrained optimization highly problematic for "large" ϵ as well.

Essentially, the only known case when none of the outlined difficulties occur is the case where ζ is a Gaussian random vector and $\epsilon<1/2$.

Indeed, when $\zeta\sim\mathcal{N}(\theta,\Theta)$, the random quantity

$$\zeta^x := [a^0]^T x + \sum_{\ell=1}^L \zeta_\ell [a^\ell]^T x - b^0 - \sum_{\ell=1}^L \zeta_\ell b^\ell \tag{2.2.5}$$

also is a Gaussian random variable with the expectation $\mathbf{E}\{\zeta^x\}=\alpha^T[x;1]$ and the variance $\mathbf{E}\left\{(\zeta^x-\alpha^T[x;1])^2\right\}=\sigma^2(x):=[x;1]^TQ^TQ[x;1]$; here α and Q are vector and matrix readily given by the data $\{[a^\ell; b_\ell]\}_{\ell=0}^L$, θ, Θ of the chance constraint. Further, the chance constraint (2.2.2) reads

$$\text{Prob}\left\{\zeta^x > 0\right\} \le \epsilon.$$

Assuming that $0 \le \epsilon \le 1/2$ and $\sigma(x) > 0$, the latter constraint is equivalent to the relation

$$\text{Erf}\left(-\alpha^T[x;1]/\sigma(x)\right) \le \epsilon,$$

where

$$\text{Erf}(s) = \frac{1}{\sqrt{2\pi}} \int_s^\infty \exp\{-r^2/2\} dr \tag{2.2.6}$$

is the error function. Introducing the inverse error function

$$\text{ErfInv}(\gamma) = \min\left\{s : \text{Erf}(s) \le \gamma\right\}, \tag{2.2.7}$$

we see that our chance constraint is equivalent to $-\alpha^T[x;1]/\sigma(x) \ge \text{ErfInv}(\epsilon)$, or, which is the same, to the conic quadratic constraint

$$\alpha^T[x;1] + \text{ErfInv}(\epsilon)\|Q[x;1]\|_2 \le 0 \tag{2.2.8}$$

in variable x.[1] It is immediately seen that this conclusion remains valid for when $\sigma(x) = 0$ as well.

Due to the severe computational difficulties associated with chance constraints, a natural course of action is to replace a chance constraint with its *computationally tractable safe approximation*. The latter notion is defined as follows:

Definition 2.1 *Let* $\{[a^\ell; b^\ell]\}_{\ell=0}^L$, P, ϵ *be the data of chance constraint* (2.2.2)*, and let* $\mathcal{S}$ *be a system of convex constraints on* x *and additional variables* v*. We say that* $\mathcal{S}$ *is a safe convex approximation of chance constraint* (2.2.2)*, if the* x *component of every feasible solution* (x, v) *of* $\mathcal{S}$ *is feasible for the chance constraint.*

A safe convex approximation $\mathcal{S}$ *of* (2.2.2) *is called computationally tractable, if the convex constraints forming* $\mathcal{S}$ *are efficiently computable.*

It is clear that by replacing the chance constraints in a given chance constrained optimization problem with their safe convex approximations, we end up with a convex optimization problem in x and additional variables that is a "safe approximation" of the chance constrained problem: the x component of every feasible solution to the approximation is feasible for the chance constrained problem. If the safe convex approximation in question is tractable, then the above approximating program is a convex program with efficiently computable constraints and as such it can be processed efficiently.

In the sequel, when speaking about safe convex approximations, we omit for the sake of brevity the adjective "convex," which should always be added "by default."

2.2.1 Ambiguous Chance Constraints

Chance constraint (2.2.2) is associated with randomly perturbed constraint (2.2.1) *and a given distribution* P *of random perturbations*, and it is reasonable to use this constraint when we do know this distribution. In reality we usually have only *partial* information on P, that is, we

[1] Since $0 < \epsilon \le 1/2$, we have $\text{ErfInv}(\epsilon) \ge 0$, so that (2.2.8) indeed is a conic quadratic constraint.

know only that P belongs to a given family $\mathcal{P}$ of distributions. When this is the case, it makes sense to pass from (2.2.2) to the *ambiguous* chance constraint

$$\forall (P \in \mathcal{P}) : \mathrm{Prob}_{\zeta\sim P}\left\{\zeta : [a^0]^T x + \sum_{\ell=1}^L \zeta_\ell [a^\ell]^T x > b^0 + \sum_{\ell=1}^L \zeta_\ell b^\ell\right\} \leq \epsilon. \tag{2.2.9}$$

Of course, the definition of a safe tractable approximation of chance constraint extends straightforwardly to the case of ambiguous chance constraint. In the sequel, we usually skip the adjective "ambiguous"; what exactly is meant depends on whether we are speaking about a partially or a fully known distribution P.

Next we present a simple scheme for the safe approximation of chance constraints.

2.3 The Generating-Function-Based Approximation Scheme

Preliminaries. Let us set

$$w = [w_0; w_1; ...; w_L] \in \mathbb{R}^{L+1}; \quad Z_w[\zeta] = w_0 + \sum_{\ell=1}^L w_\ell \zeta_\ell$$

The random variable ζ^x appearing in (2.2.5) can be represented as $Z_{w[x]}[\zeta]$, where $x \mapsto w[x]$ is an *affine* mapping readily given by the data of (2.2.2) and independent of the distribution of ζ. All we need in order to get a safe (or safe and tractable) convex approximation of the chance constraint (2.2.2) is a similar approximation of the chance constraint

$$p(w) := \mathrm{Prob}_{\zeta\sim P}\left\{Z_w[\zeta] > 0\right\} \leq \epsilon \tag{2.3.1}$$

in variables w. Indeed, given such an approximation (which is a system of convex constraints in variables w and additional variables) and carrying out the affine substitution of variables $w \mapsto w[x]$, we end up with a desired approximation of (2.2.2). Thus, from now on we focus on building a safe (or safe and tractable) convex approximation of the chance constraint (2.3.1).

2.3.1 The approach

Conceptually, the approach we are about to develop is extremely simple. The chance constraint of interest (2.3.1) reads

$$p(w) := \mathrm{Prob}_{\zeta\sim P}\left\{Z_w[\zeta] > 0\right\} \equiv \int \chi(Z_w[\zeta])dP(\zeta) \leq \epsilon, \tag{2.3.2}$$

where $\chi(s)$ is the characteristic function of the positive ray:

$$\chi(s) = \begin{cases} 0, & s \leq 0 \\ 1, & s > 0 \end{cases}$$

If $\chi(\cdot)$ were convex (which is not the case), the left hand side in (2.3.2) would be a convex function of w (since $Z_w[\cdot]$ is affine in w), and thus (2.3.2) would be a convex constraint on w. Now let $\gamma(\cdot)$ be a *convex* function on the axis which is everywhere $\geq \chi(x)$. Then we clearly have

$$p(w) \leq \Psi(w) := \int \gamma(Z_w[\zeta])dP(\zeta).$$

so that the constraint

$$\Psi(w) \leq \epsilon \tag{2.3.3}$$

is a safe approximation of (2.3.1); since $\gamma(\cdot)$ is convex, the function $\Psi(w)$ is convex in w, so that (2.3.3) is safe *convex* approximation of the chance constraint of interest. If, in addition, $\Psi(\cdot)$ is efficiently computable, this safe convex approximation is tractable.

The next step, which usually reduces dramatically the conservatism of the above approximation, is given by *scaling*. Specifically, observing that $\chi(s) = \chi(\alpha^{-1}s)$ for every $\alpha > 0$, we conclude that if $\gamma(\cdot)$ is as above, then $\gamma(\alpha^{-1}s) \geq \chi(s)$ for all s, whence

$$p(w) \leq \Psi(\alpha^{-1}w)$$

and thus *the constraint*

$$\alpha\Psi(\alpha^{-1}w) - \alpha\epsilon \leq 0 \qquad [\Leftrightarrow \Psi(\alpha^{-1}x) \leq \epsilon]$$

in variables $\alpha > 0$ *and* w *also is a safe convex approximation of the chance constraint* (2.3.1) — *whenever* w *can be extended by some* $\alpha > 0$ *to a feasible solution of this constraint,* w *is feasible for* (2.3.1). The punch line is that the constraint

$$\alpha\Psi(\alpha^{-1}w) - \alpha\epsilon \leq 0 \tag{2.3.4}$$

we end up with is convex in variables w and $\alpha > 0$.[2]

When implementing this scheme, it is convenient to impose on $\gamma(\cdot)$ *and* on the distribution of ζ natural restrictions as follows:

> (!) $\gamma(\cdot) : \mathbb{R} \to \mathbb{R}$ *is a nonnegative monotone function such that* $\gamma(0) \geq 1$ *and* $\gamma(s) \to 0$ *as* $s \to -\infty$.

These properties of γ clearly imply that $\gamma(\cdot)$ is a convex majorant of $\chi(\cdot)$, so that γ indeed can be used in the above approximation scheme. From now on, we refer to functions $\gamma(\cdot)$ satisfying (!) as to *generators*, or *generating functions*.

We have seen that the condition

$$\exists \alpha > 0 : \alpha\Psi(\alpha^{-1}w) - \alpha\epsilon \leq 0$$

is sufficient for w to be feasible for the chance constraint (2.3.1). A simple technical exercise (which exploits the fact that $\Psi(\cdot)$, due to its origin, is lower semicontinuous) shows that when $\gamma(\cdot)$ is a generator, a *weaker* condition

$$G(w) := \inf_{\alpha>0} \left[\alpha\Psi(\alpha^{-1}w) - \alpha\epsilon\right] \leq 0 \tag{2.3.5}$$

also is sufficient for w to be feasible for (2.3.1). Note that the function $G(w)$ by its origin is convex in w, so that (2.3.5) also is a safe convex approximation of (2.3.1). This approximation is tractable, provided that $G(w)$ is efficiently computable, which indeed is the case when $\Psi(\cdot)$ is efficiently computable.

Our last observation is as follows: let $\gamma(\cdot)$ be a generator, and let, as above, $\Psi(w) = \int \gamma(Z_w[\zeta])dP(\zeta)$, so that Ψ is convex and lower semicontinuous. Assume that instead of $\Psi(\cdot)$

---

[2] This is a well known fact: whenever $f(w) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function, its *projective transformation* $F(w, \alpha) = \alpha f(\alpha^{-1}w)$ is convex in the domain $\alpha > 0$.

we use in the above approximation scheme a *convex and lower semicontinuous* function $\Psi^+(\cdot)$ which is an *upper bound* on $\Psi(\cdot)$:

$$\Psi(w) \leq \Psi^+(w) \ \forall w.$$

Replacing in (2.3.4) and in (2.3.5) the function Ψ with its upper bound Ψ^+, we arrive at the convex constraint

$$\alpha\Psi^+(\alpha^{-1}w) - \alpha\epsilon \leq 0 \tag{2.3.6}$$

in variables w and $\alpha > 0$ and the convex constraint

$$G^+(w) := \inf_{\alpha>0} \left[\alpha\Psi^+(\alpha^{-1}w) - \alpha\epsilon\right] \leq 0 \tag{2.3.7}$$

which clearly also are safe convex approximations of the chance constraint of interest (2.3.1).

2.3.2 Main result

The summary of our findings (which we formulate in a form applicable to the case of ambiguous version of (2.3.1)) is as follows:

Theorem 2.1 *Consider an ambiguous chance constraint in variables w:*

$$\forall P \in \mathcal{P} : \mathrm{Prob}_{\zeta\sim P}\left\{Z_w[\zeta] := w_0 + \sum_{\ell=1}^{L} w_\ell\zeta_\ell > 0\right\} \leq \epsilon. \tag{2.3.8}$$

Let $\gamma(\cdot)$ be a generator, and let $\Psi^+(w) : \mathbb{R}^{L+1} \to \mathbb{R} \cup \{+\infty\}$ be a convex function such that

$$\forall (P \in \mathcal{P}, w) : \Psi^+(w) \geq \int \gamma(Z_w[\zeta])dP(\zeta). \tag{2.3.9}$$

Then both the constraint (2.3.6) *in variables w and $\alpha > 0$ and the constraint* (2.3.7) *in variables w are safe convex approximations of the ambiguous chance constraint* (2.3.8)*. These approximations are tractable, provided that Ψ^+ is efficiently computable.*

2.3.3 Relations to Robust Optimization

We are about to demonstrate that under mild regularity assumptions the safe convex approximation (2.3.7) of the ambiguous chance constraint (2.3.8) admits a *Robust Counterpart form:*

> (!!) *There exists a convex nonempty compact set $\mathcal{Z} \in \mathbb{R}^L$ such that*
>
> $$G^+(w) \leq 0 \Leftrightarrow \forall \zeta \in \mathcal{Z} : w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell \leq 0. \tag{2.3.10}$$

Example 2.1 Let $\mathcal{P}$ be the set of all *product-type* probability distributions on $\mathbb{R}^L$ (that is, $\zeta_1, ..., \zeta_L$ are independent whenever $\zeta \sim P \in \mathcal{P}$) with zero mean marginal distributions P_ℓ supported on $[-1, 1]$. Choosing as the generator the function $\gamma(s) = \exp\{s\}$, let us compute an appropriate Ψ^+. This should be a convex function such that

$$\forall w \in \mathbb{R}^{L+1} : \Psi^+(w) \geq \mathbf{E}_{\zeta\sim P}\left\{\exp\{w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell\}\right\} \ \forall P \in \mathcal{P}.$$

Given $P \in \mathcal{P}$, denoting by P_ℓ the marginal distributions of P and taking into account that P is product-type, we get

$$\mathbf{E}_{\zeta\sim P}\left\{\exp\{w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell\}\right\} = \exp\{w_0\}\prod_{\ell=1}^L \mathbf{E}_{\zeta_\ell\sim P_\ell}\{\exp\{w_\ell\zeta_\ell\}\}.$$

Now let us use the following simple and well known fact:

Lemma 2.1 *Let Q be a zero mean probability distribution on the axis supported on $[-1,1]$. Then for every real t one has*

$$\mathbf{E}_{s\sim Q}\{\exp\{ts\}\} \leq \cosh(t). \tag{2.3.11}$$

Proof. Given t, let us set $f(s) = \exp\{ts\} - \sinh(t)s$. The function f is convex, so that its maximum on $[-1,1]$ is attained at the endpoint and thus is equal to $\cosh(t)$ (since $f(1) = f(-1) = \cosh(t)$). We now have

$$\begin{array}{l}\int \exp\{ts\}dQ(s) = \int f(s)dQ(s) \text{ [since } Q \text{ is with zero mean]}\\ \leq \max_{|s|\leq 1} f(s) \text{ [since } Q \text{ is supported on } [-1,1]]\\ = \cosh(s).\end{array}$$ □

Note that the bound in Lemma is sharp – the inequality in (2.3.11) becomes equality when Q is the uniform distribution on $\{-1,1\}$.

Invoking Lemma, we get

$$\forall (P\in\mathcal{P}, w\in\mathbb{R}^{L+1}) : \mathbf{E}_{\zeta\sim P}\left\{\gamma(Z_w[\zeta])\right\} \leq \exp\{w_0\}\prod_{\ell=1}^L \cosh(w_\ell) \leq \Psi^+(w) := \exp\{w_0 + \frac{1}{2}\sum_{\ell=1}^L w_\ell^2\}, \tag{2.3.12}$$

where the concluding $\leq$ is given by the evident inequality $\cosh(t) \leq \exp\{t^2/2\}$ (look at the Taylor expansions of both sides).

Now, if not all w_ℓ, $\ell \geq 1$, are zeros, we have $\alpha\Psi^+(\alpha^{-1}w) - \alpha\epsilon \to +\infty$ when $\alpha \to +0$ and when $\alpha \to +\infty$, that is, our w satisfies (2.3.7) is and only if $\alpha\Psi^+(\alpha^{-1}w) - \alpha\epsilon \leq 0$ for certain $\alpha > 0$, that is, if and only if $w_0\alpha^{-1} + \frac{\alpha^{-2}}{2}\sum_{\ell=1}^L w_\ell^2 - \ln(\epsilon) \leq 0$ for certain $\alpha > 0$. The latter is the case if and only if

$$w_0 + \sqrt{2\ln(1/\epsilon)}\sqrt{\sum_{\ell=1}^L w_\ell^2} \leq 0. \tag{2.3.13}$$

When $w_1 = w_2 = ... = w_L = 0$, the relation (2.3.7) clearly takes place if and only if $w_0 \leq 0$. The bottom line is that *with our* Ψ^+, *the condition* (2.3.7) *is equivalent to* (2.3.13), *and implies that*

$$\text{Prob}\left\{w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell > 0\right\} \leq \epsilon.$$

Note that we have established a useful proposition:

Proposition 2.1 [Azuma's inequality] *If $\xi_1, ...\xi_L$ are independent zero mean random variables taking values in respective segments $[-\sigma_\ell, \sigma_\ell]$, $1 \leq \ell \leq L$, then for every $\Omega > 0$*

$$\text{Prob}\left\{\sum_{\ell=1}^L \xi_\ell > \Omega\sqrt{\sum_{\ell=1}^L \sigma_\ell^2}\right\} \leq \exp\{-\Omega^2/2\}.$$

Indeed, setting $\zeta_\ell = \xi_\ell/\sigma_\ell$ (so that $\zeta_\ell \in [-1,1]$ are independent zero mean), $\epsilon = \exp\{-\Omega^2/2\}$ (so that $\Omega = \sqrt{2\ln(1/\epsilon)}$) and $w_0 = -\Omega\sqrt{\sum_{\ell=1}^L \sigma_\ell^2}$, we get

$$w_0 + \sqrt{2\ln(1/\epsilon)}\sqrt{\sum_{\ell=1} \sigma_\ell^2} = 0,$$

whence, as we have demonstrated, $\mathrm{Prob}\{\underbrace{w_0 + \sum_{\ell=1}^L \zeta_\ell\sigma_\ell > 0}_{\Leftrightarrow \sum_\ell \xi_\ell > \Omega\sqrt{\sum_\ell \sigma_\ell^2}}\} \leq \epsilon = \exp\{-\Omega^2/2\}$, as claimed in Azuma'z inequality.

Now note that (2.3.13) is nothing but

$$w_0 + \zeta^T[w_1;...;w_l] \leq 0 \ \forall (\zeta : \|\zeta\|_2 \leq \sqrt{2\ln(1/\epsilon)}),$$

we see that *in the case in question (!!) holds true with*

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_2 \leq \sqrt{2\ln(1/\epsilon)}\}. \tag{2.3.14}$$

In particular, we have justified the usefulness of seemingly "artificial" perturbation sets – Euclidean balls: these sets arise naturally when we intend to immunize solutions to a *randomly perturbed* scalar linear constraint against "nearly all" random perturbations ζ, provided that all we know about the distribution P of ζ is the inclusion $P \in \mathcal{P}$. q

Before proving (!!), let us understand what this claim actually means. (!!) says that in order to immunize a candidate solution to the randomly perturbed scalar linear constraint

$$Z_w[\zeta] := w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell \leq 0 \tag{2.3.15}$$

in variables w against "nearly all" realizations of random perturbation ζ (that is, to ensure the validity of the constraint, as evaluated at our candidate solution, with probability $\geq 1-\epsilon$, whatever be a distribution P of ζ belonging to a given family $\mathcal{P}$), it suffices to immunize the solution against *all* realizations of ζ from a properly chosen convex compact set $\mathcal{Z}$.

By itself, this fact is of no much value. Indeed, under mild assumptions on $\mathcal{P}$ we can point out a compact convex set $\mathcal{Z}$ which is $(1-\epsilon)$-support of every distribution $P \in \mathcal{P}$, that is, $P(\mathbb{R}^L\backslash\mathcal{Z}) \leq \epsilon$ for every $P \in \mathcal{P}$. Whenever this is the case, every solution to the semi-infinite linear inequality

$$Z_w[\zeta] \leq 0 \ \forall \zeta \in \mathcal{Z}$$

clearly satisfies the ambiguous chance constraint (2.3.8). The importance of the outlined approach is that *it produces perturbation sets which,* in principle, *have nothing in common with $(1-\epsilon)$-supports of distributions from $\mathcal{P}$ and can be much smaller than these supports.* The latter is a good news, since the smaller is $\mathcal{Z}$, the less conservative is the associates safe approximation of the chance constraint (2.3.8).

What can be gained from our approach as compared to the naive "$(1-\epsilon)$-support" one, can be well seen in the situation of Example 2.1, where we are speaking about product-type distributions with zero mean supported on the unit box $B_\infty = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1\}$. As we know, in this case our generator-based approximation scheme allows to take, as $\mathcal{Z}$, the centered at the origin Euclidean ball of radius $\sqrt{2\ln(1/\epsilon)}$. When L is large, this set is much smaller than any set $\widehat{\mathcal{Z}}$ which "$(1-\epsilon)$-supports" all distributions from $\mathcal{P}$. For example, when $\epsilon < 1/2$ and $\widehat{\mathcal{Z}}$ $(1-\epsilon)$-supports just two distribution from $\mathcal{P}$, specifically, the uniform distribution on the vertices of B_∞ and the uniform distribution on the entire B_∞, the diameter of $\widehat{\mathcal{Z}}$ must be at least $2\sqrt{L}$ (why?), while the ratio of the diameters of $\widehat{\mathcal{Z}}$ and $\mathcal{Z}$ is at least $\kappa_L = \sqrt{\frac{L}{2\ln(1/\epsilon)}}$. It is easily seen that the ratio of the *average* linear sizes[3] of $\widehat{\mathcal{Z}}$ and $\mathcal{Z}$ is at least $O(1)\kappa_L$ with an absolute constant $O(1)$. Thus, when $L \gg \sqrt{2\ln(1/\epsilon)}$, $\mathcal{Z}$ is "much less" than (any) $(1-\epsilon)$-support of the distributions from $\mathcal{P}$. It should be added that when ζ is uniformly distributed on the vertices of B_∞ and $L > \sqrt{2\ln(1/\epsilon)}$, the probability for ζ to take value in $\mathcal{Z}$ is just 0...

One can argue that while for $\epsilon \ll 1$ fixed and L large, our artificial uncertainty set – the ball $\mathcal{Z}$ – while being much less than any $(1-\epsilon)$-support of the distributions from $\mathcal{P}$ in terms of diameter, average linear sizes, etc., is nevertheless larger than B_∞ (which is the common support of all distributions from $\mathcal{P}$) in some directions. Well, we shall see in a while that in the situation of Example 2.1 we lose nothing when passing from the ball $\mathcal{Z}$ to its intersection with B_∞; this intersection is, of course, smaller than B_∞, whatever be the interpretation of "smaller."

Justifying (!!)

Assumptions. To proceed, we need to make mild regularity assumptions on the functions $\Psi^+(w)$ we intend to use in the approximation scheme suggested by Theorem 2.1. Recall that the theorem itself requires from Ψ^+ to be a convex upper bound on the function $\Psi(w) = \sup\limits_{P\in P} \Psi_P(w)$, $\Psi_P(w) = \mathbf{E}_{\xi\sim P}\{\gamma(Z_w[\zeta])\}$. Observe that $\Psi(\cdot)$ possesses the following properties:

- $\Psi : \mathbb{R}^{L+1} \to \mathbb{R}_+ \cup \{+\infty\}$ is convex and lower semicontinuous;
- $0 \in \mathrm{Dom}\,\Psi$ and $\Psi(0) \geq 1$;
- We have $\infty > \Psi([-t;0;...;0]) \to 0$ as $0 \leq t \to \infty$.

From now on, we assume that the function Ψ^+ we intend to use possesses similar properties, specifically, that

A. $[\Psi \leq]\Psi^+ : \mathbb{R}^{L+1} \to \mathbb{R}_+ \cup \{+\infty\}$ is convex and lower semicontinuous;

B. $0 \in \mathrm{int}\mathrm{Dom}\,\Psi^+$ and $\Psi^+(0) \geq 1$ (note that we have strengthened $0 \in \mathrm{Dom}\,\Psi$ to $0 \in \mathrm{int}\mathrm{Dom}\,\Psi^+$);

C. $\infty > \Psi^+([-t;0;...;0]) \to 0$ as $0 \leq t \to \infty$.

We refer to functions Ψ^+ with the above properties as to *regular*.

Key fact. The following result is similar to [3, Lemma B.1.1]:

Lemma 2.2 *Consider the situation of Theorem 2.1, and let Ψ^+ be regular. Then the feasible set $W \subset \mathbb{R}^{L+1}$ of the constraint (2.3.7) is a closed convex cone with a nonempty interior, and this interior contains the vector $e = [-1;0;...;0] \in \mathbb{R}^{L+1}$.*

[3] The average linear size of a body $Q \subset \mathbb{R}^L$ is, by definition $(\mathrm{mes}_L(Q))^{1/L}$.

Proof. We know that

(+) The set
$$W^o = \{w : \exists \alpha > 0 \alpha\Psi^+(w/\alpha) - \alpha\epsilon \leq 0\}$$
which clearly is conic, is convex and is contained in the feasible set W_+ of the ambiguous chance constraint (2.3.8),

and we want to prove that the set

$$W = \{w : G^+(w) := \inf_{\alpha>0} \left\{\alpha\Psi^+(w/\alpha) - \alpha\epsilon\right\} \leq 0\}$$

(a) is a closed convex cone contained in W_+, and (b) contains a neighbourhood of the point $e = [-1; 0; ...; 0]$.
In order to derive (a), (b) from (+), it suffices to prove that

$$(c) \quad W = \mathrm{cl}\, W^o \text{ and } (d) \quad e \in \mathrm{int} W^o.$$

1^0. Let us first prove that $W \subset \mathrm{cl}\, W^o$. Indeed, let $w \in W$, and let us prove that $w \in \mathrm{cl}\, W_0$. Since $w \in W$, we have for some sequence $\{\alpha_i > 0\}$:

$$\lim_{i\to\infty} \alpha_i[\Psi^+(w/\alpha_i) - \epsilon] \leq 0 \qquad (*)$$

W.l.o.g. we can assume that either
A. $\exists \bar\alpha = \lim_{i\to\infty} \alpha_i \in (0, \infty)$, or
B. $\alpha_i \to +\infty$ as $i \to \infty$, or
C. $\alpha_i \to +0$ as $i \to \infty$.

In the case of A we have

$$0 \geq \lim_{i\to\infty} \alpha_i[\Psi^+(w/\alpha_i) - \epsilon] \geq \bar\alpha[\Psi^+(w/\bar\alpha) - \epsilon]$$

(recall that Ψ^+ is lower semicontinuous), so that $w \in W^o$, as claimed.

In the case of B, by lower semicontinuity of Ψ^+ we have $\liminf_{i\to\infty} \Psi^+(w/\alpha_i) \geq \Psi^+(0) \geq 1 > \epsilon$, whence $\lim_{i\to\infty} \alpha_i[\Psi^+(w/\alpha_i) - \epsilon] = +\infty$, which is impossible.

In the case of C $(*)$ says that $\mathbb{R}_+w \subset \mathrm{Dom}\,\Psi^+$ and that Ψ^+, by convexity, is nonincreasing along the ray $\mathbb{R}_+w$ and thus is bounded on this ray: $\exists M \in [1, \infty) : \Psi^+(tw) \leq M \ \forall t \geq 0$. Besides, $\Psi^+(se) \to 0$ as $s \to \infty$, so that we can find $\bar w = \bar s e \in \mathrm{Dom}\,\Psi^+$ such that $\Psi^+(\bar w) \leq \epsilon/3$. Setting

$$z_t = \frac{t\epsilon}{2M} w + \frac{2M - \epsilon}{2M} \bar w = \frac{\epsilon}{2M}[tw] + \left[1 - \frac{\epsilon}{2M}\right] \bar w,$$

we have $\Psi^+(z_t) \leq \frac{\epsilon}{2M}\Psi^+(tw) + \left[1 - \frac{\epsilon}{2M}\right]\Psi^+(\bar w) \leq \frac{\epsilon}{2} + \Psi^+(\bar w) < \epsilon$, whence $z_t \in W^o$ for all $t > 0$. Since W^o is conic, we get

$$W^o \ni \left[\frac{t\epsilon}{2M}\right]^{-1} z_t = w + \frac{2M - \epsilon}{t\epsilon} \bar z \to w, \ t \to \infty,$$

and thus $w \in \mathrm{cl}\, W^o$, as claimed.
2^0. We have already seen that $W \subset \mathrm{cl}\, W^o$. To prove that $W = \mathrm{cl}\, W^o$, it remains to verify that if $w = \lim_{i\to\infty} w_i$ and $w_i \in W^o$, then $w \in W$. In other words, given that $\Psi^+(w_i/\alpha_i) \leq \epsilon$ for some $\alpha_i > 0$ and that $w = \lim_i w_i$, we should prove that

$$\inf_{\alpha>0} \alpha[\Psi^+(w/\alpha) - \epsilon] \leq 0. \tag{2.3.16}$$

There is nothing to prove when $w_i = w$ for some i; thus, we can assume that $\delta_i := \|w - w_i\| > 0$ for all i. By assumption, $0 \in \mathrm{intDom}\,\Psi^+$ and Ψ^+ is convex; thus, Ψ^+ is bounded in a neighborhood of 0:

$$\exists (\rho > 0, M < \infty) : \|z\| \leq \rho \Rightarrow \Psi^+(z) \leq M.$$

Setting

$$\alpha_i = \frac{\alpha_i \rho}{\delta_i + \alpha_i \rho},\ z_i = \alpha_i(w_i/\alpha_i) + (1-\alpha_i)\frac{w - w_i}{\delta_i}\rho,$$

direct computation shows that

$$z_i = \frac{\rho}{\delta_i + \alpha_i \rho} w := w/\gamma_i,\ \gamma_i = \frac{\delta_i + \alpha_i \rho}{\rho}.$$

Besides this,

$$\begin{array}{l} \Psi^+(z_i) \le \alpha_i \Psi^+(w_i/\alpha_i) + (1-\alpha_i)\Psi^+(\rho\frac{w-w_i}{\delta_i}) \\ \le \alpha_i \epsilon + (1-\alpha_i) M, \\ \Rightarrow \Psi^+(z_i) - \epsilon = (1-\alpha_i)[M-\epsilon] = \frac{\delta_i}{\delta_i + \alpha_i \rho}[M-\epsilon] \\ \Rightarrow \gamma_i[\Psi^+(w/\gamma_i) - \epsilon] = \gamma_i[\Psi^+(z_i) - \epsilon] \le \gamma_i \frac{\delta_i}{\delta_i + \alpha_i \rho}[M-\epsilon] \\ = \frac{\delta_i}{\rho}[M-\epsilon] \to 0,\ i \to \infty, \end{array}$$

and the conclusion in $(*)$ follows.

3^0. We have proved (c). What remains to prove is that $e \in \mathrm{int} W$. Indeed, Ψ^+ is finite in a neighborhood U of the origin and $\Psi^+(se) \to 0$ as $s \to \infty$. It follows that for large enough value of s and small enough value of $\alpha > 0$ one has

$$w \in S := \alpha U + (1-\alpha) se \Rightarrow \Psi^+(w) \le \epsilon,$$

that is, $S \subset W^o$. Since S contains a neighborhood of $s_+ e$ for an appropriate $s_+ > 0$, $s_+ e \in \mathrm{int} W^o$, and since W^o is conic, we get $e \in \mathrm{int} W^o \subset \mathrm{int} W$. □

Lemma 2.2 is the key to the proof of (!!). Indeed, since W is a closed convex cone and its interior contains e, the set

$$Z = \left\{ z \in \mathbb{R}^{L+1} : z^T w \le 0\ \forall w \in W, e^T z = -1 \right\}$$

that is, the intersection of the cone $W_- := \{z : z^T w \le 0 \forall w \in W\}$ anti-dual to W with the hyperplane $\Pi = \{z : e^T z = -1\}$, is a nonempty convex compact set such that

$$W = \left\{ w : w^T z \le 0\ \forall z \in Z \right\}. \tag{2.3.17}$$

Indeed, $W \subset \mathbb{R}^{L+1}$ is a closed convex cone and as such it is anti-dual of its anti-dual cone W_-. Besides this, W clearly is proper, i.e., differs from the entire $\mathbb{R}^{L+1}$, since otherwise we would have $-e = [1; 0; ...; 0] \in W$, that is, $[1; 0; ...; 0]$ would be a feasible solution to the ambiguous chance constraint (2.3.8) (since all points from W are so), which of course is not the case. Since W is proper and $W = (W_-)_-$, the anti-dual cone W_- of W cannot be $\{0\}$. Further, the hyperplane Π intersects all rays $\mathbb{R}_+ f$ spanned by nonzero vectors $f \in W_-$.

Indeed, for such an f, $f^T e' \le 0$ for all vectors e' close enough to e (since $f \in W_-$ and all vectors e' close enough to e belong to W); since $f \ne 0$, since $f \ne 0$, the relation $f^T e' \le 0$ for all close enough to e vectors e' implies that $f^T e < 0$. The latter, in turn, means that $(tf)^T e = -1$ for properly chosen $t > 0$, that is, Π indeed intersects $\mathbb{R}_+ f$.

Since $W_- \ne \{0\}$, the set of nonzero $f \in W_-$ is nonempty, and since for every f from this set the ray $R_+ f$ is contained in W_- and intersects Π, the intersection Z of W_- and Π is convex and nonempty. Since both $\Pi$and W_- are closed, Z is closed as well. To prove that the closed set Z is compact, it suffices to prove that it is bounded, which is readily given by the fact that $e \in \mathrm{int} W$.

Indeed, assuming that Z is unbounded, there exists a sequence $f_1, f_2, ...$ of points in Z such that $\|f_i\|_2 \to \infty$, $i \to \infty$. Since $W_- \supset Z$ is a cone, the unit vectors $g_i = f_i/\|f_i\|_2$

form a bounded sequence in W_-, and passing to a subsequence, we can assume that this sequence converges as $i \to \infty$ to a (clearly unit) vector g. Since $g_i \in W_-$ and W_- is closed, we have $g \in W_-$. Further, we have $e^T f_i = -1$ for all i due to $f_i \in \Pi$, whence $e^T g = \lim\limits_{i\to\infty} e^T g_i = 0$. This is in a clear contradiction with the facts that $e \in \mathrm{int} W$ and $0 \neq g \in W_-$, since from these facts it follows that $g^T e' \leq 0$ for all e' close to e, and the latter in the case of $0 \neq g$ and $g^T e = 0$ clearly is impossible. Thus, assuming Z unbounded, we arrive at a contradiction.

We have verified that Z is a nonempty convex compact set, and the only missing fact is (2.3.17). The latter relation is nearly evident. Indeed, every $w \in W$ clearly satisfies $w^T z \leq 0\, \forall z \in Z$ due to the fact that $Z \subset W_-$. Vice versa, if $w^T z \leq 0 \,\forall z \in Z$, then $w^T f \leq 0$ for all $0 \neq f \in W_-$ (since, as we have already seen, a nonnegative multiple of such an f belongs to Z). But then $w \in (W_-) - = W$. Thus, the right hand side in (2.3.17) is exactly W.

Note that (2.3.17) is all we need to prove (!!). Indeed, since $e = [-1; 0; ...; 0]$, we have $\Pi = \{z = [z_0; z_1; ...; z_L] : e^T z = -1\} = \{z = [1; z_1; ...; z_L]\}$. Consequently, (2.3.17) reads

$$W = \{w = [w_0; w_1; ...; w_L] : w^T z \leq 0 \ \forall z \in Z\} = \{w = [w_0; ...; w_L] : w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell \leq 0 \ \forall \zeta \in \mathcal{Z}\},$$

where $\mathcal{Z} = \{\zeta \in \mathbb{R}^L : [1; \zeta] \in Z\}$. It remains to note that $\mathcal{Z}$ is a nonempty convex compact set along with Z.

Building $\mathcal{Z}$

We have justified (!!); note, however, that this claim is a kind of existence theorem – it just states that a desired $\mathcal{Z}$ exists; at the same time, the construction of $\mathcal{Z}$, as offered by the justification of (!!), is rather complicated and, in general, difficult to carry out explicitly in the general case. In fact, this is not a big deal: all that we finally interested in is the resulting convex approximation (2.3.7) of the ambiguous chance constraint (2.3.8), along with conditions under which it is tractable, and this information is readily given by Theorem 2.1: the approximation is (2.3.7), and it is tractable when Ψ^+ is efficiently computable. Whether this approximation is a Robust Counterpart one and what is the associated uncertainty set – these issues are more academic and aesthetical than practical. For the sake of aesthetics, which is important by its own right, here is a rather general case where $\mathcal{Z}$ can be pointed out explicitly. This is the case where the function Ψ^+ is regular and in addition we have in our disposal a *Fenchel-type* representation of Ψ^+, that is, representation of the form

$$\Psi^+(w) = \sup_u \left\{ w^T [Bu + b] - \phi(u) \right\}, \tag{2.3.18}$$

where $\phi(\cdot)$ is a lower semicontinuous convex function on certain $\mathbb{R}^m$ possessing bounded level sets $\{u : \phi(u) \leq c\}$, $c \in \mathbb{R}$, and B, b are a matrix and a vector of appropriate dimensions.

It is well known that every *proper* (i.e., with a nonempty domain) convex lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ admits *Fenchel representation*:

$$f(w) = \sup_z \left\{ w^T z - \phi(z) \right\},$$

ϕ being the Fenchel transform of f:

$$\phi(z) = \sup_w \left\{ z^T w - f(w) \right\}.$$

The the latter function indeed is proper, convex and lower semicontinuous; it has bounded level sets, provided that $\mathrm{Dom}\, f = \mathbb{R}^n$. It should be added that Fenchel-type (note: Fenchel-type, not the Fenchel!) representations admit an "algorithmic calculus" and thus usually are available in a closed form.

Proposition 2.2 [3, section 4.3.2] *In the situation of Theorem 2.1, let Ψ^+ be regular and given by* (2.3.18). *Given ϵ, let us set*

$$\mathcal{U} = \{u : \phi(u) \leq -\epsilon\}.$$

Then $\mathcal{U}$ is a nonempty compact convex set, and

$$G^+(w) \leq 0 \Leftrightarrow \max_{u\in\mathcal{U}} w^T(Bu + b) \leq 0 \;\forall u \in \mathcal{U}. \tag{2.3.19}$$

In other words, the safe convex approximation (2.3.7) *of the ambiguous chance constraint* (2.3.8) *is nothing but the Robust Counterpart of the uncertain linear constraint*

$$\sum_{\ell=0}^{L} z_\ell w_\ell \leq 0, \tag{2.3.20}$$

in variables w, the uncertainty set being

$$\mathcal{Z} = \{z = Bu + b : u \in \mathcal{U}\}.$$

The Proposition is an immediate corollary of the following

Lemma 2.3 *Let $H(w) : \mathbb{R}^{L+1} \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function such that $0 \in \mathrm{Dom}\, H$, and a be a real such that $H(0) > a$, and let H admit a representation*

$$H(w) = \sup_u \left[w^T[Bu + b] - h(u)\right]$$

with lower semicontinuous convex function h with bounded level sets. Then the set

$$\mathcal{U} = \{u : h(u) + a \leq 0\}$$

is a nonempty convex compact set, and

$$\{w : \inf_{\alpha>0} \alpha[H(w/\alpha) - a] \leq 0\} = \{w : w^T[Bu + b] \leq 0 \;\forall u \in \mathcal{U}\}. \tag{2.3.21}$$

Proof. 1^0. We have $H(0) = -\inf_u h(u)$ and this infimum is achieved (since h has bounded level sets and is lower semicontinuous). Thus

$$\exists u_* : h(u_*) = -H(0) < -a,$$

whence the set $\mathcal{U}$ is nonempty. Its convexity and compactness follow from the fact that h is lower semicontinuous, convex and has bounded level sets.
2^0. Let us denote by W_ℓ, W_r the left- and the right hand side sets in (2.3.21), and let us prove first that $W_\ell \subset W_r$. In other words, we should prove that if $w \notin W_r$, then $w \notin W_\ell$. To this end note that for $w \notin W_r$ there exists $\bar{u}$ such that $h(\bar{u}) + a \leq 0$ and $w^T[B\bar{u} + b] > 0$. It follows that

$$\alpha > 0 \Rightarrow \alpha[H(w/\alpha) - a] \geq \alpha[(w/\alpha)^T[B\bar{u} + b] - h(\bar{u}) - a] = w^T[B\bar{u} + b] - \alpha[h(\bar{u}) + a] \geq w^T[B\bar{u} + b],$$

and thus $\inf_{\alpha>0} \alpha[H(w/\alpha) - a] \geq w^T[B\bar{u} + b] > 0$, so that $w \notin W_\ell$, as claimed.
3^0. It remains to prove that if $w \in W_r$, then $w \in W_\ell$. Let us fix $w \in W_r$; we should prove that for every $\delta > 0$ there exists $\bar{\alpha} > 0$ such that

$$\bar{\alpha}[H(w/\bar{\alpha}) - a] \leq \delta. \tag{2.3.22}$$

Observe, first, that whenever $u \in \mathcal{U}$, there exists $\alpha_u > 0$ such that

$$w^T[Bu + b] - \alpha_u(h(u) + a) < \delta/2$$

(since $w^T[Bu+b] \leq 0$ due to $w \in W_r$). Since $-h(\cdot)$ is upper semicontinuous, there exists a neighborhood V_u of u such that

$$\forall (u' \in V_u) : w^T[Bu' + b] - \alpha_u(h(u') + a) < \delta/2.$$

Since $\mathcal{U}$ is a compact set, we see that there exist finitely many positive reals $\alpha_1, ..., \alpha_N$ such that

$$\forall u \in \mathcal{U} : \min_{1 \leq i \leq N} \left[w^T[Bu + b] - \alpha_i[h(u) + a] \right] \leq \delta/2. \tag{2.3.23}$$

Now let $R < \infty$ be such that $\|u - u_*\|_2 \leq R$ whenever $u \in \mathcal{U}$, and let $\alpha_+ > \frac{R\|B^T w\|_2}{H(0)-a}$. We claim that

$$u \notin \mathcal{U} \Rightarrow w^T[Bu + b] - \alpha_+[h(u) + a] \leq 0. \tag{2.3.24}$$

Indeed, let us fix $u \notin \mathcal{U}$, and let us prove that the conclusion in (2.3.24) holds true for our u. There is nothing to prove when $u \notin \operatorname{Dom} h$, thus assume that $u \in \operatorname{Dom} h$. Let $u_t = u_* + t(u - u_*)$, and let $\eta(t) = h(u_t) + a$, $0 \leq t \leq 1$. The function $\eta(\cdot)$ is convex, finite and lower semicontinuous on $[0, 1]$ and thus is continuous on this segment. Since $\eta(0) = -H(0) + a < 0$ and $\eta(1) > 0$ due to $u \notin \mathcal{U}$, there exists $\bar{t} \in (0, 1)$ such that $\eta(\bar{t}) = 0$, meaning that $u_{\bar{t}} \in \mathcal{U}$ and thus $w^T[Bu_{\bar{t}} + b] \leq 0$. Since $\|u_{\bar{t}} - u_*\|_2 \leq R$, we have

$$w^T[Bu + b] = w^T \left[b + B \left[u_{\bar{t}} + \frac{1 - \bar{t}}{\bar{t}}(u_{\bar{t}} - u_*) \right] \right] \leq \frac{1 - \bar{t}}{\bar{t}} w^T B[u_{\bar{t}} - u_*] \leq \frac{1 - \bar{t}}{\bar{t}} R\|B^T w\|_2.$$

On the other hand, we have $0 = \eta(\bar{t}) \leq \bar{t}\eta(1) + (1 - \bar{t})\eta(0)$, whence

$$\eta(1) \geq -\frac{1 - \bar{t}}{\bar{t}} \eta(0) = \frac{1 - \bar{t}}{\bar{t}} [H(0) - a].$$

We conclude that $w^T[Bu+b] - \alpha_+[h(u)+a] \leq \frac{1-\bar{t}}{\bar{t}} R\|B^T w\|_2 - \alpha_+ \frac{1-\bar{t}}{\bar{t}} [H(0) - a] \leq 0$, where the concluding inequality is given by the definition of α_+. We have proved (2.3.24).
3^0. Setting $\alpha_{N+1} = \alpha_+$ and invoking (2.3.23), we see that

$$\min_{1 \leq i \leq N+1} \left[w^T[Bu + b] - \alpha_i[h(u) + a] \right] \leq \delta/2 \ \forall u \in \operatorname{Dom} h,$$

meaning that

$$\sup_{u \in \operatorname{Dom} h} \min_{\substack{\lambda \geq 0, \\ \sum_{i=1}^{N+1} \lambda_i = 1}} \sum_{i=1}^{N+1} \lambda_i \left[w^T[Bu + b] - \alpha_i[h(u) + a] \right] \leq \delta/2;$$

invoking Sion-Kakutani Theorem,[4] we conclude that there exists a convex combination

$$\sum_{i=1}^{N+1} \lambda_i^* \left[w^T[Bu + b] - \alpha_i[h(u) + a] \right] = w^T[Bu + b] - \bar{\alpha}[h(u) + a],$$

$\bar{\alpha} = \sum_{i=1}^{N+1} \lambda_i \alpha_i > 0$, such that $w^T[Bu + b] - \bar{\alpha}[h(u) + a] \leq \delta$ on the domain of h, meaning that

$$\bar{\alpha}[H(w/\bar{\alpha}) - a] = \sup_u \left[w^T[Bu + b] - \bar{\alpha}[h(u) + a] \right] \leq \delta,$$

as required in (2.3.22). □

The above Proposition has an "aesthetical drawback:" in (2.3.20), the coefficients of *all* variables $w_0, ..., w_L$ are uncertain, while in (2.3.8) the coefficient of w_0 is certain – it is equal to 1. In [3, section 4.3.2] it is explained how to get rid of this drawback.

[4] This standard theorem of Convex Analysis reads as follows: *Let X, Y be nonempty convex sets with X being compact, and $f(x, y) : X \times Y \to \mathbb{R}$ be a function which is convex and lower semicontinuous in $x \in X$ and concave and upper semicontinuous in $y \in Y$. Then* $\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y)$.

2.4 Implementing the Approximation Scheme

2.4.1 "Ideal implementation" and Conditional Value at Risk

A natural question in the situation of Theorem 2.1 is how to choose the best, if any, generator $\gamma(\cdot)$ and the best Ψ^*. When all we are interested in is to minimize the conservatism of the approximation (2.3.7) of the ambiguous chance constraint (2.3.8), the answer is clear: under extremely mild boundedness assumptions, *the best* $\gamma(\cdot)$ *and* $\Psi^+(\cdot)$ *are*

$$\gamma_*(s) = \max[1+s, 0], \quad \Psi_*^+(w) = \sup_{P\in\mathcal{P}} \mathbf{E}_{\zeta\sim P}\left\{\gamma_*(w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell)\right\}. \tag{2.4.1}$$

The regularity assumptions in question merely require the expectations $\mathbf{E}_{\zeta\sim P}\{\|\zeta\|_2\}$ to be bounded uniformly in $P \in \mathcal{P}$. Note that in this case (which we assume to take place from now on) Ψ_*^+ is regular and $\mathrm{Dom}\,\Psi_*^+ = \mathbb{R}^{L+1}$.

Indeed, after a generator is chosen, the best possible, in terms of the conservatism of the resulting approximation of (2.3.8), choice of Φ^+ clearly is

$$\Phi_\gamma^+ = \sup_{P\in\mathcal{P}} \mathbf{E}_{\zeta\sim P}\left\{\gamma(Z_w[\zeta])\right\}.$$

Now let γ be a generator. Since γ is convex and nonnegative, $\gamma(0) \geq 1$ and $\gamma(s) \to 0$ as $s \to -\infty$, we have $a := \gamma'(0+) > 0$. By convexity and due to $\gamma(0) \geq 1$, we have $\gamma(s) \geq 1 + as$ for all s, and since γ is nonnegative, it follows that

$$\gamma(s) \geq \bar{\gamma}(s) := \max[1 + as, 0] = \gamma_*(as).$$

By this inequality, replacing γ with $\gamma_*(as)$ (the latter function is a generator due to $a > 0$), we can only Φ_γ^+, thus reducing the conservatism of (2.3.7). It remains to note $\Phi_{\bar{\gamma}}^+(w) = \Phi_*^+(aw)$, meaning that $\Phi_{\bar{\gamma}}^+$ and Φ_*^+ lead to proportional to each other, with a positive coefficient, functions G^+ and thus – to the equivalent to each other approximations (2.3.7).

The case of known P. In the case when $\mathcal{P} = \{P\}$ is a singleton, the approximation (2.3.7) associated with (2.4.1) after simple manipulations (carry them out!) takes the form

$$\mathrm{CVaR}_\epsilon(w) := \inf_{a\in\mathbb{R}}\left[a + \frac{1}{\epsilon}\mathbf{E}\left\{[\max[Z_w[\zeta] - a, 0]\right\}\right] \leq 0. \tag{2.4.2}$$

The left hand side in this relation is nothing but the famous *Conditional Value at Risk*, the level of risk being ϵ, of the random variable $Z_w[\zeta]$. This is the best known so far convex in w upper bound on the *Value at Risk*

$$\mathrm{VaR}_\epsilon(w) = \min_{a\in\mathbb{R}}\left\{a : \mathrm{Prob}_{\zeta\sim P}\{Z_w[\zeta] > a\} \leq \epsilon\right\}.$$

of this random variable. Note that in the non-ambiguous case, the chance constraint (2.3.8) is nothing but the constraint $\mathrm{VaR}_\epsilon(w) \leq 0$, the CVaR *approximation* (2.4.2) is therefore obtained from the "true" constraint by replacing $\mathrm{VaR}_\epsilon(w)$ with its convex upper bound $\mathrm{CVaR}_\epsilon(w)$.

While being safe and convex, the CVaR approximation typically is intractable: computing $\mathrm{CVaR}_\epsilon(w)$ reduces to multi-dimensional integration; the only computationally meaningful general technique for solving the latter problem is Monte Carlo simulation, and this technique becomes prohibitively time consuming when ϵ is small, like 10^{-5} or less. Seemingly, the only generic case when the CVaR approximation is tractable is the one where the distribution P of ζ is supported on a finite set $\{\zeta^1, ..., \zeta^N\}$ of moderate cardinality N, so that

$$\Psi_z^+(w) = \sum_{i=1}^N \pi_i \max\left[0, 1 + w_0 + \sum_{\ell=1}^L \zeta_\ell^i w_\ell\right] \qquad [\pi_i = \mathrm{Prob}\{\zeta = \zeta^i\}]$$

In this case, the RC representation of the CVaR approximation is

$$\begin{array}{c} \mathrm{CVaR}_\epsilon(w) \le 0 \Leftrightarrow w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell \le 0\ \forall \zeta \in \mathcal{Z}, \\ \mathcal{Z} = \{\zeta = \sum_{i=1}^N u_i \zeta^i : 0 \le u_i \le \pi_i/\epsilon, \sum_i u_i = 1\}, \end{array}$$

see [3, Proposition 4.3.3].

2.4.2 "Tractable case:" Bernstein Approximation

We are about to demonstrate that *under reasonable assumptions on $\mathcal{P}$, the generator $\gamma(s) =$* $\exp\{s\}$ which we have used in Example 2.1 *results in computationally tractable approximation* (2.3.7) *of the ambiguous chance constraint* (2.3.8).

The assumptions in question are as follows:

Brn.1. *$\mathcal{P}$ is comprised of all product-type probability distributions $P = P_1 \times ... \times P_L$ with marginals P_ℓ running, independently of each other, in respective families $\mathcal{P}_\ell$, $1 \le \ell \le L$;* here $\mathcal{P}_\ell$ is a given family of probability distributions on $\mathbb{R}$.

Brn.2. *The functions*

$$\Phi_\ell^*(t) := \sup_{P_\ell \in \mathcal{P}_\ell} \ln\left(\mathbf{E}_{s\sim P_\ell}\{\exp\{ts\}\}\right)$$

are convex and lower semicontinuous (this is automatic) *and* $0 \in \mathrm{int}\mathrm{Dom}\,\Phi_\ell^*$ (this indeed is an assumption), and we have in our disposal efficiently computable lower semicontinuous convex functions

$$\Phi_\ell^+(\cdot) \ge \Phi_\ell^*(\cdot)$$

with $0 \in \mathrm{int}\mathrm{Dom}\,\Phi_\ell^+$.

Under these assumptions, for every $P = P_1 \times ... \times P_L \in \mathcal{P}$ we have

$$\mathbf{E}_{\zeta\sim P}\left\{\exp\{w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell\}\right\} = \exp\{w_0\}\prod_{\ell=1}^L \mathbf{E}_{\zeta_\ell\sim P_\ell}\{\exp\{w_\ell\zeta_\ell\}\} \le \exp\{w_0\}\prod_{\ell=1}^L \exp\{\Phi_\ell^*(w_\ell)\}.$$

It follows that setting

$$\Psi^+(w) = \exp\{w_0\}\prod_{\ell=1}^L \exp\{\Phi_\ell^+(w_\ell)\},$$

we meet the requirements imposed on Ψ^+ in Theorem 2.1, so that the condition

$$\exists \alpha > 0 : \Psi^+(\alpha^{-1}w) \le \epsilon \qquad (2.4.3)$$

is sufficient for w to satisfy (2.3.8).

Now let us proceed similarly to the derivation of (2.3.7), but not exactly so; it is now convenient to rewrite (2.4.3) in the equivalent form

$$\exists \alpha > 0 : \ln \Psi^{+}(\alpha^{-1} w) \leq \ln(\epsilon).$$

Taking into account the structure of Ψ^{+}, the latter condition reads

$$\begin{array}{c} \exists \alpha > 0 : w_0 + \alpha \Phi(\alpha^{-1}[w_1;...;w_L]) + \alpha \ln(1/\epsilon) \leq 0 \\ \Phi(z) = \sum\limits_{\ell=1}^{L} \Phi_\ell^{+}(z_\ell) \end{array} \tag{2.4.4}$$

Observe that $\Phi(w)$ by construction is convex and lower semicontinuous, $0 \in \mathrm{int}\mathrm{Dom}\,\Phi$, and that $\Phi(te) \to -\infty$ as $t \to +\infty$; here, as above, $e = [-1; 0; ...; 0] \in \mathbb{R}^{L+1}$. It is not difficult to see (see [3] for all proofs we are skipping here) that the sufficient condition (2.4.4) for w to satisfy (2.3.8) can be weakened to

$$\begin{array}{l} H(w) := w_0 + \inf_{\alpha>0} \left\{ \alpha \Phi(\alpha^{-1}[w_1;...;w_L]) + \alpha \ln(1/\epsilon) \right\} \\ \equiv w_0 + \inf\limits_{\alpha>0} \left[\sum_{\ell=1}^{L} \alpha \Phi_\ell^{+}(\alpha^{-1} w_\ell) + \alpha \ln(1/\epsilon) \right] \leq 0 \end{array} \tag{2.4.5}$$

We shall call the latter sufficient condition for the validity of (2.3.8) the *Bernstein approximation* of the ambiguous chance constraint (2.3.8). The main properties of this approximation are "parallel" to those of (2.3.7). These properties are summarized in the following result (the proof is completely similar to those of Lemma 2.2 and Proposition 2.2):

Theorem 2.2 *Let assumptions Brn.1 – Brn.2 take place. Then*

(i) (2.4.5) *is a safe tractable convex approximation of the ambiguous chance constraint* (2.3.8)*, and the feasible set W of this approximation is a closed convex cone which contains the vector* $e = [-1; 0; ...; 0]$ *in its interior.*

(ii) *The approximation* (2.4.5) *is of the Robust Counterpart form:*

$$W = \{w : w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell \leq 0 \ \forall \zeta \in \mathcal{Z}\} \tag{2.4.6}$$

with properly chosen nonempty convex compact uncertainty set $\mathcal{Z}$. Assuming that Φ (see (2.4.4)) *is given by Fenchel-type representation*

$$\Phi([w_1;...;w_L]) = \sup_u \left[[w_1;...;w_L]^T (Bu + b) - \phi(u) \right]$$

where ϕ is convex, lower semicontinuous and possesses bounded level sets, one can take

$$\mathcal{Z} = \{\zeta = Bu + b : \phi(u) \leq \ln(1/\epsilon)\}.$$

2.5 Bernstein Approximation: Examples and Illustrations

In this section, we consider the Bernstein approximations of the ambiguous chance constraint (2.3.8), the assumptions Brn.1 – Brn.2 being in force all the time. Thus, we all the time speak about the case when $\zeta_1, ..., \zeta_L$ are independent, changing from example to example our assumptions on what are the families $\mathcal{P}_\ell$ where the marginal distributions of ζ_ℓ run.

2.5.1 Example 2.1 revisited: Entropy, Ball, Ball-Box, Budgeted and Box approximations

Everywhere in this subsection we assume that the families $\mathcal{P}_\ell$, $q \leq \ell \leq L$, are comprised of all zero mean probability distributions supported on $[-1,1]$, cf. Example 2.1.[5]

Entropy approximation

When processing Example 2.1, we have seen that

$$\Phi_\ell^* := \sup_{P_\ell \in \mathcal{P}_\ell} \ln\left(\mathbf{E}_{\zeta_\ell \sim P_\ell}\{\exp\{t\zeta_\ell\}\}\right) = \ln(\cosh(t)).$$

Thus, the best – the least conservative under the circumstances – Bernstein approximation of (2.3.8) is

$$w_0 + \inf_{\alpha>0} \alpha \left[\sum_{\ell=1}^{L} \ln(\cosh(\alpha^{-1} w_\ell)) + \ln(1/\epsilon)\right] \leq 0. \tag{2.5.1}$$

The corresponding function Φ, see Theorem 2.2, is $\Phi(z) = \sum_{\ell=1}^{L} \ln(\cosh(z_\ell))$. It is easy to find the Fenchel representation of Φ:

$$\Phi(z) = \sup_{u:\|u\|_\infty<1} \left[z^T u - \frac{1}{2}\sum_{\ell=1}^{L}[(1+u_\ell)\ln(1+u_\ell) + (1-u_\ell)\ln(1-u_\ell)]\right].$$

It follows that the set $\mathcal{Z}$ from the Robust Counterpart representation (2.4.6) of (2.5.1) is

$$\mathcal{Z}^{\rm Entr} = \left\{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1, \sum_{\ell=1}^{L}[(1+\zeta_\ell)\ln(1+\zeta_\ell) + (1-\zeta_\ell)\ln(1-\zeta_\ell)] \leq 2\ln(1/\epsilon)\right\}. \tag{2.5.2}$$

Ball approximation

When processing Example 2.1, we, in fact, were applying the Bernstein approximation, but with the majorant $\Phi^+(z) = \frac{1}{2}\sum\limits_{\ell=1}^{L} z_\ell^2$ of the above Φ in the role of Φ, and ended up with the Robust Counterpart type safe tractable approximation of (2.3.8) with the Euclidean ball $\mathcal{Z}^{\rm Ball} = \{z : \|z\|_2 \leq \sqrt{2\ln(1/\epsilon)}$ in the role of $\mathcal{Z}$. We can recover the latter approximation directly from the Entropy one, by noting that $(1+s)\ln(1+s) + (1-s)\ln(1-s) \geq s^2$ when $|s| \leq 1$, that is, $\mathcal{Z}^{\rm Entr} \subset \mathcal{Z}^{\rm Ball}$, so that the Ball approximation of (2.3.8) – the one with $\mathcal{Z} = \mathcal{Z}^{\rm ball}$ – is more conservative than the Entropy one and thus is safe along with the latter. The explicit form of the Ball approximation is

$$w_0 + \sqrt{2\ln(1/\epsilon)}\sqrt{\sum_{\ell=1}^{L} w_\ell^2} \leq 0,$$

and its Robust counterpart form is

$$w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell \leq 0 \ \forall \zeta \in \mathcal{Z}^{\rm Ball}$$

[5]Note that by scalings $\zeta_\ell \mapsto a_\ell + b_\ell\zeta_\ell$ we can reduce to this case the one where ζ_ℓ take values from a known segment, perhaps depending on ℓ, and the expectation of ζ_ℓ is known to be the midpoint of the segment.

Ball-Box approximation

As we have seen, the set $\mathcal{Z}^{\text{Entr}}$, which is clearly contained in the box $\mathcal{Z}^{\text{Box}} = \{z \in \mathbb{R}^L : \|z\|_\infty \leq 1\}$, is contained also in the ball $\mathcal{Z}^{\text{Ball}}$; it follows that when replacing the uncertainty set $\mathcal{Z}^{\text{Entr}}$ participating in the RC representation the entropy approximation with the larger set

$$\mathcal{Z}^{\text{BallBox}} = \mathcal{Z}^{\text{Ball}} \cap \mathcal{Z}^{\text{Box}} = \{z \in \mathbb{R}^L : \|z\|_\infty \leq 1 \,\&\|z\|_2 \leq \sqrt{2\ln(1/\epsilon)}\},$$

we get a safe tractable approximation of (2.3.8) which is more conservative than the Entropy one and is less conservative than the plain Box approximation. An explicit representation of the Ball-Box approximation of (2.3.8) is given by the conic quadratic constraint

$$w_0 + \sum_{\ell=1}^{L} |w_\ell - u_\ell| + \sqrt{2\ln(1/\epsilon)}\sqrt{\sum_{\ell=1}^{L} u_\ell^2} \leq 0$$

in variables w and additional variables u.

The latter claim is an immediate conclusion of the following well known fact (prove it!): if B, C are two convex compact sets with intersecting interiors in $\mathbb{R}^n$, then for every c one has

$$\max_{x\in A\cap B} c^T x = \min_{a,b:a+b=c}\left[\max_{x\in A} a^T x + \max_{x\in B} b^T x\right].$$

Budgeted approximation

For every vector $x \in \mathbb{R}^L$ one has $\|x\|_1 \leq \sqrt{L}\|x\|_2$; this, $\mathcal{Z}^{\text{BallBox}}$ is contained in the set $\mathcal{Z}^{\text{Budg}} = \mathcal{Z}^{Box} \cap \{z : \|z\|_1 \leq \sqrt{2L\ln(1/\epsilon)}\}$. Using this set in the role of $\mathcal{Z}$, we get what is called *Budgeted* approximation of (2.3.8) which is more conservative than the Ball-Box one and thus is safe. An explicit polyhedral representation of the Budgeted approximation is given by the constraint

$$w_0 + \sum_{\ell=1}^{L} |w_\ell - u_\ell| + \sqrt{2L\ln(1/\epsilon)} \max_{1\leq\ell\leq L} |u_\ell| \leq 0$$

in variables w and in additional variables u. While being more conservative than the Ball-Box one, the Budgeted approximation, introduced by Bertsimas and Sim, is very popular, since the approximating constraint can be represented by a system of *linear* inequality constraints, and thus we do not leave the realm of Linear Optimization.

Box approximation

This is the extreme case of the above constructions, where we use, as $\mathcal{Z}$, the box $\mathcal{Z}^{\text{box}}$, thus ignoring all information on the stochastic nature of perturbations and utilizing the only the assumption that ζ always takes its values in $\mathcal{Z}^{\text{box}}$. The polyhedral representation of the resulting *Box* approximation of (2.3.8) is

$$w_0 + \sum_{\ell=1}^{L} |w_\ell| \leq 0,$$

and this approximation is more conservative (and can be *much* more conservative when L is large) than all other approximations we have developed, except, perhaps the Ball one. As a compensation, the Box approximation is independent of ϵ and on our assumption of independence ζ_ℓ. When ϵ is "extremely small", so that $\sqrt{2\ln(1/\epsilon)} \geq \sqrt{L}$ (for not too small L, these values of ϵ are by far too small to be of any practical interest), all approximations we have built, except for the Ball one, coincide with the Box approximation.

How it works: Single Period Portfolio Selection

Let us apply the outlined techniques to the following single-period portfolio selection problem:

> There are 200 assets. Asset # 200 ("money in the bank") has yearly return $r_{200} = 1.05$ and zero variability. The yearly returns r_ℓ, $\ell = 1, ..., 199$ of the remaining assets are independent random variables taking values in the segments $[\mu_\ell - \sigma_\ell, \mu_\ell + \sigma_\ell]$ with expected values μ_ℓ; here
>
> $$\mu_\ell = 1.05 + 0.3\frac{200-\ell}{199},\ \sigma_\ell = 0.05 + 0.6\frac{200-\ell}{199},\ \ell = 1, ..., 199.$$
>
> The goal is to distribute \$1 between the assets in order to maximize the value-at-risk of the resulting portfolio, the required risk level being $\epsilon = 0.5\%$.

We want to solve the uncertain LO problem

$$\max_{y,t}\left\{t : \sum_{\ell=1}^{199} r_\ell y_\ell + r_{200}y_{200} - t \geq 0,\ \sum_{\ell=0}^{200} y_\ell = 1, y_\ell \geq 0\,\forall \ell\right\},$$

where y_ℓ is the capital to be invested in asset # ℓ. The uncertain data are the returns r_ℓ, $\ell = 1, ..., 199$; their natural parameterization is

$$r_\ell = \mu_\ell + \sigma_\ell\zeta_\ell,$$

where ζ_ℓ, $\ell = 1, ..., 199$, are independent random perturbations with zero mean varying in the segments $[-1, 1]$. Setting $x = [y; -t] \in \mathbb{R}^{201}$, the problem becomes

$$\begin{array}{lll}
\text{minimize} & x_{201} & \\
\text{subject to} & & \\
(a) & [a^0 + \sum\limits_{\ell=1}^{199}\zeta_\ell a^\ell]^T x - [b^0 + \sum\limits_{\ell=1}^{199}\zeta_\ell b^\ell] \leq 0 & \\
(b) & \sum\limits_{j=1}^{200} x_\ell = 1 & \\
(c) & x_\ell \geq 0,\ \ell = 1, ..., 200 &
\end{array} \tag{2.5.3}$$

where

$$\begin{array}{l}
a^0 = [-\mu_1; -\mu_2; ...; -\mu_{199}; -r_{200}; -1];\\
a^\ell = \sigma_\ell \cdot [0_{\ell-1,1}; 1; 0_{201-\ell,1}],\ 1 \leq \ell \leq 199;\\
b^\ell = 0,\ 0 \leq \ell \leq 199.
\end{array} \tag{2.5.4}$$

The only uncertain constraint in the problem is the inequality (2.5.3.a), and this constraint fits the framework of Example 2.1. We consider 3 of the above safe tractable approximations of the chance version of this constraint, with ϵ set to 0.005, which results in $\Omega := \sqrt{2\ln(1/\epsilon)} \approx 3.255$. We consider three of the above safe tractable approximations of the uncertain constraint in question – the Box, the Ball-Box and the Budgeted ones.

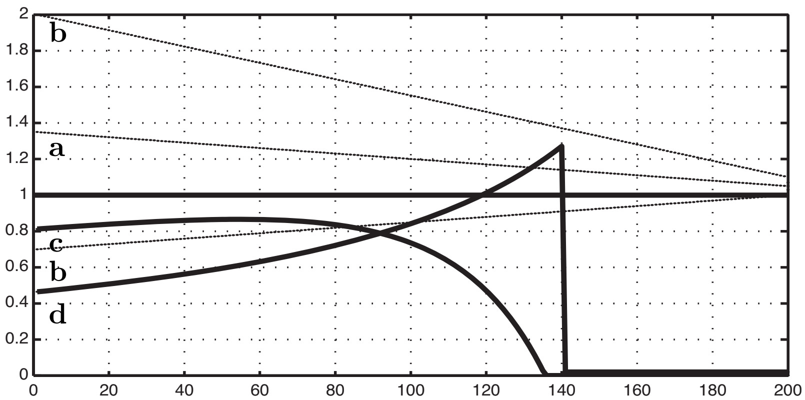


Figure 2.1: Robust solutions to portfolio selection problem. Along the x-axis: indices 1,2,...,200 of the assets. **a**: expected returns, **b**: upper and lower endpoints of the return ranges, **c**: invested capital for ball-box RC, %, **d**: invested capital for Budgeted RC, %.

Box approximation. With this approximation, the resulting robust version of the problem of interest after straightforward computations becomes the LO program

$$\max_{y,t}\left\{t:\begin{array}{l}\sum\limits_{\ell=1}^{199}(\mu_\ell-\sigma_\ell)y_\ell+1.05y_{200}\geq t\\ \sum\limits_{\ell=1}^{200}y_\ell=1,\ y\geq 0\end{array}\right\}; \tag{2.5.5}$$

as it should be expected, this is nothing but the instance of our uncertain problem corresponding to the worst possible values $r_\ell=\mu_\ell-\sigma_\ell$, $\ell=1,...,199$, of the uncertain returns. Since these values are less than the guaranteed return for money, the robust optimal solution prescribes to keep our initial capital in the bank, with a guaranteed yearly return of 1.05, that is, a guaranteed profit of 5%.

Ball-Box approximation. Here the robust version of the problem of interest is the conic quadratic program

$$\max_{y,z,w,t}\left\{t:\begin{array}{l}\sum\limits_{\ell=1}^{199}\mu_\ell y_\ell+1.05y_{200}-\sum\limits_{\ell=1}^{199}|z_\ell|-3.255\sqrt{\sum\limits_{\ell=1}^{199}w_\ell^2}\geq t\\ z_\ell+w_\ell=\sigma_\ell y_\ell,\ \ell=1,...,199,\ \sum\limits_{\ell=1}^{200}y_\ell=1,\ y\geq 0\end{array}\right\}. \tag{2.5.6}$$

The robust optimal value is 1.1200, meaning 12.0% profit with risk as low as $\epsilon=0.5\%$. The distribution of capital between assets is depicted in figure 2.1.

Budgeted RC. Here the robust version of the problem of interest is the LO problem

$$\max_{y,z,w,t}\left\{t:\begin{array}{l}\sum\limits_{\ell=1}^{199}\mu_\ell y_\ell+1.05y_{200}-\sum\limits_{\ell=1}^{199}|z_\ell|-45.921\max\limits_{1\leq\ell\leq 199}|w_\ell|\geq t\\ z_\ell+w_\ell=\sigma_\ell y_\ell,\ \ell=1,...,199,\ \sum\limits_{\ell=1}^{200}y_\ell=1,\ y\geq 0\end{array}\right\}. \tag{2.5.7}$$

The robust optimal value is 1.1014, meaning 10.1% profit with risk as low as $\epsilon=0.5\%$. The distribution of capital between assets is depicted in figure 2.1.

Discussion. First, we see how useful stochastic information might be — with risk as low as 0.5%, the value-at-risk of the portfolio profits yielded by the Ball-Box (12%) and the Budgeted (10%) approximations are twice as large as the profit guaranteed by the box approximation (5%). Note also that both the Ball-Box and the Budgeted approximations suggest "active" investment decisions, while the box RC suggests keeping the initial capital in bank. Second, the Budgeted RC, as it should be, is more conservative than the ball-box one. Finally, we should remember that the actual risk associated with the portfolio designs offered by the Ball-Box and the Budgeted approximations (that is, the probability for the actual total yearly return to be less than the corresponding robust optimal value) is *at most* the required 0.5%, and is likely to be less than this amount; indeed, all our approximations are safe and thus conservative.

It is interesting to find out how small the actual risk is. The answer, of course, depends on the actual probability distributions of uncertain returns (recall that in our model, we postulated only partial knowledge of these distributions, specifically, knowledge of their supports and expectations). Assuming that "in reality" ζ_ℓ, $\ell = 1, ..., 199$, take only their extreme values ± 1, with probability 1/2 each, and carrying out a Monte-Carlo simulation with a sample of 1,000,000 realizations, we found that the actual risk for the "Ball-Box" portfolio is less than the required risk 0.5% by factor 10, and for the "Budgeted" portfolio, by factor 50. Based on this observation, it seems plausible that we can reduce our conservatism by "tuning," that is, by replacing the required risk in the approximations with a larger quantity, in hope that the resulting actual risk, (which can be evaluated via simulation), will still be below the required level. With this tuning, reducing the coefficient 3.255 in (2.5.6) to 2.589, one ends up with the robust optimal value 1.1470 (that is, with a profit of 14.7% instead of the initial 12.0%), while keeping the empirical risk (as evaluated over 500,000 realization sample) still as low as 0.47%. Similarly, reducing the "uncertainty budget" 45.921 in (2.5.7) to 30.349, we increase the robust optimal value from 1.1012 to 1.1395 (i.e., increase profit from 10.12% to 13.95%), with the empirical risk as low as 0.42%.

2.5.2 More examples

Under assumption Brn.1, which is in force in what follows, all we need in order to apply to (2.3.8) the Bernstein approximation scheme in a computationally efficient fashion are efficiently computable convex upper bounds $\Phi_\ell^+(\cdot) : \mathbb{R} \to \mathbb{R}$ on the worst-case, over $\mathcal{P}_\ell$, logarithmic moment-generating functions

$$\Phi_\ell^*(t) = \sup_{P_\ell \in \mathcal{P}_\ell} \ln\left(\mathbf{E}_{\zeta_\ell \sim P_\ell}\{\exp\{t\zeta_\ell\}\}\right).$$

The "ideal case" here is the one when we can choose as Φ_ℓ^+ the functions Φ_ℓ^* themselves, thus arriving at the least conservative, under circumstances, versions of the Bernstein approximation. We are about to list a number of these "ideal cases." Below ℓ is a *fixed* index from $\{1, ..., L\}$. We skip the proofs of the claims to follow; all these proofs can be found in [3, Chapter 2]. More to the point, finding the proofs can be considered as a series of instructive exercises for the reader. Note that in the examples to follow, to avoid messy formulas, we normalize the situation "to the largest possible extent;" e.g., when speaking about the case when all distributions from $\mathcal{P}_\ell$ are supported on a common finite segment, we restrict ourselves with the case when this segment is $[-1, 1]$, keeping in mind that the results can be extended straightforwardly from this normalized case to the general one by appropriate scaling $\zeta_\ell \mapsto a_\ell \zeta_\ell + b_\ell$.

1. Gaussian distributions. Let $\mathcal{P}_\ell$ be the set of all Gaussian probability distributions on the axis with expectation μ varying in a given finite segment $[\mu_\ell^-, \mu_\ell^+]$ and the standard devi-

ation bounded by a given finite and positive quantity ν_ℓ. In this case

$$\Phi_\ell^*(t) = \max_{\mu_\ell^- \le \mu \le \mu_\ell^+} \mu t + \frac{\nu_\ell^2}{2} t^2.$$

2. <u>*Bounded support:*</u> $\mathcal{P}_\ell$ is comprised of all probability distributions supported on $[-1,1]$. Here

$$\Phi_\ell^*(t) = |t|.$$

3. <u>*Bounded support plus unimodality:*</u> $\mathcal{P}_\ell$ is comprised of all probability distributions supported on $[-1,1]$ which are *unimodal*, that is, are convex combinations of two probability distributions: one is the unit mass at the origin, and the other one has density $p(\cdot)$ which vanishes outside $[-1,1]$ and is nondecreasing to the left of 0 and nonincreasing to the right of 0. Here

$$\Phi_\ell^*(t) = \ln\left(\frac{\exp\{|t|\} - 1}{|t|}\right).$$

4. <u>*Bounded support plus unimodality plus symmetry:*</u> $\mathcal{P}_\ell$ is comprised of all *symmetric w.r.t. 0* distributions from the previous item. Here

$$\Phi_\ell^*(t) = \ln\left(\frac{\sinh(t)}{t}\right).$$

5. <u>*Bounded support plus symmetry:*</u> $\mathcal{P}_\ell$ is comprised of all symmetric probability distributions supported on $[-1,1]$. Here

$$\Phi_\ell^*(t) = \ln(\cosh(t)),$$

cf. Example 2.1.

6. <u>*Range and expectation information:*</u> $\mathcal{P}_\ell$ is comprised of all probability distributions supported on $[-1,1]$ with expectations belonging to a given segment $[\mu_\ell^-, \mu_\ell^+] \subset [-1,1]$. Here

$$\Phi_\ell^*(t) = \max_{\mu_\ell^- \le \mu \le \mu_\ell^+} \ln\left(\cosh(t) + \mu \sinh(t)\right).$$

7. <u>*Range, expectation and variance information:*</u> $\mathcal{P}_\ell$ is comprised of all probability distributions P_ℓ supported on $[-1,1]$ with expectations belonging to a given segment $[\mu_\ell^-, \mu_\ell^+]$ and with variances $\mathbf{E}_{\zeta_\ell \sim P_\ell}\{\zeta_\ell^2\}$ not exceeding a given quantity $\nu_\ell^2 > 0$. Assuming that $|\mu_\ell^\pm| \le \nu_\ell \le 1$, which is w.l.o.g. (why?), we have

$$\Phi_\ell^*(t) = \begin{cases} \max\limits_{\mu_\ell^- \le \mu \le \mu_\ell^+} \ln\left(\frac{(1-\mu)^2 \exp\{t\frac{\mu-\nu_\ell^2}{1-\mu}\} + (\nu_\ell^2 - \mu^2)\exp\{t\}}{1-2\mu+\nu_\ell^2}\right), & t \ge 0 \\ \max\limits_{\mu_\ell^- \le \mu \le \mu_\ell^+} \ln\left(\frac{(1+\mu)^2 \exp\{t\frac{\mu+\nu_\ell^2}{1+\mu}\} + (\nu_\ell^2 - \mu^2)\exp\{-t\}}{1+2\mu+\nu_\ell^2}\right), & t \le 0 \end{cases}$$

8. <u>*Range, symmetry and variance information:*</u> $\mathcal{P}_\ell$ is comprised of all symmetric w.r.t. 0 probability distributions supported on $[-1,1]$ with variance not exceeding a given quantity ν_ℓ^2. Assuming w.l.o.g. that $0 \le \nu_\ell \le 1$, we have

$$\Phi_\ell^*(t) = \ln\left(\nu^2 \cosh(t) + 1 - \nu^2\right).$$

case in the list	χ_ℓ^-	χ_ℓ^+	σ_ℓ
1	μ_ℓ^-	μ_ℓ^+	ν_ℓ
2	-1	1	0
3	$-\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{1/12}$
4	0	0	$\sqrt{1/3}$
5	0	0	1
6	μ_ℓ^-	μ_ℓ^+	≤ 1
7	μ_ℓ^-	μ_ℓ^+	≤ 1
8	0	0	≤ 1
9	0	0	$\leq \sqrt{1/3}$

Table 2.1: Parameters $\chi_\ell^\pm$, σ_ℓ for the listed cases

9. Range, symmetry, unimodality and variance information: $\mathcal{P}_\ell$ is comprised of all symmetric and unimodal w.r.t. 0 probability distributions supported on $[-1,1]$ with variance not exceeding a given quantity ν_ℓ^2. Assuming w.l.o.g. that $0 \leq \nu_\ell \leq 1/\sqrt{3}$ (this is the largest possible variance of a symmetric and unimodal w.r.t. 0 random variable supported on $[-1,1]$), we have

$$\Phi_\ell^*(t) = \ln\left(1 - 3\nu_\ell^2 + 3\nu_\ell^2 \frac{\sinh(t)}{t}\right).$$

Note that the above list seemingly is rich enough for typical applications. When implementing Bernstein approximation, one can use the above "knowledge" as it is, thus arriving at safe tractable approximations of the structure depending on what we know about every one of $\mathcal{P}_\ell$. There exists also another way to utilize the above knowledge, slightly more conservative, but with the definite advantage that the resulting approximation always is of the same nice structure – it is a conic quadratic optimization program. Specifically, it is easily seen that in the situations we have listed, the functions $\Phi_\ell^*(t)$ admits a majorant of the form

$$\Phi_\ell^+(t) = \max\left[\chi_\ell^- t, \chi_\ell^+ t\right] + \frac{\sigma_\ell^2}{2} t^2,\ 1 \leq \ell \leq L, \tag{2.5.8}$$

with easy-to-compute parameters $\chi_\ell^\pm$, $\sigma_\ell > 0$. The values of $\chi_\ell^\pm$ and σ_ℓ are presented in table 2.1 (in all cases from the above list where σ_ℓ is given by an efficient computation rather than by an explicit value, we replace σ_ℓ with its valid upper bound). Now, we are in our right to use in the Bernstein approximation scheme in the role of Φ_ℓ^+ the functions (2.5.8), thus arriving at the approximation

$$H(w) := \inf_{\alpha > 0}\left[w_0 + \alpha \sum_{\ell=1}^L \left[\max[\chi_\ell^- w_\ell/\alpha, \chi_\ell^+ w_\ell/\alpha] + \frac{\sigma_\ell^2 w_\ell^2}{2\alpha^2}\right] + \alpha \ln(1/\epsilon)\right] \leq 0,$$

which is nothing but the explicit convex constraint

$$w_0 + \sum_{\ell=1}^L \max[\chi_\ell^- w_\ell, \chi_\ell^+ w_\ell] + \sqrt{2\ln(1/\epsilon)} \sqrt{\sum_{\ell=1}^L \sigma_\ell^2 w_\ell^2} \leq 0 \tag{2.5.9}$$

in variables w; by its origin this constraint is a safe tractable approximation of the ambiguous chance constraint (2.3.8) in question. Note also that this approximation can be straightforwardly represented by a system of conic quadratic inequalities.
The Robust Counterpart from of (2.5.9) is

$$\begin{array}{c} w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell \leq 0 \ \forall \zeta \in \mathcal{Z}, \\ \mathcal{Z} = \{\zeta = u + v : \chi_\ell^- \leq u_\ell \leq \chi_\ell^+, \ell = 1, ..., L, \sum_{\ell=1}^{L} v_\ell^2/\sigma_\ell^2 \leq 2\ln(1/\epsilon)\}, \end{array} \tag{2.5.10}$$

that is, the corresponding perturbation set is the arithmetic sum of the box $\{u : [\chi_1^-; ...; \chi_L^-] \leq u \leq [\chi_1^+; ...; \chi_L^+]\}$ and the ellipsoid $\{v : \sum_{\ell=1}^{L} v_\ell^2/\sigma_\ell^2 \leq 2\ln(1/\epsilon)\}$; here, as always, $v_\ell^2/0^2$ is either $+\infty$ or 0, depending on whether or not $v_\ell > 0$).

Refinements in the bounded case

Assume that we are in the situation of the previous item and that, in addition, all $\mathcal{P}_\ell$ are comprised of distributions supported on $[-1, 1]$ (or, equivalently, for every $\ell \leq 1$, one of cases 2 – 9 takes place). In this case, all distributions $P \in \mathcal{P}$ are supported on the unit box $B_\infty = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1\}$. It is natural to guess, *and is indeed true* that in this case the above $\mathcal{Z}$ can be reduced to $\mathcal{Z}_- = \mathcal{Z} \cap B_\infty$, so that the Robust Counterpart of the uncertain linear constraint

$$w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell \leq 0,$$

the uncertainty set being $\mathcal{Z}_-$, still is a safe tractable approximation of (2.3.8).

When proving the latter claim, assume for the sake of simplicity that for every ℓ either $\chi_\ell^- < \chi_\ell^+$, or $\sigma_\ell > 0$, or both (on a closest inspection, the general case easily reduces to the latter one). Taking into account that $1 \leq \chi_\ell^- \leq \chi_\ell^+ \leq 1$ for all ℓ, it is easily seen that in the case in question the interiors of the convex compact sets $\mathcal{Z}$ and B_∞ have a point in common. This, in view of the basic fact of convex analysis we have already mentioned, implies the equivalence

$$\begin{array}{l} \forall w = [w_0; \bar{w}] \in \mathbb{R}^{L+1} : \\ \overline{\zeta^T \bar{w} \leq -w_0 \, \forall \zeta \in \mathcal{Z}_-}^{\mathcal{A}(w)} \Leftrightarrow \overline{\exists a : \max\limits_{\zeta \in \mathcal{Z}} a^T \zeta + \max\limits_{\zeta \in B_\infty} [\bar{w} - a]^T \zeta \leq -w_0}^{\mathcal{B}(w,a)}. \end{array}$$

Now, all we want to prove is that whenever $w = [w_0; \bar{w}]$ satisfies $\mathcal{A}(w)$, w is feasible for (2.3.8). Assuming that w satisfies $\mathcal{A}(w)$, the above equivalence says that there exists a such that w, a satisfy $\mathcal{B}(w, a)$, that is, such that

$$\max_{\zeta \in \mathcal{Z}} a^T \zeta \leq \widehat{w}_0 := -\|\bar{w} - a\|_1 - w_0.$$

As we remember from our previous considerations, the latter inequality implies that

$$\forall P \in \mathcal{P} : \text{Prob}_{\zeta \sim P} \left\{ w_0 + \|\bar{w} - a\|_1 + \zeta^T a > 0 \right\} \leq \epsilon. \tag{2.5.11}$$

When ζ belongs to the support of P, we have $\|\zeta\|_\infty \leq 1$ and thus $\zeta^T[\bar{w} - a] \leq \|\bar{w} - a\|_1$, so that

$$w_0 + \zeta^T \bar{w} = w_0 + \zeta^T[\bar{w} - a] + \zeta^T a \leq w_0 + \|\bar{w} - a\|_1 + \zeta^T a,$$

which combines with (2.5.11) to imply that w is feasible for (2.3.8), as claimed.

2.5.3 Approximating Quadratically Perturbed Linear Constraint

Till now, we were focusing on Bernstein approximation of (the chance constrained version of) a scalar linear constraint $f(x, \zeta) \leq 0$ *affinely* perturbed by random perturbation ζ (i.e., with f bi-affine in x and in ζ). Note that *in principle* the generating-function-based approximation scheme (and in particular Bernstein approximation) could handle non-affine random perturbations; what is difficult in the non-affine case (same as in the affine one, aside of the situation covered by assumptions Brn.1-2), is to get a computationally tractable and a reasonably tight upper bound on the function $\Psi^*(x) = \sup_{P \in \mathcal{P}} \mathbf{E}\{\gamma(f(x, \zeta))\}$. There is, however, a special case where the latter difficulty does not occur – this is the case when $f(x, \zeta)$ is affine in x and quadratic in ζ, and ζ is a Gaussian random vector. We are about to consider this special case. Thus, we assume that the constraint in question is

$$\zeta^T W \zeta + 2[w_1; ...; w_L]\zeta + w_0 \leq 0 \tag{2.5.12}$$

where the symmetric matrix W and vector $w = [w_0; w_1; ...; w_L]$ are affine in the decision variables x, and $\zeta \in \mathbb{R}^L$ is a Gaussian random vector. We lose nothing by assuming that the decision variables are (W, w) themselves, and that $\zeta \sim \mathcal{N}(0, I_L)$.

We start with the following observation (which can be justified by a straightforward computation):

Lemma 2.4 *Let $\zeta \sim \mathcal{N}(0, I)$, and let*

$$\xi = \xi^{W,w} = \zeta^T W \zeta + 2[w_1; ...; w_L]^T \zeta + w_0.$$

Then $\ln\left(\mathbf{E}\left\{\exp\{\xi^{W,w}\}\right\}\right) = F(W, w)$, *where*

$$\begin{array}{rcl} F(W,w) & = & w_0 - \frac{1}{2}\ln \mathrm{Det}(I - 2W) + 2b^T(I - 2W)^{-1}[w_1; ...; w_L] \\ & & \mathrm{Dom}\, F = \{(W, w) \in \mathbf{S}^L \times \mathbb{R}^{L+1} : 2W \prec I\} \end{array} \tag{2.5.13}$$

Applying the Bernstein approximation scheme, we arrive at the following result:

Theorem 2.3 [3, Theorem 4.5.9] *Let*

$$\begin{array}{l} \Phi(\alpha, W, w) = \alpha F(\alpha^{-1}(W, w)) \\ = \alpha \left[-\frac{1}{2}\ln \mathrm{Det}(I - 2\alpha^{-1}W) + 2\alpha^{-2}[w_1; ...; w_L]^T(I - 2\alpha^{-1}W)^{-1}[w_1; ...; w_L]\right] + w_0 \\ \qquad\qquad [\mathrm{Dom}\,\Phi = \{(\alpha, W, w) : \alpha > 0, 2W \prec \alpha I\}], \\ \mathcal{W}^o = \{(W, w) : \exists \alpha > 0 : \Phi(\alpha, W, w) + \alpha \ln(1/\epsilon) \leq 0\}, \quad \mathcal{W} = \mathrm{cl}\, \mathcal{W}^o. \end{array} \tag{2.5.14}$$

Then $\mathcal{W}$ is the solution set of the convex inequality

$$H(W, w) \equiv \inf_{\alpha > 0} [\Phi(\alpha, W, w) + \alpha \ln(1/\epsilon)] \leq 0. \tag{2.5.15}$$

If $\zeta \sim \mathcal{N}(0, I)$, then this inequality is a safe tractable approximation of the chance constraint

$$\mathrm{Prob}_{\zeta \sim \mathcal{N}(0, I_L)}\left\{\zeta^T W \zeta + 2[w_1; ...; w_L]^T \zeta + w_0 > 0\right\} \leq \epsilon. \tag{2.5.16}$$

Application: A useful inequality

Let W be a symmetric $L \times L$ matrix and w be an L-dimensional vector. Consider the quadratic form

$$f(s) = s^T W s + 2w^T s,$$

and let $\zeta \sim \mathcal{N}(0, I)$. We clearly have $\mathbf{E}\{f(\zeta)\} = \mathrm{Tr}(W)$. Our goal is to establish a simple bound on $\mathrm{Prob}\{f(\zeta) - \mathrm{Tr}(W) > t\}$, and here is this bound:

Proposition 2.3 *Let λ be the vector of eigenvalues of W. Then*

$$\begin{array}{l} \forall \Omega > 0 : \mathrm{Prob}_{\zeta\sim\mathcal{N}(0,I)}\left\{[\zeta^T W\zeta + 2w^T\zeta] - \mathrm{Tr}(W) > \Omega\sqrt{\lambda^T\lambda + w^T w}\right\} \\ \leq \exp\left\{-\frac{\Omega^2\sqrt{\lambda^T\lambda+w^Tw}}{4\left(2\sqrt{\lambda^T\lambda+w^Tw}+\|\lambda\|_\infty\Omega\right)}\right\} \end{array} \tag{2.5.17}$$

(by definition, the right hand side is 0 when $W = 0$, $w = 0$).

Proof. The claim is clearly true in the trivial case of $W = 0$, $w = 0$, thus assume that f is not identically zero. Passing to the orthonormal eigenbasis of W, we can w.l.o.g. assume that W is diagonal with diagonal entries $\lambda_1, ..., \lambda_L$. Given $\Omega > 0$, let us set $s = \Omega\sqrt{\lambda^T\lambda + w^T w}$ and let

$$\gamma = \frac{s}{2\left(2(\lambda^T\lambda + w^T w) + \|\lambda\|_\infty s\right)},$$

so that

$$0 < \gamma \ \&\ 2\gamma W \prec I \ \&\ \frac{4\gamma(\lambda^T\lambda + w^T w)}{1 - 2\gamma\|\lambda\|_\infty} = s. \tag{2.5.18}$$

Applying Theorem 2.3 with $w_0 = -[\mathrm{Tr}(W) + s]$ and specifying β as $1/\gamma$, we get

$$\begin{array}{l} \mathrm{Prob}\{f(\zeta) > \mathrm{Tr}(W) + s\} \\ \leq \exp\left\{-\gamma s + \sum_{\ell=1}^L \left(-\frac{1}{2}\ln(1 - 2\gamma\lambda_\ell) + 2\gamma^2\frac{w_\ell^2}{1-2\gamma\lambda_\ell} - \gamma\lambda_\ell\right)\right\} \\ \leq \exp\left\{-\gamma s + \sum_{\ell=1}^L \left(\frac{\gamma\lambda_\ell}{1-2\gamma\lambda_\ell} + 2\gamma^2\frac{w_\ell^2}{1-2\gamma\lambda_\ell} - \gamma\lambda_\ell\right)\right\} \\ \qquad [\text{since } \ln(1-\delta) + \frac{\delta}{1-\delta} \geq \ln(1) = 0 \text{ by the concavity of } \ln(\cdot)] \\ = \exp\left\{-\gamma s + \sum_{\ell=1}^L \left(\frac{2\gamma^2(\lambda_\ell^2+w_\ell^2)}{1-2\gamma\lambda_\ell}\right)\right\} \leq \exp\left\{-\gamma s + \frac{2\gamma^2(\lambda^T\lambda+w^Tw)}{1-2\gamma\|\lambda\|_\infty}\right\} \\ \leq \exp\{-\frac{\gamma s}{2}\} \\ \qquad\qquad\qquad\qquad\qquad\qquad [\text{by (2.5.18)}]\ . \end{array}$$

Substituting the values of γ and s, we arrive at (2.5.17). □

Application: Linearly perturbed Least Squares inequality

Consider a chance constrained linearly perturbed Least Squares inequality

$$\mathrm{Prob}\left\{\|A[x]\zeta + b[x]\|_2 \leq c[x]\right\} \geq 1 - \epsilon, \tag{2.5.19}$$

where $A[x]$, $b[x]$, $c[x]$ are affine in the variables x and $\zeta \sim \mathcal{N}(0, I)$. Taking squares of both sides in the body of the constraint, this inequality is equivalent to

$$\begin{array}{l} \exists U, u, u_0 : \\ \left[\begin{array}{c|c} U & u^T \\ \hline u & u_0 \end{array}\right] \succeq \left[\begin{array}{c|c} A^T[x]A[x] & A^T[x]b[x] \\ \hline b^T[x]A[x] & b^T[x]b[x] - c^2[x] \end{array}\right] \ \&\ \mathrm{Prob}\left\{\zeta^T U\zeta + 2\sum_{\ell=1}^L u_\ell\zeta_\ell + u_0 > 0\right\} \leq \epsilon. \end{array}$$

Assuming $c[x] > 0$, passing from U, u variables to $W = c^{-1}[x]U$, $w = c^{-1}[x]u$, and dividing both sides of the LMI by $c[x]$, this can be rewritten equivalently as

$$\exists (W, w, w_0): \left[\begin{array}{c|c} W & w^T \\ \hline w & w_0 + c[x] \end{array}\right] \succeq c^{-1}[x][A[x], b[x]]^T[A[x], b[x]] \ \&\ \mathrm{Prob}\left\{\zeta^T W\zeta + 2\sum_{\ell=1}^{L} w_\ell \zeta_\ell + w_0 > 0\right\} \leq \epsilon.$$

The constraint linking W, w and x is, by the Schur Complement Lemma, nothing but the Linear Matrix Inequality

$$\left[\begin{array}{c|c|c} W & [w_1; ...; w_L] & A^T[x] \\ \hline [w_1, ..., w_L] & w_0 + c[x] & b^T[x] \\ \hline A[x] & b[x] & c[x]I \end{array}\right] \succeq 0. \tag{2.5.20}$$

Invoking Theorem 2.3, we arrive at the following

Corollary 2.1 *The system of convex constraints* (2.5.20) *and* (2.5.15) *in variables W, w, x is a safe tractable approximation of the chance constrained Least Squares Inequality* (2.5.19).

Note that while we have derived this Corollary under the assumption that $c[x] > 0$, the result is trivially true when $c[x] = 0$, since in this case (2.5.20) already implies that $A[x] = 0$, $b[x] = 0$ and thus (2.5.19) holds true.

2.6 Beyond the Case of Independent Linear Perturbations

Situation and goal

Until now, we were mainly focusing on the chance constrained version of a scalar linear constraint *affinely* affected random perturbations *with independent components*. Now let us consider the situation where

A. The uncertainty-affected constraint still is scalar and affine in the decision variables, but is quadratic in perturbations.

A generic form of such a constraint is

$$[\zeta; 1]^T W[\zeta; 1] \leq 0, \tag{2.6.1}$$

where $W \in \mathbf{S}^{L+1}$ is affine in the decision variables, and $\zeta \in \mathbb{R}^L$ is random perturbation vector. We lose nothing by assuming that our decision variable is the symmetric matrix W itself.

B. We are interested in the ambiguously chance constrained version of (2.6.1), specifically, in the constraint

$$\sup_{P \in \mathcal{P}} \mathrm{Prob}_{\zeta \sim P}\left\{[\zeta; 1]^T W[\zeta; 1] > 0\right\} \leq \epsilon \tag{2.6.2}$$

in variable W, where $\epsilon \in (0, 1)$ is a given tolerance, and $\mathcal{P}$ is a given family of probability distributions on $\mathbb{R}^L$. We assume that our a priori knowledge of $\mathcal{P}$ reduces to

B.1: partial knowledge of marginal distributions of $\zeta = [\zeta_1; ...; \zeta_L]$. Specifically, we are given families $\mathcal{P}_\ell$, $1 \leq \ell \leq L$, of probability distributions on the axis such that $\mathcal{P}_\ell$ contains the distribution P_ℓ of ζ_ℓ. We assume that $\sup\limits_{P \in \mathcal{P}_\ell} \int s^2 dP_\ell(s) < \infty$ for all ℓ;

B.2: partial knowledge of the expectation and the covariance matrix of ζ. Specifically, we are given a closed convex set $\mathcal{V}$ in the cone of symmetric $(L+1)\times(L+1)$ positive semidefinite matrices which contains the matrix

$$V_\zeta = \mathbf{E}\left\{\left[\begin{array}{c|c}\zeta\zeta^T & \zeta\\ \hline \zeta^T & 1\end{array}\right]\right\};$$

B.3: partial knowledge of the support of ζ. Specifically, we assume that we are given a system of quadratic (not necessarily convex) constraints

$$\mathcal{A}(u) \equiv \left[\begin{array}{c} u^TA_1u + 2a_1^Tu + \alpha_1\\ \vdots\\ u^TA_mu + 2a_m^Tu + \alpha_m\end{array}\right] \le 0 \qquad (2.6.3)$$

such that ζ is supported on the solution set of this system.

Given this information, we want to build a safe tractable approximation of (2.6.2).

The approach

The approach we intend to use can be traced to [27, 28] and resembles the distance-generating-function approximation scheme from section 2.3; by reasons to become clear soon, we call this approach *Lagrangian approximation*. Specifically, let P be the distribution of ζ. Our a priori information allows to bound from above the expectation over P of a function of the form

$$\gamma(u) = [u;1]^T\Gamma[u;1] + \sum_{\ell=1}^{L}\gamma_\ell(u_\ell); \qquad (2.6.4)$$

indeed, we have

$$\mathbf{E}_{\zeta\sim P}\{\gamma(\zeta)\} = \mathrm{Tr}(V_\zeta\Gamma) + \sum_{\ell=1}^{L}\mathbf{E}_{\zeta\sim P}\gamma_\ell(\zeta_\ell) \le \sup_{V\in\mathcal{V}}\mathrm{Tr}(V\Gamma) + \sum_{\ell=1}^{L}\sup_{P_\ell\in\mathcal{P}_\ell}\mathbf{E}_{\zeta_\ell\sim P_\ell}\left\{\gamma_\ell(\zeta_\ell)\right\}. \qquad (2.6.5)$$

Assuming that

γ.1: $\gamma(\cdot)\ge 0$ on the support of ζ,

and that for some $\lambda > 0$ we have

γ.2: $\gamma(\cdot)\ge\lambda$ on the part of the support of ζ where $[\zeta;1]^TW[\zeta;1] > 0$,

the left hand side in (2.6.5) is an upper bound on $\lambda\mathrm{Prob}\{[\zeta;1]^TW[\zeta;1] > 0\}$. It follows that under the conditions γ.1-2 we have

$$\sup_{V\in\mathcal{V}}\mathrm{Tr}(V\Gamma) + \sum_{\ell=1}^{L}\sup_{P_\ell\in\mathcal{P}_\ell}\mathbf{E}_{\zeta_\ell\sim P_\ell}\left\{\gamma_\ell(\zeta_\ell)\right\} \le \lambda\epsilon \Rightarrow \sup_{P\in\mathcal{P}}\mathrm{Prob}_{\zeta\sim P}\left\{[\zeta;1]^TW[\zeta;1] > 0\right\} \le \epsilon. \qquad (2.6.6)$$

Now, the simplest way to impose on a function $\gamma(\cdot)$ of the form (2.6.4) condition γ.1 is to require from it to satisfy the relation

$$\gamma(u) + \mu^T\mathcal{A}(u) \ge 0\ \forall u$$

with some $\mu \geq 0$ (this is called Lagrangian relaxation). Now,

$$\gamma(u) + \mu^T \mathcal{A}(u) = \sum_{\ell=1}^{L} \gamma_\ell(u_\ell) + \left[[u;1]^T \Gamma [u;1] + \mu^T \mathcal{A}(u)\right]$$

is a separable perturbation of a quadratic function. The simplest (and seemingly the only tractable) way to enforce global nonnegativity of such a perturbation is to require from the perturbation term $2\sum_\ell \gamma_\ell(u_\ell)$ to majorate a *separable quadratic* function $f(u)$ of u such that $f(u) + \left[[u;1]^T \Gamma [u;1] + \mu^T \mathcal{A}(u)\right] \geq 0$ for all u. Thus, a *sufficient* condition for $\gamma(\cdot)$ to satisfy the condition $\gamma.1$ is
$\widehat{\gamma}.1$: there exist p_ℓ, q_ℓ, r_ℓ such that

$$\gamma_\ell(s) \geq p_\ell + 2q_\ell s + r_\ell s^2 \;\forall s \in \mathbb{R}, 1 \leq \ell \leq L \tag{2.6.7}$$

and

$$\sum_\ell (p_\ell + 2q_\ell u_\ell + r_\ell u_\ell^2) + [u;1]^T \Gamma [u;1] + \mu^T \mathcal{A}(u) \geq 0 \;\forall u. \tag{2.6.8}$$

Observing that

$$\mu^T \mathcal{A}(u) = [u;1]^T A(\mu)[u;1],\;\; A(\mu) = \left[\begin{array}{c|c} \sum_{i=1}^m \mu_i A_i & \sum_{i=1}^m \mu_i a_i \\ \hline \sum_{i=1}^m \mu_i a_i^T & \sum_{i=1}^m \alpha_i \end{array}\right],$$

relation (2.6.8) is equivalent to

$$\Gamma + A(\mu) + \left[\begin{array}{c|c} \mathrm{Diag}\{r_1,...,r_L\} & [q_1;...;q_L] \\ \hline [q_1,...,q_L] & \sum_{\ell=1}^L p_\ell \end{array}\right] \succeq 0. \tag{2.6.9}$$

The bottom line is that *the existence of* $\{p_\ell, q_\ell, r_\ell\}_{\ell=1}^L$ *and* $\mu \geq 0$ *making* (2.6.7) *and* (2.6.9) *valid is sufficient for* $\gamma(\cdot)$ *to satisfy* $\gamma.1$.

By similar reasons, a sufficient condition for $(\gamma, \lambda > 0)$ to satisfy $\gamma.2$ is given by the existence of $\{p'_\ell, q'_\ell, r'_\ell\}_{\ell=1}^L$ and $\mu' \geq 0$ such that

$$\gamma_\ell(s) \geq p'_\ell + 2q'_\ell s + r'_\ell s^2 \;\forall s \in \mathbb{R}, 1 \leq \ell \leq L \tag{2.6.10}$$

and

$$\sum_\ell (p'_\ell + 2q'_\ell u_\ell + r'_\ell u_\ell^2) + [u;1]^T \Gamma [u;1] + [\mu']^T \mathcal{A}(u) \geq \lambda + [u;1]^T W [u;1] \;\forall u.$$ [6]

The latter restriction can be represented equivalently by the matrix inequality

$$\Gamma + A(\mu') + \left[\begin{array}{c|c} \mathrm{Diag}\{r'_1,...,r'_L\} & [q'_1;...;q'_L] \\ \hline [q'_1,...,q'_L] & \sum_{\ell=1}^L p'_\ell - \lambda \end{array}\right] - W \succeq 0. \tag{2.6.11}$$

[6] In fact we could assign the term $[u;1]^T W[u;1]$ with a positive coefficient. On the closest inspection, however, this does not increase flexibility of our construction.

The result

Observing that the best – resulting in the smallest possible $\gamma_\ell(\cdot)$ – way to ensure (2.6.7), (2.6.10) is to set

$$\gamma_\ell(s) = \max[p_\ell + 2q_\ell s + r_\ell s^2, p'_\ell + 2q'_\ell s + r'_\ell s^2],$$

we arrive at the following result:

Theorem 2.4 *In the situation* **A**, **B** *the system of convex constraints*

$$\begin{array}{ll}
(a_1) & \Gamma + A(\mu) + \left[\begin{array}{c|c} \mathrm{Diag}\{r_1,...,r_L\} & [q_1;...;q_L] \\ \hline [q_1,...,q_L] & \sum_{\ell=1}^L p_\ell \end{array}\right] \succeq 0 \\
(a_2) & \mu \geq 0 \\
(b.1) & \Gamma + A(\mu') + \left[\begin{array}{c|c} \mathrm{Diag}\{r'_1,...,r'_L\} & [q'_1;...;q'_L] \\ \hline [q'_1,...,q'_L] & \sum_{\ell=1}^L p'_\ell - \lambda \end{array}\right] - W \succeq 0 \\
(b.2) & \mu' \geq 0 \\
(c.1) & \sup\limits_{V\in\mathcal{V}} \mathrm{Tr}(V\Gamma) + \sum_{\ell=1}^L \sup\limits_{P_\ell \in \mathcal{P}_\ell} \int \max[p_\ell + 2q_\ell s + r_\ell s^2, p'_\ell + 2q'_\ell s + r'_\ell s^2] dP_\ell(s) \leq \lambda\epsilon \\
(c.2) & \left[\begin{array}{c|c} \lambda & 1 \\ \hline 1 & \nu \end{array}\right] \succeq 0 \quad [\text{this just says that } \lambda > 0]
\end{array} \tag{2.6.12}$$

in variables $W \in \mathbf{S}^{L+1}, \Gamma \in \mathbf{S}^{L+1}, \{p_\ell, q_\ell, r_\ell, p'_\ell, q'_\ell, r'_\ell\}_{\ell=1}^L, \mu \in \mathbb{R}^m, \mu' \in \mathbb{R}^m, \lambda \in \mathbb{R}, \nu \in \mathbb{R}$ *is a safe approximation of the ambiguous chance constraint* (2.6.1)*. This approximation is tractable, provided that all suprema in* (c.1) *are efficiently computable.*

How it works

To illustrate the techniques we have developed, consider an instance of the Portfolio Selection problem as follows:

> There are $L = 15$ assets with random yearly returns $r_\ell = 1 + \mu_\ell + \sigma_\ell\zeta_\ell$, $1 \leq \ell \leq L$, where $\mu_\ell \geq 0$ and $\sigma_\ell \geq 0$ are expected gains and their variabilities given by
>
> $$\mu_\ell = 0.001 + 0.9\frac{\ell-1}{14},\ \sigma_\ell = \left[0.9 + 0.2\frac{\ell-1}{14}\right]\mu_\ell,\ 1 \leq \ell \leq 15,$$
>
> and ζ_ℓ are random perturbations taking values in $[-1,1]$. Given partial information on the distribution of $\zeta = [\zeta_1;...;\zeta_L]$, we want to distribute \$1 between the assets in order to maximize the guaranteed value-at-ϵ-risk of the profit.

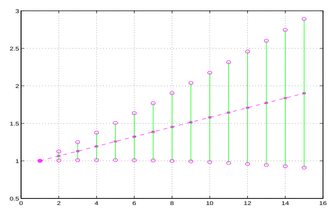
This is the ambiguously chance constrained problem

$$\mathrm{Opt} = \max_{t,x}\left\{t : \mathrm{Prob}_{\zeta\sim P}\left\{\sum_{\ell=1}^{15}\mu_\ell x_\ell + \sum_{\ell=1}^{15}\zeta_\ell\sigma_\ell x_\ell \geq t\right\} \geq 0.99\ \forall P \in \mathcal{P},\ x \geq 0,\ \sum_{\ell=1}^{15} x_\ell = 1\right\} \tag{2.6.13}$$

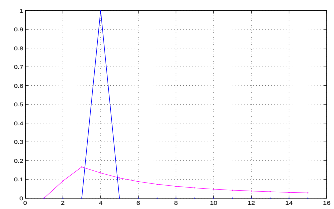
Consider three hypotheses A, B, C about $\mathcal{P}$. In all of them, ζ_ℓ are zero mean and supported on $[-1,1]$, so that the domain information is given by the quadratic inequalities $\zeta_\ell^2 \leq 1$, $1 \leq \ell \leq 15$; this is exactly what is stated by C. In addition, A says that ζ_ℓ are independent, and B says that the covariance matrix of ζ is proportional to the unit matrix. Thus, the sets $\mathcal{V}$ associated with the hypotheses are, respectively, $\{V \in \mathbf{S}_+^{L+1} : V_{\ell\ell} \leq V_{00} = 1, V_{k\ell} = 0,\ k \neq \ell\}$, $\{V \in \mathbf{S}_+^{L+1} : 1 = V_{00} \geq$

Hypothesis	Approximation	Guaranteed profit-at-1%-risk
A	Bernstein	0.0552
B, C	Lagrangian	0.0101

Table 2.2: Optimal values in various approximations of (2.6.13).



Expectations and ranges of returns



Optimal portfolios: diversified (hypothesis A) and single-asset (hypotheses B, C)

Figure 2.2: Data and results for portfolio allocation.

$V_{11} = V_{22} = \ldots = V_{LL}, V_{k\ell} = 0, k \neq \ell\}$, and $\{V \in \mathbf{S}^{L+1}_+ : V_{\ell\ell} \leq V_{00} = 1, V_{0\ell} = 0, 1 \leq \ell \leq L\}$, where $\mathbf{S}^k_+$ is the cone of positive semidefinite symmetric $k \times k$ matrices. Solving the associated safe tractable approximations of the problem, specifically, the Bernstein approximation in the case of A, and the Lagrangian approximations in the cases of B, C, we arrive at the results displayed in table 2.2 and on figure 2.2.

Note that in our illustration, the (identical to each other) single-asset portfolios yielded by the Lagrangian approximation under hypotheses B, C are *exactly optimal* under circumstances. Indeed, on a closest inspection, there exists a distribution P_* compatible with hypothesis B (and therefore – with C as well) such that the probability of "crisis," where all ζ_ℓ simultaneously are equal to -1, is ≥ 0.01. It follows that under hypotheses B, C, the worst-case, over $P \in \mathcal{P}$, profit at 1% risk of *any* portfolio cannot be better than the profit of this portfolio in the case of crisis, and the latter quantity is maximized by the single-asset portfolio depicted on figure 2.2. Note that the Lagrangian approximation turns out to be "intelligent enough" to discover this phenomenon and to infer its consequences. A couple of other instructive observations is as follows:

- the diversified portfolio yielded by the Bernstein approximation in the case of crisis exhibits *negative* profit, meaning that under hypotheses B, C its worst-case profit at 1% risk is negative;

- assume that the yearly returns are observed on a year-by-year basis, and the year-by-year realizations of ζ are independent and identically distributed. It turns out that it takes over 100 years to distinguish, with reliability 0.99, between hypothesis A and the "bad" distribution P_* via the historical data.

To put these observations into proper perspective, note that it is extremely time-consuming to identify, to reasonable accuracy and with reasonable reliability, a multi-dimensional distribution directly from historical data, so that in applications one usually postulates certain parametric form of the distribution with a relatively small number of parameters to be estimated from the historical data. When $\dim\zeta$ is large, the requirement on the distribution to admit a low-dimensional parameterization usually results in postulating some kind of independence. While in some applications (e.g., in telecommunications) this independence in many cases can be justified via the "physics" of the uncertain data, in Finance and other decision-making applications postulating independence typically is an "act of faith" which is difficult to justify experimentally, and we believe a decision-maker should be well aware of the dangers related to these "acts of faith."

2.6.1 A Modification

Situation and Goal

Assume that our losses in certain uncertainty-affected decision process are of the form

$$L(x,\zeta) = f([\zeta;1]^T W[x][\zeta;1]),$$

where x is the decision vector, $\zeta = [\zeta_1;...;\zeta_L] \in \mathbb{R}^L$ is a random perturbation, $W[x]$ is affine in x and $f(s)$ is a convex piecewise linear function:

$$f(s) = \max_{1\le j\le J}[a_j + b_j s],$$

and we want the expected loss $\mathbf{E}\{L(x,\zeta)\}$ to be $\le$ a given bound τ.

> For example, let
>
> $$[\zeta;1]^T W[x][\zeta;1] \equiv \sum_{\ell=1}^{L} \zeta_\ell x_\ell$$
>
> be the would-be value of a portfolio (that is, x_ℓ is the initial capital invested in asset ℓ, and ζ_ℓ is the return of this asset), and let $-f(s)$ be our utility function; in this case, the constraint
>
> $$\mathbf{E}\{L(x,\zeta)\} \le \tau$$
>
> models the requirement that the expected utility of an investment x is at least a given quantity $-\tau$.

Our goal is to build a safe convex approximation of the ambiguous form of our bound:

$$\sup_{P\in\mathcal{P}} \mathbf{E}_{\zeta\sim P}\{L(x,\zeta)\} \equiv \sup_{P\in\mathcal{P}} \mathbf{E}_{\zeta\sim\mathcal{P}}\left\{\max_{1\le j\le J}[a_j + b_j[\zeta;1]^T W[x][\zeta;1]]\right\} \le \tau, \tag{2.6.14}$$

that is, to find a system S of convex constraints in variables x,τ and, perhaps, additional variables u such that whenever (x,τ,u) is feasible for S, (x,τ) satisfy (2.6.14).

When achieving this goal, we lose nothing when assuming that our variable is W itself, and we can further extend our setting by allowing different W in different terms $a_j + b_j[\zeta;1]^T W[\zeta;1]$. With these modifications, our goal becomes to find a safe convex approximation to the constraint

$$\sup_{P\in\mathcal{P}} \mathbf{E}_{\zeta\sim P}\left\{\max_{1\le j\le J}[a_j + b_j[\zeta;1]^T W_j[\zeta;1]]\right\} \le \tau \tag{2.6.15}$$

in matrix variables $W_1,...,W_J$ and scalar variable τ.

As about the set $\mathcal{P}$ of possible distributions of ζ, we keep the assumptions **B.1-3**.

The Approximation

We follow the same approach as above. Specifically, if $\gamma(\cdot)$ is a function of the form (2.6.4) satisfying the conditions

$\gamma_j.3$: $\gamma(u) \geq a_j + b_j[u;1]^T W_j[u;1]$ for every u from the support of ζ

for all $j \leq J$, then the quantity

$$\mathbf{E}_{\zeta\sim P}\{\gamma(\zeta)\} = \mathrm{Tr}(V_\zeta \Gamma) + \sum_{\ell=1}^{L} \mathbf{E}_{\zeta_\ell \sim P_\ell}\{\gamma_\ell(\zeta_\ell)\}$$

(P_ℓ are the marginals of the distribution P of ζ) is an upper bound on

$$\mathbf{E}_{\zeta\sim P}\left\{\max_{1\leq j\leq J}[a_j + b_j[\zeta;1]^T W_j[\zeta;1]]\right\},$$

so that the validity of the inequality

$$\sup_{V\in\mathcal{V}} \mathrm{Tr}(V\Gamma) + \sum_{\ell=1}^{L} \sup_{P_\ell\in\mathcal{P}_\ell} \mathbf{E}_{\zeta_\ell\sim P_\ell}\{\gamma_\ell(\zeta_\ell)\} \leq \tau \tag{2.6.16}$$

is a sufficient condition for the validity of the target inequality (2.6.15). In order to "optimize this condition in γ", we, same as above, observe that a simple sufficient condition for $\gamma(\cdot)$ to satisfy the condition $\gamma_j.3$ for a given j is the existence of reals $p_{j\ell}, q_{j\ell}, r_{j\ell}$, $1 \leq \ell \leq L$, and nonnegative vector $\mu^j \in \mathbb{R}^m$ such that

$$\gamma_\ell(s) \geq p_{j\ell} + 2q_{j\ell}s + r_{j\ell}s^2 \;\forall s \in \mathbb{R}, 1 \leq \ell \leq L,$$

and

$$[u;1]^T\Gamma[u;1] + \sum_{\ell=1}^{L}[p_{j\ell} + 2q_{j\ell}u_\ell + r_{j\ell}u_\ell^2] + [\mu^j]^T\mathcal{A}(u) \geq a_j + b_j[u;1]^T W_j[u;1] \;\forall u;$$

the latter restriction is equivalent to the matrix inequality

$$\Gamma + A(\mu^j) + \left[\begin{array}{c|c} \mathrm{Diag}\{r_{j1},...,r_{jL}\} & [q_{j1};...;q_{jL}] \\ \hline [q_{j1},...,q_{jL}] & \sum_{\ell=1}^{L} p_{j\ell} - a_j \end{array}\right] - b_j W_j \succeq 0. \tag{M_j}$$

We have arrived at the following result:

Theorem 2.5 *The system of convex constraints*

$$\begin{array}{ll} (a.1) & (M_1)\&(M_2)\&...\&(M_J) \\ (a.2) & \mu^j \geq 0,\, 1 \leq j \leq J \\ (b) & \sup\limits_{V\in\mathcal{V}} \mathrm{Tr}(V\Gamma) + \sum\limits_{\ell=1}^{L} \sup\limits_{P_\ell\in\mathcal{P}_\ell} \mathbf{E}_{s\sim P_\ell}\left\{\max\limits_{1\leq j\leq J}[p_{j\ell} + 2q_{j\ell}s + r_{j\ell}s^2]\right\} \leq \tau \end{array} \tag{2.6.17}$$

in variables τ, $\{W_j\}_{j\leq J}$, $\Gamma \in \mathbf{S}^{L+1}$, $\{p_{j\ell}, q_{j\ell}, r_{j\ell}\}_{j\leq J, \ell\leq L}$, $\{\mu^j \in \mathbb{R}^m\}_{j\leq J}$ *is a safe convex approximation of the constraint* (2.6.15). *This approximation is tractable, provided that all suprema in* (*b*) *are efficiently computable.*

A special case. Assume that we are in the "utility case:"

$$[\zeta;1]^T W_1[\zeta;1] \equiv ... \equiv [\zeta_1]^T W_J[\zeta;1] \equiv \sum_{\ell=1}^{L} \zeta_j w_j$$

and that all $\mathcal{P}_\ell$ are singletons – we know *exactly* the distributions of ζ_ℓ. Let also $\mathcal{V}$ be comprised of all positive semidefinite matrices, which is the same as to say that we have no information on inter-dependencies different entries in ζ. Finally, assume that we have no domain information on ζ, so that $\mathcal{A}(\cdot) \equiv 0$. It turns out that in this special case our approximation is *precise*: given a collection of decision variables $w_1, ..., w_L$ and setting $W[w] = \left[\begin{array}{c|c} & \frac{1}{2}[w_1; ...; w_L] \\ \hline \frac{1}{2}[w_1, ..., w_L] & \end{array}\right]$, $1 \leq j \leq J$ (which ensures that $[\zeta;1]^T W[\zeta;1] \equiv \sum_\ell \zeta_\ell w_\ell$), the collection $(\{W_j = W[w]\}_{j \leq L}, \tau)$ can be extended to a feasible solution of (2.6.17) if and only if (2.6.15) holds true. In fact, this conclusion remains true when we replace (2.6.17) with the simpler system of constraints

$$\begin{array}{ll} (a) & \sum_{\ell=1}^{L} p_{j\ell} \geq a_j,\ 1 \leq j \leq J \\ (b) & \sum\limits_{\ell=1}^{L} \mathbf{E}_{s \sim P_\ell} \left\{ \max\limits_{1 \leq j \leq J} [p_{j\ell} + b_j w_j s] \right\} \leq \tau \end{array}$$

in variables τ, w, $\{p_{i\ell}, q_{j\ell}\}_{\ell=1}^{L}$. This system is obtained from (2.6.17) by setting $\Gamma = 0$, $r_{j\ell} \equiv 0$ and eliminating the (immaterial now) terms with μ^j; the resulting system of constraints implies that $2q_{j\ell} = w_j$ for all ℓ, which allows to eliminate q-variables. This result of [3, Chapter 4] is a reformulation of the surprising result, established in [42], which states that for every piecewise linear convex function $f(\cdot)$ on the axis the maximum of the quantity $\int f(\sum_{\ell=1}^{L} \eta_\ell) dP(\eta)$ over all distributions P on $\mathbb{R}^L$ with given marginals $P_1, ..., P_L$ is achieved when $\eta_1, ..., \eta_L$ are *comonotone*, that is, they are deterministic nonincreasing transformations of a *single* random variable uniformly distributed on $[0, 1]$.

2.7 More...

In the first reading this section can be skipped.

2.7.1 Bridging the Gap between the Bernstein and the CVaR Approximations

The Bernstein approximation of a chance constraint (2.3.8) is a particular case of the general generating-function-based scheme for building a safe convex approximation of the constraint, and we know that this particular approximation is not the best in terms of conservatism. What makes it attractive, is that under certain structural assumptions (namely, those of independence of $\zeta_1, ..., \zeta_L$ plus availability of efficiently computable convex upper bounds on the functions $\ln(\mathbf{E}\{\exp\{s\zeta_\ell\}\})$) this approximation is computationally tractable. The question we now address is how to reduce, to some extent, the conservatism of the Bernstein approximation without sacrificing computational tractability. The idea is as follows. Assume that

> **A.** *We know exactly the distribution P of ζ (that is, $\mathcal{P}$ is a singleton), $\zeta_1, ..., \zeta_L$ are independent, and we can compute efficiently the associated moment-generating functions*

$$\Psi_\ell(s) = \mathbf{E}\{\exp\{s\zeta_\ell\}\} : \mathbb{C} \to \mathbb{C}.$$

Under this assumption, whenever $\gamma(s)=\sum\limits_{\nu=0}^{d} c_\nu \exp\{\lambda_\nu s\}$ is an exponential polynomial, we can efficiently compute the function

$$\Psi(w)=\mathbf{E}\left\{\gamma(w_0+\sum_{\ell=1}^{L}\zeta_\ell w_\ell)\right\}=\sum_{\nu=0}^{d} c_\nu \exp\{\lambda_\nu w_0\}\prod_{\ell=1}^{L}\Psi_\ell(\lambda_\nu w_\ell).$$

In other words,

> (!) *Whenever a generator* $\gamma(\cdot):\mathbb{R}\to\mathbb{R}$ *is an exponential polynomial, the associated upper bound*
>
> $$\inf_{\alpha>0}\Psi(\alpha w),\;\;\Psi(w)=\mathbf{E}\left\{\gamma(w_0+\sum_{\ell=1}^{L}\zeta_\ell w_\ell)\right\}$$
>
> *on the quantity* $p(w)=\mathrm{Prob}\{w_0+\sum_{\ell=1}^{L}\zeta_\ell w_\ell>0\}$ *is efficiently computable.*

We now can utilize (!) in the following construction:

> *Given design parameters* $T>0$ *("window width") and* d *("degree of approximation"), we build the trigonometric polynomial*
>
> $$\chi_{c_*}(s)\equiv\sum_{\nu=0}^{d}\left[c_{*\nu}\exp\{\imath\pi\nu s/T\}+\overline{c_{*\nu}}\exp\{-\imath\pi\nu s/T\}\right]$$
>
> *by solving the following problem of the best uniform approximation:*
>
> $$\begin{array}{l} c_*\in\mathop{\mathrm{Argmin}}\limits_{c\in\mathbb{C}^{d+1}}\big\{\max_{-T\le s\le T}|\exp\{s\}\chi_c(s)-\max[1+s,0]|:\\ \qquad 0\le\chi_c(s)\le\chi_c(0)=1\,\forall s\in\mathbb{R},\ \exp\{s\}\chi_c(s)\text{ is convex}\\ \qquad\text{and nondecreasing on }[-T,T]\big\}\end{array}$$
>
> *and use in* (!) *the exponential polynomial*
>
> $$\gamma_{d,T}(s)=\exp\{s\}\chi_{c_*}(s). \tag{2.7.1}$$

It can be immediately verified that

(i) The outlined construction is well defined and results in an exponential polynomial $\gamma_{d,T}(s)$ which is a generator and thus induces an efficiently computable convex upper bound on $p(w)$.

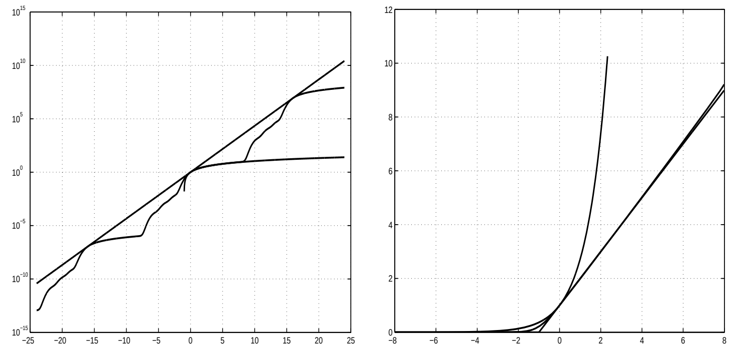
(ii) The resulting upper bound on $p(w)$ is $\le$ the Bernstein upper bound associated, according to (!), with $\gamma(s)=\exp\{s\}$.

The generator $\gamma_{11,8}(\cdot)$ is depicted in figure 2.3.

The case of ambiguous chance constraint

A disadvantage of the improved Bernstein approximation as compared to the plain one is that the improved approximation requires *precise* knowledge of the moment-generating functions $\mathbf{E}\{\exp\{s\zeta_\ell\}\}$, $s\in\mathbb{C}$, of the independent random variables ζ_ℓ, while the original approximation requires knowledge of *upper bounds* on these functions and thus is applicable in the case of ambiguous chance constraints, those with only partially known distributions of ζ_ℓ. Such partial information is equivalent to the fact that the distribution P of ζ belongs to a given family $\mathcal{P}$ in the space of product probability distributions on $\mathbb{R}^L$. All we need in this situation is a possibility to compute efficiently the convex function

$$\Psi^*_{\mathcal{P}}(w)=\sup_{P\in\mathcal{P}}\mathbf{E}_{\zeta\sim P}\{\gamma(w_0+\sum_{\ell=1}^{L}\zeta_\ell w_\ell)\}$$



(a) (b)

Figure 2.3: Generating function $\gamma_{11,8}(s)$ (middle curve) vs. $\exp\{s\}$ (top curve) and $\max[1+s,0]$ (bottom curve). (a): $-24 \le s \le 24$, logarithmic scale along the y-axis; (b): $-8 \le s \le 8$, natural scale along the y-axis.

associated with $\mathcal{P}$ and with a given generator $\gamma(\cdot)$. When $\Psi^*_{\mathcal{P}}(\cdot)$ is available, a computationally tractable safe approximation of the ambiguous chance constraint

$$\forall (P \in \mathcal{P}) : \mathrm{Prob}_{\zeta\sim P}\{w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell > 0\} \le \epsilon \tag{2.7.2}$$

is

$$H(w) := \inf_{\alpha>0} \left[\alpha \Psi^*_{\mathcal{P}}(\alpha^{-1} z) - \alpha\epsilon\right] \le 0.$$

Now, in all applications of the "plain" Bernstein approximation we have considered so far the family $\mathcal{P}$ was comprised of all product distributions $P = P_1 \times ... \times P_L$ with P_ℓ running through given families $\mathcal{P}_\ell$ of probability distributions on the axis, and these families $\mathcal{P}_\ell$ were "simple," specifically, allowing us to compute explicitly the functions

$$\Psi^*_\ell(t) = \sup_{P_\ell \in \mathcal{P}_\ell} \int \exp\{t\zeta_\ell\} dP_\ell(\zeta_\ell).$$

With these functions at our disposal and with $\gamma(s) = \exp\{s\}$, the function

$$\Psi^*_{\mathcal{P}}(w) = \sup_{P\in\mathcal{P}} \mathbf{E}\left\{\exp\{w_0 + \sum_{\ell=1}^{L} \zeta_\ell w_\ell\}\right\}$$

is readily available — it is merely $\exp\{w_0\} \prod_{\ell=1}^{L} \Psi^*_\ell(w_\ell)$. Note, however, that when $\gamma(\cdot)$ is an exponential polynomial rather than the exponent, the associated function $\Psi^*_{\mathcal{P}}(w)$ does *not* admit a simple representation via the functions $\Psi^*_\ell(\cdot)$. Thus, it is indeed unclear how to implement the improved Bernstein approximation in the case of an ambiguous chance constraint.

Our current goal is to implement the improved Bernstein approximation in the case of a particular ambiguous chance constraint (2.7.2), namely, in the case when $\mathcal{P}$ is comprised of all product distributions $P = P_1 \times ... \times P_L$ with the marginal distributions P_ℓ satisfying the restrictions

$$\mathrm{supp}\, P_\ell \subset [-1, 1] \ \&\ \mu^-_\ell \le \mathbf{E}_{\zeta_\ell \sim P_\ell}\{\zeta_\ell\} \le \mu^+_\ell \tag{2.7.3}$$

with known $\mu^\pm_\ell \in [-1, 1]$ (cf. item 6 on p. 51).

The result is as follows:

Proposition 2.4 [3, Proposition 4.5.4] *For the just defined family $\mathcal{P}$ and for every generator $\gamma(\cdot)$ one has*

$$\Psi^*_{\mathcal{P}}(w) = \mathbf{E}_{\zeta\sim P^w}\left\{\gamma(w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell)\right\},$$

where $P^w = P_1^{w_1}\times ... \times P_L^{w_L}$ and P_ℓ^s is the distribution supported at the endpoints of $[-1,1]$ given by

$$P_\ell^s\{1\} = 1 - P_\ell^s\{-1\} = \begin{cases} \frac{1+\mu_\ell^+}{2}, & s\geq 0 \\ \frac{1+\mu_\ell^-}{2}, & s<0 \end{cases};$$

In particular, when $\gamma(\cdot)\equiv\gamma_{d,T}(\cdot)$, the function $\Psi^*_{\mathcal{P}}(w)$ is efficiently computable.

Illustration I

To illustrate our findings, assume that all our a priori information on the random perturbations ζ_ℓ in (2.7.2) is that they are independent, supported on $[-1,1]$ and with zero means (cf. Example 2.1. Let us overview the safe approximations to the corresponding ambiguous chance constraint

$$\forall (P = P_1\times ... \times P_L \in \mathcal{P}) : \mathrm{Prob}_{\zeta\sim P}\left\{w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell > 0\right\} \leq \epsilon, \tag{2.7.4}$$

where $\mathcal{P}$ is the family of all product distributions with zero mean marginals supported on $[-1,1]$. Note that on a closest inspection, the results yielded by all approximation schemes to be listed below remain intact when instead of the ambiguous chance constraint we were speaking about the usual one, with ζ distributed uniformly on the vertices of the unit box $\{\zeta : \|\zeta\|_\infty \leq 1\}$.

We are about to outline the approximations, ascending in their conservatism and descending in their complexity. When possible, we present approximations in both the "inequality form" (via an explicit system of convex constraints) and in the "Robust Counterpart form"

$$\{w : w_0 + \zeta^T[w_1;...;w_L] \leq 0 \ \forall \zeta\in\mathcal{Z}\}.$$

• CVaR *approximation* [section 2.4.1]

$$\inf_{\alpha>0}\left[\max_{P_1\times...\times P_L\in\mathcal{P}}\int \max[\alpha + w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell, 0]dP_1(\zeta_1)...dP_L(\zeta_L) - \alpha\epsilon\right] \leq 0 \tag{2.7.5}$$

While being the least conservative among all generation-function-based approximations, the CVaR approximation is in general intractable. It remains intractable already when passing from the ambiguous chance constraint case to the case where ζ_ℓ are, say, uniformly distributed on $[-1,1]$ (which corresponds to replacing $\max\limits_{P_1\times...\times P_L\in\mathcal{P}}\int \ldots dP_1(\zeta_1)...dP_L(\zeta_\ell)$ in (2.7.5) with $\int\limits_{\|\zeta\|_\infty\leq 1} \ldots d\zeta$).

We have "presented" the inequality form of the CVaR approximation. By the results of section 2.3.3, this approximation admits a Robust Counterpart form; the latter "exists in the nature," but is computationally intractable, and thus of not much use.

• *Bridged Bernstein-*CVaR *approximation* [p. 64 and Proposition 2.4]

$$\begin{array}{l}\inf\limits_{\alpha>0}\left[\alpha\Psi_{d,T}(\alpha^{-1}w) - \alpha\epsilon\right] \leq 0,\\ \Psi_{d,T}(w) = \sum\limits_{\epsilon_\ell=\pm 1, 1\leq\ell\leq L} 2^{-L}\gamma_{d,T}\left(w_0 + \sum_{\ell=1}^L \epsilon_\ell w_\ell\right),\end{array} \tag{2.7.6}$$

where d,T are parameters of the construction and $\gamma_{d,T}$ is the exponential polynomial (2.7.1). Note that we used Proposition 2.4 to cope with the ambiguity of the chance constraint of interest.

In spite of the disastrous complexity of the representation (2.7.6), the function $\Psi_{d,T}$ is efficiently computable (via the recipe from Proposition 2.4, and *not* via the formula in (2.7.6)). Thus, our approximation is computationally tractable. Recall that this tractable safe approximation is less conservative than the plain Bernstein one.

Due to the results of section 2.3.3, approximation (2.7.5) admits a Robust Counterpart representation that now involves a computationally tractable uncertainty set $\mathcal{Z}^{\text{BCV}}$; this set, however, seems to have no explicit representation.

• *Bernstein approximation* [section 2.5.1]

$$\begin{array}{l}\inf\limits_{\alpha>0}\left[w_0+\sum_{\ell=1}^L \alpha\ln\left(\cosh(\alpha^{-1}w_\ell)\right)+\alpha\ln(1/\epsilon)\right]\leq 0\\ \Leftrightarrow w_0+\sum_{\ell=1}^L\zeta_\ell w_\ell\leq 0\ \forall\zeta\in\mathcal{Z}^{\text{Entr}}=\{\zeta:\sum_{\ell=1}^L\phi(\zeta_\ell)\leq 2\ln(1/\epsilon)\}\\ {[\phi(u)=(1+u)\ln(1+u)+(1-u)\ln(1-u),\ \text{Dom}\,\phi=[-1,1]]}.\end{array} \tag{2.7.7}$$

• *Ball-Box approximation* [section 2.5.1]

$$\begin{array}{l}\exists u: w_0+\sum_{\ell=1}^L|w_\ell-u_\ell|+\sqrt{2\ln(1/\epsilon)}\sqrt{\sum_{\ell=1}^L u_\ell^2}\leq 0\\ \Leftrightarrow w_0+\sum_{\ell=1}^L\zeta_\ell w_\ell\leq 0\ \forall\zeta\in\mathcal{Z}^{\text{BallBox}}:=\left\{\zeta\in\mathbb{R}^L:\ \begin{array}{l}|\zeta_\ell|\leq 1,\ \ell=1,...,L,\\ \sqrt{\sum\limits_{\ell=1}^L\zeta_\ell^2}\leq\sqrt{2\ln(1/\epsilon)}\end{array}\right\}.\end{array} \tag{2.7.8}$$

• *Budgeted approximation* [section 2.5.1]

$$\begin{array}{l}\exists u,v: z=u+v,\ v_0+\sum_{\ell=1}^L|v_\ell|\leq 0,\ u_0+\sqrt{2L\ln(1/\epsilon)}\max\limits_\ell|u_\ell|\leq 0\\ \Leftrightarrow w_0+\sum_{\ell=1}^L\zeta_\ell w_\ell\leq 0\ \forall\zeta\in\mathcal{Z}^{\text{Bdg}}:=\left\{\zeta\in\mathbb{R}^L:\ \begin{array}{l}|\zeta_\ell|\leq 1,\ \ell=1,...,L,\\ \sum_{\ell=1}^L|\zeta_\ell|\leq\sqrt{2L\ln(1/\epsilon)}\end{array}\right\}.\end{array} \tag{2.7.9}$$

The computationally tractable uncertainty sets we have listed form a chain:

$$\mathcal{Z}^{\text{BCV}}\subset\mathcal{Z}^{\text{Brn}}\subset\mathcal{Z}^{\text{BlBx}}\subset\mathcal{Z}^{\text{Bdg}}.$$

Figure 2.4, where we plot a random 2-D cross-section of our nested uncertainty sets, gives an impression of the "gaps" in this chain.

Illustration II

In this illustration we use the above approximation schemes to build safe approximations of the ambiguously chance constrained problem

$$\text{Opt}(\epsilon)=\max\left\{w_0:\max_{P_1\times\ldots\times P_L\in\mathcal{P}}\text{Prob}\{w_0+\sum_{\ell=1}^L\zeta_\ell w_\ell>0\}\leq\epsilon, w_1=...=w_L=1\right\} \tag{2.7.10}$$

where, as before, $\mathcal{P}$ is the set of L-element tuples of probability distributions supported on $[-1,1]$ and possessing zero means. Due to the simplicity of our chance constraint, here we can build efficiently the CVaR-approximation of the problem. Moreover, we can solve *exactly* the chance constrained problem

$$\text{Opt}^+(\epsilon)=\max\left\{w_0:\max_{\zeta\sim U}\text{Prob}\{w_0+\sum_{\ell=1}^L\zeta_\ell w_\ell>0\}\leq\epsilon, w_1=...=w_L=1\right\}$$

where U is the uniform distribution on the vertices of the unit box $\{\zeta:\|\zeta\|_\infty\leq 1\}$. Clearly, $\text{Opt}^+(\epsilon)$ is an upper bound on the true optimal value $\text{Opt}(\epsilon)$ of the ambiguously chance constrained problem (2.7.10), while the optimal values of our approximations are lower bounds on $\text{Opt}(\epsilon)$. In our experiment, we used $L=128$. The results are depicted in figure 2.5 and are displayed in table 2.3.

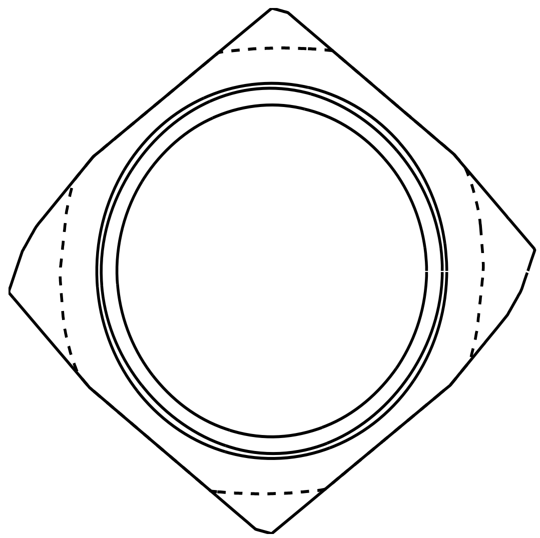


Figure 2.4: Intersection of uncertainty sets, underlying various approximation schemes in Illustration I, with a random 2-D plane. From inside to outside:
– Bridged Bernstein-CVaR approximation, $d = 11$, $T = 8$;
– Bernstein approximation;
– Ball-Box approximation;
– Budgeted approximation;
– "worst-case" approximation with the support $\{\|\zeta\|_\infty \le 1\}$ of ζ in the role of the uncertainty set.

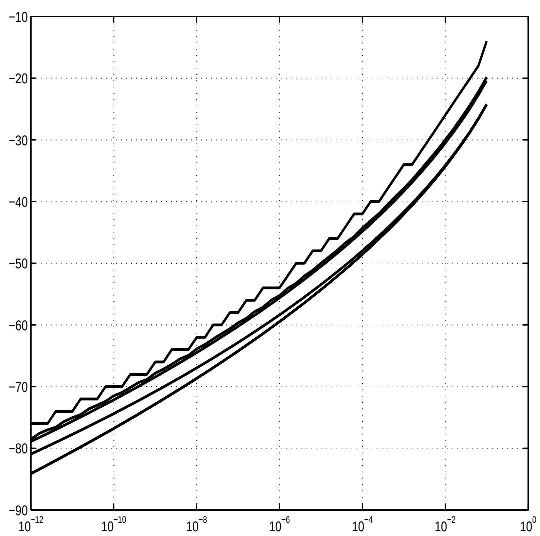


Figure 2.5: Optimal values of various approximations of (2.7.10) with $L = 128$ vs. ϵ.
From bottom to top:
– Budgeted and Ball-Box approximations
– Bernstein approximation
– Bridged Bernstein-CVaR approximation, $d = 11$, $T = 8$
– CVaR-approximation
– $\mathrm{Opt}^+(\epsilon)$

ϵ	$\mathrm{Opt}^+(\epsilon)$	$\mathrm{Opt}_V(\epsilon)$	$\mathrm{Opt}_{IV}(\epsilon)$	$\mathrm{Opt}_{III}(\epsilon)$	$\mathrm{Opt}_{II}(\epsilon)$	$\mathrm{Opt}_I(\epsilon)$
10^{-12}	−76.00	−78.52 (−3.3%)	−78.88 (−0.5%)	−80.92 (−3.1%)	−84.10 (−7.1%)	−84.10 (−7.1%)
10^{-11}	−74.00	−75.03 (−1.4%)	−75.60 (−0.8%)	−77.74 (−3.6%)	−80.52 (−7.3%)	−80.52 (−7.3%)
10^{-10}	−70.00	−71.50 (−2.1%)	−72.13 (−0.9%)	−74.37 (−4.0%)	−76.78 (−7.4%)	−76.78 (−7.4%)
10^{-9}	−66.00	−67.82 (−2.8%)	−68.45 (−0.9%)	−70.80 (−4.4%)	−72.84 (−7.4%)	−72.84 (−7.4%)
10^{-8}	−62.00	−63.88 (−3.0%)	−64.49 (−1.0%)	−66.97 (−4.8%)	−68.67 (−7.5%)	−68.67 (−7.5%)
10^{-7}	−58.00	−59.66 (−2.9%)	−60.23 (−1.0%)	−62.85 (−5.4%)	−64.24 (−7.7%)	−64.24 (−7.7%)
10^{-6}	−54.00	−55.25 (−2.3%)	−55.60 (−0.6%)	−58.37 (−5.7%)	−59.47 (−7.6%)	−59.47 (−7.6%)
10^{-5}	−48.00	−49.98 (−4.1%)	−50.52 (−1.1%)	−53.46 (−7.0%)	−54.29 (−8.6%)	−54.29 (−8.6%)
10^{-4}	−42.00	−44.31 (−5.5%)	−44.85 (−1.2%)	−47.97 (−8.3%)	−48.56 (−9.6%)	−48.56 (−9.6%)
10^{-3}	−34.00	−37.86(−11.4%)	−38.34 (−1.2%)	−41.67(−10.1%)	−42.05(−11.1%)	−42.05(−11.1%)
10^{-2}	−26.00	−29.99(−15.4%)	−30.55 (−1.9%)	−34.13(−13.8%)	−34.34(−14.5%)	−34.34(−14.5%)
10^{-1}	−14.00	−19.81(−41.5%)	−20.43 (−3.1%)	−24.21(−22.2%)	−24.28(−22.5%)	−24.28(−22.5%)

Table 2.3: Comparing various safe approximations of the ambiguously chance constrained problem (2.7.10). $\mathrm{Opt}_I(\epsilon)$ through $\mathrm{Opt}_V(\epsilon)$ are optimal values of the Ball, Ball-Box (or, which in the case of (2.7.10) is the same, the Budgeted), Bernstein, Bridged Bernstein-CVaR and the CVaR approximations, respectively. Numbers in parentheses in column "$\mathrm{Opt}_V(\epsilon)$" refer to the conservativeness of the CVaR-approximation as compared to $\mathrm{Opt}^+(\cdot)$, and in remaining columns to the conservativeness of the corresponding approximation as compared to the CVaR approximation.

2.7.2 Majorization

One way to bound from above the probability

$$\mathrm{Prob}\left\{w_0 + \sum_{\ell=1}^{L} w_\ell \zeta_\ell > 0\right\}$$

for independent random variables ζ_ℓ is to replace ζ_ℓ with "more diffused" random variables ξ_ℓ (meaning that the probability in question increases when we replace ζ_ℓ with ξ_ℓ) such that the quantity $\mathrm{Prob}\left\{w_0 + \sum_{\ell=1}^{L} w_\ell \xi_\ell > 0\right\}$, (which now is an upper bound on probability in question), is easy to handle. Our goal here is to investigate the outlined approach in the case of random variables with symmetric and unimodal w.r.t. 0 probability distributions. In other words, the probability distributions P in question are such that for any measurable set A on the axis

$$P(A) = \chi(A) + \int_A p(s)ds,$$

where p is an even nonnegative and nonincreasing on the nonnegative ray *density* of P, $\int p(s)ds := 1 - \alpha \leq 1$, and $\chi(A)$ is either α (when $0 \in A$), or 0 (otherwise). We say that such a distribution P is regular, if there is no mass at the origin: $\alpha = 0$.

In what follows, we denote the family of densities of all symmetric and unimodal w.r.t. 0 random variables by $\mathcal{P}$, and the family of these random variables themselves by Π.

If we want the outlined scheme to work, the notion of a "more diffused" random variable should imply the following: If $p, q \in \mathcal{P}$ and q is "more diffused" than p, then, for every $a \geq 0$, we should have $\int_a^\infty p(s)ds \leq \int_a^\infty q(s)ds$. We make this requirement the definition of "more diffused":

Definition 2.2 *Let $p, q \in \mathcal{P}$. We say that q is more diffused than p* (notation: $q \succeq_{\mathrm{m}} p$, or $p \preceq_{\mathrm{m}} q$) *if*

$$\forall a \geq 0 : P(a) := \int_a^\infty p(s)ds \leq Q(a) := \int_a^\infty q(s)ds.$$

When $\xi, \eta \in \Pi$, we say that η is more diffuse than ξ (notation: $\eta \succeq_{\rm m} \xi$), *if the corresponding densities are in the same relation.*

It is immediately seen that the relation $\succeq_{\rm m}$ is a partial order on $\mathcal{P}$; this order is called "monotone dominance." It is well known that an equivalent description of this order is given by the following

Proposition 2.5 *Let $\pi, \theta \in \Pi$, let ν, q be the probability distribution of θ and the density of θ, and let μ, p be the probability distribution and the density of π. Finally, let $\mathcal{M}_b$ be the family of all continuously differentiable even and bounded functions on the axis that are nondecreasing on $\mathbb{R}_+$. Then $\theta \succeq_{\rm m} \pi$ if and only if*

$$\int f(s)d\nu(s) \geq \int f(s)d\mu(s)ds \;\forall f \in \mathcal{M}_b, \tag{2.7.11}$$

same as if and only if

$$\int f(s)q(s)ds \geq \int f(s)p(s)ds \;\forall f \in \mathcal{M}_b. \tag{2.7.12}$$

Moreover, when (2.7.11) *takes place, the inequalities in* (2.7.11), (2.7.12) *hold true for every even function on the axis that is nondecreasing on $\mathbb{R}_+$.*

Example 2.2 Let $\xi \in \Pi$ be a random variable that is supported on $[-1,1]$, ζ be uniformly distributed on $[-1,1]$ and $\eta \sim \mathcal{N}(0, 2/\pi)$. Then $\xi \preceq_{\rm m} \zeta \preceq_{\rm m} \eta$.

Indeed, let $p(\cdot)$, $q(\cdot)$ be the densities of random variables $\pi, \theta \in \Pi$. Then the functions $P(t) = \int\limits_t^\infty p(s)ds$, $Q(t) = \int\limits_t^\infty q(s)ds$ of $t \geq 0$ are convex, and $\pi \preceq_{\rm m} \theta$ iff $P(t) \leq Q(t)$ for all $t \geq 0$. Now let $\pi \in \Pi$ be supported on $[-1,1]$ and θ be uniform on $[-1,1]$. Then $P(t)$ is convex on $[0,\infty)$ with $P(0) \leq 1/2$ and $P(t) \equiv P(1) = 0$ for $t \geq 1$, while $Q(t) = \frac{1}{2}\max[1-t, 0]$ when $t \geq 0$. Since $Q(0) \geq P(0)$, $Q(1) = P(1)$, P is convex, and Q is linear on $[0,1]$, we have $P(t) \leq Q(t)$ for all $t \in [0,1]$, whence $P(t) \leq Q(t)$ for all $t \geq 0$, and thus $\pi \preceq_{\rm m} \theta$. Now let π be uniform on $[-1,1]$, so that $P(t) = \frac{1}{2}\max[1-t,0]$, and θ be $\mathcal{N}(0, 2/\pi)$, so that $Q(t)$ is a convex function and therefore $Q(t) \geq Q(0) + Q'(0)t = (1-t)/2$ for all $t \geq 0$. This combines with $Q(t) \geq 0$, $t \geq 0$, to imply that $P(t) \leq Q(t)$ for all $t \geq 0$ and thus $\pi \preceq_{\rm m} \theta$.

Our first majorization result is as follows:

Proposition 2.6 [3, Proposition 4.4.5] *Let $w_0 \leq 0$, $w_1, ..., w_L$ be deterministic reals, $\{\zeta_\ell\}_{\ell=1}^L$ be independent random variables with unimodal and symmetric w.r.t. 0 distributions, and $\{\eta_\ell\}_{\ell=1}^L$ be a similar collection of independent random variables such that $\eta_\ell \succeq_{\rm m} \zeta_\ell$ for every ℓ. Then*

$$\mathrm{Prob}\{w_0 + \sum_{\ell=1}^L w_\ell \zeta_\ell > 0\} \leq \mathrm{Prob}\{w_0 + \sum_{\ell=1}^L w_\ell \eta_\ell > 0\}. \tag{2.7.13}$$

If, in addition, $\eta_\ell \sim \mathcal{N}(0, \sigma_\ell^2)$, $\ell = 1, ..., L$, then, for every $\epsilon \in (0, 1/2]$, one has

$$w_0 + \mathrm{ErfInv}(\epsilon)\sqrt{\sum_{\ell=1}^L \sigma_\ell^2 w_\ell^2} \leq 0 \Rightarrow \mathrm{Prob}\{w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell > 0\} \leq \epsilon, \tag{2.7.14}$$

where $\mathrm{ErfInv}(\cdot)$ *is the inverse error function* (2.2.7).

Relation (2.7.14) seems to be the major "yield" we can extract from Proposition 2.6, since the case of independent $\mathcal{N}(0, \sigma_\ell^2)$ random variables η_ℓ is, essentially, the only interesting case for which we can easily compute $\mathrm{Prob}\{w_0 + \sum\limits_{\ell=1}^L w_\ell \eta_\ell > 0\}$ and the chance constraint $\mathrm{Prob}\{w_0 + \sum\limits_{\ell=1}^L w_\ell \eta_\ell > 0\} \leq \epsilon$ for $\epsilon \leq 1/2$ is equivalent to an explicit convex constraint, specifically,

$$w_0 + \mathrm{ErfInv}(\epsilon)\sqrt{\sum_{\ell=1}^L \sigma_\ell^2 w_\ell^2} \leq 0. \tag{2.7.15}$$

Comparison with (2.5.9). Assume that independent random variables $\zeta_\ell \in \Pi$, $\ell = 1, ..., L$, admit "Gaussian upper bounds" $\eta_\ell \succeq_{\rm m} \zeta_\ell$ with $\eta_\ell \sim \mathcal{N}(0, \sigma_\ell^2)$. Then

$$\begin{array}{l} \mathbf{E}\{\exp\{t\zeta_e ll\}\} = \mathbf{E}\{\cosh(t\zeta_\ell)\} \text{ [since } \zeta_\ell \text{ is symmetrically distributed]} \\ \leq \mathbf{E}\{\cosh(t\eta_\ell)\} \text{ [since } \eta_\ell \succeq_{\rm m} \zeta_\ell] \\ = \exp\{t^2\sigma_\ell^2/2\}, \end{array}$$

meaning that the logarithmic moment-generating functions $\ln\left(\mathbf{E}\left\{\exp\{t\zeta_\ell\}\right\}\right)$ of ζ_ℓ admit upper bounds (2.5.8) with $\chi_\ell^\pm = 0$ and the above σ_ℓ. Invoking the results of section (2.5.2), the chance constraint

$$\mathrm{Prob}\{w_0 + \sum_{\ell=1}^L \zeta_\ell w_\ell > 0\} \leq \epsilon$$

admits a safe tractable approximation

$$w_0 + \sqrt{2\ln(1/\epsilon)}\sqrt{\sum_{\ell=1}^L \sigma_\ell^2 w_\ell^2} \leq 0$$

obtained from (2.5.9) by setting $\chi_\ell^\pm = 0$. This approximation is slightly more conservative than (2.7.14), since $\mathrm{ErfInv}(\epsilon) \leq \sqrt{2\ln(1/\epsilon)}$; note, however, that the ratio of the latter two quantities goes to 1 as $\epsilon \to 0$.

Majorization Theorem

Proposition 2.6 can be rephrased as follows:

> *Let $\{\zeta_\ell\}_{\ell=1}^L$ be independent random variables with unimodal and symmetric w.r.t. 0 distributions, and $\{\eta_\ell\}_{\ell=1}^L$ be a similar collection of independent random variables such that $\eta_\ell \succeq_{\rm m} \zeta_\ell$ for every ℓ. Given a deterministic vector $z \in \mathbb{R}^L$ and $w_0 \leq 0$, consider the "strip"*
>
> $$S = \{x \in \mathbb{R}^L : |z^T x| \leq -w_0\}.$$
>
> *Then*
> $$\mathrm{Prob}\{[\zeta_1; ...; \zeta_L] \in S\} \geq \mathrm{Prob}\{[\eta_1; ...; \eta_L] \in S\}.$$

It turns out that the resulting inequality holds true for every closed convex set S that is symmetric w.r.t. the origin.

Theorem 2.6 [3, Theorem 4.4.6] *Let $\{\zeta_\ell\}_{\ell=1}^L$ be independent random variables with unimodal and symmetric w.r.t. 0 distributions, and $\{\eta_\ell\}_{\ell=1}^L$ be a similar collection of independent random variables such that $\eta_\ell \succeq_{\rm m} \zeta_\ell$ for every ℓ. Then for every closed convex set $S \subset \mathbb{R}^L$ that is symmetric w.r.t. the origin one has*

$$\mathrm{Prob}\{[\zeta_1; ...; \zeta_L] \in S\} \geq \mathrm{Prob}\{[\eta_1; ...; \eta_L] \in S\}. \tag{2.7.16}$$

Example 2.3 Let $\xi \sim \mathcal{N}(0, \Sigma)$ and $\eta \sim \mathcal{N}(0, \Theta)$ be two Gaussian random vectors taking values in $\mathbb{R}^n$ and let $\Sigma \preceq \Theta$. We claim that for every closed convex set $S \subset \mathbb{R}^n$ symmetric w.r.t. 0 one has

$$\mathrm{Prob}\{\xi \in S\} \geq \mathrm{Prob}\{\eta \in S\}.$$

Indeed, by continuity reasons, it suffices to consider the case when Θ is nondegenerate. Passing from random vectors ξ, η to random vectors $A\xi, A\eta$ with properly defined nonsingular A, we can reduce the situation to the one where $\Theta = I$ and Σ is diagonal, meaning that the densities $p(\cdot)$ of ξ and q of η are of the forms

$$p(x) = p_1(x_1)...p_n(x_n),\ q(x) = q_1(x_1)...q_n(x_n),$$

with $p_i(s)$ being the $\mathcal{N}(0, \Sigma_{ii})$ densities, and $q_i(s)$ being the $\mathcal{N}(0, 1)$ densities. Since $\Sigma \preceq \Theta = I$, we have $\Sigma_{ii} \leq 1$, meaning that $p_i \preceq_{\rm m} q_i$ for all i. It remains to apply the Majorization Theorem.

2.8 Exercises

Exercise 2.1 Prove the claim in example 8, section 2.5.2.

Exercise 2.2 Consider a toy chance constrained LO problem:

$$\min_{x,t}\{t : \mathrm{Prob}\{\underbrace{\sum_{j=1}^{n}\zeta_j x_j}_{\xi^n[x]} \le t\} \ge 1-\epsilon, 0 \le x_i \le 1, \sum_j x_j = n\} \tag{2.8.1}$$

where $\zeta_1, ..., \zeta_n$ are independent random variables uniformly distributed in $[-1, 1]$.

1. Find a way to solve the problem exactly, and find the true optimal value t_{tru} of the problem for $n = 16, 256$ and $\epsilon = 0.05, 0.0005, 0.000005$.
   Hint: The deterministic constraints say that $x_1 = ... = x_n = 1$. All we need is an efficient way to compute the probability distribution $\mathrm{Prob}\{\xi^n < t\}$ of the sum ξ^n of n independent random variables uniformly distributed on $[-1, 1]$. The density of ξ^n clearly is supported on $[-n, n]$ and is a polynomial of degree $n-1$ in every one of the segments $[-n+2i, -n+2i+2]$, $0 \le i < n$. The coefficients of these polynomials can be computed via a simple recursion in n.

2. For the same pairs (n, ϵ) as in i), compute the optimal values of the tractable approximations of the problem as follows:
   (a) t_{Nrm} — the optimal value of the problem obtained from (2.8.1) when replacing the "true" random variable $\xi^n[x]$ with its "normal approximation" — a Gaussian random variable with the same mean and standard deviation as those of $\xi^n[x]$;
   (b) t_{Bll} — the optimal value of the Ball approximation of (2.8.1), p. 46, section 2.5.1;
   (c) t_{BllBx} — the optimal value of the Ball-Box approximation of (2.8.1), p. 47, section 2.5.1;
   (d) t_{Bdg} — the optimal value of the Budgeted approximation of (2.8.1), p. 47, section 2.5.1;
   (e) $t_{\mathrm{E.7}}$ — the optimal value of the safe tractable approximation of (2.8.1) suggested by example 7 in section 2.5.2, where you set $\mu^{\pm} = 0$ and $\nu = 1/\sqrt{3}$;
   (f) $t_{\mathrm{E.8}}$ — the optimal value of the safe tractable approximation of (2.8.1) suggested by example 8 in section 2.5.2, where you set $\nu = 1/\sqrt{3}$;
   (g) $t_{\mathrm{E.9}}$ — the optimal value of the safe tractable approximation of (2.8.1) suggested by example 9 in section 2.5.2, where you set $\nu = 1/\sqrt{3}$;
   (h) t_{Unim} — the optimal value of the safe tractable approximation of (2.8.1) suggested by example 3 in section 2.5.2.

   Think of the results as compared to each other and to those of i).

Exercise 2.3 Consider the chance constrained LO problem (2.8.1) with independent $\zeta_1, ..., \zeta_n$ taking values ± 1 with probability 0.5.

1. Find a way to solve the problem exactly, and find the true optimal value t_{tru} of the problem for $n = 16, 256$ and $\epsilon = 0.05, 0.0005, 0.000005$.

2. For the same pairs (n, ϵ) as in i), compute the optimal values of the tractable approximations of the problem as follows:

   (a) $t_{\rm Nrm}$ — the optimal value of the problem obtained from (2.8.1) when replacing the "true" random variable ξ^n with its "normal approximation" — a Gaussian random variable with the same mean and standard deviation as those of ξ^n;

   (b) $t_{\rm Bll}$ — the optimal value of the Ball approximation of (2.8.1), p. 46, section 2.5.1;

   (c) $t_{\rm BllBx}$ — the optimal value of the Ball-Box approximation of (2.8.1), p. 47, section 2.5.1;

   (d) $t_{\rm Bdg}$ — the optimal value of the Budgeted approximation of (2.8.1), p. 47, section 2.5.1;

   (e) $t_{\rm E.7}$ — the optimal value of the safe tractable approximation of (2.8.1) suggested by example 7, section 2.5.2, where you set $\mu^{\pm} = 0$ and $\nu = 1$;

   (f) $t_{\rm E.8}$ — the optimal value of the safe tractable approximation of (2.8.1) suggested by example 8, section 2.5.2, where you set $\nu = 1$.

   Think of the results as compared to each other and to those of i).

Exercise 2.4 A) Verify that whenever $n = 2^k$ is an integral power of 2, one can build an $n \times n$ matrix B_n with all entries ± 1, all entries in the first column equal to 1, and with rows that are orthogonal to each other.

Hint: Use recursion $B_{2^0} = [1]$; $B_{2^{k+1}} = \left[\begin{array}{cc} B_{2^k} & B_{2^k} \\ B_{2^k} & -B_{2^k} \end{array}\right]$.

B) Let $n = 2^k$ and $\widehat{\zeta} \in \mathbb{R}^n$ be the random vector as follows. We fix a matrix B_n from A). To get a realization of ζ, we generate random variable $\eta \sim \mathcal{N}(0,1)$ and pick at random (according to uniform distribution on $\{1, ..., n\}$) a column in the matrix ηB_n; the resulting vector is a realization of $\widehat{\zeta}$ that we are generating.

B.1) Prove that the marginal distributions of ζ_j and the covariance matrix of $\widehat{\zeta}$ are exactly the same as for the random vector $\widetilde{\zeta} \sim \mathcal{N}(0, I_n)$. It follows that most primitive statistical tests cannot distinguish between the distributions of $\widehat{\zeta}$ and $\widetilde{\zeta}$.

B.2) Consider problem (2.8.1) with $\epsilon < 1/(2n)$ and compute the optimal values in the cases when (a) ζ is $\widetilde{\zeta}$, and (b) ζ is $\widehat{\zeta}$. Compare the results for $n = 10, \epsilon = 0.01$; $n = 100, \epsilon = 0.001$; $n = 1000, \epsilon = 0.0001$.

Exercise 2.5 Let ζ_ℓ, $1 \le \ell \le L$, be independent Poisson random variables with parameters λ_ℓ, (i.e., ζ_ℓ takes nonnegative integer value k with probability $\frac{\lambda_\ell^k}{k!} {\rm e}^{-\lambda_\ell}$). Build Bernstein approximation of the chance constraint

$$\mathrm{Prob}\{z_0 + \sum_{\ell=1}^{L} w_\ell \zeta_\ell \le 0\} \ge 1 - \epsilon.$$

What is the associated uncertainty set $\mathcal{Z}_\epsilon$ as given by Theorem 2.2?

Exercise 2.6 The stream of customers of an ATM can be split into L groups, according to the amounts of cash c_ℓ they are withdrawing. The per-day number of customers of type ℓ is a realization of Poisson random variable ζ_ℓ with parameter λ_ℓ, and these variables are independent of each other. What is the minimal amount of cash $w(\epsilon)$ to be loaded in the ATM in the morning in order to ensure service level $1-\epsilon$, (i.e., the probability of the event that not all customers arriving during the day are served should be $\leq \epsilon$)?

Consider the case when

$$L = 7, c = [20; 40; 60; 100; 300; 500; 1000], \lambda_\ell = 1000/c_\ell$$

and compute and compare the following quantities:

1. The expected value of the per-day customer demand for cash.

2. The true value of $w(\epsilon)$ and its CVaR-upper bound (utilize the integrality of c_ℓ to compute these quantities efficiently).

3. The bridged Bernstein - CVaR, and the pure Bernstein upper bounds on $w(\epsilon)$.

4. The $(1-\epsilon)$-reliable empirical upper bound on $w(\epsilon)$ built upon a 100,000-element simulation sample of the per day customer demands for cash.

   The latter quantity is defined as follows. Assume that given an N-element sample $\{\eta_i\}_{i=1}^N$ of independent realizations of a random variable η, and a tolerance $\delta \in (0,1)$, we want to infer from the sample a "$(1-\delta)$-reliable" upper bound on the upper ϵ quantile $q_\epsilon = \min\{q : \mathrm{Prob}\{\eta > q\} \leq \epsilon\}$ of η. It is natural to take, as this bound, the M-th order statistics S_M of the sample, (i.e., M-th element in the non-descending rearrangement of the sample), and the question is, how to choose M in order for S_M to be $\geq q_\epsilon$ with probability at least $1-\delta$. Since $\mathrm{Prob}\{\eta \geq q_\epsilon\} \geq \epsilon$, the relation $S_M < q_\epsilon$ for a given M implies that in our sample of N independent realizations η_i of η the relation $\{\eta_i \geq q_\epsilon\}$ took place at most $N-M$ times, and the probability of this event is at most $p_M = \sum_{k=0}^{N-M} \binom{N}{k} \epsilon^k (1-\epsilon)^{N-k}$. It follows that if M is such that $p_M \leq \delta$, then the event in question takes place with probability at most δ, i.e., S_M is an upper bound on q_ϵ with probability at least $1-\delta$. Thus, it is natural to choose M as the smallest integer $\leq N$ such that $p_M \leq \delta$. Note that such an integer not necessarily exists — it may happen that already $p_N > \delta$, meaning that the sample size N is insufficient to build a $(1-\delta)$-reliable upper bound on q_ϵ.

Carry out the computation for $\epsilon = 10^{-k}$, $1 \leq k \leq 6$. [7]

Exercise 2.7 Consider the same situation as in Exercise 2.6, with the only difference that now we do not assume the Poisson random variables ζ_ℓ to be independent, and make no assumptions whatsoever on how they relate to each other. Now the minimal amount of "cash input" to the ATM that guarantees service level $1-\epsilon$ is the optimal value $\widehat{w}(\epsilon)$ of the "ambiguously chance constrained" problem

$$\min\left\{w_0 : \mathrm{Prob}_{\zeta\sim P}\{\sum_\ell c_\ell\zeta_\ell \leq w_0\} \geq 1-\epsilon\, \forall P \in \mathcal{P}\right\},$$

where $\mathcal{P}$ is the set of all distributions P on $\mathbb{R}^L$ with Poisson distributions with parameters $\lambda_1, ..., \lambda_L$ as their marginals.

By which margin can $\widehat{w}(\epsilon)$ be larger than $w(\epsilon)$? To check your intuition, use the same data as in Exercise 2.6 to compute

[7] Of course, in our ATM story the values of ϵ like 0.001 and less make no sense. Well, you can think about an emergency center and requests for blood transfusions instead of an ATM and dollars.

– the upper bound on $\widehat{w}(\epsilon)$ given by Theorem 2.4;
– the lower bound on $\widehat{w}(\epsilon)$ corresponding to the case where $\zeta_1, ..., \zeta_L$ are comonotone, (i.e., are deterministic nondecreasing functions of the same random variable η uniformly distributed on $[0, 1]$, cf. p. 63).
Carry out computations for $\epsilon = 10^{-k}$, $1 \leq k \leq 6$.

Exercise 2.8 1) Consider the same situation as in Exercise 2.6, but assume that the nonnegative vector $\lambda = [\lambda_1; ...; \lambda_L]$ is known to belong to a given convex compact set $\Lambda \subset \{\lambda \geq 0\}$. Prove that with

$$B_\Lambda(\epsilon) = \max_{\lambda\in\Lambda} \inf \left\{ w_0 : \inf_{\beta>0} \left[-w_0 + \beta \sum_\ell \lambda_\ell (\exp\{c_\ell/\beta\} - 1) - \beta \ln(1/\epsilon) \right] \leq 0 \right\}$$

one has

$$\forall \lambda \in \Lambda : \mathrm{Prob}_{\zeta\sim P_{\lambda_1}\times...\times P_{\lambda_L}} \left\{ \sum_\ell c_\ell \zeta_\ell > B_\Lambda(\epsilon) \right\} \leq \epsilon,$$

where P_μ stands for the Poisson distribution with parameter μ. In other words, initial charge of $B_\Lambda(\epsilon)$ dollars is enough to ensure service level $1 - \epsilon$, whatever be the vector $\lambda \in \Lambda$ of parameters of the (independent of each other) Poisson streams of customers of different types.

2) In 1), we have considered the case when λ runs through a given "uncertainty set" Λ, and we want the service level to be at least $1 - \epsilon$, whatever be $\lambda \in \Lambda$. Now consider the case when we impose a chance constraint on the service level, specifically, assume that λ is picked at random every morning, according to a certain distribution P on the nonnegative orthant, and we want to find a once and forever fixed morning cash charge w_0 of the ATM such that the probability for a day to be "bad" (such that the service level in this day drops below the desired level $1 - \epsilon$) is at most a given $\delta \in (0, 1)$. Now consider the chance constraint

$$\mathrm{Prob}_{\lambda\sim P}\{z_0 + \sum_\ell \lambda_\ell z_\ell > 0\} \leq \delta$$

in variables $z_0, ..., z_L$, and assume that we have in our disposal a Robust Counterpart type safe convex approximation of this constraint, i.e., we know a convex compact set $\Lambda \subset \{\lambda \geq 0\}$ such that

$$\forall (z_0, ..., z_L) : z_0 + \max_{\lambda\in\Lambda} \sum_\ell \lambda_\ell z_\ell \leq 0 \Rightarrow \mathrm{Prob}_{\lambda\sim P}\{z_0 + \sum_\ell \lambda_\ell z_\ell > 0\} \leq \delta.$$

Prove that by loading the ATM with $B_\Lambda(\epsilon)$ dollars in the morning, we ensure that the probability of a day to be bad is $\leq \delta$.

Lecture 3

Robust Conic Quadratic and Semidefinite Optimization

In this lecture, we extend the RO methodology onto *non-linear* convex optimization problems, specifically, *conic* ones.

3.1 Uncertain Conic Optimization: Preliminaries

3.1.1 Conic Programs

A *conic* optimization (CO) problem (also called *conic program*) is of the form

$$\min_x \left\{ c^T x + d : Ax - b \in \mathbf{K} \right\}, \tag{3.1.1}$$

where $x \in \mathbb{R}^n$ is the decision vector, $\mathbf{K} \subset \mathbb{R}^m$ is a closed pointed convex cone with a nonempty interior, and $x \mapsto Ax - b$ is a given affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. Conic formulation is one of the universal forms of a Convex Programming problem; among the many advantages of this specific form is its "unifying power." An extremely wide variety of convex programs is covered by just three types of cones:

1. Direct products of nonnegative rays, i.e., $\mathbf{K}$ is a non-negative orthant $\mathbb{R}^m_+$. These cones give rise to Linear Optimization problems

$$\min_x \left\{ c^T x + d : a_i^T x - b_i \geq 0,\, 1 \leq i \leq m \right\}.$$

2. Direct products of *Lorentz* (or *Second-order*, or Ice-cream) cones $\mathbf{L}^k = \{x \in \mathbb{R}^k : x_k \geq \sqrt{\sum_{j=1}^{k-1} x_j^2}\}$. These cones give rise to Conic Quadratic Optimization (called also Second Order Conic Optimization). The Mathematical Programming form of a CO problem is

$$\min_x \left\{ c^T x + d : \|A_i x - b_i\|_2 \leq c_i^T x - d_i,\, 1 \leq i \leq m \right\};$$

here i-th scalar constraint (called *Conic Quadratic Inequality*) (CQI) expresses the fact that the vector $[A_i x; c_i^T x] - [b_i; d_i]$ that depends affinely on x belongs to the Lorentz cone $\mathbf{L}_i$ of appropriate dimension, and the system of all constraints says that the affine mapping

$$x \mapsto \left[[A_1 x; c_1^T x]; ...; [A_m x; c_m^T x]\right] - [[b_1; d_1]; ..., ; [b_m; d_m]]$$

maps x into the direct product of the Lorentz cones $\mathbf{L}_1 \times ... \times \mathbf{L}_m$.

3. Direct products of *semidefinite* cones $\mathbf{S}^k_+$.

   $\mathbf{S}^k_+$ is the cone of positive semidefinite $k \times k$ matrices; it "lives" in the space $\mathbf{S}^k$ of symmetric $k \times k$ matrices. We treat $\mathbf{S}^k$ as Euclidean space equipped with the *Frobenius* inner product $\langle A, B\rangle = \mathrm{Tr}(AB) = \sum\limits_{i,j=1}^{k} A_{ij}B_{ij}$.

   The family of semidefinite cones gives rise to *Semidefinite Optimization* (SDO) — optimization programs of the form

$$\min_x \left\{ c^T x + d : \mathcal{A}_i x - B_i \succeq 0,\ 1 \leq i \leq m \right\},$$

   where

$$x \mapsto \mathcal{A}_i x - B_i \equiv \sum_{j=1}^{n} x_j A^{ij} - B_i$$

   is an affine mapping from $\mathbb{R}^n$ to $\mathbf{S}^{k_i}$ (so that A^{ij} and B_i are symmetric $k_i \times k_i$ matrices), and $A \succeq 0$ means that A is a symmetric positive semidefinite matrix. The constraint of the form "a symmetric matrix affinely depending on the decision vector should be positive semidefinite" is called an LMI — Linear Matrix Inequality. Thus, a Semidefinite Optimization problem (called also *semidefinite program*) is the problem of minimizing a linear objective under finitely many LMI constraints. One can rewrite an SDO program in the Mathematical Programming form, e.g., as

$$\min_x \left\{ c^T x + d : \lambda_{\min}(\mathcal{A}_i x - B_i) \geq 0,\ 1 \leq i \leq m \right\},$$

   where $\lambda_{\min}(A)$ stands for the minimal eigenvalue of a symmetric matrix A, but this reformulation usually is of no use.

Keeping in mind our future needs related to *Globalized Robust Counterparts*, it makes sense to modify slightly the format of a conic program, specifically, to pass to programs of the form

$$\min_x \left\{ c^T x + d : A_i x - b_i \in \mathbf{Q}_i,\ 1 \leq i \leq m \right\}, \tag{3.1.2}$$

where $\mathbf{Q}_i \subset \mathbb{R}^{k_i}$ are nonempty closed convex sets given by finite lists of conic inclusions:

$$\mathbf{Q}_i = \{u \in \mathbb{R}^{k_i} : Q_{i\ell} u - q_{i\ell} \in \mathbf{K}_{i\ell},\ \ell = 1, ..., L_i\}, \tag{3.1.3}$$

with closed convex pointed cones $\mathbf{K}_{i\ell}$. We will restrict ourselves to the cases where $\mathbf{K}_{i\ell}$ are nonnegative orthants, or Lorentz, or Semidefinite cones. Clearly, a problem in the form (3.1.2) is equivalent to the conic problem

$$\min_x \left\{ c^T x + d : Q_{i\ell} A_i x - [Q_{i\ell} b_i + q_{i\ell}] \in \mathbf{K}_{i\ell}\ \forall (i, \ell \leq L_i) \right\}$$

We treat the collection $(c, d, \{A_i, b_i\}_{i=1}^m)$ as *natural data* of problem (3.1.2). The collection of sets $\mathbf{Q}_i$, $i = 1, ..., m$, is interpreted as the *structure* of problem (3.1.2), and thus the quantities $Q_{i\ell}, q_{i\ell}$ specifying these sets are considered as certain data.

3.1.2 Uncertain Conic Problems and their Robust Counterparts

Uncertain conic problem (3.1.2) is a problem with fixed structure and uncertain natural data *affinely* parameterized by a *perturbation* vector $\zeta \in \mathbb{R}^L$

$$(c, d, \{A_i, b_i\}_{i=1}^m) = (c^0, d^0, \{A_i^0, b_i^0\}_{i=1}^m) + \sum_{\ell=1}^{L} \zeta_\ell (c^\ell, d^\ell, \{A_i^\ell, b_i^\ell\}_{i=1}^m). \tag{3.1.4}$$

running through a given perturbation set $\mathcal{Z} \subset \mathbb{R}^L$.

Robust Counterpart of an uncertain conic problem

The notions of a robust feasible solution and the *Robust Counterpart* (RC) of uncertain problem (3.1.2) are defined exactly as in the case of an uncertain LO problem (see Definition 1.4):

Definition 3.1 *Let an uncertain problem* (3.1.2), (3.1.4) *be given and let* $\mathcal{Z} \subset \mathbb{R}^L$ *be a given perturbation set.*

(i) *A candidate solution* $x \in \mathbb{R}^n$ *is robust feasible, if it remains feasible for all realizations of the perturbation vector from the perturbation set:*

$$\begin{gathered} x \text{ is robust feasible} \\ \Updownarrow \\ [A_i^0 + \sum_{\ell=1}^{L} \zeta_\ell A_i^\ell]x - [b_i^0 + \sum_{\ell=1}^{L} \zeta_\ell b_i^\ell] \in \mathbf{Q}_i \ \forall \, (i, 1 \le i \le m, \zeta \in \mathcal{Z}) \, . \end{gathered}$$

(ii) *The Robust Counterpart of* (3.1.2), (3.1.4) *is the problem*

$$\min_{x,t} \left\{ t : \left. \begin{array}{l} [c^0 + \sum_{\ell=1}^{L} \zeta_\ell c^\ell]^T x + [d^0 + \sum_{\ell=1}^{L} \zeta_\ell d^\ell] - t \in \mathbf{Q}_0 \equiv \mathbb{R}_-, \\ [A_i^0 + \sum_{\ell=1}^{L} \zeta_\ell A_i^\ell]x - [b_i^0 + \sum_{\ell=1}^{L} \zeta_\ell b_i^\ell] \in \mathbf{Q}_i, \ 1 \le i \le m \end{array} \right\} \forall \zeta \in \mathcal{Z} \right\} \tag{3.1.5}$$

of minimizing the guaranteed value of the objective over the robust feasible solutions.

As in the LO case, it is immediately seen that the RC remains intact when the perturbation set $\mathcal{Z}$ is replaced with its closed convex hull; so, from now on we assume the perturbation set to be closed and convex. Note also that the case when the entries of the uncertain data $[A; b]$ are affected by perturbations in a *non-affine* fashion in principle could be reduced to the case of affine perturbations (see section 1.5); however, we do not know meaningful cases beyond uncertain LO where such a reduction leads to a tractable RC.

3.1.3 Robust Counterpart of Uncertain Conic Problem: Tractability

In contrast to uncertain LO, where the RC turn out to be computationally tractable whenever the perturbation set is so, uncertain conic problems with computationally tractable RCs are a "rare commodity." The ultimate reason for this phenomenon is rather simple: the RC (3.1.5) of an uncertain conic problem (3.1.2), (3.1.4) is a convex problem with linear objective and constraints of the generic form

$$P(y, \zeta) = \pi(y) + \Phi(y)\zeta = \phi(\zeta) + \Phi(\zeta)y \in \mathbf{Q}, \tag{3.1.6}$$

where $\pi(y), \Phi(y)$ are affine in the vector y of the decision variables, $\phi(\zeta), \Phi(\zeta)$ are affine in the perturbation vector ζ, and $\mathbf{Q}$ is a "simple" closed convex set. For such a problem, its computational tractability is, essentially, equivalent to the possibility to check efficiently whether a given candidate solution y is or is not feasible. The latter question, in turn, is whether the image of the perturbation set $\mathcal{Z}$ under an affine mapping $\zeta \mapsto \pi(y) + \Phi(y)\zeta$ is or is not contained in a given convex set $\mathbf{Q}$. This question is easy when $\mathbf{Q}$ is a polyhedral set given by an explicit list of scalar linear inequalities $a_i^T u \le b_i$, $i = 1, ..., I$ (in particular, when $\mathbf{Q}$ is a nonpositive ray, that is what we deal with in LO), in which case the required verification consists in checking whether the maxima of I affine functions $a_i^T(\pi(y) + \Phi(y)\zeta) - b_i$ of ζ over $\zeta \in \mathcal{Z}$ are or are not nonnegative. Since the maximization of an affine (and thus concave!) function over a computationally tractable convex set $\mathcal{Z}$ is easy, so is the required verification. When $\mathbf{Q}$ is given by *nonlinear* convex inequalities $a_i(u) \le 0$, $i = 1, ..., I$, the verification in question requires checking whether the *maxima* of *convex* functions $a_i(\pi(y) + \Phi(y)\zeta)$ over $\zeta \in \mathcal{Z}$ are or are not nonpositive. A problem of maximizing a convex function $f(\zeta)$ over a convex set $\mathcal{Z}$ can be computationally intractable already in the case of $\mathcal{Z}$ as simple as the unit box and f as simple as a convex quadratic form $\zeta^T Q \zeta$. Indeed, it is known that the problem

$$\max_{\zeta} \left\{ \zeta^T B \zeta : \|\zeta\|_\infty \le 1 \right\}$$

with positive semidefinite matrix B is NP-hard; in fact, it is already NP-hard to approximate the optimal value in this problem within a relative accuracy of 4%, even when probabilistic algorithms are allowed [55]. This example immediately implies that the RC of a generic uncertain conic quadratic problem with a perturbation set as simple as a box is computationally intractable.

Indeed, consider a simple-looking uncertain conic quadratic inequality

$$\|0 \cdot y + Q\zeta\|_2 \le 1$$

(Q is a given square matrix) along with its RC, the perturbation set being the unit box:

$$\|0 \cdot y + Q\zeta\|_2 \le 1 \ \forall (\zeta : \|\zeta\|_\infty \le 1). \tag{RC}$$

The feasible set of the RC is either the entire space of y-variables, or is empty, which depends on whether or not one has

$$\max_{\|\zeta\|_\infty \le 1} \zeta^T B \zeta \le 1. \qquad [B = Q^T Q]$$

Varying Q, we can get, as B, an arbitrary positive semidefinite matrix of a given size. Now, assuming that we can process (RC) efficiently, we can check efficiently whether the feasible set of (RC) is or is not empty, that is, we can compare efficiently the maximum of a positive semidefinite quadratic form over the unit box with the value 1. If we can do it, we can compute the maximum of a general-type positive semidefinite quadratic form $\zeta^T B \zeta$ over the unit box within relative accuracy ϵ in time polynomial in the dimension of ζ and $\ln(1/\epsilon)$ (by comparing $\max_{\|\zeta\|_\infty \le 1} \lambda \zeta^T B \zeta$ with 1 and applying bisection in $\lambda > 0$). Thus, the NP-hard problem of computing $\max_{\|\zeta\|_\infty \le 1} \zeta^T B \zeta$, $B \succ 0$, within relative accuracy $\epsilon = 0.04$ reduces to checking feasibility of the RC of a CQI with a box perturbation set, meaning that it is NP-hard to process the RC in question.

The unpleasant phenomenon we have just outlined leaves us with only two options:

A. To identify meaningful particular cases where the RC of an uncertain conic problem is computationally tractable; and

B. To develop *tractable approximations* of the RC in the remaining cases.

Note that the RC, same as in the LO case, is a "constraint-wise" construction, so that investigating tractability of the RC of an uncertain conic problem reduces to the same question for the RCs of the conic constraints constituting the problem. Due to this observation, from now on we focus on tractability of the RC

$$\forall(\zeta \in \mathcal{Z}) : A(\zeta)x + b(\zeta) \in \mathbf{Q}$$

of a *single* uncertain conic inequality.

3.1.4 Safe Tractable Approximations of RCs of Uncertain Conic Inequalities

In sections 3.2, 3.4 we will present a number of special cases where the RC of an uncertain CQI/LMI is computationally tractable; these cases have to do with rather specific perturbation sets. The question is, what to do when the RC is *not* computationally tractable. A natural course of action in this case is to look for a *safe tractable approximation* of the RC, defined as follows:

Definition 3.2 *Consider the RC*

$$\underbrace{A(\zeta)x + b(\zeta)}_{\equiv \alpha(x)\zeta + \beta(x)} \in \mathbf{Q} \ \forall \zeta \in \mathcal{Z} \tag{3.1.7}$$

of an uncertain constraint

$$A(\zeta)x + b(\zeta) \in \mathbf{Q}. \tag{3.1.8}$$

($A(\zeta) \in \mathbb{R}^{k\times n}, b(\zeta) \in \mathbb{R}^k$ are affine in ζ, so that $\alpha(x)$, $\beta(x)$ are affine in the decision vector x). We say that a system $\mathcal{S}$ of convex constraints in variables x and, perhaps, additional variables u is a safe approximation of the RC (3.1.7)*, if the projection of the feasible set of $\mathcal{S}$ on the space of x variables is contained in the feasible set of the RC:*

$$\forall x : (\exists u : \ (x, u) \ \textit{satisfies}\ \mathcal{S}) \Rightarrow x \ \textit{satisfies}\ (3.1.7).$$

This approximation is called tractable, provided that $\mathcal{S}$ is so, (e.g., $\mathcal{S}$ is an explicit system of CQIs/LMIs or, more generally, the constraints in $\mathcal{S}$ are efficiently computable).

The rationale behind the definition is as follows: assume we are given an uncertain conic problem (3.1.2) with vector of design variables x and a certain objective $c^T x$ (as we remember, the latter assumption is w.l.o.g.) and we have at our disposal a safe tractable approximation $\mathcal{S}_i$ of i-th constraint of the problem, $i = 1, ..., m$. Then the problem

$$\min_{x,u^1,...,u^m} \left\{ c^T x : (x, u^i) \text{ satisfies } \mathcal{S}_i,\ 1 \le i \le m \right\}$$

is a computationally tractable *safe approximation* of the RC, meaning that the x-component of every feasible solution to the approximation is feasible for the RC, and thus an optimal solution to the approximation is a *feasible* suboptimal solution to the RC.

In principle, there are many ways to build a safe tractable approximation of an uncertain conic problem. For example, assuming $\mathcal{Z}$ bounded, which usually is the case, we could find a simplex $\Delta = \mathrm{Conv}\{\zeta^1, ..., \zeta^{L+1}\}$ in the space $\mathbb{R}^L$ of perturbation vectors that is large enough to contain the actual perturbation set $\mathcal{Z}$. The RC of our uncertain problem, the perturbation set being Δ, is computationally tractable (see section 3.2.1) and is a safe approximation of the RC associated with the actual perturbation set $\mathcal{Z}$ due to $\Delta \supset \mathcal{Z}$. The essence of the matter is, of course, how conservative an approximation is: how much it "adds" to the built-in conservatism of the worst-case-oriented RC. In order to answer the latter question, we should quantify the "conservatism" of an approximation. There is no evident way to do it. One possible way could be to look by how much the optimal value of the approximation is larger than the optimal value of the true RC, but here we run into a severe difficulty. It may well happen that the feasible set of an approximation is empty, while the true feasible set of the RC is not so. Whenever this is the case, the optimal value of the approximation is "infinitely worse" than the true optimal value. It follows that comparison of optimal values makes sense only when the approximation scheme in question guarantees that the approximation inherits the feasibility properties of the true problem. On a closer inspection, such a requirement is, in general, not less restrictive than the requirement for the approximation to be precise.

The way to quantify the conservatism of an approximation to be used in this book is as follows. Assume that $0 \in \mathcal{Z}$ (this assumption is in full accordance with the interpretation of vectors $\zeta \in \mathcal{Z}$ as data perturbations, in which case $\zeta = 0$ corresponds to the nominal data). With this assumption, we can embed our closed convex perturbation set $\mathcal{Z}$ into a single-parametric family of perturbation sets

$$\mathcal{Z}_\rho = \rho\mathcal{Z},\ 0 < \rho \leq \infty, \tag{3.1.9}$$

thus giving rise to a single-parametric family

$$\underbrace{A(\zeta)x + b(\zeta)}_{\equiv \alpha(x)\zeta + \beta(x)} \in \mathbf{Q}\ \forall \zeta \in \mathcal{Z}_\rho \tag{RC$_\rho$}$$

of RCs of the uncertain conic constraint (3.1.8). One can think about ρ as *perturbation level*; the original perturbation set $\mathcal{Z}$ and the associated RC (3.1.7) correspond to the perturbation level 1. Observe that the feasible set X_ρ of (RC$_\rho$) shrinks as ρ grows. This allows us to quantify the conservatism of a safe approximation to (RC) by "positioning" the feasible set of $\mathcal{S}$ with respect to the scale of "true" feasible sets X_ρ, specifically, as follows:

Definition 3.3 *Assume that we are given an approximation scheme that puts into correspondence to* (3.1.9), (RC$_\rho$) *a finite system $\mathcal{S}_\rho$ of efficiently computable convex constraints on variables x and, perhaps, additional variables u, depending on $\rho > 0$ as on a parameter, in such a way that for every ρ the system $\mathcal{S}_\rho$ is a safe tractable approximation of (RC$_\rho$), and let $\widehat{X}_\rho$ be the projection of the feasible set of $\mathcal{S}_\rho$ onto the space of x variables.*

We say that the conservatism (or "tightness factor") of the approximation scheme in question does not exceed $\vartheta \geq 1$ if, for every $\rho > 0$, we have

$$X_{\vartheta\rho} \subset \widehat{X}_\rho \subset X_\rho.$$

Note that the fact that $\mathcal{S}_\rho$ is a safe approximation of (RC$_\rho$) tight within factor ϑ is equivalent to the following pair of statements:

1. [safety] *Whenever a vector x and $\rho > 0$ are such that x can be extended to a feasible solution of $\mathcal{S}_\rho$, x is feasible for* (RC$_\rho$);

2. [tightness] *Whenever a vector x and $\rho > 0$ are such that x can<u>not</u> be extended to a feasible solution of $\mathcal{S}_\rho$, x is <u>not</u> feasible for* (RC$_{\vartheta\rho}$).

Clearly, a tightness factor equal to 1 means that the approximation is precise: $\widehat{X}_\rho = X_\rho$ for all ρ. In many applications, especially in those where the level of perturbations is known only "up to an order of magnitude," a safe approximation of the RC with a moderate tightness factor is almost as useful, from a practical viewpoint, as the RC itself.

An important observation is that *with a bounded perturbation set* $\mathcal{Z} = \mathcal{Z}_1 \subset \mathbb{R}^L$ *that is symmetric w.r.t. the origin, we can always point out a safe computationally tractable approximation scheme for* (3.1.9), (RC_ρ) *with tightness factor* $\leq L$.

Indeed, w.l.o.g. we may assume that $\mathrm{int}\mathcal{Z} \neq \emptyset$, so that $\mathcal{Z}$ is a closed and bounded convex set symmetric w.r.t. the origin. It is known that for such a set, there always exist two similar ellipsoids, centered at the origin, with the similarity ratio at most $\sqrt{L}$, such that the smaller ellipsoid is contained in $\mathcal{Z}$, and the larger one contains $\mathcal{Z}$. In particular, one can choose, as the smaller ellipsoid, the largest volume ellipsoid contained in $\mathcal{Z}$; alternatively, one can choose, as the larger ellipsoid, the smallest volume ellipsoid containing $\mathcal{Z}$. Choosing coordinates in which the smaller ellipsoid is the unit Euclidean ball B, we conclude that $B \subset \mathcal{Z} \subset \sqrt{L}B$. Now observe that B, and therefore $\mathcal{Z}$, contains the convex hull $\underline{\mathcal{Z}} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_1 \leq 1\}$ of the $2L$ vectors $\pm e_\ell$, $\ell = 1, ..., L$, where e_ℓ are the basic orths of the axes in question. Since $\underline{\mathcal{Z}}$ clearly contains $L^{-1/2}B$, the convex hull $\widehat{\mathcal{Z}}$ of the vectors $\pm L e_\ell$, $\ell = 1, ..., L$, contains $\mathcal{Z}$ and is contained in $L\mathcal{Z}$. Taking, as $\mathcal{S}_\rho$, the RC of our uncertain constraint, the perturbation set being $\rho\widehat{\mathcal{Z}}$, we clearly get an L-tight safe approximation of (3.1.9), (RC_ρ), and this approximation is merely the system of constraints

$$A(\rho L e_\ell)x + b(\rho L e_\ell) \in \mathbf{Q},\ A(-\rho L e_\ell)x + b(-\rho L e_\ell) \in \mathbf{Q},\ \ell = 1, ..., L,$$

that is, our approximation scheme is computationally tractable.

3.2 Uncertain Conic Quadratic Problems with Tractable RCs

In this section we focus on uncertain conic quadratic problems (that is, the sets $\mathbf{Q}_i$ in (3.1.2) are given by explicit lists of conic quadratic inequalities) for which the RCs are computationally tractable.

3.2.1 A Generic Solvable Case: Scenario Uncertainty

We start with a simple case where the RC of an uncertain conic problem (not necessarily a conic quadratic one) is computationally tractable — the case of *scenario uncertainty*.

Definition 3.4 *We say that a perturbation set $\mathcal{Z}$ is scenario generated, if $\mathcal{Z}$ is given as the convex hull of a given finite set of scenarios $\zeta^{(\nu)}$:*

$$\mathcal{Z} = \mathrm{Conv}\{\zeta^{(1)}, ..., \zeta^{(N)}\}. \tag{3.2.1}$$

Theorem 3.1 *The RC* (3.1.5) *of uncertain problem* (3.1.2), (3.1.4) *with scenario perturbation set* (3.2.1) *is equivalent to the explicit conic problem*

$$\min_{x,t} \left\{ t : \left. \begin{array}{c} [c^0 + \sum_{\ell=1}^{L} \zeta_\ell^{(\nu)} c^\ell]^T x + [d^0 + \sum_{\ell=1}^{L} \zeta_\ell^{(\nu)} d^\ell] - t \leq 0 \\ [A_i^0 + \sum_{\ell=1}^{L} \zeta_\ell^{(\nu)} A_i^\ell]^T x - [b^0 + \sum_{\ell=1}^{L} \zeta_\ell^{(\nu)} b^\ell] \in \mathbf{Q}_i, \\ 1 \leq i \leq m \end{array} \right\}, 1 \leq \nu \leq N \right\} \tag{3.2.2}$$

with a structure similar to the one of the instances of the original uncertain problem.

Proof. This is evident due to the convexity of $\mathbf{Q}_i$ and the affinity of the left hand sides of the constraints in (3.1.5) in ζ. $\square$

The situation considered in Theorem 3.1 is "symmetric" to the one considered in lecture 1, where we spoke about problems (3.1.2) with the simplest possible sets $\mathbf{Q}_i$ — just nonnegative rays, and the RC turns out to be computationally tractable whenever the perturbation set is so. Theorem 3.1 deals with another extreme case of the tradeoff between the geometry of the right hand side sets $\mathbf{Q}_i$ and that of the perturbation set. Here the latter is as simple as it could be — just the convex hull of an explicitly listed finite set, which makes the RC computationally tractable for rather general (just computationally tractable) sets $\mathbf{Q}_i$. Unfortunately, the second extreme is not too interesting: in the large scale case, a "scenario approximation" of a reasonable quality for typical perturbation sets, like boxes, requires an astronomically large number of scenarios, thus preventing listing them explicitly and making problem (3.2.2) computationally intractable. This is in sharp contrast with the first extreme, where the simple sets were $\mathbf{Q}_i$ — Linear Optimization is definitely interesting and has a lot of applications.

In what follows, we consider a number of less trivial cases where the RC of an uncertain conic quadratic problem is computationally tractable. As always with RC, which is a constraint-wise construction, we may focus on computational tractability of the RC of a *single* uncertain CQI

$$\| \underbrace{A(\zeta)y + b(\zeta)}_{\equiv \alpha(y)\zeta + \beta(y)} \|_2 \leq \underbrace{c^T(\zeta)y + d(\zeta)}_{\equiv \sigma^T(y)\zeta + \delta(y)}, \tag{3.2.3}$$

where $A(\zeta) \in \mathbb{R}^{k\times n}, b(\zeta) \in \mathbb{R}^k, c(\zeta) \in \mathbb{R}^n, d(\zeta) \in \mathbb{R}$ are affine in ζ, so that $\alpha(y),\ \beta(y), \sigma(y), \delta(y)$ are affine in the decision vector y.

3.2.2 Solvable Case I: Simple Interval Uncertainty

Consider uncertain conic quadratic constraint (3.2.3) and assume that:

1. The uncertainty is *side-wise*: the perturbation set $\mathcal{Z} = \mathcal{Z}^{\rm left} \times \mathcal{Z}^{\rm right}$ is the direct product of two sets (so that the perturbation vector $\zeta \in \mathcal{Z}$ is split into blocks $\eta \in \mathcal{Z}^{\rm left}$ and $\chi \in \mathcal{Z}^{\rm right}$), with the left hand side data $A(\zeta), b(\zeta)$ depending solely on η and the right hand side data $c(\zeta), d(\zeta)$ depending solely on χ, so that (3.2.3) reads

$$\| \underbrace{A(\eta)y + b(\eta)}_{\equiv \alpha(y)\eta + \beta(y)} \|_2 \leq \underbrace{c^T(\chi)y + d(\chi)}_{\equiv \sigma^T(y)\chi + \delta(y)}, \tag{3.2.4}$$

   and the RC of this uncertain constraint reads

$$\|A(\eta)y + b(\eta)\|_2 \leq c^T(\chi)y + d(\chi) \ \ \forall(\eta \in \mathcal{Z}^{\rm left}, \chi \in \mathcal{Z}^{\rm right}); \tag{3.2.5}$$

2. The right hand side perturbation set is as described in Theorem 1.1, that is,

$$\mathcal{Z}^{\rm right} = \left\{\chi : \exists u : P\chi + Qu + p \in \mathbf{K}\right\},$$

   where either $\mathbf{K}$ is a closed convex pointed cone, and the representation is strictly feasible, or $\mathbf{K}$ is a polyhedral cone given by an explicit finite list of linear inequalities;

3. The left hand side uncertainty is a simple interval one:

$$\begin{array}{rcl}\mathcal{Z}^{\rm left} & = & \big\{\eta=[\delta A,\delta b]:|(\delta A)_{ij}|\leq\delta_{ij},1\leq i\leq k,1\leq j\leq n,\\ & & \quad |(\delta b)_i|\leq\delta_i,\ 1\leq i\leq k\big\},\\ [A(\zeta),b(\zeta)] & = & [A^{\rm n},b^{\rm n}]+[\delta A,\delta b].\end{array}$$

In other words, every entry in the left hand side data $[A, b]$ of (3.2.3), independently of all other entries, runs through a given segment centered at the nominal value of the entry.

Proposition 3.1 *Under assumptions 1 – 3 on the perturbation set $\mathcal{Z}$, the RC of the uncertain CQI* (3.2.3) *is equivalent to the following explicit system of conic quadratic and linear constraints in variables y, z, τ, v:*

$$\begin{array}{ll}(a) & \begin{array}{l}\tau+p^Tv\leq\delta(y),\ P^Tv=\sigma(y),\\ Q^Tv=0,\ v\in\mathbf{K}_*\end{array}\\ (b) & \begin{array}{l}z_i\geq|(A^{\rm n}y+b^{\rm n})_i|+\delta_i+\sum\limits_{j=1}^n|\delta_{ij}y_j|,\ i=1,...,k\\ \|z\|_2\leq\tau\end{array}\end{array}\tag{3.2.6}$$

where $\mathbf{K}_*$ is the cone dual to $\mathbf{K}$.

Proof. Due to the side-wise structure of the uncertainty, a given y is robust feasible if and only if there exists τ such that

$$\begin{array}{ll}(a) & \tau\leq\min\limits_{\chi\in\mathcal{Z}^{\rm right}}\big\{\sigma^T(y)\chi+\delta(y)\big\}\\ & =\min\limits_{\chi,u}\big\{\sigma^T(y)\chi:P\chi+Qu+p\in\mathbf{K}\big\}+\delta(y),\\ (b) & \tau\geq\max\limits_{\eta\in\mathcal{Z}^{\rm left}}\|A(\eta)y+b(\eta)\|_2\\ & =\max\limits_{\delta A,\delta b}\{\|[A^{\rm n}y+b^{\rm n}]+[\delta Ay+\delta b]\|_2:|\delta A|_{ij}\leq\delta_{ij},|\delta b_i|\leq\delta_i\}.\end{array}$$

By Conic Duality, a given τ satisfies (a) if and only if τ can be extended, by properly chosen v, to a solution of (3.2.6.a); by evident reasons, τ satisfies (b) if and only if there exists z satisfying (3.2.6.b). □

3.2.3 Solvable Case II: Unstructured Norm-Bounded Uncertainty

Consider the case where the uncertainty in (3.2.3) is still side-wise ($\mathcal{Z}=\mathcal{Z}^{\rm left}\times\mathcal{Z}^{\rm right}$) with the right hand side uncertainty set $\mathcal{Z}^{\rm right}$ as in section 3.2.2, while the left hand side uncertainty is *unstructured norm-bounded*, meaning that

$$\mathcal{Z}^{\rm left}=\big\{\eta\in\mathbb{R}^{p\times q}:\|\eta\|_{2,2}\leq 1\big\}\tag{3.2.7}$$

and either

$$A(\eta)y+b(\eta)=A^{\rm n}y+b^{\rm n}+L^T(y)\eta R\tag{3.2.8}$$

with $L(y)$ affine in y and $R\neq 0$, or

$$A(\eta)y+b(\eta)=A^{\rm n}y+b^{\rm n}+L^T\eta R(y)\tag{3.2.9}$$

with $R(y)$ affine in y and $L\neq 0$. Here

$$\|\eta\|_{2,2}=\max_u\{\|\eta u\|_2:u\in\mathbb{R}^q,\|u\|_2\leq 1\}$$

is the usual matrix norm of a $p\times q$ matrix η (the maximal singular value),

Example 3.1

(i) Imagine that some $p \times q$ submatrix P of the left hand side data $[A, b]$ of (3.2.4) is uncertain and differs from its nominal value $P^{\rm n}$ by an additive perturbation $\Delta P = M^T \Delta N$ with Δ having matrix norm at most 1, and all entries in $[A, b]$ outside of P are certain. Denoting by I the set of indices of the rows in P and by J the set of indices of the columns in P, let U be the natural projector of $\mathbb{R}^{n+1}$ on the coordinate subspace in $\mathbb{R}^{n+1}$ given by J, and V be the natural projector of $\mathbb{R}^k$ on the subspace of $\mathbb{R}^k$ given by I (e.g., with $I = \{1, 2\}$ and $J = \{1, 5\}$, $Uu = [u_1; u_5] \in \mathbb{R}^2$ and $Vu = [u_1; u_2] \in \mathbb{R}^2$). Then the outlined perturbations of $[A, b]$ can be represented as

$$[A(\eta), b(\eta)] = [A^{\rm n}, b^{\rm n}] + \underbrace{V^T M^T}_{L^T} \eta \underbrace{(NU)}_{R},\ \|\eta\|_{2,2} \le 1,$$

whence, setting $Y(y) = [y; 1]$,

$$A(\eta)y + b(\eta) = [A^{\rm n}y + b^{\rm n}] + L^T \eta \underbrace{[RY(y)]}_{R(y)},$$

and we are in the situation (3.2.7), (3.2.9).

(ii) [Simple ellipsoidal uncertainty] Assume that the left hand side perturbation set $\mathcal{Z}^{\rm left}$ is a p-dimensional ellipsoid; w.l.o.g. we may assume that this ellipsoid is just the unit Euclidean ball $B = \{\eta \in \mathbb{R}^p : \|\eta\|_2 \le 1\}$. Note that for vectors $\eta \in \mathbb{R}^p = \mathbb{R}^{p\times 1}$ their usual Euclidean norm $\|\eta\|_2$ and their matrix norm $\|\eta\|_{2,2}$ are the same. We now have

$$A(\eta)y + b(\eta) = [A^0 y + b^0] + \sum_{\ell=1}^{p} \eta_\ell [A^\ell y + b^\ell] = [A^{\rm n}y + b^{\rm n}] + L^T(y)\eta R,$$

where $A^{\rm n} = A^0$, $b^{\rm n} = b^0$, $R = 1$ and $L(y)$ is the matrix with the rows $[A^\ell y + b^\ell]^T$, $\ell = 1, ..., p$. Thus, we are in the situation (3.2.7), (3.2.8).

Theorem 3.2 *The RC of the uncertain CQI* (3.2.4) *with unstructured norm-bounded uncertainty is equivalent to the following explicit system of LMIs in variables y, τ, u, λ:*
(i) *<u>In the case of left hand side perturbations (3.2.7), (3.2.8):</u>*

$$\begin{array}{ll} (a) & \tau + p^T v \le \delta(y),\ P^T v = \sigma(y),\ Q^T v = 0,\ v \in \mathbf{K}_* \\ (b) & \left[\begin{array}{c|c|c} \tau I_k & L^T(y) & A^{\rm n}y + b^{\rm n} \\ \hline L(y) & \lambda I_p & \\ \hline [A^{\rm n}y + b^{\rm n}]^T & & \tau - \lambda R^T R \end{array}\right] \succeq 0. \end{array} \quad (3.2.10)$$

(ii) *<u>In the case of left hand side perturbations (3.2.7), (3.2.9):</u>*

$$\begin{array}{ll} (a) & \tau + p^T v \le \delta(y),\ P^T v = \sigma(y),\ Q^T v = 0,\ v \in \mathbf{K}_* \\ (b) & \left[\begin{array}{c|c|c} \tau I_k - \lambda L^T L & & A^{\rm n}y + b^{\rm n} \\ \hline & \lambda I_q & R(y) \\ \hline [A^{\rm n}y + b^{\rm n}]^T & R^T(y) & \tau \end{array}\right] \succeq 0. \end{array} \quad (3.2.11)$$

Here $\mathbf{K}_*$ is the cone dual to $\mathbf{K}$.

Proof. Same as in the proof of Proposition 3.1, y is robust feasible for (3.2.4) if and only if there exists τ such that

$$\begin{array}{ll}(a) & \tau \leq \min\limits_{\chi\in\mathcal{Z}^{\rm right}} \left\{\sigma^T(y)\chi + \delta(y)\right\} \\ & = \min\limits_{\chi,u} \left\{\sigma^T(y)\chi : P\chi + Qu + p \in \mathbf{K}\right\}, \\ (b) & \tau \geq \max\limits_{\eta\in\mathcal{Z}^{\rm left}} \|A(\eta)y + b(\eta)\|_2,\end{array} \tag{3.2.12}$$

and a given τ satisfies (a) if and only if it can be extended, by a properly chosen v, to a solution of (3.2.10.a)⇔(3.2.11.a). It remains to understand when τ satisfies (b). This requires two basic facts.

Lemma 3.1 [Semidefinite representation of the Lorentz cone] *A vector* $[y;t] \in \mathbb{R}^k \times \mathbb{R}$ *belongs to the Lorentz cone* $\mathbf{L}^{k+1} = \{[y;t] \in \mathbb{R}^{k+1} : t \geq \|y\|_2\}$ *if and only if the "arrow matrix"*

$$\mathrm{Arrow}(y,t) = \left[\begin{array}{c|c} t & y^T \\ \hline y & tI_k \end{array}\right]$$

is positive semidefinite.

Proof of Lemma 3.1: We use the following fundamental fact:

Lemma 3.2 [Schur Complement Lemma] *A symmetric block matrix*

$$A = \left[\begin{array}{c|c} P & Q^T \\ \hline Q & R \end{array}\right]$$

with $R \succ 0$ *is positive (semi)definite if and only if the matrix*

$$P - Q^T R^{-1} Q$$

is positive (semi)definite.

Schur Complement Lemma ⇒ Lemma 3.1: When $t = 0$, we have $[y;t] \in \mathbf{L}^{k+1}$ iff $y = 0$, and $\mathrm{Arrow}(y,t) \succeq 0$ iff $y = 0$, as claimed in Lemma 3.1. Now let $t > 0$. Then the matrix tI_k is positive definite, so that by the Schur Complement Lemma we have $\mathrm{Arrow}(y,t) \succeq 0$ if and only if $t \geq t^{-1}y^Ty$, or, which is the same, iff $[y;t] \in \mathbf{L}^{k+1}$. When $t < 0$, we have $[y;t] \notin \mathbf{L}^{k+1}$ and $\mathrm{Arrow}(y,t) \not\succeq 0$. □

Proof of the Schur Complement Lemma: Matrix $A = A^T$ is $\succeq 0$ iff $u^TPu + 2u^TQ^Tv + v^TRv \geq 0$ for all u, v, or, which is the same, iff

$$\forall u : 0 \leq \min_v \left\{u^TPu + 2u^TQ^Tv + v^TRv\right\} = u^TPu - u^TQ^TR^{-1}Qu$$

(indeed, since $R \succ 0$, the minimum in v in the last expression is achieved when $v = R^{-1}Qu$). The concluding relation $\forall u : u^T[P - Q^TR^{-1}Q]u \geq 0$ is valid iff $P - Q^TR^{-1}Q \succeq 0$. Thus, $A \succeq 0$ iff $P - Q^TR^{-1}Q \succeq 0$. The same reasoning implies that $A \succ 0$ iff $P - Q^TR^{-1}Q \succ 0$. □

We further need the following fundamental result:

Lemma 3.3 **[$\mathcal{S}$-Lemma]**

(i) [homogeneous version] *Let A, B be symmetric matrices of the same size such that $\bar{x}^T A\bar{x} > 0$ for some $\bar{x}$. Then the implication*

$$x^T Ax \geq 0 \Rightarrow x^T Bx \geq 0$$

holds true if and only if

$$\exists \lambda \geq 0 : B \succeq \lambda A.$$

(ii) [inhomogeneous version] *Let A, B be symmetric matrices of the same size, and let the quadratic form $x^T Ax + 2a^T x + \alpha$ be strictly positive at some point. Then the implication*

$$x^T Ax + 2a^T x + \alpha \geq 0 \Rightarrow x^T Bx + 2b^T x + \beta \geq 0$$

holds true if and only if

$$\exists \lambda \geq 0 : \left[\begin{array}{c|c} B - \lambda A & b^T - \lambda a^T \\ \hline b - \lambda a & \beta - \lambda\alpha \end{array}\right] \succeq 0.$$

For proof of this fundamental Lemma, see, e.g., [9, section 4.3.5].

Coming back to the proof of Theorem 3.2, we can now understand when a given pair τ, y satisfies (3.2.12.b). Let us start with the case (3.2.8). We have

$$\begin{array}{rl}
& (y,\tau) \text{ satisfies } (3.2.12.b) \\
\Leftrightarrow & [\overbrace{[A^{\mathrm{n}}y + b^{\mathrm{n}}]}^{\widehat{y}} + L^T(y)\eta R; \tau] \in \mathbf{L}^{k+1} \quad \forall (\eta : \|\eta\|_{2,2} \leq 1) \\
& \hfill [\text{by } (3.2.8)] \\
\Leftrightarrow & \left[\begin{array}{c|c} \tau & \widehat{y}^T + R^T\eta^T L(y) \\ \hline \widehat{y} + L^T(y)\eta R & \tau I_k \end{array}\right] \succeq 0 \quad \forall (\eta : \|\eta\|_{2,2} \leq 1) \\
& \hfill [\text{by Lemma 3.1}] \\
\Leftrightarrow & \tau s^2 + 2sr^T[\widehat{y} + L^T(y)\eta R] + \tau r^T r \geq 0 \quad \forall [s;r] \ \forall (\eta : \|\eta\|_{2,2} \leq 1) \\
\Leftrightarrow & \tau s^2 + 2s\widehat{y}^T r + 2 \min\limits_{\eta:\|\eta\|_{2,2}\leq 1} \left[s(\eta^T L(y) r)^T R\right] + \tau r^T r \geq 0 \quad \forall [s;r] \\
\Leftrightarrow & \tau s^2 + 2s\widehat{y}^T r - 2\|L(y)r\|_2 \|sR\|_2 + \tau r^T r \geq 0 \quad \forall [s;r] \\
\Leftrightarrow & \tau r^T r + 2(L(y)r)^T \xi + 2sr^T \widehat{y} + \tau s^2 \geq 0 \quad \forall (s, r, \xi : \xi^T\xi \leq s^2 R^T R) \\
\Leftrightarrow & \exists \lambda \geq 0 : \left[\begin{array}{c|c|c} \tau I_k & L^T(y) & \widehat{y} \\ \hline L(y) & \lambda I_p & \\ \hline \widehat{y}^T & & \tau - \lambda R^T R \end{array}\right] \succeq 0 \\
& [\text{by the homogeneous } \mathcal{S}\text{-Lemma; note that } R \neq 0].
\end{array}$$

The requirement $\lambda \geq 0$ in the latter relation is implied by the LMI in the relation and is therefore redundant. Thus, in the case of (3.2.8) relation (3.2.12.b) is equivalent to the possibility to extend (y, τ) to a solution of (3.2.10.b).

Now let (3.2.9) be the case. We have

$$
\begin{array}{ll}
& (y,\tau) \text{ satisfies } (3.2.12.b) \\
\Leftrightarrow & [\overbrace{[A^{\rm n}y+b^{\rm n}]}^{\widehat{y}}+L^T\eta R(y);\tau]\in \mathbf{L}^{k+1}\ \ \forall(\eta:\|\eta\|_{2,2}\leq 1)\ [\text{by } (3.2.9)] \\
\Leftrightarrow & \left[\begin{array}{c|c}\tau & \widehat{y}^T+R^T(y)\eta^T L\\ \hline \widehat{y}+L^T\eta R(y) & \tau I_k\end{array}\right]\succeq 0\ \ \forall(\eta:\|\eta\|_{2,2}\leq 1) \\
& \hfill [\text{by Lemma 3.1}] \\
\Leftrightarrow & \tau s^2+2sr^T[\widehat{y}+L^T\eta R(y)]+\tau r^Tr\geq 0\ \ \forall[s;r]\ \forall(\eta:\|\eta\|_{2,2}\leq 1) \\
\Leftrightarrow & \tau s^2+2s\widehat{y}^Tr+2\min\limits_{\eta:\|\eta\|_{2,2}\leq 1}\left[s(\eta^TLr)^TR(y)\right]+\tau r^Tr\geq 0\ \ \forall[s;r] \\
\Leftrightarrow & \tau s^2+2s\widehat{y}^Tr-2\|Lr\|_2\|sR(y)\|_2+\tau r^Tr\geq 0\ \ \forall[s;r] \\
\Leftrightarrow & \tau r^Tr+2sR^T(y)\xi+2sr^T\widehat{y}+\tau s^2\geq 0\ \ \forall(s,r,\xi:\xi^T\xi\leq r^TL^TLr) \\
\Leftrightarrow & \exists\lambda\geq 0:\left[\begin{array}{c|c|c}\tau I_k-\lambda L^TL & & \widehat{y}\\ \hline & \lambda I_q & R(y)\\ \hline \widehat{y}^T & R^T(y) & \tau\end{array}\right]\succeq 0 \\
& \qquad [\text{by the homogeneous } \mathcal{S}\text{-Lemma; note that } L\neq 0].
\end{array}
$$

As above, the restriction $\lambda \geq 0$ is redundant. We see that in the case of (3.2.9) relation (3.2.12.b) is equivalent to the possibility to extend (y,τ) to a solution of (3.2.11.b). □

3.2.4 Solvable Case III: Convex Quadratic Inequality with Unstructured Norm-Bounded Uncertainty

A special case of an uncertain conic quadratic constraint (3.2.3) is a convex quadratic constraint

$$
\begin{array}{ll}
(a) & y^TA^T(\zeta)A(\zeta)y\leq 2y^Tb(\zeta)+c(\zeta)\\
& \Updownarrow \\
(b) & \|[2A(\zeta)y;1-2y^Tb(\zeta)-c(\zeta)]\|_2\leq 1+2y^Tb(\zeta)+c(\zeta).
\end{array}
\tag{3.2.13}
$$

Here $A(\zeta)$ is $k\times n$.

Assume that the uncertainty affecting this constraint is an unstructured norm-bounded one, meaning that

$$
\begin{array}{ll}
(a) & \mathcal{Z}=\{\zeta\in\mathbb{R}^{p\times q}:\|\zeta\|_{2,2}\leq 1\},\\
(b) & \left[\begin{array}{c}A(\zeta)y\\ y^Tb(\zeta)\\ c(\zeta)\end{array}\right]=\left[\begin{array}{c}A^{\rm n}y\\ y^Tb^{\rm n}\\ c^{\rm n}\end{array}\right]+L^T(y)\zeta R(y),
\end{array}
\tag{3.2.14}
$$

where $L(y)$, $R(y)$ are matrices of appropriate sizes affinely depending on y and such that at least one of the matrices is constant. We are about to prove that the RC of (3.2.13), (3.2.14) is computationally tractable. Note that the just defined unstructured norm-bounded uncertainty in the data of convex quadratic constraint (3.2.13.a) implies similar uncertainty in the left hand side data of the equivalent uncertain CQI (3.2.13.a). Recall that Theorem 3.2 ensures that the RC of a general-type uncertain CQI with side-wise uncertainty and unstructured norm-bounded perturbations in the left hand side data is tractable. The result to follow removes the requirement of "side-wiseness" of the uncertainty at the cost of restricting the structure of the CQI in question — now it should come from an uncertain convex quadratic constraint. Note also that the case we are about to consider covers in particular the one when the data $(A(\zeta),b(\zeta),c(\zeta))$ of (3.2.13.a) are affinely parameterized by ζ varying in an ellipsoid (cf. Example 3.1.(ii)).

Proposition 3.2 *Let us set $L(y) = [L_A(y), L_b(y), L_c(y)]$, where $L_b(y)$, $L_c(y)$ are the last two columns in $L(y)$, and let*

$$\begin{array}{c}\widehat{L}^T(y) = \left[L_b^T(y) + \frac{1}{2}L_c^T(y); L_A^T(y)\right],\ \widehat{R}(y) = [R(y), 0_{q\times k}],\\ \mathcal{A}(y) = \left[\begin{array}{c|c} 2y^T b^{\rm n} + c^{\rm n} & [A^{\rm n} y]^T \\ \hline A^{\rm n} y & I_k \end{array}\right],\end{array} \tag{3.2.15}$$

so that $\mathcal{A}(y)$, $\widehat{L}(y)$ and $\widehat{R}(y)$ are affine in y and at least one of the latter two matrices is constant.

The RC of (3.2.13), (3.2.14) *is equivalent to the explicit LMI $\mathcal{S}$ in variables y, λ as follows:*

(i) *In the case when $\widehat{L}(y)$ is independent of y and is nonzero, $\mathcal{S}$ is*

$$\left[\begin{array}{c|c} \mathcal{A}(y) - \lambda \widehat{L}^T\widehat{L} & \widehat{R}^T(y) \\ \hline \widehat{R}(y) & \lambda I_q \end{array}\right] \succeq 0; \tag{3.2.16}$$

(ii) *In the case when $\widehat{R}(Y)$ is independent of y and is nonzero, $\mathcal{S}$ is*

$$\left[\begin{array}{c|c} \mathcal{A}(y) - \lambda \widehat{R}^T\widehat{R} & \widehat{L}^T(y) \\ \hline \widehat{L}(y) & \lambda I_p \end{array}\right] \succeq 0; \tag{3.2.17}$$

(iii) *In all remaining cases (that is, when either $\widehat{L}(y) \equiv 0$, or $\widehat{R}(y) \equiv 0$, or both), $\mathcal{S}$ is*

$$\mathcal{A}(y) \succeq 0. \tag{3.2.18}$$

Proof. We have

$$\begin{array}{rl} & y^T A^T(\zeta) A(\zeta) y \le 2y^T b(\zeta) + c(\zeta)\ \ \forall \zeta \in \mathcal{Z} \\ \Leftrightarrow & \left[\begin{array}{c|c} 2y^T b(\zeta) + c(\zeta) & [A(\zeta)y]^T \\ \hline A[\zeta]y & I_k \end{array}\right] \succeq 0\ \ \forall \zeta \in \mathcal{Z} \\ & \hfill \text{[Schur Complement Lemma]} \\ \Leftrightarrow & \overbrace{\left[\begin{array}{c|c} 2y^T b^{\rm n} + c^{\rm n} & [A^{\rm n} y]^T \\ \hline A^{\rm n} y & I \end{array}\right]}^{\mathcal{A}(y)} \\ & + \overbrace{\left[\begin{array}{c|c} 2L_b^T(y)\zeta R(y) + L_c^T(y)\zeta R(y) & R^T(y)\zeta^T L_A(y) \\ \hline L_A^T(y)\zeta R(y) & \end{array}\right]}^{\mathcal{B}(y,\zeta)} \succeq 0\ \ \forall (\zeta : \|\zeta\|_{2,2} \le 1) \\ & \hfill \text{[by (3.2.14)]} \\ \Leftrightarrow & \mathcal{A}(y) + \widehat{L}^T(y)\zeta\widehat{R}(y) + \widehat{R}^T(y)\zeta^T\widehat{L}(y) \succeq 0\ \ \forall (\zeta : \|\zeta\|_{2,2} \le 1) \qquad \text{[by (3.2.15)].} \end{array}$$

Now the reasoning can be completed exactly as in the proof of Theorem 3.2. Consider, e.g., the case of (i). We have

$$\begin{array}{ll} & y^T A^T(\zeta)A(\zeta)y \leq 2y^T b(\zeta) + c(\zeta) \quad \forall \zeta \in \mathcal{Z} \\ \Leftrightarrow & \mathcal{A}(y) + \widehat{L}^T \zeta \widehat{R}(y) + \widehat{R}^T(y)\zeta^T \widehat{L} \succeq 0 \ \forall (\zeta : \|\zeta\|_{2,2} \leq 1) \text{ [already proved]} \\ \Leftrightarrow & \xi^T \mathcal{A}(y)\xi + 2(\widehat{L}\xi)^T \zeta \widehat{R}(y)\xi \geq 0 \quad \forall \xi \quad \forall (\zeta : \|\zeta\|_{2,2} \leq 1) \\ \Leftrightarrow & \xi^T \mathcal{A}(y)\xi - 2\|\widehat{L}\xi\|_2 \|\widehat{R}(y)\xi\|_2 \geq 0 \quad \forall \xi \\ \Leftrightarrow & \xi^T \mathcal{A}(y)\xi + 2\eta^T \widehat{R}(y)\xi \geq 0 \quad \forall (\xi, \eta : \eta^T \eta \leq \xi^T \widehat{L}^T \widehat{L}\xi) \\ \Leftrightarrow & \exists \lambda \geq 0 : \left[\begin{array}{c|c} \mathcal{A}(y) - \lambda \widehat{L}^T \widehat{L} & \widehat{R}^T(y) \\ \hline \widehat{R}(y) & \lambda I_q \end{array}\right] \succeq 0 \text{ [}\mathcal{S}\text{-Lemma]} \\ \Leftrightarrow & \exists \lambda : \left[\begin{array}{c|c} \mathcal{A}(y) - \lambda \widehat{L}^T \widehat{L} & \widehat{R}^T(y) \\ \hline \widehat{R}(y) & \lambda I_q \end{array}\right] \succeq 0, \end{array}$$

and we arrive at (3.2.16). □

3.2.5 Solvable Case IV: CQI with Simple Ellipsoidal Uncertainty

The last solvable case we intend to present is of uncertain CQI (3.2.3) with an ellipsoid as the perturbation set. Now, unlike the results of Theorem 3.2 and Proposition 3.2, we neither assume the uncertainty side-wise, nor impose specific structural restrictions on the CQI in question. However, whereas in all tractability results stated so far we ended up with a "well-structured" tractable reformulation of the RC (mainly in the form of an explicit system of LMIs), now the reformulation will be less elegant: we shall prove that the feasible set of the RC admits an efficiently computable *separation oracle* — an efficient computational routine that, given on input a candidate decision vector y, reports whether this vector is robust feasible, and if it is not the case, returns a *separator* — a linear form $e^T z$ on the space of decision vectors such that

$$e^T y > \sup_{z \in Y} e^T z,$$

where Y is the set of all robust feasible solutions. Good news is that equipped with such a routine, one can optimize efficiently a linear form over the intersection of Y with any convex compact set Z that is itself given by an efficiently computable separation oracle. On the negative side, the family of "theoretically efficient" optimization algorithms available in this situation is much more restricted than the family of algorithms available in the situations we encountered so far. Specifically, in these past situations, we could process the RC by high-performance Interior Point polynomial time methods, while in our present case we are forced to use slower black-box-oriented methods, like the Ellipsoid algorithm. As a result, the design dimensions that can be handled in a realistic time can drop considerably.

We are about to describe an efficient separation oracle for the feasible set

$$Y = \{y : \|\alpha(y)\zeta + \beta(y)\|_2 \leq \sigma^T(y)\zeta + \delta(y) \quad \forall (\zeta : \zeta^T \zeta \leq 1)\} \tag{3.2.19}$$

of the uncertain CQI (3.2.3) with the unit ball in the role of the perturbation set; recall that $\alpha(y)$, $\beta(y)$, $\sigma(y)$, $\delta(y)$ are affine in y.

Observe that $y \in Y$ if and only if the following two conditions hold true:

$$
\begin{array}{|ll|c|}
\hline
& 0 \le \sigma^T(y)\zeta + \delta(y) \quad \forall(\zeta : \|\zeta\|_2 \le 1) & \\
\Leftrightarrow & \|\sigma(y)\|_2 \le \delta(y) & (a) \\
\hline
& (\sigma^T(y)\zeta + \delta(y))^2 - [\alpha(y)\zeta + \beta(y)]^T[\alpha(y)\zeta + \beta(y)] \ge 0 & \\
& \qquad\qquad\qquad\qquad \forall(\zeta : \zeta^T\zeta \le 1) & \\
\Leftrightarrow & \exists \lambda \ge 0 : & \\
& A_y(\lambda) \equiv \left[\begin{array}{c|c} \lambda I_L + \sigma(y)\sigma^T(y) & \delta(y)\sigma^T(y) \\ -\alpha^T(y)\alpha(y) & -\beta^T(y)\alpha(y) \\ \hline \delta(y)\sigma(y) & \delta^2(y) - \beta^T(y)\beta(y) \\ -\alpha^T(y)\beta(y) & -\lambda \end{array}\right] \succeq 0 & (b) \\
\hline
\end{array}
\qquad (3.2.20)
$$

where the second $\Leftrightarrow$ is due to the inhomogeneous $\mathcal{S}$-Lemma. Observe that *given y, it is easy to verify the validity of* (3.2.20). Indeed,

1. Verification of (3.2.20.a) is trivial.

2. To verify (3.2.20.b), we can use bisection in λ as follows.
   First note that any $\lambda \ge 0$ satisfying the matrix inequality (MI) in (3.2.20.b) clearly should be $\le \lambda_+ \equiv \delta^2(y) - \beta^T(y)\beta(y)$. If $\lambda_+ < 0$, then (3.2.20.b) definitely does not take place, and we can terminate our verification. When $\lambda_+ \ge 0$, we can build a shrinking sequence of localizers $\Delta_t = [\underline{\lambda}_t, \overline{\lambda}_t]$ for the set Λ_* of solutions to our MI, namely, as follows:

   • We set $\underline{\lambda}_0 = 0$, $\overline{\lambda}_0 = \lambda_+$, thus ensuring that $\Lambda_* \subset \Delta_0$.

   • Assume that after $t-1$ steps we have in our disposal a segment Δ_{t-1}, $\Delta_{t-1} \subset \Delta_{t-2} \subset ... \subset \Delta_0$, such that $\Lambda_* \subset \Delta_{t-1}$. Let λ_t be the midpoint of Δ_{t-1}. At step t, we check whether the matrix $A_y(\lambda_t)$ is $\succeq 0$; to this end we can use any one from the well-known Linear Algebra routines capable to check in $O(k^3)$ operations positive semidefiniteness of a $k \times k$ matrix A, and if it is not the case, to produce a "certificate" for the fact that $A \not\succeq 0$ — a vector z such that $z^T A z < 0$. If $A_y(\lambda_t) \succeq 0$, we are done, otherwise we get a vector z_t such that the affine function $f_t(\lambda) \equiv z_t^T A_y(\lambda) z_t$ is negative when $\lambda = \lambda_t$. Setting $\Delta_t = \{\lambda \in \Delta_{t-1} : f_t(\lambda) \ge 0\}$, we clearly get a new localizer for Λ_* that is at least twice shorter than Δ_{t-1}; if this localizer is nonempty, we pass to step $t+1$, otherwise we terminate with the claim that (3.2.20.b) is not valid.

   Since the sizes of subsequent localizers shrink at each step by a factor of at least 2, the outlined procedure rapidly converges: for all practical purposes[1] we may assume that the procedure terminates after a small number of steps with either a λ that makes the MI in (3.2.20) valid, or with an empty localizer, meaning that (3.2.20.b) is invalid.

So far we built an efficient procedure that checks whether or not y is robust feasible (i.e., whether or not $y \in Y$). To complete the construction of a separation oracle for Y, it remains to build a separator of y and Y when $y \notin Y$. Our "separation strategy" is as follows. Recall that $y \in Y$ if and only if all vectors $v_y(\zeta) = [\alpha(y)\zeta + \beta(y); \sigma^T(y)\zeta + \delta(y)]$ with $\|\zeta\|_2 \le 1$ belong to the Lorentz cone $\mathbf{L}^{k+1}$, where $k = \dim \beta(y)$. Thus, $y \notin Y$ if there exists $\bar{\zeta}$ such that $\|\bar{\zeta}\|_2 \le 1$ and $v_y(\bar{\zeta}) \notin \mathbf{L}^{k+1}$. Given such a $\bar{\zeta}$, we can immediately build a separator of y and Y as follows:

[1] We could make our reasoning precise, but it would require going into tedious technical details that we prefer to skip.

1. Since $v_y(\bar\zeta) \not\in \mathbf{L}^{k+1}$, we can easily separate $v_y(\bar\zeta)$ and $\mathbf{L}^{k+1}$. Specifically, setting $v_y(\bar\zeta) = [a;b]$, we have $b < \|a\|_2$, so that setting $e = [a/\|a\|_2; -1]$, we have $e^T v_y(\bar\zeta) = \|a\|_2 - b > 0$, while $e^T u \leq 0$ for all $u \in \mathbf{L}^{k+1}$.

2. After a separator e of $v_y(\bar\zeta)$ and $\mathbf{L}^{k+1}$ is built, we look at the function $\phi(z) = e^T v_z(\bar\zeta)$. This is an affine function of z such that

$$\sup_{z\in Y}\phi(z) \leq \sup_{u\in\mathbf{L}^{k+1}} e^T u < e^T v_y(\bar\zeta) = \phi(y)$$

where the first $\leq$ is given by the fact that $v_z(\bar\zeta) \in \mathbf{L}^{k+1}$ when $z \in Y$. Thus, the homogeneous part of $\phi(\cdot)$, (which is a linear form readily given by e), separates y and Y.

In summary, all we need is an efficient routine that, in the case when $y \not\in Y$, i.e.,

$$\widehat{\mathcal{Z}}_y \equiv \{\bar\zeta : \|\bar\zeta\|_2 \leq 1,\, v_y(\bar\zeta) \not\in \mathbf{L}^{k+1}\} \neq \emptyset,$$

finds a point $\bar\zeta \in \widehat{\mathcal{Z}}_y$ ("an infeasibility certificate"). Here is such a routine. First, recall that our algorithm for verifying robust feasibility of y reports that $y \not\in Y$ in two situations:

• $\|\sigma(y)\|_2 > \delta(y)$. In this case we can without any difficulty find a $\bar\zeta$, $\|\bar\zeta\|_2 \leq 1$, such that $\sigma^T(y)\bar\zeta + \delta(y) < 0$. In other words, the vector $v_y(\bar\zeta)$ has a negative last coordinate and therefore it definitely does not belong to $\mathbf{L}^{k+1}$. Such a $\bar\zeta$ is an infeasibility certificate.

• We have discovered that (a) $\lambda_+ < 0$, or (b) got $\Delta_t = \emptyset$ at a certain step t of our bisection process. In this case building an infeasibility certificate is more tricky.

Step 1: Separating the positive semidefinite cone and the "matrix ray" $\{A_y(\lambda) : \lambda \geq 0\}$. Observe that with z_0 defined as the last basic orth in $\mathbb{R}^{L+1}$, we have $f_0(\lambda) \equiv z_0^T A_y(\lambda) z_0 < 0$ when $\lambda > \lambda_+$. Recalling what our bisection process is, we conclude that in both cases (a), (b) we have at our disposal a collection $z_0, ..., z_t$ of $(L+1)$-dimensional vectors such that with $f_s(\lambda) = z_s^T A_y(\lambda) z_s$ we have $f(\lambda) \equiv \min\left[f_0(\lambda), f_1(\lambda), ..., f_t(\lambda)\right] < 0$ for all $\lambda \geq 0$. By construction, $f(\lambda)$ is a piecewise linear concave function on the nonnegative ray; looking at what happens at the maximizer of f over $\lambda \geq 0$, we conclude that an appropriate convex combination of just two of the "linear pieces" $f_0(\lambda), ..., f_t(\lambda)$ of f is negative everywhere on the nonnegative ray. That is, with properly chosen and easy-to-find $\alpha \in [0,1]$ and $\tau_1, \tau_2 \leq t$ we have

$$\phi(\lambda) \equiv \alpha f_{\tau_1}(\lambda) + (1-\alpha) f_{\tau_2}(\lambda) < 0 \quad \forall \lambda \geq 0.$$

Recalling the origin of $f_\tau(\lambda)$ and setting $z^1 = \sqrt{\alpha} z_{\tau_1}$, $z^2 = \sqrt{1-\alpha} z_{\tau_2}$, $Z = z^1[z^1]^T + z^2[z^2]^T$, we have

$$0 > \phi(\lambda) = [z^1]^T A_y(\lambda) z^1 + [z^2]^T A_y(\lambda) z^2 = \mathrm{Tr}(A_y(\lambda) Z) \quad \forall \lambda \geq 0. \tag{3.2.21}$$

This inequality has a simple interpretation: the function $\Phi(X) = \mathrm{Tr}(XZ)$ is a linear form on $\mathbf{S}^{L+1}$ that is nonnegative on the positive semidefinite cone (since $Z \succeq 0$ by construction) and is negative everywhere on the "matrix ray" $\{A_y(\lambda) : \lambda \geq 0\}$, thus certifying that this ray does not intersect the positive semidefinite cone (the latter is exactly the same as the fact that (3.2.20.b) is false).

Step 2: from Z to $\bar\zeta$. Relation (3.2.21) says that an affine function $\phi(\lambda)$ is negative everywhere on the nonnegative ray, meaning that the slope of the function is nonpositive, and the value at the origin is negative. Taking into account (3.2.20), we get

$$Z_{L+1,L+1} \geq \sum_{i=1}^{L} Z_{ii},\ \mathrm{Tr}(Z \underbrace{\left[\begin{array}{c|c} \begin{array}{c}\sigma(y)\sigma^T(y)\\ -\alpha^T(y)\alpha(y)\end{array} & \begin{array}{c}\delta(y)\sigma^T(y)\\ -\beta^T(y)\alpha(y)\end{array} \\ \hline \begin{array}{c}\delta(y)\sigma(y)\\ -\alpha^T(y)\beta(y)\end{array} & \delta^2(y)-\beta^T(y)\beta(y) \end{array}\right]}_{A_y(0)}) < 0. \tag{3.2.22}$$

Besides this, we remember that Z is given as $z^1[z^1]^T + z^2[z^2]^T$. We claim that

(!) *We can efficiently find a representation* $Z = ee^T + ff^T$ *such that* $e, f \in \mathbf{L}^{L+1}$.

Taking for the time being (!) for granted, let us build an infeasibility certificate. Indeed, from the second relation in (3.2.22) it follows that either $\mathrm{Tr}(A_y(0)ee^T) < 0$, or $\mathrm{Tr}(A_y(0)ff^T) < 0$, or both. Let us check which one of these inequalities indeed holds true; w.l.o.g., let it be the first one. From this inequality, in particular, $e \neq 0$, and since $e \in \mathbf{L}^{L+1}$, we have $e_{L+1} > 0$. Setting $\bar{e} = e/e_{L+1} = [\bar{\zeta}; 1]$, we have $\mathrm{Tr}(A_y(0)\bar{e}\bar{e}^T) = \bar{e}^T A_y(0)\bar{e} < 0$, that is,

$$\begin{aligned}&\delta^2(y) - \beta^T(y)\beta(y) + 2\delta(y)\sigma^T(y)\bar{\zeta} - 2\beta^T(y)\alpha(y)\bar{\zeta} + \bar{\zeta}^T\sigma(y)\sigma^T(y)\bar{\zeta}\\ &-\bar{\zeta}^T\alpha^T(y)\alpha(y)\bar{\zeta} < 0,\end{aligned}$$

or, which is the same,

$$(\delta(y) + \sigma^T(y)\bar{\zeta})^2 < (\alpha(y)\bar{\zeta} + \beta(y))^T(\alpha(y)\bar{\zeta} + \beta(y)).$$

We see that the vector $v_y(\bar{\zeta}) = [\alpha(y)\bar{\zeta} + \beta(y); \sigma^T(y)\bar{\zeta} + \delta(y)]$ does not belong to $\mathbf{L}^{L+1}$, while $\bar{e} = [\bar{\zeta}; 1] \in \mathbf{L}^{L+1}$, that is, $\|\bar{\zeta}\|_2 \leq 1$. We have built a required infeasibility certificate.

It remains to justify (!). Replacing, if necessary, z^1 with $-z^1$ and z^2 with $-z^2$, we can assume that $Z = z^1[z^1]^T + z^2[z^2]^T$ with $z^1 = [p; s]$, $z^2 = [q; r]$, where $s, r \geq 0$. It may happen that $z^1, z^2 \in \mathbf{L}^{L+1}$ — then we are done. Assume now that not both z^1, z^2 belong to $\mathbf{L}^{L+1}$, say, $z^1 \notin \mathbf{L}^{L+1}$, that is, $0 \leq s < \|p\|_2$. Observe that $Z_{L+1,L+1} = s^2 + r^2$ and $\sum\limits_{i=1}^{L} Z_{ii} = p^Tp + q^Tq$; therefore the first relation in (3.2.22) implies that $s^2 + r^2 \geq p^Tp + q^Tq$. Since $0 \leq s < \|p\|_2$ and $r \geq 0$, we conclude that $r > \|q\|_2$. Thus, $s < \|p\|_2$, $r > \|q\|_2$, whence there exists (and can be easily found) $\alpha \in (0,1)$ such that for the vector $e = \sqrt{\alpha}z^1 + \sqrt{1-\alpha}z^2 = [u; t]$ we have $e_{L+1} = \sqrt{e_1^2 + ... + e_L^2}$. Setting $f = -\sqrt{1-\alpha}z^1 + \sqrt{\alpha}z^2$, we have $ee^T + ff^T = z^1[z^1]^T + z^2[z^2]^T = Z$. We now have

$$0 \leq Z_{L+1,L+1} - \sum_{i=1}^{L} Z_{ii} = e_{L+1}^2 + f_{L+1}^2 - \sum_{i=1}^{L}[e_i^2 + f_i^2] = f_{L+1}^2 - \sum_{i=1}^{L} f_i^2;$$

thus, replacing, if necessary, f with $-f$, we see that $e, f \in \mathbf{L}^{L+1}$ and $Z = ee^T + ff^T$, as required in (!).

Semidefinite Representation of the RC of an Uncertain CQI with Simple Ellipsoidal Uncertainty

This book was nearly finished when the topic considered in this section was significantly advanced by R. Hildebrand [56, 57] who discovered an explicit SDP representation of the cone of "Lorentz-positive" $n \times m$ matrices (real $m \times n$ matrices that map the Lorentz cone $\mathbf{L}^m$ into the Lorentz cone $\mathbf{L}^n$). Existence of such a representation was a long-standing open question. As a byproduct of answering this question, the construction of Hildebrand offers an explicit SDP reformulation of the RC of an uncertain conic quadratic inequality with ellipsoidal uncertainty.

The RC of an uncertain conic quadratic inequality with ellipsoidal uncertainty and Lorentz-positive matrices. Consider the RC of an uncertain conic quadratic inequality with simple ellipsoidal uncertainty; w.l.o.g., we assume that the uncertainty set $\mathcal{Z}$ is the unit Euclidean ball in some $\mathbb{R}^{m-1}$, so that the RC is the semi-infinite constraint of the form

$$B[x]\zeta + b[x] \in \mathbf{L}^n \ \forall(\zeta \in \mathbb{R}^{m-1} : \zeta^T\zeta \leq 1), \tag{3.2.23}$$

with $B[x]$, $b[x]$ affinely depending on x. This constraint is clearly exactly the same as the constraint

$$B[x]\xi + \tau b[x] \in \mathbf{L}^n \ \forall([\xi;\tau] \in \mathbf{L}^m).$$

We see that *x is feasible for the RC in question if and only if the $n \times m$ matrix $M[x] = [B[x], b[x]]$ affinely depending on x is Lorentz-positive, that is, maps the cone $\mathbf{L}^m$ into the cone $\mathbf{L}^n$.* It follows that *in order to get an explicit SDP representation of the RC, is suffices to know an explicit SDP representation of the set $P_{n,m}$ of $n \times m$ matrices mapping $\mathbf{L}^m$ into $\mathbf{L}^n$.*

SDP representation of $P_{n,m}$ as discovered by R. Hildebrand (who used tools going far beyond those used in this book) is as follows.

A. Given m, n, we define a linear mapping $A \mapsto \mathcal{W}(A)$ from the space $\mathbb{R}^{n\times m}$ of real $n \times m$ matrices into the space $\mathbf{S}^N$ of symmetric $N \times N$ matrices with $N = (n-1)(m-1)$, namely, as follows.

Let $W_n[u] = \begin{bmatrix} u_n + u_1 & u_2 & \cdots & u_{n-1} \\ u_2 & u_n - u_1 & & \\ \vdots & & \ddots & \\ u_{n-1} & & & u_n - u_1 \end{bmatrix}$, so that W_n is a symmetric $(n-1) \times (n-1)$ matrix depending on a vector u of n real variables. Now consider the *Kronecker product* $W[u,v] = W_n[u] \bigotimes W_m[v]$. [2] W is a symmetric $N \times N$ matrix with entries that are bilinear functions of u and v variables, so that an entry is of the form "weighted sum of pair products of the u and the v-variables." Now, given an $n \times m$ matrix A, let us replace pair products $u_i v_k$ in the representation of the entries in $W[u,v]$ with the entries A_{ik} of A. As a result of this formal substitution, W will become a symmetric $(n-1) \times (m-1)$ matrix $\mathcal{W}(A)$ that depends linearly on A.

B. We define a linear subspace $\mathcal{L}_{m,n}$ in the space $\mathbf{S}^N$ as the linear span of the Kronecker products $S \bigotimes T$ of all skew-symmetric real $(n-1)\times(n-1)$ matrices S and skew-symmetric real $(m-1)\times(m-1)$ matrices T. Note that the Kronecker product of two skew-symmetric matrices is a symmetric matrix, so that the definition makes sense. Of course, we can easily build a basis in $\mathcal{L}_{m,n}$ — it is comprised of pairwise Kronecker products of the basic $(n-1)$-dimensional and $(m-1)$-dimensional skew-symmetric matrices.

The Hildebrand SDP representation of $P_{n,m}$ is given by the following:

Theorem 3.3 [Hildebrand [57, Theorem 5.6]] *Let* $\min[m,n] \geq 3$. *Then an $n \times m$ matrix A maps $\mathbf{L}^m$ into $\mathbf{L}^n$ if and only if A can be extended to a feasible solution to the explicit system of LMIs*

$$\mathcal{W}(A) + X \succeq 0, \ X \in \mathcal{L}_{m,n}$$

in variables A, X.

[2]Recall that the Kronecker product $A \bigotimes B$ of a $p \times q$ matrix A and an $r \times s$ matrix B is the $pr \times qs$ matrix with rows indexed by pairs (i,k), $1 \leq i \leq p$, $1 \leq k \leq r$, and columns indexed by pairs (j,ℓ), $1 \leq j \leq q$, $1 \leq \ell \leq s$, and the $((i,k),(j,\ell))$-entry equal to $A_{ij}B_{k\ell}$. Equivalently, $A \bigotimes B$ is a $p \times q$ block matrix with $r \times s$ blocks, the (i,j)-th block being $A_{ij}B$.

As a corollary,

When $m-1 := \dim\zeta \geq 2$ *and* $n := \dim b[x] \geq 3$, *the explicit* $(n-1)(m-1)\times(n-1)(m-1)$ *LMI*

$$\mathcal{W}([B[x], b[x]]) + X \succeq 0 \tag{3.2.24}$$

in variables x *and* $X \in \mathcal{L}_{m,n}$ *is an equivalent SDP representation of the semi-infinite conic quadratic inequality* (3.2.23) *with ellipsoidal uncertainty set.*

The lower bounds on the dimensions of ζ and $b[x]$ in the corollary do not restrict generality — we can always ensure their validity by adding zero columns to $B[x]$ and/or adding zero rows to $[B[x], b[x]]$.

3.2.6 Illustration: Robust Linear Estimation

Consider the situation as follows: we are given noisy observations

$$w = (I_p + \Delta)z + \xi \tag{3.2.25}$$

of a signal z that, in turn, is the result of passing an unknown input signal v through a given linear filter: $z = Av$ with known $p\times q$ matrix A. The measurements contain errors of two kinds:

- bias Δz linearly depending on z, where the only information on the bias matrix Δ is given by a bound $\|\Delta\|_{2,2} \leq \rho$ on its norm;
- random noise ξ with zero mean and known covariance matrix $\Sigma = \mathbf{E}\{\xi\xi^T\}$.

The goal is to estimate a given linear functional $f^T v$ of the input signal. We restrict ourselves with estimators that are linear in w:

$$\widehat{f} = x^T w,$$

where x is a fixed weight vector. For a linear estimator, the mean squares error is

$$\begin{array}{rcl}\mathrm{EstErr} & = & \sqrt{\mathbf{E}\{(x^T[(I+\Delta)Av+\xi] - f^T v)^2\}} \\ & = & \sqrt{([A^T(I+\Delta^T)x - f]^T v)^2 + x^T\Sigma x}.\end{array}$$

Now assume that our a priori knowledge of the true signal is that $v^T Q v \leq R^2$, where $Q \succ 0$ and $R > 0$. In this situation it makes sense to look for the *minimax optimal* weight vector x that minimizes the worst, over v and Δ compatible with our a priori information, mean squares estimation error. In other words, we choose x as the optimal solution to the following optimization problem

$$\min_x \max_{\substack{v: v^T Q v \leq R^2 \\ \Delta: \|\Delta\|_{2,2}\leq\rho}} \left(([\underbrace{A^T(I+\Delta^T)}_{S} x - f]^T v)^2 + x^T\Sigma x\right)^{1/2}. \tag{P}$$

Now,

$$\begin{array}{l}\max\limits_{v: v^T Q v \leq R^2} [Sx - f]^T v = \max\limits_{u: u^T u \leq 1} [Sx-f]^T (RQ^{-1/2}u) \\ = R\|Q^{-1/2}Sx - \underbrace{Q^{-1/2}f}_{\widehat{f}}\|_2,\end{array}$$

so that (P) reduces to the problem

$$\min_x \sqrt{x^T\Sigma x + R^2 \max_{\|\Delta\|_{2,2}\leq\rho} \|\underbrace{Q^{-1/2}A^T(I+\Delta^T)}_{B} x - \widehat{f}\|_2^2},$$

which is exactly the RC of the uncertain conic quadratic program

$$\min_{x,t,r,s}\left\{t: \begin{array}{c}\sqrt{r^2+s^2}\le t,\ \|\Sigma^{1/2}x\|_2\le r,\\ \|Bx-\widehat{f}\|_2\le R^{-1}s\end{array}\right\}, \tag{3.2.26}$$

where the only uncertain element of the data is the matrix $B = Q^{-1/2}A^T(I+\Delta^T)$ running through the uncertainty set

$$\mathcal{U} = \{B = \underbrace{Q^{-1/2}A^T}_{B_{\rm n}} + \rho Q^{-1/2}A^T\zeta, \zeta\in\mathcal{Z} = \{\zeta\in\mathbb{R}^{p\times p}:\|\zeta\|_{2,2}\le 1\}\}. \tag{3.2.27}$$

The uncertainty here is the unstructured norm-bounded one; the RC of (3.2.26), (3.2.27) is readily given by Theorem 3.2 and Example 3.1.(i). Specifically, the RC is the optimization program

$$\min_{x,t,r,s,\lambda}\left\{t: \begin{array}{c}\sqrt{r^2+s^2}\le t,\ \|\Sigma^{1/2}x\|_2\le r,\\ \left[\begin{array}{c|c|c} R^{-1}sI_q-\lambda\rho^2 B_{\rm n}B_{\rm n}^T & & B_{\rm n}x-\widehat{f}\\ \hline & \lambda I_p & x\\ \hline [B_{\rm n}x-\widehat{f}]^T & x^T & R^{-1}s\end{array}\right]\succeq 0\end{array}\right\}, \tag{3.2.28}$$

which can further be recast as an SDP.

Next we present a numerical illustration.

Example 3.2 Consider the problem as follows:

> A thin homogeneous iron plate occupies the 2-D square $D=\{(x,y):0\le x,y\le 1\}$. At time $t=0$ it was heated to temperature $T(0,x,y)$ such that $\int_D T^2(0,x,y)dxdy\le T_0^2$ with a given T_0, and then was left to cool; the temperature along the perimeter of the plate is kept at the level 0^o all the time. At a given time 2τ we measure the temperature $T(2\tau,x,y)$ along the 2-D grid
>
> $$\Gamma=\{(u_\mu,u_\nu):1\le\mu,\nu\le N\},\ u_k = frack-1/2N$$
>
> The vector w of measurements is obtained from the vector
>
> $$z=\{T(2\tau,u_\mu,u_\nu):1\le\mu,\nu\le N\}$$
>
> according to (3.2.25), where $\|\Delta\|_{2,2}\le\rho$ and $\xi_{\mu\nu}$ are independent Gaussian random variables with zero mean and standard deviation σ. Given the measurements, we need to estimate the temperature $T(\tau,1/2,1/2)$ at the center of the plate at time τ.

It is known from physics that the evolution in time of the temperature $T(t,x,y)$ of a homogeneous plate occupying a 2-D domain Ω, with no sources of heat in the domain and heat exchange solely via the boundary, is governed by the *heat equation*

$$\frac{\partial}{\partial t}T = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)T$$

(In fact, in the right hand side there should be a factor γ representing material's properties, but by an appropriate choice of the time unit, this factor can be made equal to 1.) For the case of $\Omega=D$ and zero boundary conditions, the solution to this equation is as follows:

$$T(t,x,y)=\sum_{k,\ell=1}^{\infty}a_{k\ell}\exp\{-(k^2+\ell^2)\pi^2t\}\sin(\pi kx)\sin(\pi\ell y), \tag{3.2.29}$$

where the coefficients $a_{k\ell}$ can be obtained by expanding the initial temperature into a series in the orthogonal basis $\phi_{k\ell}(x,y) = \sin(\pi kx)\sin(\pi\ell y)$ in $L_2(D)$:

$$a_{k\ell} = 4\int_D T(0,x,y)\phi_{k\ell}(x,y)dxdy.$$

In other words, the Fourier coefficients of $T(t,\cdot,\cdot)$ in an appropriate orthogonal spatial basis decrease exponentially as t grows, with the "decay time" (the smallest time in which every one of the coefficients is multiplied by factor ≤ 0.1) equal to

$$\Delta = \frac{\ln(10)}{2\pi^2}.$$

Setting $v_{k\ell} = a_{k\ell}\exp\{-(k^2+\ell^2)\pi^2\tau\}$, the problem in question becomes to estimate

$$T(\tau,1/2,1/2) = \sum_{k,\ell} v_{k\ell}\phi_{k\ell}(1/2,1/2)$$

given observations

$$w = (I+\Delta)z + \xi,\ z = \{T(2\tau,u_\mu,u_\nu) : 1 \leq \mu,\nu \leq N\},$$
$$\xi = \{\xi_{\mu\nu} \sim \mathcal{N}(0,\sigma^2) : 1 \leq \mu,\nu \leq N\}$$

($\xi_{\mu\nu}$ are independent).

Finite-dimensional approximation. Observe that

$$a_{k\ell} = \exp\{\pi^2(k^2+\ell^2)\tau\}v_{k\ell}$$

and that

$$\sum_{k,\ell} v_{k\ell}^2\exp\{2\pi^2(k^2+\ell^2)\tau\} = \sum_{k,\ell} a_{k\ell}^2 = 4\int_D T^2(0,x,y)dxdy \leq 4T_0^2. \tag{3.2.30}$$

It follows that

$$|v_{k\ell}| \leq 2T_0\exp\{-\pi^2(k^2+\ell^2)\tau\}.$$

Now, given a tolerance $\epsilon > 0$, we can easily find L such that

$$\sum_{k,\ell:k^2+\ell^2>L^2} \exp\{-\pi^2(k^2+\ell^2)\tau\} \leq \frac{\epsilon}{2T_0},$$

meaning that when replacing by zeros the actual (unknown!) $v_{k\ell}$ with $k^2+\ell^2 > L^2$, we change temperature at time τ (and at time 2τ as well) at every point by at most ϵ. Choosing ϵ really small (say, $\epsilon = 1.\text{e-}16$), we may assume for all practical purposes that $v_{k\ell} = 0$ when $k^2+\ell^2 > L^2$, which makes our problem a finite-dimensional one, specifically, as follows:

Given the parameters L, N, ρ, σ, T_0 and observations

$$w = (I+\Delta)z + \xi, \tag{3.2.31}$$

where $\|\Delta\|_{2,2} \leq \rho$, $\xi_{\mu\nu} \sim \mathcal{N}(0,\sigma^2)$ are independent, $z = Av$ is defined by the relations

$$z_{\mu\nu} = \sum_{k^2+\ell^2\leq L^2} \exp\{-\pi^2(k^2+\ell^2)\tau\}v_{k\ell}\phi_{k\ell}(u_\mu,u_\nu),\ 1 \leq \mu,\nu \leq N,$$

and $v = \{v_{k\ell}\}_{k^2+\ell^2 \le L^2}$ is known to satisfy the inequality

$$v^T Q v \equiv \sum_{k^2+\ell^2 \le L^2} v_{k\ell}^2 \exp\{2\pi^2(k^2+\ell^2)\tau\} \le 4T_0^2,$$

estimate the quantity

$$\sum_{k^2+\ell^2 \le L^2} v_{k\ell}\phi_{k\ell}(1/2, 1/2),$$

where $\phi_{k\ell}(x,y) = \sin(\pi k x)\sin(\pi \ell y)$ and $u_\mu = \frac{\mu - 1/2}{N}$.

The latter problem fits the framework of robust estimation we have built, and we can recover $T = T(\tau, 1/2, 1/2)$ by a linear estimator

$$\widehat{T} = \sum_{\mu,\nu} x_{\mu\nu} w_{\mu\nu}$$

with weights $x_{\mu\nu}$ given by an optimal solution to the associated problem (3.2.28).

Assume, for example, that τ is half of the decay time of our system:

$$\tau = \frac{1}{2}\frac{\ln(10)}{2\pi^2} \approx 0.0583,$$

and let

$$T_0 = 1000, N = 4.$$

With $\epsilon = 1.\text{e-}15$, we get $L = 8$ (this corresponds to just 41-dimensional space for v's). Now consider four options for ρ and σ:

(a) $\rho = 1.\text{e-}9$, $\sigma = 1.\text{e-}9$
(b) $\rho = 0$, $\sigma = 1.\text{e-}3$
(c) $\rho = 1.\text{e-}3$, $\sigma = 1.\text{e-}3$
(d) $\rho = 1.\text{e-}1$, $\sigma = 1.\text{e-}1$

In the case of (a), the optimal value in (3.2.28) is 0.0064, meaning that the expected squared error of the minimax optimal estimator never exceeds $(0.0064)^2$. The minimax optimal weights are

$$\begin{bmatrix} 6625.3 & -2823.0 & -2.8230 & 6625.3 \\ -2823.0 & 1202.9 & 1202.9 & -2823.0 \\ -2823.0 & 1202.9 & 1202.9 & -2823.0 \\ 6625.3 & -2823.0 & -2823.0 & 6625.3 \end{bmatrix} \tag{A}$$

(we represent the weights as a 2-D array, according to the natural structure of the observations).

In the case of (b), the optimal value in (3.2.28) is 0.232, and the minimax optimal weights are

$$\begin{bmatrix} -55.6430 & -55.6320 & -55.6320 & -55.6430 \\ -55.6320 & 56.5601 & 56.5601 & -55.6320 \\ -55.6320 & 56.5601 & 56.5601 & -55.6320 \\ -55.6430 & -55.6320 & -55.6320 & -55.6430 \end{bmatrix}. \tag{B}$$

In the case of (c), the optimal value in (3.2.28) is 8.92, and the minimax optimal weights are

$$\begin{bmatrix} -0.4377 & -0.2740 & -0.2740 & -0.4377 \\ -0.2740 & 1.2283 & 1.2283 & -0.2740 \\ -0.2740 & 1.2283 & 1.2283 & -0.2740 \\ -0.4377 & -0.2740 & -0.2740 & -0.4377 \end{bmatrix}. \tag{C}$$

In the case of (d), the optimal value in (3.2.28) is 63.9, and the minimax optimal weights are

$$\begin{bmatrix} 0.1157 & 0.2795 & 0.2795 & 0.1157 \\ 0.2795 & 0.6748 & 0.6748 & 0.2795 \\ 0.2795 & 0.6748 & 0.6748 & 0.2795 \\ 0.1157 & 0.2795 & 0.2795 & 0.1157 \end{bmatrix}. \tag{D}$$

Now, in reality we can hardly know exactly the bounds ρ, σ on the measurement errors. What happens when we under- or over-estimate these quantities? To get an orientation, let us use every one of the weights given by (A), (B), (C), (D) in every one of the situations (a), (b), (c), (d). This is what happens with the errors (obtained as the average of observed errors over 100 random simulations using the "nearly worst-case" signal v and "nearly worst-case" perturbation matrix Δ):

	(a)	(b)	(c)	(d)
(A)	0.001	18.0	6262.9	6.26e5
(B)	0.063	0.232	89.3	8942.7
(C)	8.85	8.85	8.85	108.8
(D)	8.94	8.94	8.94	63.3

We clearly see that, first, in our situation taking into account measurement errors, even pretty small ones, is a must (this is so in all *ill-posed* estimation problems — those where the condition number of $B_{\rm n}$ is large). Second, we see that underestimating the magnitude of measurement errors seems to be much more dangerous than overestimating them.

3.3 Approximating RCs of Uncertain Conic Quadratic Problems

In this section we focus on *tight* tractable approximations of uncertain CQIs — those with tightness factor independent (or nearly so) of the "size" of the description of the perturbation set. Known approximations of this type deal with side-wise uncertainty and two types of the left hand side perturbations: the first is the case of *structured norm-bounded perturbations* to be considered in section 3.3.1, while the second is the case of $\cap$-*ellipsoidal* left hand side perturbation sets to be considered in section 3.3.2.

3.3.1 Structured Norm-Bounded Uncertainty

Consider the case where the uncertainty in CQI (3.2.3) is side-wise with the right hand side uncertainty as in section 3.2.2, and with *structured norm-bounded* left hand side uncertainty, meaning that

1. The left hand side perturbation set is

$$\mathcal{Z}_\rho^{\rm left} = \rho \mathcal{Z}_1^{\rm left} = \left\{ \eta = (\eta^1, ..., \eta^N) : \begin{array}{c} \eta^\nu \in \mathbb{R}^{p_\nu \times q_\nu} \, \forall \nu \le N \\ \|\eta^\nu\|_{2,2} \le \rho \, \forall \nu \le N \\ \eta^\nu = \theta_\nu I_{p_\nu}, \theta_\nu \in \mathbb{R}, \, \nu \in \mathcal{I}_{\rm S} \end{array} \right\} \tag{3.3.1}$$

   Here $\mathcal{I}_{\rm S}$ is a given subset of the index set $\{1, ..., N\}$ such that $p_\nu = q_\nu$ for $\nu \in \mathcal{I}_{\rm S}$.

Thus, the left hand side perturbations $\eta \in \mathcal{Z}_1^{\rm left}$ are block-diagonal matrices with $p_\nu \times q_\nu$ diagonal blocks $\eta^\nu, \nu = 1,...,N$. All of these blocks are of matrix norm not exceeding 1, and, in addition, prescribed blocks should be proportional to the unit matrices of appropriate sizes. The latter blocks are called *scalar*, and the remaining — *full* perturbation blocks.

2. We have

$$A(\eta)y + b(\eta) = A^{\rm n}y + b^{\rm n} + \sum_{\nu=1}^{N} L_\nu^T(y)\eta^\nu R_\nu(y), \tag{3.3.2}$$

where all matrices $\mathbf{L}_\nu(y) \not\equiv 0$, $R_\nu(y) \not\equiv 0$ are affine in y and for every ν, either $L_\nu(y)$, or $R_\nu(y)$, or both are independent of y.

Remark 3.1 *W.l.o.g., we assume from now on that all scalar perturbation blocks are of the size* 1×1*:* $p_\nu = q_\nu = 1$ *for all* $\nu \in \mathcal{I}_{\rm S}$.

To see that this assumption indeed does not restrict generality, note that if $\nu \in \mathcal{I}_{\rm S}$, then in order for (3.3.2) to make sense, $R_\nu(y)$ should be a $p_\nu \times 1$ vector, and $L_\nu(y)$ should be a $p_\nu \times k$ matrix, where k is the dimension of $b(\eta)$. Setting $\bar{R}_\nu(y) \equiv 1$, $\bar{L}_\nu(y) = R_\nu^T(y)L_\nu(y)$, observe that $\bar{L}_\nu(y)$ is affine in y, and the contribution $\theta_\nu L_\nu^T(y)R_\nu(y)$ of the ν-th scalar perturbation block to $A(\eta)y + b(\eta)$ is exactly the same as if this block were of size 1×1, and the matrices $L_\nu(y)$, $R_\nu(y)$ were replaced with $\bar{L}_\nu(y)$, $\bar{R}_\nu(y)$, respectively.

Note that Remark 3.1 is equivalent to the assumption that *there are no scalar perturbation blocks at all* — indeed, 1×1 scalar perturbation blocks can be thought of as full ones as well. [3].

Recall that we have already considered the particular case $N = 1$ of the uncertainty structure. Indeed, with a single perturbation block, that, as we just have seen, we can treat as a full one, we find ourselves in the situation of side-wise uncertainty with unstructured norm-bounded left hand side perturbation (section 3.2.3). In this situation the RC of the uncertain CQI in question is computationally tractable. The latter is not necessarily the case for general ($N > 1$) structured norm-bounded left hand side perturbations. To see that the general structured norm-bounded perturbations are difficult to handle, note that they cover, in particular, the case of *interval uncertainty*, where $\mathcal{Z}_1^{\rm left}$ is the box $\{\eta \in \mathbb{R}^L : \|\eta\|_\infty \leq 1\}$ and $A(\eta)$, $b(\eta)$ are arbitrary affine functions of η.

Indeed, the interval uncertainty

$$\begin{array}{rcl} A(\eta)y + b(\eta) &=& [A^{\rm n}y + b^{\rm n}] + \sum\limits_{\nu=1}^{N} \eta_\nu [A^\nu y + b^\nu] \\ &=& [A^{\rm n}y + b^{\rm n}] \quad + \sum\limits_{\nu=1}^{N} \underbrace{[A^\nu y + b^\nu]}_{L_\nu^T(y)} \cdot \eta_\nu \cdot \underbrace{1}_{R_\nu(y)}, \end{array} \tag{3.3.3}$$

is nothing but the structured norm-bounded perturbation with 1×1 perturbation blocks.

From the beginning of section 3.1.3 we know that the RC of uncertain CQI with side-wise uncertainty and interval uncertainty in the left hand side in general is computationally intractable, meaning that structural norm-bounded uncertainty can be indeed difficult.

[3]A reader could ask, why do we need the scalar perturbation blocks, given that finally we can get rid of them without loosing generality. The answer is, that we intend to use the same notion of structured norm-bounded uncertainty in the case of uncertain LMIs, where Remark 3.1 does not work.

Approximating the RC of Uncertain Least Squares Inequality

We start with deriving a safe tractable approximation of the RC of an *uncertain Least Squares constraint*

$$\|A(\eta)y + b(\eta)\|_2 \le \tau, \tag{3.3.4}$$

with structured norm-bounded perturbation (3.3.1), (3.3.2).

Step 1: reformulating the RC of (3.3.4), (3.3.1), (3.3.2) as a semi-infinite LMI. Given a k-dimensional vector u (k is the dimension of $b(\eta)$) and a real τ, let us set

$$\mathrm{Arrow}(u,t) = \left[\begin{array}{c|c} \tau & u^T \\ \hline u & \tau I_k \end{array}\right].$$

Recall that by Lemma 3.1 $\|u\|_2 \le \tau$ if and only if $\mathrm{Arrow}(u,\tau) \succeq 0$. It follows that the RC of (3.3.4), (3.3.1), (3.3.2), which is the semi-infinite Least Squares inequality

$$\|A(\eta)y + b(\eta)\|_2 \le \tau \;\forall \eta \in \mathcal{Z}_\rho^{\mathrm{left}},$$

can be rewritten as

$$\mathrm{Arrow}(A(\eta)y + b(\eta), \tau) \succeq 0 \;\forall \eta \in \mathcal{Z}_\rho^{\mathrm{left}}. \tag{3.3.5}$$

Introducing $k \times (k+1)$ matrix $\mathcal{L} = [0_{k\times 1}, I_k]$ and $1 \times (k+1)$ matrix $\mathcal{R} = [1, 0, ..., 0]$, we clearly have

$$\begin{array}{l} \mathrm{Arrow}(A(\eta)y + b(\eta), \tau) = \mathrm{Arrow}(A^{\mathrm{n}}y + b^{\mathrm{n}}, \tau) \\ \quad + \sum\limits_{\nu=1}^{N} \left[\mathcal{L}^T L_\nu^T(y)\eta^\nu R_\nu(y)\mathcal{R} + \mathcal{R}^T R_\nu^T(y)[\eta^\nu]^T L_\nu(y)\mathcal{L}\right]. \end{array} \tag{3.3.6}$$

Now, since for every ν, either $L_\nu(y)$, or $R_\nu(y)$, or both, are independent of y, renaming, if necessary $[\eta^\nu]^T$ as η^ν, and swapping $L_\nu(y)\mathcal{L}$ and $R_\nu(y)\mathcal{R}$, we may assume w.l.o.g. that in the relation (3.3.6) all factors $L_\nu(y)$ are independent of y, so that the relation reads

$$\begin{array}{l} \mathrm{Arrow}(A(\eta)y + b(\eta), \tau) = \mathrm{Arrow}(A^{\mathrm{n}}y + b^{\mathrm{n}}, \tau) \\ \quad + \sum\limits_{\nu=1}^{N} \big[\underbrace{\mathcal{L}^T L_\nu^T}_{\widehat{L}_\nu^T}\eta^\nu \underbrace{R_\nu(y)\mathcal{R}}_{\widehat{R}_\nu(y)} + \widehat{R}_\nu^T(y)[\eta^\nu]^T \widehat{L}_\nu\big] \end{array}$$

where $\widehat{R}_\nu(y)$ are affine in y and $\widehat{L}_\nu \ne 0$. Observe also that all the symmetric matrices

$$B_\nu(y,\eta^\nu) = \widehat{L}_\nu^T \eta^\nu \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)[\eta^\nu]^T \widehat{L}_\nu$$

are differences of two matrices of the form $\mathrm{Arrow}(u,\tau)$ and $\mathrm{Arrow}(u',\tau)$, so that these are matrices of rank at most 2. The intermediate summary of our observations is as follows:

(#): *The RC of* (3.3.4), (3.3.1), (3.3.2) *is equivalent to the semi-infinite LMI*

$$\begin{array}{c} \underbrace{\mathrm{Arrow}(A^{\mathrm{n}}y + b^{\mathrm{n}}, \tau)}_{B_0(y,\tau)} + \sum\limits_{\nu=1}^{N} B_\nu(y,\eta^\nu) \succeq 0 \;\forall \left(\eta : \begin{array}{c} \eta^\nu \in \mathbb{R}^{p_\nu \times q_\nu}, \\ \|\eta^\nu\|_{2,2} \le \rho \,\forall \nu \le N \end{array}\right) \\ \left[\begin{array}{c} B_\nu(y,\eta^\nu) = \widehat{L}_\nu^T \eta^\nu \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)[\eta^\nu]^T \widehat{L}_\nu,\; \nu = 1, ..., N \\ p_\nu = q_\nu = 1 \,\forall \nu \in \mathcal{I}_{\mathrm{S}} \end{array}\right] \end{array} \tag{3.3.7}$$

Here $\widehat{R}(y)$ are affine in y, and for all y, all $\nu \ge 1$ and all η^ν the ranks of the matrices $B_\nu(y,\eta^\nu)$ do not exceed 2.

Step 2. Approximating (3.3.7). Observe that an evident *sufficient* condition for the validity of (3.3.7) for a given y is the existence of symmetric matrices Y_ν, $\nu = 1, ..., N$, such that

$$Y_\nu \succeq B_\nu(y, \eta^\nu) \, \forall (\eta^\nu \in \mathcal{Z}_\nu = \{\eta^\nu : \|\eta^\nu\|_{2,2} \leq 1; \nu \in \mathcal{I}_{\rm S} \Rightarrow \eta^\nu \in \mathbb{R}I_{p_\nu}\}) \tag{3.3.8}$$

and

$$B_0(y, \tau) - \rho \sum_{\nu=1}^{N} Y_\nu \succeq 0. \tag{3.3.9}$$

We are about to demonstrate that the semi-infinite LMIs (3.3.8) in variables Y_ν, y, τ can be represented by explicit finite systems of LMIs, so that the system $\mathcal{S}^0$ of semi-infinite constraints (3.3.8), (3.3.9) on variables $Y_1, ..., Y_N, y, \tau$ is equivalent to an explicit finite system $\mathcal{S}$ of LMIs. Since $\mathcal{S}^0$, due to its origin, is a safe approximation of (3.3.7), so will be $\mathcal{S}$, (which, in addition, is tractable). Now let us implement our strategy.

1^0. Let us start with $\nu \in \mathcal{I}_{\rm S}$. Here (3.3.8) clearly is equivalent to just two LMIs

$$Y_\nu \succeq B_\nu(y) \equiv \widehat{L}_\nu^T \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y) \widehat{L}_\nu \;\&\; Y_\nu \succeq -B_\nu(y). \tag{3.3.10}$$

2^0. Now consider relation (3.3.8) for the case $\nu \notin \mathcal{I}_{\rm S}$. Here we have

$$\begin{array}{rc} & (Y_\nu, y) \text{ satisfies (3.3.8)} \\ \Leftrightarrow & u^T Y_\nu u \geq u^T B_\nu(y, \eta^\nu) u \;\forall u \forall (\eta^\nu : \|\eta^\nu\|_{2,2} \leq 1) \\ \Leftrightarrow & u^T Y_\nu u \geq u^T \widehat{L}_\nu^T \eta^\nu \widehat{R}_\nu(y) u + u^T \widehat{R}_\nu^T(y) [\eta^\nu]^T \widehat{L}_\nu u \;\forall u \forall (\eta^\nu : \|\eta^\nu\|_{2,2} \leq 1) \\ \Leftrightarrow & u^T Y_\nu u \geq 2 u^T \widehat{L}_\nu^T \eta^\nu \widehat{R}_\nu(y) u \;\forall u \forall (\eta^\nu : \|\eta^\nu\|_{2,2} \leq 1) \\ \Leftrightarrow & u^T Y_\nu u \geq 2 \|\widehat{L}_\nu u\|_2 \|\widehat{R}(y) u\|_2 \;\forall u \\ \Leftrightarrow & u^T Y_\nu u - 2\xi^T \widehat{R}_\nu(y) u \;\forall (u, \xi : \xi^T \xi \leq u^T \widehat{L}_\nu^T \widehat{L}_\nu u) \end{array}$$

Invoking the $\mathcal{S}$-Lemma, the concluding condition in the latter chain is equivalent to

$$\exists \lambda_\nu \geq 0 : \left[\begin{array}{c|c} Y_\nu - \lambda_\nu \widehat{L}_\nu^T \widehat{L}_\nu & -\widehat{R}_\nu^T(y) \\ \hline -\widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu} \end{array}\right] \succeq 0, \tag{3.3.11}$$

where k_ν is the number of rows in $\widehat{R}_\nu(y)$.

We have proved the first part of the following statement:

Theorem 3.4 *The explicit system of LMIs*

$$\begin{array}{c} Y_\nu \succeq \pm(\widehat{L}_\nu^T \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y) \widehat{L}_\nu), \; \nu \in \mathcal{I}_{\rm S} \\ \left[\begin{array}{c|c} Y_\nu - \lambda_\nu \widehat{L}_\nu^T \widehat{L}_\nu & \widehat{R}_\nu^T(y) \\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu} \end{array}\right] \succeq 0, \; \nu \notin \mathcal{I}_{\rm S} \\ \mathrm{Arrow}(A^{\rm n} y + b^{\rm n}, \tau) - \rho \sum_{\nu=1}^{N} Y_\nu \succeq 0 \end{array} \tag{3.3.12}$$

(for notation, see (3.3.7)*) in variables* $Y_1, ..., Y_N, \lambda_\nu, y, \tau$ *is a safe tractable approximation of the RC of the uncertain Least Squares inequality* (3.3.4), (3.3.1), (3.3.2)*. The tightness factor of this approximation never exceeds* $\pi/2$*, and equals to 1 when* $N = 1$*.*

Proof. By construction, (3.3.12) indeed is a safe tractable approximation of the RC of (3.3.4), (3.3.1), (3.3.2) (note that a matrix of the form $\left[\begin{array}{c|c} A & B \\ \hline B^T & A \end{array}\right]$ is $\succeq 0$ if and only if the matrix $\left[\begin{array}{c|c} A & -B \\ \hline -B^T & A \end{array}\right]$ is so). By Remark and Theorem 3.2, our approximation is exact when $N = 1$. The fact that the tightness factor never exceeds $\pi/2$ is an immediate corollary of the real case Matrix Cube Theorem (Theorem A.7), and we use the corresponding notation in the rest of the proof. Observe that a given pair (y, τ) is robust feasible for (3.3.4), (3.3.1), (3.3.2) if and only if the matrices $B_0 = B_0(y, \tau)$, $B_i = B_{\nu_i}(y, 1)$, $i = 1, ..., p$, $L_j = \widehat{L}_{\mu_j}, R_j = \widehat{R}_{\mu_j}(y)$, $j = 1, ..., q$, satisfy $\mathcal{A}(\rho)$; here $\mathcal{I}_{\rm S} = \{\nu_1 < ... < \nu_p\}$ and $\{1, ..., L\} \backslash \mathcal{I}_{\rm S} = \{\mu_1 < ... < \mu_q\}$. At the same time, the validity of the corresponding predicate $\mathcal{B}(\rho)$ is equivalent to the possibility to extend y to a solution of (3.3.12) due to the origin of the latter system. Since all matrices B_i, $i = 1, ..., p$, are of rank at most 2 by (#), the Matrix Cube Theorem implies that if (y, τ) cannot be extended to a feasible solution to (3.3.12), then (y, τ) is not robust feasible for (3.3.4), (3.3.1), (3.3.2) when the uncertainty level is increased by the factor $\vartheta(2) = \frac{\pi}{2}$. □

Illustration: Antenna Design revisited. Consider the Antenna Design example (Example 1.1) and assume that instead of measuring the closeness of a synthesized diagram to the target one in the uniform norm, as was the case in section 1.1.3, we want to use the Euclidean norm, specifically, the weighted 2-norm

$$\|f(\cdot)\|_{2,w} = \left(\sum_{i=1}^{m} f^2(\theta_i)\mu_i\right)^{1/2} \qquad [\theta_i = \tfrac{i\pi}{2m}, 1 \leq i \leq m = 240,\ \mu_i = \tfrac{\cos(\theta_i)}{\sum_{s=1}^{m}\cos(\theta_s)}]$$

To motivate the choice of weights, recall that the functions $f(\cdot)$ we are interested in are restrictions of diagrams (and their differences) on the equidistant L-point grid of altitude angles. The diagrams in question are, physically speaking, functions of a 3D direction from the upper half-space (a point on the unit 2D hemisphere) which depend solely on the altitude angle and are independent of the longitude angle. A "physically meaningful" L_2-norm here corresponds to uniform distribution on the hemisphere; after discretization of the altitude angle, this L_2 norm becomes our $\|\cdot\|_2$.

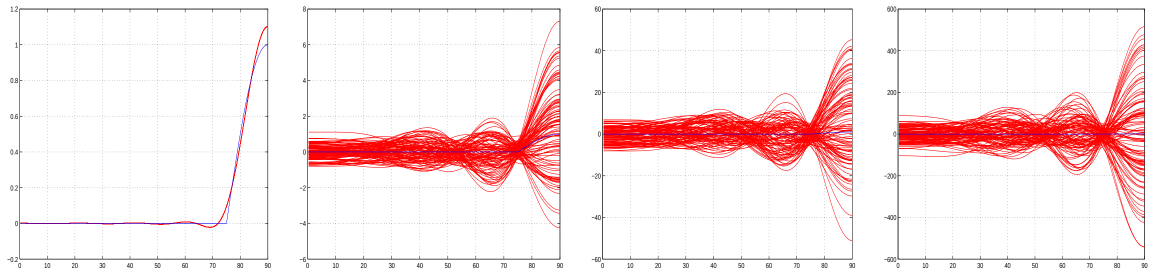
with this measure of discrepancy between a synthesized and the target diagram, the problem of interest becomes the uncertain problem

$$\left\{\min_{y,\tau}\{\tau : \|WD[I + \mathrm{Diag}\{\eta\}]y - b\|_2 \leq \tau\} : \eta \in \rho\mathcal{Z}\right\},\ \mathcal{Z} = \{\eta \in \mathbb{R}^{L=10} : \|\eta\|_\infty \leq 1\}, \quad (3.3.13)$$

where

- $D = [D_{ij} = D_j(\theta_i)]_{\substack{1\leq i\leq m=240,\\ 1\leq j\leq L=10}}$ is the matrix comprised of the diagrams of $L = 10$ antenna elements (central circle and surrounding rings), see section 1.1.3,
- $W = \mathrm{Diag}\{\sqrt{\mu_1}, ..., \sqrt{\mu_m}\}$, so that $\|Wz\|_2 = \|z\|_{2,w}$, and $b = W[D_*(\theta_1); ...; D_*(\theta_m)]$ comes from the target diagram $D_*(\cdot)$, and
- η is comprised of actuation errors, and ρ is the uncertainty level.

Nominal design. Solving the nominal problem (corresponding to $\rho = 0$), we end up with the nominal optimal design which "in the dream" – with no actuation errors – is really nice (figure



$\rho = 0$ $\rho = 0.0001$ $\rho = 0.001$ $\rho = 0.01$

Figure 3.1: "Dream and reality," nominal optimal design: samples of 100 actual diagrams (red) for different uncertainty levels. Blue: the target diagram

	Dream	Reality								
	$\rho = 0$	$\rho = 0.0001$			$\rho = 0.001$			$\rho = 0.01$		
	value	min	mean	max	min	mean	max	min	mean	max
$\lVert \cdot \rVert_{2,w}$-distance to target	0.011	0.077	0.424	0.957	1.177	4.687	9.711	8.709	45.15	109.5
energy concentration	99.4%	0.23%	20.1%	77.7%	0.70%	19.5%	61.5%	0.53%	18.9%	61.5%

Table 3.1: Quality of nominal antenna design: dream and reality. Data over 100 samples of actuation errors per each uncertainty level ρ.

3.1, case of $\rho = 0$): the $\|\cdot\|_{2,w}$-distance of the nominal diagram to the target is as small as 0.0112, and the energy concentration for this diagram is as large as 99.4%. Unfortunately, the data in figure 3.1 and table 3.1 show that "in reality," with the uncertainty level as small as $\rho = 0.01\%$, the nominal design is a complete disaster.

Robust design. Let us build a robust design. The set $\mathcal{Z}$ is a unit box, that is, we are in the case of interval uncertainty, or, which the same, structured norm-bounded uncertainty with $L = 10$ scalar perturbation blocks η_ℓ. Denoting ℓ-th column of WD by $[WD]_\ell$, we have

$$WD[I + \mathrm{Diag}\{\eta\}]y - b = [WDy - b] + \sum_{\ell=1}^{L} \eta_\ell \left[y_\ell [WD]_\ell\right],$$

that is, taking into account (3.3.3), we have in the notation of (3.3.2):

$$A^{\mathrm{n}}y + b^{\mathrm{n}} = WDy - b,\ L_\nu(y) = y_\nu [WD]_\nu^T,\ \eta^\nu \equiv \eta_\nu,\ R_\nu(y) \equiv 1,\ 1 \le \nu \le N \equiv L = 10,$$

so that in the notation of Theorem 3.4 we have

$$\widehat{R}_\nu(y) = [0, y_\nu [WD]_\nu^T],\ \widehat{L}_\nu = [1, 0_{1\times n}], k_\nu = 1, ., 1 \le \nu \le N \equiv L.$$

Since we are in our right to treat all perturbation blocks as scalar, a tight within the factor $\pi/2$ safe tractable approximation, given by Theorem 3.4, of the RC of our uncertain Least Squares problem reads

$$\min_{\tau,y,Y_1,\dots,Y_L}\left\{\tau:\mathrm{Arrow}(WDy-b,\tau)-\rho\sum_{\nu=1}^{L}Y_\nu\succeq 0,\ Y_\nu\succeq\pm\left[\widehat{L}_\nu^T\widehat{R}_\nu(y)+\widehat{R}_\nu^T(y)\widehat{L}_\nu\right],1\le\nu\le L\right\}. \tag{3.3.14}$$

This problem simplifies dramatically due to the following simple fact (see Exercise 3.6):

> (!) *Let a,b be two vectors of the same dimension with $a\neq 0$. Then $Y\succeq ab^T+ba^T$ if and only if there exists $\lambda\ge 0$ such that $Y\succeq\lambda aa^T+\frac{1}{\lambda}bb^T$.*
> Here, by definition, $\frac{1}{0}bb^T$ is undefined when $b\neq 0$ and is the zero matrix when $b=0$.

By (!), a pair (τ,y) can be extended, by properly chosen Y_ν, $\nu=1,\dots,L$, to a feasible solution of (3.3.14) if and only if there exist $\lambda_\nu\ge 0$, $1\le\nu\le L$, such that $\mathrm{Arrow}(WDy-b,\tau)-\rho\sum_\nu\left[\lambda_\nu\widehat{L}_\nu^T\widehat{L}_\nu+\lambda_\nu^{-1}\widehat{R}_\nu^T(y)\widehat{R}_\nu(y)\right]\succeq 0$, which, by the Schur Complement Lemma is equivalent to

$$\left[\begin{array}{c|c}\mathrm{Arrow}(WDy-b,\tau)-\sum\limits_\nu\rho\lambda_\nu\widehat{L}_\nu^T\widehat{L}_\nu & \rho[\widehat{R}_1^T(y),\dots,\widehat{R}_L^T(y)]\\ \hline \rho[\widehat{R}_1(y);\dots;\widehat{R}_L(y)] & \rho\mathrm{Diag}\{\lambda_1,\dots,\lambda_L\}\end{array}\right]\succeq 0.$$

Thus, problem (3.3.14) is equivalent to

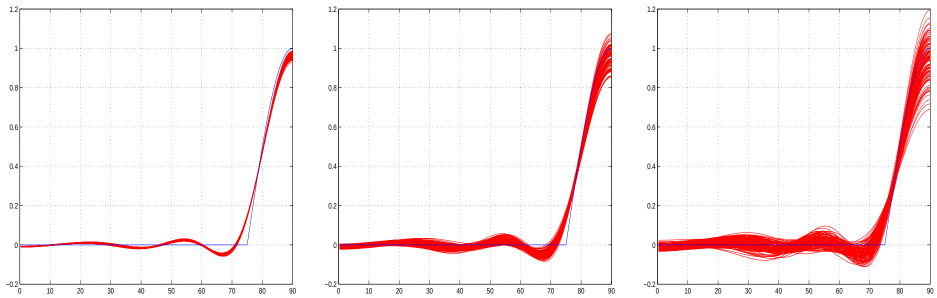
$$\min_{\tau,y,\gamma}\left\{\tau:\left[\begin{array}{c|c}\mathrm{Arrow}(WDy-b,\tau)-\sum\limits_\nu\gamma_\nu\widehat{L}_\nu^T\widehat{L}_\nu & \rho[\widehat{R}_1^T(y),\dots,\widehat{R}_L^T(y)]\\ \hline \rho[\widehat{R}_1(y);\dots;\widehat{R}_L(y)] & \mathrm{Diag}\{\gamma_1,\dots,\gamma_L\}\end{array}\right]\succeq 0\right\} \tag{3.3.15}$$

(we have set $\gamma_\nu=\rho\lambda_\nu$). Note that we managed to replace every matrix variable Y_ν in (3.3.14) with a *single scalar* variable λ_ν in (3.3.15). Note that this dramatic simplification is possible whenever all perturbation blocks are scalar.

With our particular $\widehat{L}_\nu$ and $\widehat{R}_\nu(y)$ the resulting problem (3.3.15) reads

$$\min_{\tau,y,\gamma}\left\{\tau:\left[\begin{array}{c|c|c}\tau-\sum_{\nu=1}^L\gamma_\nu & [WDy-b]^T & \\ \hline WDy-b & \tau I_m & \rho[y_1[WD]_1,\dots,y_L[WD]_L]\\ \hline & \rho[y_1[WD]_1,\dots,y_L[WD]_L]^T & \mathrm{Diag}\{\gamma_1,\dots,\gamma_L\}\end{array}\right]\succeq 0\right\}. \tag{3.3.16}$$

We have solved (3.3.16) at the uncertainty level $\rho=0.01$, thus getting a robust design. The optimal value in (3.3.16) is 0.02132 – while being approximately 2 times worse than the nominal optimal value, it still is pretty small. We then tested the robust design against actuation errors of magnitude $\rho=0.01$ and larger. The results, summarized in figure 3.2 and table 3.2, allow for the same conclusions as in the case of LP-based design, see p. 22. Recall that (3.3.16) is not the "true" RC of our uncertain problem, is just a safe approximation, tight within the factor $\pi/2$, of this RC. All we can conclude from this is that the value $\tau_*=0.02132$ of τ yielded by the approximation (that is, the guaranteed value of the objective at our robust design, the uncertainty level being 0.01) is in-between the true robust optimal values $\mathrm{Opt}_*(0.01)$ and $\mathrm{Opt}_*(0.01\pi/2)$ at the uncertainty levels 0.01 and $0.01\pi/2$, respectively. This information does *not* allow for meaningful conclusions on how far away is τ_* from the true robust optimal value $\mathrm{Opt}_*(0.01)$; at this point, all we can say in this respect is that $\mathrm{Opt}_*(0,01)$ is at least the nominal optimal value $\mathrm{Opt}_*(0)=0.0112$, and thus the loss in optimality caused by our approximation is at most by factor $\tau_*/\mathrm{Opt}_*(0)=1.90$. In particular, we cannot exclude that with our approximation, we lose as much as 90% in the value of the objective. The reality, however, is by far not so bad. Note that our perturbation set – the 10-dimensional box – is a



$\rho = 0.01$ $\rho = 0.05$ $\rho = 0.1$

Figure 3.2: "Dream and reality," robust optimal design: samples of 100 of actual diagrams (red) for different uncertainty levels. Blue: the target diagram.

	Reality								
	$\rho = 0.01$			$\rho = 0.05$			$\rho = 0.1$		
	min	mean	max	min	mean	max	min	mean	max
$\|\cdot\|_{2,w}$-distance to target	0.021	0.021	0.021	0.021	0.023	0.030	0.021	0.030	0.048
energy concentration	96.5%	96.7%	96.9%	93.0%	95.8%	96.8%	80.6%	92.9%	96.7%

Table 3.2: Quality of robust antenna design. Data over 100 samples of actuation errors per each uncertainty level ρ.

For comparison: for nominal design, with the uncertainty level as small as $\rho = 0.001$, the average $\|\cdot\|_{2,w}$-distance of the actual diagram to target is as large as 4.69, and the expected energy concentration is as low as 19.5%.

convex hull of 1024 vertices, so we can think about our uncertainty as of the scenario one (section 3.2.1) generated by 1024 scenarios. This number is still within the grasp of the straightforward scheme proposed in section 3.2.1, and thus we can find in a reasonable time the true robust optimal value $\mathrm{Opt}_*(0.01)$, which turns out to be 0.02128. We see that the actual loss in the value of the objective caused by approximation its really small – it is less than 0.2%.

Least Squares Inequality with Structured Norm-Bounded Uncertainty, Complex Case

The uncertain Least Squares inequality (3.3.4) with structured norm-bounded perturbations makes sense in the case of complex left hand side data as well as in the case of real data. Surprisingly, in the complex case the RC admits a better in tightness factor safe tractable approximation than in the real case (specifically, the tightness factor $\frac{\pi}{2} = 1.57...$ stated in Theorem 3.4 in the complex case improves to $\frac{4}{\pi} = 1.27...$). Consider an uncertain Least Squares inequality (3.3.4) where $A(\eta) \in \mathbb{C}^{m\times n}$, $b(\eta) \in \mathbb{C}^m$ and the perturbations are structured norm-bounded and *complex*, meaning that (cf. (3.3.1), (3.3.2))

$$\begin{array}{ll}(a) & \mathcal{Z}_\rho^{\mathrm{left}} = \rho\mathcal{Z}_1^{\mathrm{left}} = \left\{\eta = (\eta^1,...,\eta^N) : \begin{array}{l}\eta^\nu \in \mathbb{C}^{p_\nu\times q_\nu},\ \nu = 1,...,N\\ \|\eta^\nu\|_{2,2} \le \rho,\ \nu = 1,...,N\\ \eta^\nu = \theta_\nu I_{p_\nu}, \theta_\nu \in \mathbb{C},\ \nu \in \mathcal{I}_{\mathrm{S}}\end{array}\right\}, \\ (b) & A(\zeta)y + b(\zeta) = [A^{\mathrm{n}}y + b^{\mathrm{n}}] + \sum\limits_{\nu=1}^N L_\nu^H(y)\eta^\nu R_\nu(y),\end{array} \tag{3.3.17}$$

where $L_\nu(y)$, $R_\nu(y)$ are affine in $[\Re(y);\Im(y)]$ matrices with complex entries such that for every ν at least one of these matrices is independent on y and is nonzero, and B^H denotes the Hermitian conjugate of a complex-valued matrix B: $(B^H)_{ij} = \overline{B_{ji}}$, where $\overline{z}$ is the complex conjugate of a complex number z.

Observe that by exactly the same reasons as in the real case, we can assume w.l.o.g. that all scalar perturbation blocks are 1×1, or, equivalently, that there are no scalar perturbation blocks at all, so that from now on we assume that $\mathcal{I}_{\mathrm{S}} = \emptyset$.

The derivation of the approximation is similar to the one in the real case. Specifically, we start with the evident observation that for a complex k-dimensional vector u and a real t the relation

$$\|u\|_2 \le t$$

is equivalent to the fact that the Hermitian matrix

$$\mathrm{Arrow}(u,t) = \left[\begin{array}{c|c} t & u^H \\ \hline u & tI_k\end{array}\right]$$

is $\succeq 0$; this fact is readily given by the complex version of the Schur Complement Lemma: *a Hermitian block matrix* $\left[\begin{array}{c|c} P & Q^H \\ \hline Q & R\end{array}\right]$ *with* $R \succ 0$ *is positive semidefinite if and only if the Hermitian matrix* $P - Q^HR^{-1}Q$ *is positive semidefinite* (cf. the proof of Lemma 3.1). It follows that (y,τ) is robust feasible for the uncertain Least Squares inequality in question if and only if

$$\begin{array}{c}\underbrace{\mathrm{Arrow}(A^{\mathrm{n}}y + b^{\mathrm{n}},\tau)}_{B_0(y,\tau)} + \sum\limits_{\nu=1}^N B_\nu(y,\eta^\nu) \succeq 0 \forall\,(\eta : \|\eta^\nu\|_{2,2} \le \rho \forall \nu \le N) \\ \left[B_\nu(y,\eta^\nu) = \widehat{L}_\nu^H\eta^\nu\widehat{R}_\nu(y) + \widehat{R}_\nu^H(y)[\eta^\nu]^H\widehat{L}_\nu,\ \nu = 1,...,N\right]\end{array} \tag{3.3.18}$$

where $\widehat{L}_\nu$ are constant matrices, and $\widehat{R}(y)$ are affine in $[\Re(y);\Im(y)]$ matrices readily given by $L_\nu(y)$, $R_\nu(y)$ (cf. (3.3.7) and take into account that we are in the situation $\mathcal{I}_{\rm S}=\emptyset$). It follows that whenever, for a given (y,τ), one can find Hermitian matrices Y_ν such that

$$Y_\nu \succeq B_\nu(y,\eta^\nu)\ \forall(\eta^\nu\in\mathbb{C}^{p_\nu\times q_\nu}:\|\eta^\nu\|_{2,2}\leq 1),\ \nu=1,...,N, \tag{3.3.19}$$

and $B_0(y,\tau)\succeq \rho\sum\limits_{\nu=1}^{N} Y_\nu$, the pair (y,τ) is robust feasible.
Same as in the real case, applying the $\mathcal{S}$-Lemma, (which works in the complex case as well as in the real one), a matrix Y_ν satisfies (3.3.19) if and only if

$$\exists\lambda_\nu\geq 0:\left[\begin{array}{c|c} Y_\nu-\lambda_\nu\widehat{L}_\nu^H\widehat{L}_\nu & -\widehat{R}_\nu^H(y)\\ \hline -\widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu}\end{array}\right],$$

where k_ν is the number of rows in $\widehat{R}_\nu(y)$. We have arrived at the first part of the following statement:

Theorem 3.5 *The explicit system of LMIs*

$$\begin{array}{c}\left[\begin{array}{c|c} Y_\nu-\lambda_\nu\widehat{L}_\nu^H\widehat{L}_\nu & \widehat{R}_\nu^H(y)\\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu}\end{array}\right]\succeq 0,\ \nu=1,...,N,\\ \mathrm{Arrow}(A^{\rm n}y+b^{\rm n},\tau)-\rho\sum\limits_{\nu=1}^{N}Y_\nu\succeq 0\end{array} \tag{3.3.20}$$

(for notation, see (3.3.18)) in the variables $\{Y_i=Y_i^H\},\lambda_\nu,y,\tau$ is a safe tractable approximation of the RC of the uncertain Least Squares inequality (3.3.4), (3.3.17). *The tightness factor of this approximation never exceeds $4/\pi$, and is equal to 1 when $N=1$.*

Proof is completely similar to the one of Theorem 3.4, modulo replacing the real case of the Matrix Cube Theorem (Theorem A.7) with its complex case (Theorem A.6).

From Uncertain Least Squares to Uncertain CQI

Let us come back to the real case. We have already built a tight approximation for the RC of a Least Squares inequality with structured norm-bounded uncertainty in the left hand side data. Our next goal is to extend this approximation to the case of uncertain CQI with side-wise uncertainty.

Theorem 3.6 *Consider the uncertain CQI* (3.2.3) *with side-wise uncertainty, where the left hand side uncertainty is the structured norm-bounded one given by* (3.3.1), (3.3.2), *and the right hand side perturbation set is given by a conic representation (cf. Theorem 1.1)*

$$\mathcal{Z}_\rho^{\rm right}=\rho\mathcal{Z}_1^{\rm right},\ \mathcal{Z}_1^{\rm right}=\left\{\chi:\exists u: P\chi+Qu+p\in\mathbf{K}\right\}, \tag{3.3.21}$$

where $0\in\mathcal{Z}_1^{\rm right}$, $\mathbf{K}$ is a closed convex pointed cone and the representation is strictly feasible unless $\mathbf{K}$ is a polyhedral cone given by an explicit finite list of linear inequalities, and $0\in\mathcal{Z}_1^{\rm right}$.

For $\rho > 0$, the explicit system of LMIs

$$\begin{array}{ll} (a) & \tau + \rho p^T v \leq \delta(y),\ P^T v = \sigma(y),\ Q^T v = 0,\ v \in \mathbf{K}_* \\ (b.1) & Y_\nu \succeq \pm(\widehat{L}_\nu^T \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)\widehat{L}_\nu),\ \nu \in \mathcal{I}_{\rm S} \\ (b.2) & \left[\begin{array}{c|c} Y_\nu - \lambda_\nu \widehat{L}_\nu^T \widehat{L}_\nu & \widehat{R}_\nu^T(y) \\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu} \end{array}\right] \succeq 0,\ \nu \notin \mathcal{I}_{\rm S} \\ (b.3) & \mathrm{Arrow}(A^{\rm n} y + b^{\rm n}, \tau) - \rho \sum\limits_{\nu=1}^{N} Y_\nu \succeq 0 \end{array} \tag{3.3.22}$$

(for notation, see (3.3.7)) in variables $Y_1, ..., Y_N, \lambda_\nu, y, \tau, v$ is a safe tractable approximation of the RC of (3.2.4)*. This approximation is exact when $N = 1$, and is tight within the factor $\frac{\pi}{2}$ otherwise.*

Proof. Since the uncertainty is side-wise, y is robust feasible for (3.2.4), (3.3.1), (3.3.2), (3.3.21), the uncertainty level being $\rho > 0$, if and only if there exists τ such that

$$\begin{array}{ll} (c) & \sigma^T(\chi)y + \delta(\chi) \geq \tau\ \forall \chi \in \rho \mathcal{Z}_1^{\rm right}, \\ (d) & \|A(\eta)y + b(\eta)\|_2 \leq \tau\ \forall \eta \in \rho \mathcal{Z}_1^{\rm left}. \end{array}$$

When $\rho > 0$, we have

$$\rho \mathcal{Z}_1^{\rm right} = \{\chi : \exists u : P(\chi/\rho) + Qu + p \in \mathbf{K}\} = \{\chi : \exists u' : P\chi + Qu' + \rho p \in \mathbf{K}\};$$

from the resulting conic representation of $\rho \mathcal{Z}_1^{\rm right}$, same as in the proof of Theorem 1.1, we conclude that the relations (3.3.22.a) represent equivalently the requirement (c), that is, (y, τ) satisfies (c) if and only if (y, τ) can be extended, by properly chosen v, to a solution of (3.3.22.a). By Theorem 3.4, the possibility to extend (y, τ) to a feasible solution of (3.3.22.b) is a sufficient condition for the validity of (d). Thus, the (y, τ) component of a feasible solution to (3.3.22) satisfies (c), (d), meaning that y is robust feasible at the level of uncertainty ρ. Thus, (3.3.22) is a safe approximation of the RC in question.

The fact that the approximation is precise when there is only one left hand side perturbation block is readily given by Theorem 3.2 and Remark 3.1 allowing us to treat this block as full. It remains to verify that the tightness factor of the approximation is at most $\frac{\pi}{2}$, that is, to check that if a given y cannot be extended to a feasible solution of the approximation for the uncertainty level ρ, then y is not robust feasible for the uncertainty level $\frac{\pi}{2}\rho$ (see comments after Definition 3.3). To this end, let us set

$$\tau_y(r) = \inf_{\chi} \left\{ \sigma^T(\chi)y + \delta(\chi) : \chi \in r\mathcal{Z}_1^{\rm right} \right\}.$$

Since $0 \in \mathcal{Z}_1^{\rm right}$ by assumption, $\tau_y(r)$ is nonincreasing in r. Clearly, y is robust feasible at the uncertainty level r if and only if

$$\|A(\eta)y + b(\eta)\|_2 \leq \tau_y(r)\ \forall \eta \in r\mathcal{Z}_1^{\rm left}. \tag{3.3.23}$$

Now assume that a given y cannot be extended to a feasible solution of (3.3.22) for the uncertainty level ρ. Let us set $\tau = \tau_y(\rho)$; then (y, τ) can be extended, by a properly chosen v, to a feasible solution of (3.3.22.a). Indeed, the latter system expresses equivalently the fact that

(y, τ) satisfies (c), which indeed is the case for our (y, τ). Now, since y cannot be extended to a feasible solution to (3.3.22) at the uncertainty level ρ, and the pair (y, τ) can be extended to a feasible solution of (3.3.22.a), we conclude that (y, τ) cannot be extended to a feasible solution of (3.3.22.b). By Theorem 3.4, the latter implies that y is *not* robust feasible for the semi-infinite Least Squares constraint

$$\|A(\eta)y + b(\eta)\|_2 \le \tau = \tau_y(\rho) \ \forall \eta \in \frac{\pi}{2}\rho\mathcal{Z}_1^{\text{left}}.$$

Since $\tau_y(r)$ is nonincreasing in r, we conclude that y does *not* satisfy (3.3.23) when $r = \frac{\pi}{2}\rho$, meaning that y is not robust feasible at the level of uncertainty $\frac{\pi}{2}\rho$. □

Convex Quadratic Constraint with Structured Norm-Bounded Uncertainty

Consider an uncertain convex quadratic constraint

$$\begin{array}{ll} (a) & y^T A^T(\zeta)A(\zeta)y \le 2y^T b(\zeta) + c(\zeta) \\ & \Updownarrow \\ (b) & \|[2A(\zeta)y; 1 - 2y^T b(\zeta) - c(\zeta)]\|_2 \le 1 + 2y^T b(\zeta) + c(\zeta), \end{array} \tag{3.2.13}$$

where $A(\zeta)$ is $k \times n$ and the uncertainty is structured norm-bounded (cf. (3.2.14)), meaning that

$$\begin{array}{ll} (a) & \mathcal{Z}_\rho = \rho\mathcal{Z}_1 = \left\{ \zeta = (\zeta^1, ..., \zeta^N) : \begin{array}{c} \zeta^\nu \in \mathbb{R}^{p_\nu \times q_\nu} \\ \|\zeta^\nu\|_{2,2} \le \rho, 1 \le \nu \le N \\ \zeta^\nu = \theta_\nu I_{p_\nu}, \theta_\nu \in \mathbb{R}, \nu \in \mathcal{I}_{\rm S} \end{array} \right\}, \\ (b) & \left[\begin{array}{c} A(\zeta)y \\ y^T b(\zeta) \\ c(\zeta) \end{array}\right] = \left[\begin{array}{c} A^{\rm n}y \\ y^T b^{\rm n} \\ c^{\rm n} \end{array}\right] + \sum\limits_{\nu=1}^{N} L_\nu^T(y)\zeta^\nu R_\nu(y) \end{array} \tag{3.3.24}$$

where, for every ν, $L_\nu(y)$, $R_\nu(y)$ are matrices of appropriate sizes depending affinely on y and such that at least one of the matrices is constant. Same as above, we can assume w.l.o.g. that all scalar perturbation blocks are 1×1: $p_\nu = k_\nu = 1$ for all $\nu \in \mathcal{I}_{\rm S}$.

Note that the equivalence in (3.2.13) means that we still are interested in an uncertain CQI with structured norm-bounded left hand side uncertainty. The uncertainty, however, is *not* side-wise, that is, we are in the situation we could not handle before. We can handle it now due to the fact that the uncertain CQI possesses a favorable structure inherited from the original convex quadratic form of the constraint.

We are about to derive a tight tractable approximation of the RC of (3.2.13), (3.3.24). The construction is similar to the one we used in the unstructured case $N = 1$, see section 3.2.4. Specifically, let us set $L_\nu(y) = [L_{\nu,A}(y), L_{\nu,b}(y), L_{\nu,c}(y)]$, where $L_{\nu,b}(y)$, $L_{\nu,c}(y)$ are the last two columns in $L_\nu(y)$, and let

$$\begin{array}{c} \widetilde{L}_\nu^T(y) = \left[L_{\nu,b}^T(y) + \frac{1}{2}L_{\nu,c}^T(y); L_{\nu,A}^T(y)\right], \ \widetilde{R}_\nu(y) = [R_\nu(y), 0_{q_\nu \times k}], \\ \mathcal{A}(y) = \left[\begin{array}{c|c} 2y^T b^{\rm n} + c^{\rm n} & [A^{\rm n}y]^T \\ \hline A^{\rm n}y & I \end{array}\right], \end{array} \tag{3.3.25}$$

so that $\mathcal{A}(y)$, $\widetilde{L}_\nu(y)$ and $\widetilde{R}_\nu(y)$ are affine in y and at least one of the latter two matrices is constant.

We have

$$
\begin{array}{cl}
& y^T A^T(\zeta)A(\zeta)y \le 2y^T b(\zeta) + c(\zeta)\ \forall \zeta \in \mathcal{Z}_\rho \\
\Leftrightarrow & \left[\begin{array}{c|c} 2y^T b(\zeta) + c(\zeta) & [A(\zeta)y]^T \\ \hline A(\zeta)y & I \end{array}\right] \succeq 0\ \forall \zeta \in \mathcal{Z}_\rho \text{ [Schur Complement Lemma]} \\
\Leftrightarrow & \overbrace{\left[\begin{array}{c|c} 2y^T b^{\rm n} + c^{\rm n} & [A^{\rm n} y]^T \\ \hline A^{\rm n} y & I \end{array}\right]}^{\mathcal{A}(y)} \\
& + \sum\limits_{\nu=1}^N \underbrace{\left[\begin{array}{c|c} [2L_{\nu,b}(y) + L_{\nu,c}(y)]^T \zeta^\nu R_\nu(y) & [L_{\nu,A}^T(y)\zeta^\nu R_\nu(y)]^T \\ \hline L_{\nu,A}^T(y)\zeta^\nu R_\nu(y) & \end{array}\right]}_{=\widetilde{L}_\nu^T(y)\zeta^\nu \widetilde{R}_\nu(y) + \widetilde{R}_\nu^T(y)[\zeta^\nu]^T \widetilde{L}_\nu(y)} \succeq 0\ \forall \zeta \in \mathcal{Z}_\rho \\
& \hfill \text{[by (3.3.24)]} \\
\Leftrightarrow & \mathcal{A}(y) + \sum\limits_{\nu=1}^N \left[\widetilde{L}_\nu^T(y)\zeta^\nu \widetilde{R}_\nu(y) + \widetilde{R}_\nu^T(y)[\zeta^\nu]^T \widetilde{L}_\nu(y)\right] \succeq 0\ \forall \zeta \in \mathcal{Z}_\rho.
\end{array}
$$

Taking into account that for every ν at least one of the matrices $\widetilde{L}_\nu(y)$, $\widetilde{R}_\nu(y)$ is independent of y and swapping, if necessary, ζ^ν and $[\zeta^\nu]^T$, we can rewrite the last condition in the chain as

$$
\mathcal{A}(y) + \sum_{\nu=1}^N \left[\widehat{L}_\nu^T \zeta^\nu \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)[\zeta^\nu]^T \widehat{L}_\nu\right] \succeq 0\ \forall (\zeta : \|\zeta^\nu\|_{2,2} \le \rho) \tag{3.3.26}
$$

where $\widehat{L}_\nu$, $\widehat{R}_\nu(y)$ are readily given matrices and $\widehat{R}_\nu(y)$ is affine in y. (Recall that we are in the situation where all scalar perturbation blocks are 1×1 ones, and we can therefore skip the explicit indication that $\zeta^\nu = \theta_\nu I_{p_\nu}$ for $\nu \in \mathcal{I}_{\rm S}$). Observe also that similarly to the case of a Least Squares inequality, all matrices $\left[\widehat{L}_\nu^T \zeta^\nu \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)[\zeta^\nu]^T \widehat{L}_\nu\right]$ are of rank at most 2. Finally, we lose nothing by assuming that $\widehat{L}_\nu$ are nonzero for all ν.

Proceeding exactly in the same fashion as in the case of the uncertain Least Squares inequality with structured norm-bounded perturbations, we arrive at the following result (cf. Theorem 3.4):

Theorem 3.7 *The explicit system of LMIs*

$$
\begin{array}{c}
Y_\nu \succeq \pm(\widehat{L}_\nu^T \widehat{R}_\nu(y) + \widehat{R}_\nu^T(y)\widehat{L}_\nu),\ \nu \in \mathcal{I}_{\rm S} \\
\left[\begin{array}{c|c} Y_\nu - \lambda_\nu \widehat{L}_\nu^T \widehat{L}_\nu & \widehat{R}_\nu^T(y) \\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu} \end{array}\right] \succeq 0,\ \nu \notin \mathcal{I}_{\rm S} \\
\mathcal{A}(y) - \rho \sum\limits_{\nu=1}^L Y_\nu \succeq 0
\end{array} \tag{3.3.27}
$$

(k_ν is the number of rows in $\widehat{R}_\nu$) in variables $Y_1, ..., Y_N, \lambda_\nu, y$ is a safe tractable approximation of the RC of the uncertain convex quadratic constraint (3.2.13), (3.3.24). *The tightness factor of this approximation never exceeds $\pi/2$, and equals 1 when $N = 1$.*

Complex case. The situation considered in section 3.3.1 admits a complex data version as well. Consider a convex quadratic constraint with complex-valued variables and a complex-valued structured norm-bounded uncertainty:

$$\begin{array}{l} y^H A^H(\zeta)A(\zeta)y \leq \Re\{2y^H b(\zeta) + c(\zeta)\} \\ \zeta \in \mathcal{Z}_\rho = \rho\mathcal{Z}_1 = \left\{ \zeta = (\zeta^1, ..., \zeta^N) : \begin{array}{l} \zeta^\nu \in \mathbb{C}^{p_\nu \times q_\nu}, 1 \leq \nu \leq N \\ \|\zeta^\nu\|_{2,2} \leq \rho, 1 \leq \nu \leq N \\ \nu \in \mathcal{I}_{\rm S} \Rightarrow \zeta^\nu = \theta_\nu I_{p_\nu}, \theta_\nu \in \mathbb{C} \end{array} \right\} \\ \begin{bmatrix} A(\zeta)y \\ y^H b(\zeta) \\ c(\zeta) \end{bmatrix} = \begin{bmatrix} A^{\rm n} y \\ y^H b^{\rm n} \\ c^{\rm n} \end{bmatrix} + \sum_{\nu=1}^N L_\nu^H(y)\zeta^\nu R_\nu(y), \end{array} \tag{3.3.28}$$

where $A^{\rm n} \in \mathbb{C}^{k\times m}$ and the matrices $L_\nu(y)$, $R_\nu(y)$ are affine in $[\Re(y); \Im(y)]$ and such that for every ν, either $L_\nu(y)$, or $R_\nu(y)$ are independent of y. Same as in the real case we have just considered, we lose nothing when assuming that all scalar perturbation blocks are 1×1, which allows us to treat these blocks as full. Thus, the general case can be reduced to the case where $\mathcal{I}_{\rm S} = \emptyset$, which we assume from now on (cf. section 3.3.1).

In order to derive a safe approximation of the RC of (3.3.28), we can act exactly in the same fashion as in the real case to arrive at the equivalence

$$\begin{array}{l} y^H A^H(\zeta)A(\zeta)y \leq \Re\{2y^H b(\zeta) + c(\zeta)\} \ \forall \zeta \in \mathcal{Z}_\rho \\ \Leftrightarrow \overbrace{\left[\begin{array}{c|c} \Re\{2y^H b^{\rm n} + c^{\rm n}\} & [A^{\rm n} y]^H \\ \hline A^{\rm n} y & I \end{array}\right]}^{\mathcal{A}(y)} \\ \quad + \sum_{\nu=1}^N \left[\begin{array}{c|c} \Re\{2y^H L_{\nu,b}(y)\zeta^\nu R_\nu(y) + L_{\nu,c}(y)\zeta^\nu R_\nu(y)\} & R_\nu^H[\zeta^\nu]^H L_{\nu,A}(y) \\ \hline L_{\nu,A}^H(y)\zeta^\nu R_\nu(y) & \end{array}\right] \succeq 0 \\ \qquad \forall (\zeta : \|\zeta^\nu\|_{2,2} \leq \rho, 1 \leq \nu \leq N) \end{array}$$

where $L_\nu(y) = [L_{\nu,A}(y), L_{\nu,b}(y), L_{\nu,c}(y)]$ and $L_{\nu,b}(y)$, $L_{\nu,c}(y)$ are the last two columns in $L_\nu(y)$.

Setting

$$\widetilde{L}_\nu^H(y) = \left[L_{\nu,b}^H(y) + \frac{1}{2}L_{\nu,c}^H(y); L_{\nu,A}^H(y)\right], \ \widetilde{R}_\nu(y) = [R_\nu(y), 0_{q_\nu \times k}]$$

(cf. (3.3.25)), we conclude that the RC of (3.3.28) is equivalent to the semi-infinite LMI

$$\begin{array}{l} \mathcal{A}(y) + \sum_{\nu=1}^N \left[\widetilde{L}_\nu^H(y)\zeta^\nu \widetilde{R}_\nu(y) + \widetilde{R}_\nu^H(y)[\zeta^\nu]^H \widetilde{L}_\nu(y)\right] \succeq 0 \\ \qquad \forall (\zeta : \|\zeta^\nu\|_{2,2} \leq \rho, 1 \leq \nu \leq N). \end{array} \tag{3.3.29}$$

As always, swapping, if necessary, ζ^ν and $[\zeta^\nu]^H$ we may rewrite the latter semi-infinite LMI equivalently as

$$\begin{array}{l} \mathcal{A}(y) + \sum_{\nu=1}^N \left[\widehat{L}_\nu^H \zeta^\nu \widehat{R}_\nu(y) + \widehat{R}_\nu^H(y)[\zeta^\nu]^H \widehat{L}_\nu\right] \succeq 0 \\ \qquad \forall (\zeta : \|\zeta^\nu\|_{2,2} \leq \rho, 1 \leq \nu \leq N), \end{array}$$

where $\widehat{R}_\nu(y)$ are affine in $[\Re(y); \Im(y)]$ and $\widehat{L}_\nu$ are nonzero. Applying the Complex case Matrix Cube Theorem (see the proof of Theorem 3.5), we finally arrive at the following result:

Theorem 3.8 *The explicit system of LMIs*

$$\begin{array}{l} \left[\begin{array}{c|c} Y_\nu - \lambda_\nu \widehat{L}_\nu^H \widehat{L}_\nu & \widehat{R}_\nu^H(y) \\ \hline \widehat{R}_\nu(y) & \lambda_\nu I_{k_\nu} \end{array}\right] \succeq 0, \ \nu = 1, ..., N, \\ \left[\begin{array}{c|c} \Re\{2y^H b^{\rm n} + c^{\rm n}\} & [A^{\rm n} y]^H \\ \hline A^{\rm n} y & I \end{array}\right] - \rho \sum_{\nu=1}^N Y_\nu \succeq 0 \end{array} \tag{3.3.30}$$

(k_ν is the number of rows in $\widehat{R}_\nu(y)$) in variables $Y_1 = Y_1^H, ..., Y_N = Y_N^H, \lambda_\nu \in \mathbb{R}, y \in \mathbb{C}^m$ is a safe tractable approximation of the RC of the uncertain convex quadratic inequality (3.3.28)*. The tightness of this approximation is $\leq \frac{4}{\pi}$, and is equal to 1 when $N = 1$.*

3.3.2 The Case of ∩-Ellipsoidal Uncertainty

Consider the case where the uncertainty in CQI (3.2.3) is side-wise with the right hand side uncertainty exactly as in section 3.2.2, and with *∩-ellipsoidal left hand side perturbation set*, that is,

$$\mathcal{Z}_\rho^{\text{left}} = \left\{\eta : \eta^T Q_j \eta \leq \rho^2,\ j = 1, ..., J\right\}, \tag{3.3.31}$$

where $Q_j \succeq 0$ and $\sum\limits_{j=1}^{J} Q_j \succ 0$. When $Q_j \succ 0$ for all j, $\mathcal{Z}_\rho^{\text{left}}$ is the intersection of J ellipsoids centered at the origin. When $Q_j = a_j a_j^T$ are rank 1 matrices, $\mathcal{Z}^{\text{left}}$ is a polyhedral set symmetric w.r.t. origin and given by J inequalities of the form $|a_j^T \eta| \leq \rho$, $j = 1, ..., J$. The requirement $\sum\limits_{j=1}^{J} Q_j \succ 0$ implies that $\mathcal{Z}_\rho^{\text{left}}$ is bounded (indeed, every $\eta \in \mathcal{Z}_\rho^{\text{left}}$ belongs to the ellipsoid $\eta^T(\sum_j Q_j)\eta \leq J\rho^2$).

We have seen in section 3.2.3 that the case $J = 1$, (i.e., of an ellipsoid $\mathcal{Z}_\rho^{\text{left}}$ centered at the origin), is a particular case of unstructured norm-bounded perturbation, so that in this case the RC is computationally tractable. The case of general ∩-ellipsoidal uncertainty includes the situation when $\mathcal{Z}_\rho^{\text{left}}$ is a box, where the RC is computationally intractable. However, we intend to demonstrate that with ∩-ellipsoidal left hand side perturbation set, the RC of (3.2.4) admits a safe tractable approximation tight within the "nearly constant" factor $\sqrt{(O(\ln J))}$.

Approximating the RC of Uncertain Least Squares Inequality

Same as in section 3.3.1, the side-wise nature of uncertainty reduces the task of approximating the RC of uncertain CQI (3.2.4) to a similar task for the RC of the uncertain Least Squares inequality (3.3.4). Representing

$$A(\zeta)y + b(\zeta) = \underbrace{[A^{\text{n}}y + b^{\text{n}}]}_{\beta(y)} + \underbrace{\sum_{\ell=1}^{L} \eta_\ell [A^\ell y + b^\ell]}_{\alpha(y)\eta} \tag{3.3.32}$$

where $L = \dim \eta$, observe that the RC of (3.3.4), (3.3.31) is equivalent to the system of constraints

$$\tau \geq 0 \ \&\ \|\beta(y) + \alpha(y)\eta\|_2^2 \leq \tau^2 \ \forall(\eta : \eta^T Q_j \eta \leq \rho^2,\ j = 1, ..., J)$$

or, which is clearly the same, to the system

$$\begin{array}{ll} (a) & \mathcal{A}_\rho \equiv \max\limits_{\eta,t} \left\{\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta : \eta^T Q_j \eta \leq \rho^2\,\forall j,\ t^2 \leq 1\right\} \\ & \leq \tau^2 - \beta^T(y)\beta(y) \\ (b) & \tau \geq 0. \end{array} \tag{3.3.33}$$

Next we use Lagrangian relaxation to derive the following result:

(!) Assume that for certain nonnegative reals γ, γ_j, $j = 1, ..., J$, the homogeneous quadratic form in variables η, t

$$\gamma t^2 + \sum_{j=1}^{J} \gamma_j \eta^T Q_j \eta - \left[\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta\right] \tag{3.3.34}$$

is nonnegative everywhere. Then

$$\begin{array}{l} \mathcal{A}_\rho \equiv \max\limits_{\eta,t} \left\{\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta : \eta^T Q_j \eta \leq \rho^2,\, t^2 \leq 1\right\} \\ \leq \gamma + \rho^2 \sum\limits_{j=1}^{J} \gamma_j. \end{array} \tag{3.3.35}$$

Indeed, let $F = \{(\eta, t) : \eta^T Q_j \eta \leq \rho^2, j = 1, ..., J, t^2 \leq 1\}$. We have

$$\begin{array}{rcl} \mathcal{A}_\rho & = & \max\limits_{(\eta,t)\in F} \left\{\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta\right\} \\ & \leq & \max\limits_{(\eta,t)\in F} \left\{\gamma t^2 + \sum\limits_{j=1}^{J} \gamma_j \eta^T Q_j \eta\right\} \\ & & \text{[since the quadratic form (3.3.34) is nonnegative everywhere]} \\ & \leq & \gamma + \rho^2 \sum\limits_{j=1}^{J} \gamma_j \\ & & \text{[due to the origin of } F \text{ and to } \gamma \geq 0,\ \gamma_j \geq 0]. \end{array}$$

From (!) it follows that if $\gamma \geq 0$, $\gamma_j \geq 0$, $j = 1, ..., J$ are such that the quadratic form (3.3.34) is nonnegative everywhere, or, which is the same, such that

$$\left[\begin{array}{c|c} \gamma & -\beta^T(y)\alpha(y) \\ \hline -\alpha^T(y)\beta(y) & \sum\limits_{j=1}^{J} \gamma_j Q_j - \alpha^T(y)\alpha(y) \end{array}\right] \succeq 0$$

and

$$\gamma + \rho^2 \sum_{j=1}^{J} \gamma_j \leq \tau^2 - \beta^T(y)\beta(y),$$

then (y, τ) satisfies (3.3.33.a). Setting $\nu = \gamma + \beta^T(y)\beta(y)$, we can rewrite this conclusion as follows: if there exist ν and $\gamma_j \geq 0$ such that

$$\left[\begin{array}{c|c} \nu - \beta^T(y)\beta(y) & -\beta^T(y)\alpha(y) \\ \hline -\alpha^T(y)\beta(y) & \sum\limits_{j=1}^{J} \gamma_j Q_j - \alpha^T(y)\alpha(y) \end{array}\right] \succeq 0$$

and

$$\nu + \rho^2 \sum_{j=1}^{J} \gamma_j \leq \tau^2,$$

then (y, τ) satisfies (3.3.33.a).

Assume for a moment that $\tau > 0$. Setting $\lambda_j = \gamma_j/\tau$, $\mu = \nu/\tau$, the above conclusion can be rewritten as follows: if there exist μ and $\lambda_j \geq 0$ such that

$$\left[\begin{array}{c|c} \mu - \tau^{-1}\beta^T(y)\beta(y) & -\tau^{-1}\beta^T(y)\alpha(y) \\ \hline -\tau^{-1}\alpha^T(y)\beta(y) & \sum\limits_{j=1}^{J} \lambda_j Q_j - \tau^{-1}\alpha^T(y)\alpha(y) \end{array}\right] \succeq 0$$

and

$$\mu + \rho^2 \sum_{j=1}^{J} \lambda_j \leq \tau,$$

then (y, τ) satisfies (3.3.33.a).

By the Schur Complement Lemma, the latter conclusion can further be reformulated as follows: if $\tau > 0$ and there exist μ, λ_j satisfying the relations

$$\begin{array}{ll} (a) & \left[\begin{array}{c|c|c} \mu & & \beta^T(y) \\ \hline & \sum\limits_{j=1}^{J} \lambda_j Q_j & \alpha^T(y) \\ \hline \beta(y) & \alpha(y) & \tau I \end{array}\right] \succeq 0 \\ (b) & \mu + \rho^2 \sum\limits_{j=1}^{J} \lambda_j \leq \tau \quad (c)\ \lambda_j \geq 0,\ j = 1, ..., J \end{array} \tag{3.3.36}$$

then (y, τ) satisfies (3.3.33.a). Note that in fact our conclusion is valid for $\tau \leq 0$ as well. Indeed, assume that $\tau \leq 0$ and μ, λ_j solve (3.3.36). Then clearly $\tau = 0$ and therefore $\alpha(y) = 0$, $\beta(y) = 0$, and thus (3.3.33.a) is valid. We have proved the first part of the following statement:

Theorem 3.9 *The explicit system of constraints* (3.3.36) *in variables y, τ, μ, $\lambda_1, ..., \lambda_J$ is a safe tractable approximation of the RC of the uncertain Least Squares constraint* (3.3.4) *with $\cap$-ellipsoidal perturbation set* (3.3.31)*. The approximation is exact when $J = 1$, and in the case of $J > 1$ the tightness factor of this approximation does not exceed*

$$\Omega(J) \leq 9.19\sqrt{\ln(J)}. \tag{3.3.37}$$

Proof. The fact that (3.3.36) is a safe approximation of the RC of (3.3.4), (3.3.31) is readily given by the reasoning preceding Theorem 3.9. To prove that the approximation is tight within the announced factor, note that the Approximate $\mathcal{S}$-Lemma (Theorem A.8) as applied to the quadratic forms in variables $x = [\eta; t]$

$$\begin{aligned} & x^T A x \equiv \left\{\eta^T \alpha^T(y)\alpha(y)\eta + 2t\beta^T(y)\alpha(y)\eta\right\},\ x^T B x \equiv t^2, \\ & x^T B_j x \equiv \eta^T Q_j \eta,\ 1 \leq j \leq J, \end{aligned}$$

states that if $J = 1$, then (y, τ) can be extended to a solution of (3.3.36) if and only if (y, τ) satisfies (3.3.33), that is, if and only if (y, τ) is robust feasible; thus, our approximation of the RC of (3.3.4), (3.3.31) is exact when $J = 1$. Now let $J > 1$, and suppose that (y, τ) cannot be extended to a feasible solution of (3.3.36). Due to the origin of this system, it follows that

$$\begin{aligned} & \mathrm{SDP}(\rho) \equiv \min_{\lambda, \{\lambda_j\}} \left\{ \lambda + \rho^2 \sum_{j=1}^{J} \lambda_j : \lambda B + \sum_j \lambda_j B_j \succeq A, \lambda \geq 0, \lambda_j \geq 0 \right\} \\ & > \tau^2 - \beta^T(y)\beta(y). \end{aligned} \tag{3.3.38}$$

By the Approximate $\mathcal{S}$-Lemma, with appropriately chosen $\Omega(J) \le 9.19\sqrt{\ln(J)}$ we have $\mathcal{A}_{\Omega(J)\rho} \ge \mathrm{SDP}(\rho)$, which combines with (3.3.38) to imply that $\mathcal{A}_{\Omega(J)\rho} > \tau^2 - \beta^T(y)\beta(y)$, meaning that (y,τ) is not robust feasible at the uncertainty level $\Omega(J)\rho$ (cf. (3.3.33)). Thus, the tightness factor of our approximation does not exceed $\Omega(J)$. □

From Uncertain Least Squares to Uncertain CQI

The next statement can obtained from Theorem 3.9 in the same fashion as Theorem 3.6 has been derived from Theorem 3.4.

Theorem 3.10 *Consider uncertain CQI* (3.2.3) *with side-wise uncertainty, where the left hand side perturbation set is the* $\cap$*-ellipsoidal set* (3.3.31)*, and the right hand side perturbation set is as in Theorem 3.6. For* $\rho > 0$*, the explicit system of LMIs*

$$
\begin{array}{ll}
(a) & \tau + \rho p^T v \le \delta(y),\ P^T v = \sigma(y),\ Q^T v = 0,\ v \in \mathbf{K}_* \\
(b.1) & \left[\begin{array}{c|c|c} \mu & & \beta^T(y) \\ \hline & \sum_{j=1}^J \lambda_j Q_j & \alpha^T(y) \\ \hline \beta(y) & \alpha(y) & I \end{array}\right] \succeq 0 \\
(b.2) & \mu + \rho^2 \sum_{j=1}^J \lambda_j \le \tau,\ \lambda_j \succeq 0\,\forall j
\end{array}
\tag{3.3.39}
$$

in variables $y, v, \mu, \lambda_j, \tau$ *is a safe tractable approximation of the RC of the uncertain CQI. This approximation is exact when* $J = 1$ *and is tight within the factor* $\Omega(J) \le 9.19\sqrt{\ln(J)}$ *when* $J > 1$.

Convex Quadratic Constraint with $\cap$-Ellipsoidal Uncertainty

Now consider approximating the RC of an uncertain convex quadratic inequality

$$
\begin{array}{c}
y^T A^T(\zeta) A(\zeta) y \le 2y^T b(\zeta) + c(\zeta) \\
\left[(A(\zeta), b(\zeta), c(\zeta)) = (A^{\mathrm{n}}, b^{\mathrm{n}}, c^{\mathrm{n}}) + \sum_{\ell=1}^L \zeta_\ell (A^\ell, b^\ell, c^\ell)\right]
\end{array}
\tag{3.3.40}
$$

with $\cap$-ellipsoidal uncertainty:

$$
\mathcal{Z}_\rho = \rho \mathcal{Z}_1 = \{\zeta \in \mathbb{R}^L : \zeta^T Q_j \zeta \le \rho^2\} \quad [Q_j \succeq 0, \sum_j Q_j \succ 0]
\tag{3.3.41}
$$

Observe that

$$
\begin{array}{rcl}
A(\zeta)y & = & \alpha(y)\zeta + \beta(y), \\
& & \alpha(y)\zeta = [A^1 y, ..., A^L y],\ \beta(y) = A^{\mathrm{n}} y \\
2y^T b(\zeta) + c(\zeta) & = & 2\sigma^T(y)\zeta + \delta(y), \\
& & \sigma(y) = [y^T b^1 + c^1; ...; y^T b^L + c^L],\ \delta(y) = y^T b^{\mathrm{n}} + c^{\mathrm{n}}
\end{array}
\tag{3.3.42}
$$

so that the RC of (3.3.40), (3.3.41) is the semi-infinite inequality

$$
\zeta^T \alpha^T(y)\alpha(y)\zeta + 2\zeta^T\left[\alpha^T(y)\beta(y) - \sigma(y)\right] \le \delta(y) - \beta^T(y)\beta(y)\ \forall \zeta \in \mathcal{Z}_\rho,
$$

or, which is the same, the semi-infinite inequality

$$\begin{array}{l}\mathcal{A}_\rho(y) \equiv \max\limits_{\zeta\in\mathcal{Z}_\rho, t^2\le 1} \zeta^T\alpha^T(y)\alpha(y)\zeta + 2t\zeta^T\left[\alpha^T(y)\beta(y)-\sigma(y)\right]\\ \le \delta(y)-\beta^T(y)\beta(y).\end{array} \tag{3.3.43}$$

Same as in section 3.3.2, we have

$$\begin{array}{rcl}\mathcal{A}_\rho(y) & \le & \inf\limits_{\lambda,\{\lambda_j\}}\left\{\lambda+\rho^2\sum\limits_{j=1}^J\lambda_j : \begin{array}{l}\lambda\ge 0, \lambda_j\ge 0, j=1,...,J\\ \forall(t,\zeta):\\ \lambda t^2+\zeta^T(\sum\limits_{j=1}^J\lambda_jQ_j)\zeta \ge \zeta^T\alpha^T(y)\alpha(y)\zeta\\ \qquad +2t\zeta^T\left[\alpha^T(y)\beta(y)-\sigma(y)\right]\end{array}\right\}\\ & = & \inf\limits_{\lambda,\{\lambda_j\}}\left\{\lambda+\rho^2\sum\limits_{j=1}^J\lambda_j : \lambda\ge 0, \lambda_j\ge 0, j=1,...,J,\right.\\ & & \left.\left[\begin{array}{c|c}\lambda & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j\lambda_jQ_j-\alpha^T(y)\alpha(y)\end{array}\right]\succeq 0\right\}.\end{array} \tag{3.3.44}$$

We conclude that the condition

$$\exists(\lambda\ge 0,\{\lambda_j\ge 0\}): \left\{\begin{array}{l}\lambda+\rho^2\sum\limits_{j=1}^J\lambda_j\le\delta(y)-\beta^T(y)\beta(y)\\ \left[\begin{array}{c|c}\lambda & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j\lambda_jQ_j-\alpha^T(y)\alpha(y)\end{array}\right]\succeq 0\end{array}\right.$$

is sufficient for y to be robust feasible. Setting $\mu=\lambda+\beta^T(y)\beta(y)$, this sufficient condition can be rewritten equivalently as

$$\exists(\{\lambda_j\ge 0\},\mu): \left\{\begin{array}{l}\mu+\rho^2\sum\limits_{j=1}^J\lambda_j\le\delta(y)\\ \left[\begin{array}{c|c}\mu-\beta^T(y)\beta(y) & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j\lambda_jQ_j-\alpha^T(y)\alpha(y)\end{array}\right]\succeq 0\end{array}\right. \tag{3.3.45}$$

We have

$$\begin{array}{l}\left[\begin{array}{c|c}\mu-\beta^T(y)\beta(y) & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j\lambda_jQ_j-\alpha^T(y)\alpha(y)\end{array}\right]\\ = \left[\begin{array}{c|c}\mu & \sigma^T(y)\\ \hline \sigma(y) & \sum\limits_{j=1}^J\lambda_jQ_j\end{array}\right]-\left[\begin{array}{c}\beta^T(y)\\ \alpha^T(y)\end{array}\right]\left[\begin{array}{c}\beta^T(y)\\ \alpha^T(y)\end{array}\right]^T,\end{array}$$

so that the Schur Complement Lemma says that

$$\begin{array}{l}\left[\begin{array}{c|c}\mu-\beta^T(y)\beta(y) & -[\beta^T(y)\alpha(y)-\sigma^T(y)]\\ \hline -[\alpha^T(y)\beta(y)-\sigma(y)] & \sum\limits_j\lambda_jQ_j-\alpha^T(y)\alpha(y)\end{array}\right]\succeq 0\\ \Leftrightarrow \left[\begin{array}{c|c|c}\mu & \sigma^T(y)] & \beta^T(y)\\ \hline \sigma(y) & \sum\limits_j\lambda_jQ_j & \alpha^T(y)\\ \hline \beta(y) & \alpha(y) & I\end{array}\right]\succeq 0.\end{array}$$

The latter observation combines with the fact that (3.3.45) is a sufficient condition for the robust feasibility of y to yield the first part of the following statement:

Theorem 3.11 *The explicit system of LMIs in variables y, μ, λ_j:*

$$\begin{array}{ll} (a) & \left[\begin{array}{c|c|c} \mu & \sigma^T(y)] & \beta^T(y) \\ \hline \sigma(y) & \sum\limits_j \lambda_j Q_j & \alpha^T(y) \\ \hline \beta(y) & \alpha(y) & I \end{array}\right] \succeq 0 \\ (b) & \mu + \rho^2 \sum\limits_{j=1}^{J} \lambda_j \leq \delta(y) \quad (c)\ \lambda_j \geq 0,\ j = 1, ..., J \end{array} \tag{3.3.46}$$

(for notation, see (3.3.42)*) is a safe tractable approximation of the RC of* (3.3.40), (3.3.41). *The tightness factor of this approximation equals 1 when $J = 1$ and does not exceed $\Omega(J) \leq 9.19\sqrt{\ln(J)}$ when $J > 1$.*

The proof of this theorem is completely similar to the proof of Theorem 3.9.

3.4 Uncertain Semidefinite Problems with Tractable RCs

In this section, we focus on uncertain Semidefinite Optimization (SDO) problems for which tractable Robust Counterparts can be derived.

3.4.1 Uncertain Semidefinite Problems

Recall that a *semidefinite program* (SDP) is a conic optimization program

$$\begin{array}{c} \min\limits_x \left\{ c^T x + d : \mathcal{A}_i(x) \equiv \sum\limits_{j=1}^{n} x_j A^{ij} - B_i \in \mathbf{S}_+^{k_i},\ i = 1, ..., m \right\} \\ \Updownarrow \\ \min\limits_x \left\{ c^T x + d : \mathcal{A}_i(x) \equiv \sum\limits_{j=1}^{n} x_j A^{ij} - B_i \succeq 0,\ i = 1, ..., m \right\} \end{array} \tag{3.4.1}$$

where A^{ij}, B_i are symmetric matrices of sizes $k_i \times k_i$, $\mathbf{S}_+^k$ is the cone of real symmetric positive semidefinite $k \times k$ matrices, and $A \succeq B$ means that A, B are symmetric matrices of the same sizes such that the matrix $A - B$ is positive semidefinite. A constraint of the form $\mathcal{A}x - B \equiv \sum\limits_j x_j A^j - B \succeq 0$ with symmetric A^j, B is called a *Linear Matrix Inequality* (LMI); thus, an SDP is the problem of minimizing a linear objective under finitely many LMI constraints. Another, sometimes more convenient, setting of a semidefinite program is in the form of (3.1.2), that is,

$$\min_x \left\{ c^T x + d : A_i x - b_i \in \mathbf{Q}_i,\ i = 1, ..., m \right\}, \tag{3.4.2}$$

where nonempty sets $\mathbf{Q}_i$ are given by explicit finite lists of LMIs:

$$\mathbf{Q}_i = \{u \in \mathbb{R}^{p_i} : \mathcal{Q}_{i\ell}(u) \equiv \sum_{s=1}^{p_i} u_s Q^{si\ell} - Q^{i\ell} \succeq 0,\ \ell = 1, ..., L_i\}.$$

Note that (3.4.1) is a particular case of (3.4.2) where $\mathbf{Q}_i = \mathbf{S}_+^{k_i}$, $i = 1, ..., m$.

The notions of the *data* of a semidefinite program, of an *uncertain* semidefinite problem and of its (exact or approximate) *Robust Counterparts* are readily given by specializing the general descriptions from sections 3.1, 3.1.4, to the case when the underlying cones are the cones of positive semidefinite matrices. In particular,

• The *natural data* of a semidefinite program (3.4.2) is the collection

$$(c, d, \{A_i, b_i\}_{i=1}^m),$$

while the right hand side sets $\mathbf{Q}_i$ are treated as the problem's structure;

• An *uncertain* semidefinite problem is a collection of problems (3.4.2) with common structure and natural data running through an *uncertainty set*; we always assume that the data are affinely parameterized by *perturbation vector* $\zeta \in \mathbb{R}^L$ running through a given closed and convex *perturbation set* $\mathcal{Z}$ such that $0 \in \mathcal{Z}$:

$$\begin{array}{rcl} [c; d] & = & [c^{\rm n}; d^{\rm n}] + \sum\limits_{\ell=1}^{L} \zeta_\ell [c^\ell; d^\ell]; \\ [A_i, b_i] & = & [A_i^{\rm n}, b_i^{\rm n}] + \sum\limits_{\ell=1}^{L} \zeta_\ell [A_i^\ell, b_i^\ell],\ i = 1, ..., m \end{array} \tag{3.4.3}$$

• The Robust Counterpart of uncertain SDP (3.4.2), (3.4.3) at a perturbation level $\rho > 0$ is the semi-infinite optimization program

$$\min_{y=(x,t)} \left\{ t : \left. \begin{array}{l} [[c^{\rm n}]^T x + d^{\rm n}] + \sum\limits_{\ell=1}^{L} \zeta_\ell [[c^\ell]^T x + d^\ell] \le t \\ [A_i^{\rm n} x + b_i^{\rm n}] + \sum\limits_{\ell=1}^{L} \zeta_\ell [A_i^\ell x + b_i^\ell] \in \mathbf{Q}_i,\ i = 1, ..., m \end{array} \right\} \forall \zeta \in \rho \mathcal{Z} \right\} \tag{3.4.4}$$

• A *safe tractable approximation* of the RC of uncertain SDP (3.4.2), (3.4.3) is a finite system $\mathcal{S}_\rho$ of explicitly computable convex constraints in variables $y = (x, t)$ (and possibly additional variables u) depending on $\rho > 0$ as a parameter, such that the projection $\widehat{Y}_\rho$ of the solution set of the system onto the space of y variables is contained in the feasible set Y_ρ of (3.4.4). Such an approximation is called *tight* within factor $\vartheta \ge 1$, if $Y_\rho \supset \widehat{Y}_\rho \supset Y_{\vartheta\rho}$. In other words, $\mathcal{S}_\rho$ is a ϑ-tight safe approximation of (3.4.4), if:

1. Whenever $\rho > 0$ and y are such that y can be extended, by a properly chosen u, to a solution of $\mathcal{S}_\rho$, y is robust feasible at the uncertainty level ρ, (i.e., y is feasible for (3.4.4)).

2. Whenever $\rho > 0$ and y are such that y can*not* be extended to a feasible solution to $\mathcal{S}_\rho$, y is *not* robust feasible at the uncertainty level $\vartheta\rho$, (i.e., y violates some of the constraints in (3.4.4) when ρ is replaced with $\vartheta\rho$).

3.4.2 Tractability of RCs of Uncertain Semidefinite Problems

Building the RC of an uncertain semidefinite problem reduces to building the RCs of the uncertain constraints constituting the problem, so that the tractability issues in Robust Semidefinite Optimization reduce to those for the Robust Counterpart

$$\mathcal{A}_\zeta(y) \equiv \mathcal{A}^{\rm n}(y) + \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}_\ell(y) \succeq 0\ \forall \zeta \in \rho\mathcal{Z} \tag{3.4.5}$$

of a single uncertain LMI

$$\mathcal{A}_\zeta(y) \equiv \mathcal{A}^{\rm n}(y) + \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}_\ell(y) \succeq 0; \tag{3.4.6}$$

here $\mathcal{A}^n(x)$, $\mathcal{A}_\ell(x)$ are symmetric matrices affinely depending on the design vector y.

More often than not the RC of an uncertain LMI is computationally intractable. Indeed, we saw in section 3 that intractability is typical already for the RCs of uncertain conic quadratic inequalities, and the latter are very special cases of uncertain LMIs (due to the fact that Lorentz cones are cross-sections of semidefinite cones, see Lemma 3.1). In the relatively simple case of uncertain CQIs, we met just 3 generic cases where the RCs were computationally tractable, specifically, the cases of

1. Scenario perturbation set (section 3.2.1);
2. Unstructured norm-bounded uncertainty (section 3.2.3);
3. Simple ellipsoidal uncertainty (section 3.2.5).

The RC associated with a scenario perturbation set is tractable for an arbitrary uncertain conic problem on a tractable cone; in particular, the RC of an uncertain LMI with scenario perturbation set is computationally tractable. Specifically, if $\mathcal{Z}$ in (3.4.5) is given as $\mathrm{Conv}\{\zeta^1, ..., \zeta^N\}$, then the RC (3.4.5) is nothing but the explicit system of LMIs

$$\mathcal{A}^{\rm n}(y) + \sum_{\ell=1}^{L} \zeta_\ell^i \mathcal{A}_\ell(y) \succeq 0,\ i = 1, ..., N. \tag{3.4.7}$$

The fact that the simple ellipsoidal uncertainty ($\mathcal{Z}$ is an ellipsoid) results in a tractable RC is specific for Conic Quadratic Optimization. In the LMI case, (3.4.5) can be NP-hard even with an ellipsoid in the role of $\mathcal{Z}$. In contrast to this, the case of unstructured norm-bounded perturbations remains tractable in the LMI situation. This is the only nontrivial tractable case we know. We are about to consider this case in full details.

Unstructured Norm-Bounded Perturbations

Definition 3.5 *We say that uncertain LMI* (3.4.6) *is with unstructured norm-bounded perturbations, if*

1. *The perturbation set $\mathcal{Z}$ (see* (3.4.3)*) is the set of all $p \times q$ matrices ζ with the usual matrix norm $\|\cdot\|_{2,2}$ not exceeding 1;*
2. *"The body" $\mathcal{A}_\zeta(y)$ of* (3.4.6) *can be represented as*

$$\mathcal{A}_\zeta(y) \equiv \mathcal{A}^{\rm n}(y) + \left[L^T(y)\zeta R(y) + R^T(y)\zeta^T L(y)\right], \tag{3.4.8}$$

where both $L(\cdot)$, $R(\cdot)$ are affine and at least one of these matrix-valued functions is in fact independent of y.

Example 3.3 Consider the situation where $\mathcal{Z}$ is the unit Euclidean ball in $\mathbb{R}^L$ (or, which is the same, the set of $L \times 1$ matrices of $\|\cdot\|_{2,2}$-norm not exceeding 1), and

$$\mathcal{A}_\zeta(y) = \left[\begin{array}{c|c} a(y) & \zeta^T B^T(y) + b^T(y) \\ \hline B(y)\zeta + b(y) & A(y) \end{array}\right], \tag{3.4.9}$$

where $a(\cdot)$ is an affine scalar function, and $b(\cdot)$, $B(\cdot)$, $A(\cdot)$ are affine vector- and matrix-valued functions with $A(\cdot) \in \mathbf{S}^M$. Setting $R(y) \equiv R = [1, 0_{1\times M}]$, $L(y) = [0_{L\times 1}, B^T(y)]$, we have

$$\mathcal{A}_\zeta(y) = \underbrace{\left[\begin{array}{c|c} a(y) & b^T(y) \\ \hline b(y) & A(y) \end{array}\right]}_{\mathcal{A}^{\rm n}(y)} + L^T(y)\zeta R(y) + R^T(y)\zeta^T L(y),$$

thus, we are in the case of an unstructured norm-bounded uncertainty.

A closely related example is given by the LMI reformulation of an uncertain Least Squares inequality with unstructured norm-bounded uncertainty, see section 3.2.3.

Let us derive a tractable reformulation of an uncertain LMI with unstructured norm-bounded uncertainty. W.l.o.g. we may assume that $R(y) \equiv R$ is independent of y (otherwise we can swap ζ and ζ^T, swapping simultaneously L and R) and that $R \neq 0$. We have

$$\begin{array}{cl}
 & y \text{ is robust feasible for (3.4.6), (3.4.8) at uncertainty level } \rho \\
\Leftrightarrow & \xi^T[\mathcal{A}^{\rm n}(y) + L^T(y)\zeta R + R^T\zeta^T L(y)]\xi \geq 0 \quad \forall \xi \quad \forall (\zeta : \|\zeta\|_{2,2} \leq \rho) \\
\Leftrightarrow & \xi^T\mathcal{A}^{\rm n}(y)\xi + 2\xi^T L^T(y)\zeta R\xi \geq 0 \quad \forall \xi \quad \forall (\zeta : \|\zeta\|_{2,2} \leq \rho) \\
\Leftrightarrow & \xi^T\mathcal{A}^{\rm n}(y)\xi + 2\underbrace{\min_{\|\zeta\|_{2,2}\leq\rho}\xi^T L^T(y)\zeta R\xi}_{=-\rho\|L(y)\xi\|_2\|R\xi\|_2} \geq 0 \quad \forall \xi \\
\Leftrightarrow & \xi^T\mathcal{A}^{\rm n}(y)\xi - 2\rho\|L(y)\xi\|_2\|R\xi\|_2 \geq 0 \ \forall \xi \\
\Leftrightarrow & \xi^T\mathcal{A}^{\rm n}(y)\xi + 2\rho\eta^T L(y)\xi \geq 0 \quad \forall (\xi, \eta : \eta^T\eta \leq \xi^T R^T R\xi) \\
\Leftrightarrow & \exists \lambda \geq 0 : \left[\begin{array}{c|c} & \rho L(y) \\ \hline \rho L^T(y) & \mathcal{A}^{\rm n}(y) \end{array}\right] \succeq \lambda \left[\begin{array}{c|c} -I_p & \\ \hline & R^T R \end{array}\right] \ \ [\mathcal{S}\text{-Lemma}] \\
\Leftrightarrow & \exists \lambda : \left[\begin{array}{c|c} \lambda I_p & \rho L(y) \\ \hline \rho L^T(y) & \mathcal{A}^{\rm n}(y) - \lambda R^T R \end{array}\right] \succeq 0.
\end{array}$$

We have proved the following statement:

Theorem 3.12 *The RC*

$$\mathcal{A}^{\rm n}(y) + L^T(y)\zeta R + R^T\zeta^T L(y) \succeq 0 \quad \forall (\zeta \in \mathbb{R}^{p\times q} : \|\zeta\|_{2,2} \leq \rho) \tag{3.4.10}$$

of uncertain LMI (3.4.6) *with unstructured norm-bounded uncertainty* (3.4.8) *(where, w.l.o.g., we assume that $R \neq 0$) can be represented equivalently by the LMI*

$$\left[\begin{array}{c|c} \lambda I_p & \rho L(y) \\ \hline \rho L^T(y) & \mathcal{A}^{\rm n}(y) - \lambda R^T R \end{array}\right] \succeq 0 \tag{3.4.11}$$

in variables y, λ.

Application: Robust Structural Design

Structural Design problem. Consider a "linearly elastic" mechanical system S that, mathematically, can be characterized by:

1. A linear space $\mathbb{R}^M$ of *virtual displacements* of the system.
2. A symmetric positive semidefinite $M \times M$ matrix A, called the *stiffness matrix* of the system.

The potential energy capacitated by the system when its displacement from the equilibrium is v is

$$E = \frac{1}{2} v^T A v.$$

An external load applied to the system is given by a vector $f \in \mathbb{R}^M$. The associated *equilibrium displacement* v of the system solves the linear equation

$$Av = f.$$

If this equation has no solutions, the load destroys the system — no equilibrium exists; if the solution is not unique, so is the equilibrium displacement. Both these "bad phenomena" can occur only when A is not positive definite.

The *compliance* of the system under a load f is the potential energy capacitated by the system in the equilibrium displacement v associated with f, that is,

$$\mathrm{Compl}_f(A) = \frac{1}{2} v^T A v = \frac{1}{2} v^T f.$$

An equivalent way to define compliance is as follows. Given external load f, consider the concave quadratic form

$$f^T v - \frac{1}{2} v^T A v$$

on the space $\mathbb{R}^M$ of virtual displacements. It is easily seen that this form either is unbounded above, (which is the case when no equilibrium displacements exist), or attains its maximum. In the latter case, the compliance is nothing but the maximal value of the form:

$$\mathrm{Compl}_f(A) = \sup_{v \in \mathbb{R}^M} \left[f^T v - \frac{1}{2} v^T A v \right],$$

and the equilibrium displacements are exactly the maximizers of the form.

There are good reasons to treat the compliance as the measure of rigidity of the construction with respect to the corresponding load — the less the compliance, the higher the rigidity. A typical *Structural Design* problem is as follows:

Structural Design: *Given*

- *the space $\mathbb{R}^M$ of virtual displacements of the construction,*
- *the stiffness matrix $A = A(t)$ affinely depending on a vector t of design parameters restricted to reside in a given convex compact set $\mathcal{T} \subset \mathbb{R}^N$,*
- *a set $\mathcal{F} \subset \mathbb{R}^M$ of external loads,*

find a construction t_* that is as rigid as possible w.r.t. the "most dangerous" load from $\mathcal{F}$, that is,

$$t_* \in \operatorname*{Argmin}_{T \in \mathcal{T}} \left\{ \mathrm{Compl}_{\mathcal{F}}(t) \equiv \sup_{f \in \mathcal{F}} \mathrm{Compl}_f(A(t)) \right\}.$$

Next we present three examples of Structural Design.

Example 3.4 Truss Topology Design. A *truss* is a mechanical construction, like railroad bridge, electric mast, or the Eiffel Tower, comprised of thin elastic *bars* linked to each other at *nodes*. Some of the nodes are partially or completely fixed, so that their virtual displacements form proper subspaces in $\mathbb{R}^2$ (for planar constructions) or $\mathbb{R}^3$ (for spatial ones). An external load is a collection of external forces acting at the nodes. Under such a load, the nodes move slightly, thus causing elongations and compressions in the bars, until the construction achieves an equilibrium, where the tensions caused in the bars as a result of their deformations compensate the external forces. The compliance is the potential energy capacitated in the truss at the equilibrium as a result of deformations of the bars.

A mathematical model of the outlined situation is as follows.

• *Nodes and the space of virtual displacements.* Let $\mathcal{M}$ be the nodal set, that is, a finite set in $\mathbb{R}^d$ ($d = 2$ for planar and $d = 3$ for spatial trusses), and let $V_i \subset \mathbb{R}^d$ be the linear space of virtual displacements of node i. (This set is the entire $\mathbb{R}^d$ for non-supported nodes, is $\{0\}$ for fixed nodes and is something in-between these two extremes for partially fixed nodes.) The space $V = \mathbb{R}^M$ of virtual displacements of the truss is the direct product $V = V_1 \times ... \times V_m$ of the spaces of virtual displacements of the nodes, so that a virtual displacement of the truss is a collection of "physical" virtual displacements of the nodes.

Now, an external load applied to the truss can be thought of as a collection of external physical forces $f_i \in \mathbb{R}^d$ acting at nodes i from the nodal set. We lose nothing when assuming that $f_i \in V_i$ for all i, since the component of f_i orthogonal to V_i is fully compensated by the supports that make the directions from V_i the only possible displacements of node i. Thus, we can always assume that $f_i \in V_i$ for all i, which makes it possible to identify a load with a vector $f \in V$. Similarly, the collection of nodal reaction forces caused by elongations and compressions of the bars can be thought of as a vector from V.

• *Bars and the stiffness matrix.* Every bar j, $j = 1, ..., N$, in the truss links two nodes from the nodal set $\mathcal{M}$. Denoting by t_j the volume of the j-th bar, a simple analysis, (where one assumes that the nodal displacements are small and neglects all terms of order of squares of these displacements), demonstrates that the collection of the reaction forces caused by a nodal displacement $v \in V$ can be represented as $A(t)v$, where

$$A(t) = \sum_{j=1}^{N} t_j b_j b_j^T \tag{3.4.12}$$

is the stiffness matrix of the truss. Here $b_j \in V$ is readily given by the characteristics of the material of the j-th bar and the "nominal," (i.e., in the unloaded truss), positions of the nodes linked by this bar.

In a typical Truss Topology Design (TTD) problem, one is given a *ground structure* — a set $\mathcal{M}$ of tentative nodes along with the corresponding spaces V_i of virtual displacements and the list $\mathcal{J}$ of N tentative bars, (i.e., a list of pairs of nodes that could be linked by bars), and the characteristics of the bar's material; these data determine, in particular, the vectors b_j. The design variables are the volumes t_j of the tentative bars. The design specifications always include the natural restrictions $t_j \geq 0$ and an upper bound w on $\sum_j t_j$, (which, essentially, is an upper bound on the total weight of the truss). Thus, $\mathcal{T}$ is always a subset of the standard simplex $\{t \in \mathbb{R}^N : t \geq 0, \sum_j t_j \leq w\}$. There could be other design specifications, like upper and lower bounds on the volumes of some bars. The scenario set $\mathcal{F}$ usually is either a singleton (*single-load TTD*) or a small collection of external loads (*multi-load TTD*). With this setup, one seeks for a design $t \in \mathcal{T}$, that results in the smallest possible worst case, i.e., maximal over the loads from $\mathcal{F}$ compliance.

When formulating a TTD problem, one usually starts with a dense nodal set and allows for all pair connections of the tentative nodes by bars. At an optimal solution to the associated TTD problem, usually a pretty small number of bars get positive volumes, so that the solution recovers not only the optimal bar sizing, but also the optimal topology of the construction.

Example 3.5 Free Material Optimization. In Free Material Optimization (FMO) one seeks to design a mechanical construction comprised of material continuously distributed over a given 2-D or 3-D domain Ω, and the mechanical properties of the material are allowed to vary from point to point. The ultimate goal of the design is to build a construction satisfying a number of constraints (most notably, an upper bound on the total weight) and most rigid w.r.t. loading scenarios from a given sample.

After finite element discretization, this (originally infinite-dimensional) optimization problem becomes a particular case of the aforementioned Structural Design problem where:

- the space $V = \mathbb{R}^M$ of virtual displacements is the space of "physical displacements" of the vertices of the finite element cells, so that a displacement $v \in V$ is a collection of displacements $v_i \in \mathbb{R}^d$ of the vertices ($d = 2$ for planar and $d = 3$ for spatial constructions). Same as in the TTD problem, displacements of some of the vertices can be restricted to reside in proper linear subspaces of $\mathbb{R}^d$;

- external loads are collections of physical forces applied at the vertices of the finite element cells; same as in the TTD case, these collections can be identified with vectors $f \in V$;

- the stiffness matrix is of the form

$$A(t) = \sum_{j=1}^{N} \sum_{s=1}^{S} b_{js} t_j b_{js}^T, \tag{3.4.13}$$

  where N is the number of finite element cells and t_j is the *stiffness tensor* of the material in the j-th cell. This tensor can be identified with a $p \times p$ symmetric positive semidefinite matrix, where $p = 3$ for planar constructions and $p = 6$ for spatial ones. The number S and the $M \times p$ matrices b_{is} are readily given by the geometry of the finite element cells and the type of finite element discretization.

In a typical FMO problem, one is given the number of the finite element cells along with the matrices b_{ij} in (3.4.13), and a collection $\mathcal{F}$ of external loads of interest. The design vectors are collections $t = (t_1, ..., t_N)$ of positive semidefinite $p \times p$ matrices, and the design specifications always include the natural restrictions $t_j \succeq 0$ and an upper bound $\sum_j c_j \mathrm{Tr}(t_j) \le w$, $c_j > 0$, on the total weighted trace of t_j; this bound reflects, essentially, an upper bound on the total weight of the construction. Along with these restrictions, the description of the feasible design set $\mathcal{T}$ can include other constraints, such as bounds on the spectra of t_j, (i.e., lower bounds on the minimal and upper bounds on the maximal eigenvalues of t_j). With this setup, one seeks for a design $t \in \mathcal{T}$ that results in the smallest worst case, (i.e., the maximal over the loads from $\mathcal{F}$) compliance.

The design yielded by FMO usually cannot be implemented "as it is" — in most cases, it would be either impossible, or too expensive to use a material with mechanical properties varying from point to point. The role of FMO is in providing an engineer with an "educated guess" of what the optimal construction could possibly be; given this guess, engineers produce something similar from composite materials, applying existing design tools that take into account finer design specifications, (which may include nonconvex ones), than those taken into consideration by the FMO design model.

Our third example, due to C. Roos, has nothing in common with mechanics — it is about design of electrical circuits. Mathematically, however, it is modeled as a Structural Design problem.

Example 3.6 Consider an electrical circuit comprised of resistances and sources of current. Mathematically, such a circuit can be thought of as a graph with nodes $1,...,n$ and a set E of oriented arcs. Every arc γ is assigned with its *conductance* $\sigma_\gamma \geq 0$ (so that $1/\sigma_\gamma$ is the resistance of the arc). The nodes are equipped with external sources of current, so every node i is assigned with a real number f_i — the current supplied by the source. The steady state functioning of the circuit is characterized by currents $\jmath_\gamma$ in the arcs and potentials v_i at the nodes, (these potentials are defined up to a common additive constant). The potentials and the currents can be found from the Kirchhoff laws, specifically, as follows. Let G be the node-arc incidence matrix, so that the columns in G are indexed by the nodes, the rows are indexed by the arcs, and $G_{\gamma i}$ is 1, -1 or 0, depending on whether the arc γ starts at node i, ends at this node, or is not incident to the node, respectively. The first Kirchhoff law states that sum of all currents in the arcs leaving a given node minus the sum of all currents in the arcs entering the node is equal to the external current at the node. Mathematically, this law reads

$$G^T \jmath = f,$$

where $f = (f_1, ..., f_n)$ and $\jmath = \{\jmath_\gamma\}_{\gamma \in E}$ are the vector of external currents and the vector of currents in the arcs, respectively. The second law states that the current in an arc γ is σ_γ times the arc voltage — the difference of potentials at the nodes linked by the arc. Mathematically, this law reads

$$\jmath = \Sigma G v, \Sigma = \mathrm{Diag}\{\sigma_\gamma, \gamma \in E\}.$$

Thus, the potentials are given by the relation

$$G^T \Sigma G v = f.$$

Now, the heat H dissipated in the circuit is the sum, over the arcs, of the products of arc currents and arc voltages, that is,

$$H = \sum_\gamma \sigma_\gamma ((Gv)_\gamma)^2 = v^T G^T \Sigma G v.$$

In other words, the heat dissipated in the circuit, the external currents forming a vector f, is the maximum of the convex quadratic form

$$2v^T f - v^T G^T \Sigma G v$$

over all $v \in \mathbb{R}^n$, and the steady state potentials are exactly the maximizers of this quadratic form. In other words, the situation is as if we were speaking about a mechanical system with stiffness matrix $A(\sigma) = G^T \Sigma G$ affinely depending on the vector $\sigma \geq 0$ of arc conductances subject to external load f, with the steady-state potentials in the role of equilibrium displacements, and the dissipated heat in this state in the role of (twice) the compliance.

It should be noted that the "stiffness matrix" in our present situation is degenerate — indeed, we clearly have $G\mathbf{1} = 0$, where $\mathbf{1}$ is the vector of ones, ("when the potentials of all nodes are equal, the currents in the arcs should be zero"), whence $A(\sigma)\mathbf{1} = 0$ as well. As a result, the necessary condition for the steady state to exist is $f^T\mathbf{1} = 0$, that is, the total sum of all external currents should be zero — a fact we could easily foresee. Whether this necessary condition is also sufficient depends on the topology of the circuit.

A straightforward “electrical” analogy of the Structural Design problem would be to build a circuit of a given topology, (i.e., to equip the arcs of a given graph with nonnegative conductances forming a design vector σ), satisfying specifications $\sigma \in \mathcal{S}$ in a way that minimizes the maximal steady-state dissipated heat, the maximum being taken over a given family $\mathcal{F}$ of vectors of external currents.

Structural Design as an uncertain Semidefinite problem. The aforementioned Structural Design problem can be easily posed as an SDP. The key element in the transformation of the problem is the following semidefinite representation of the compliance:

$$\mathrm{Compl}_f(A) \leq \tau \Leftrightarrow \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A \end{array}\right] \succeq 0. \tag{3.4.14}$$

Indeed,

$$\begin{array}{rl} & \mathrm{Compl}_f(A) \leq \tau \\ \Leftrightarrow & f^T v - \frac{1}{2} v^T A v \geq \tau \quad \forall v \in \mathbb{R}^M \\ \Leftrightarrow & 2\tau s^2 - 2 s f^T v + v^T A v \geq 0 \quad \forall ([v,s] \in \mathbb{R}^{M+1}) \\ \Leftrightarrow & \left[\begin{array}{c|c} 2\tau & -f^T \\ \hline -f & A \end{array}\right] \succeq 0 \\ \Leftrightarrow & \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A \end{array}\right] \succeq 0 \end{array}$$

where the last $\Leftrightarrow$ follows from the fact that

$$\left[\begin{array}{c|c} 2\tau & -f^T \\ \hline -f & A \end{array}\right] = \left[\begin{array}{c|c} 1 & \\ \hline & -I \end{array}\right] \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A \end{array}\right] \left[\begin{array}{c|c} 1 & \\ \hline & -I \end{array}\right]^T.$$

Thus, the Structural Design problem can be posed as

$$\min_{\tau,t} \left\{ \tau : \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \quad \forall f \in \mathcal{F}, t \in \mathcal{T} \right\}. \tag{3.4.15}$$

Assuming that the set $\mathcal{T}$ of feasible designs is LMI representable, problem (3.4.15) is nothing but the RC of the uncertain semidefinite problem

$$\min_{\tau,t} \left\{ \tau : \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0,\, t \in \mathcal{T} \right\}, \tag{3.4.16}$$

where the only uncertain data is the load f, and this data varies in a given set $\mathcal{F}$ (or, which is the same, in its closed convex hull $\mathrm{cl}\,\mathrm{Conv}(\mathcal{F})$). Thus, in fact we are speaking about the RC of a *single-load* Structural Design problem, with the load in the role of uncertain data varying in the uncertainty set $\mathcal{U} = \mathrm{cl}\,\mathrm{Conv}(\mathcal{F})$.

In actual design the set $\mathcal{F}$ of loads of interest is finite and usually quite small. For example, when designing a bridge for cars, an engineer is interested in a quite restricted family of scenarios, primarily in the load coming from many cars uniformly distributed along the bridge (this is, essentially, what happens in rush hours), and, perhaps, in a few other scenarios (like loads coming from a single heavy car in various positions). With finite $\mathcal{F} = \{f^1, ..., f^k\}$, we are in the situation of a scenario uncertainty, and the RC of (3.4.16) is the explicit semidefinite program

$$\min_{\tau,t} \left\{ \tau : \left[\begin{array}{c|c} 2\tau & [f^i]^T \\ \hline f^i & A(t) \end{array}\right] \succeq 0, i = 1, ..., k,\, t \in \mathcal{T} \right\}.$$

Note, however, that in reality the would-be construction will be affected by small "occasional" loads (like side wind in the case of a bridge), and the construction should be stable with respect to these loads. It turns out, however, that the latter requirement is not necessarily satisfied by the "nominal" construction that takes into consideration only the loads of primary interest. As an instructive example, consider the design of a console.

Example 3.7 Figure 3.3.(c) represents optimal single-load design of a console with a 9×9 nodal grid on 2-D plane; nodes from the very left column are fixed, the remaining nodes are free, and the single scenario load is the unit force f acting down and applied at the mid-node of the very right column (see figure 3.3.(a)). We allow nearly all tentative bars (numbering 2,039), except for (clearly redundant) bars linking fixed nodes or long bars that pass through more than two nodes and thus can be split into shorter ones (figure 3.3.(b)). The set $\mathcal{T}$ of admissible designs is given solely by the weight restriction:

$$\mathcal{T} = \{t \in \mathbb{R}^{2039} : t \geq 0, \sum_{i=1}^{2039} t_i \leq 1\}$$

(compliance is homogeneous of order 1 w.r.t. t: $\mathrm{Compl}_f(\lambda t) = \lambda \mathrm{Compl}_f(t)$, $\lambda > 0$, so we can normalize the weight bound to be 1).

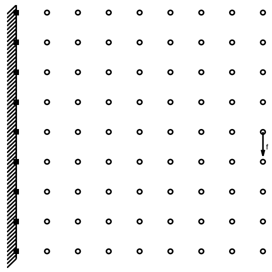
The compliance, in an appropriate scale, of the resulting nominally optimal truss (12 nodes, 24 bars) w.r.t. the scenario load f is 1.00. At the same time, the construction turns out to be highly unstable w.r.t. small "occasional" loads distributed along the 10 free nodes used by the nominal design. For example, the mean compliance of the nominal design w.r.t. a random load $h \sim \mathcal{N}(0, 10^{-9} I_{20})$ is 5.406 (5.4 times larger than the nominal compliance), while the "typical" norm $\|h\|_2$ of this random load is $10^{-4.5}\sqrt{20}$ — more than three orders of magnitude less than the norm $\|f\|_2 = 1$ of the scenario load. The compliance of the nominally optimal truss w.r.t. a "bad" load g that is 10^4 times smaller than f ($\|g\|_2 = 10^{-4}\|f\|_2$) is 27.6 — by factor 27 larger than the compliance w.r.t. f! Figure 3.3.(e) shows the deformation of the nominal design under the load $10^{-4}g$ (that is, the load that is 10^8 (!) times smaller than the scenario load). One can compare this deformation with the one under the load f (figure 3.3.(d)). Figure 3.3.(f) depicts shifts of the nodes under a sample of 100 random loads $h \sim \mathcal{N}(0, 10^{-16} I_{20})$ — loads of norm by 7 plus orders of magnitude less than $\|f\|_2 = 1$.

To prevent the optimal design from being crushed by a small load that is outside of the set $\mathcal{F}$ of loading scenarios, it makes sense to extend $\mathcal{F}$ to a more "massive" set, primarily by adding to $\mathcal{F}$ all loads of magnitude not exceeding a given "small" uncertainty level ρ. A challenge here is to decide where the small loads can be applied. In problems like TTD, it does not make sense to require the would-be construction to be capable of carrying small loads distributed along *all* nodes of the ground structure; indeed, not all of these nodes should be present in the final design, and of course there is no reason to bother about forces acting at non-existing nodes. The difficulty is that we do not know in advance which nodes will be present in the final design. One possibility to resolve this difficulty to some extent is to use a two-stage procedure as follows:

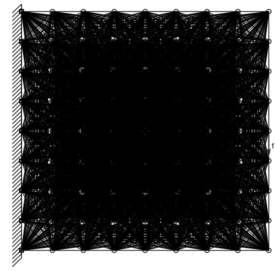
• at the first stage, we seek for the "nominal" design — the one that is optimal w.r.t. the "small" set $\mathcal{F}$ comprised of the scenario loads and, perhaps, all loads of magnitude $\leq \rho$ acting along the same nodes as the scenario loads — these nodes definitely will be present in the resulting design;

• at the second stage, we solve the problem again, with the nodes actually used by the nominal design in the role of our new nodal set $\mathcal{M}^+$, and extend $\mathcal{F}$ to the set $\mathcal{F}^+$ by taking the union of $\mathcal{F}$ and the Euclidean ball B_ρ of all loads g, $\|g\|_2 \leq \rho$, acting along $\mathcal{M}^+$.

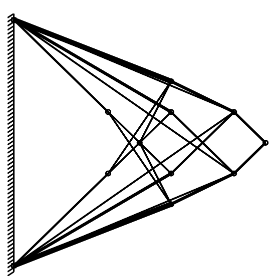
We have arrived at the necessity to solve (3.4.15) in the situation where $\mathcal{F}$ is the union of a finite set $\{f^1, ..., f^k\}$ and a Euclidean ball. This is a particular case of the situation when $\mathcal{F}$ is



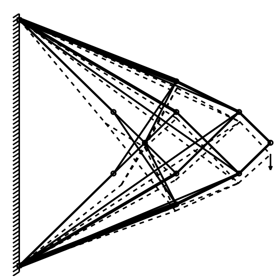
(a): 9×9 nodal grid with most left nodes fixed and the load of interest. $M =$ 144 degrees of freedom.



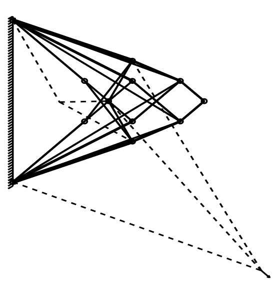
(b): 2,039 tentative bars



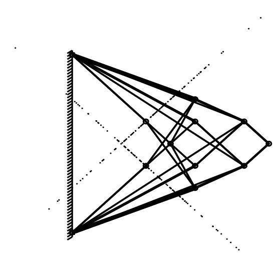
(c): Single-load optimal design, 12 nodes, 24 bars. Compliance w.r.t. load of interest 1.00.



(d): Deformation of nominal design under the load of interest.



(e): Deformation of nominal design under "occasional" load 10^8 times less than the load of interest.



(f): "Dotted lines": positions of nodes in deformed nominal design, sample of 100 loads $\sim \mathcal{N}(0, 10^{-16} I_{20})$

Figure 3.3: Nominal design.

the union of $S < \infty$ ellipsoids

$$E_s = \{f = f^s + B_s\zeta^s : \zeta^s \in \mathbb{R}^{k_s}, \|\zeta^s\|_2 \le 1\}$$

or, which is the same, $\mathcal{Z}$ is the convex hull of the union of S ellipsoids $E_1, ..., E_S$. The associated "uncertainty-immunized" Structural Design problem (3.4.15) — the RC of (3.4.16) with $\mathcal{Z}$ in the role of $\mathcal{F}$ — is clearly equivalent to the problem

$$\min_{t,\tau}\left\{\tau : \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t)\end{array}\right] \succeq 0 \quad \forall f \in E_s,\, s = 1, ..., S; t \in \mathcal{T}\right\}. \tag{3.4.17}$$

In order to build a tractable equivalent of this semi-infinite semidefinite problem, we need to build a tractable equivalent to a semi-infinite LMI of the form

$$\left[\begin{array}{c|c} 2\tau & \zeta^T B^T + f^T \\ \hline B\zeta + f & A(t)\end{array}\right] \succeq 0 \quad \forall (\zeta \in \mathbb{R}^k : \|\zeta\|_2 \le \rho). \tag{3.4.18}$$

But such an equivalent is readily given by Theorem 3.12 (cf. Example 3.3). Applying the recipe described in this Theorem, we end up with a representation of (3.4.18) as the following LMI in variables τ, t, λ:

$$\left[\begin{array}{c|c|c} \lambda I_k & & \rho B^T \\ \hline & 2\tau - \lambda & f^T \\ \hline \rho B & f & A(t)\end{array}\right] \succeq 0. \tag{3.4.19}$$

Observe that when $f = 0$, (3.4.19) simplifies to

$$\left[\begin{array}{c|c} 2\tau I_k & \rho B^T \\ \hline \rho B & A(t)\end{array}\right] \succeq 0. \tag{3.4.20}$$

Example 3.7 continued. Let us apply the outlined methodology to the Console example (Example 3.7). In order to immunize the design depicted on figure 3.3.(c) against small occasional loads, we start with reducing the initial 9×9 nodal set to the set of 12 nodes $\mathcal{M}^+$ (figure 3.4.(a)) used by the nominal design, and allow for $N = 54$ tentative bars on this reduced nodal set (figure 5.1.(b)) (we again allow for all pair connections of nodes, except for connections of two fixed nodes and for long bars passing through more than two nodes). According to the outlined methodology, we should then extend the original singleton $\mathcal{F} = \{f\}$ of scenario loads to the larger set $\mathcal{F}^+ = \{f\} \cup B_\rho$, where B_ρ is the Euclidean ball of radius ρ, centered at the origin in the ($M = 20$)-dimensional space of virtual displacements of the reduced planar nodal set. With this approach, an immediate question would be how to specify ρ. In order to avoid an ad hoc choice of ρ, we modify our approach as follows. Recalling that the compliance of the nominally optimal design w.r.t. the scenario load is 1.00, let us impose on our would-be "immunized" design the restriction that its worst case compliance w.r.t. the extended scenario set $\mathcal{F}_\rho = \{f\} \cup B_\rho$ should be at most $\tau_* = 1.025$, (i.e., 2.5% more than the optimal nominal compliance), and maximize under this restriction the radius ρ. In other words, we seek for a truss of the same unit weight as the nominally optimal one with "nearly optimal" rigidity w.r.t. the scenario load f and as large as possible worst-case rigidity w.r.t. occasional loads of a given magnitude. The resulting problem is the semi-infinite semidefinite program

$$\max_{t,\rho}\left\{\rho : \begin{array}{l} \left[\begin{array}{c|c} 2\tau_* & f^T \\ \hline f & A(t)\end{array}\right] \succeq 0 \\ \left[\begin{array}{c|c} 2\tau_* & \rho h^T \\ \hline \rho h & A(t)\end{array}\right] \succeq 0 \quad \forall (h : \|h\|_2 \le 1) \\ t \succeq 0, \sum_{i=1}^N t_i \le 1 \end{array}\right\}.$$

This semi-infinite program is equivalent to the usual semidefinite program

$$\max_{t,\rho}\left\{\rho:\ \begin{array}{c}\left[\begin{array}{c|c}2\tau_* & f^T\\ \hline f & A(t)\end{array}\right]\succeq 0\\ \left[\begin{array}{c|c}2\tau_* I_M & \rho I_M\\ \hline \rho I_M & A(t)\end{array}\right]\succeq 0\\ t\succeq 0,\sum_{i=1}^N t_i\leq 1\end{array}\right\} \tag{3.4.21}$$

(cf. (3.4.20)).

Computation shows that for Example 3.7, the optimal value in (3.4.21) is $\rho_* = 0.362$; the *robust design* yielded by the optimal solution to the problem is depicted in figure 3.4.(c). Along with the differences in sizing of bars, note the difference in the structures of the robust and the nominal design (figure 3.5). Observe that passing from the nominal to the robust design, we lose just 2.5% in the rigidity w.r.t. the scenario load and gain a dramatic improvement in the capability to carry occasional loads. Indeed, the compliance of the robust truss w.r.t. *every* load g of the magnitude $\|g\|_2 = 0.36$ (36% of the magnitude of the load of interest) is at most 1.025; the similar quantity for the nominal design is as large as 1.65×10^9 ! An additional evidence of the dramatic advantages of the robust design as compared to the nominal one can be obtained by comparing the pictures (d) through (f) in figure 3.3 with their counterparts in figure 3.4.

Applications in Robust Control

A major source of uncertain Semidefinite problems is Robust Control. An instructive example is given by Lyapunov Stability Analysis/Synthesis.

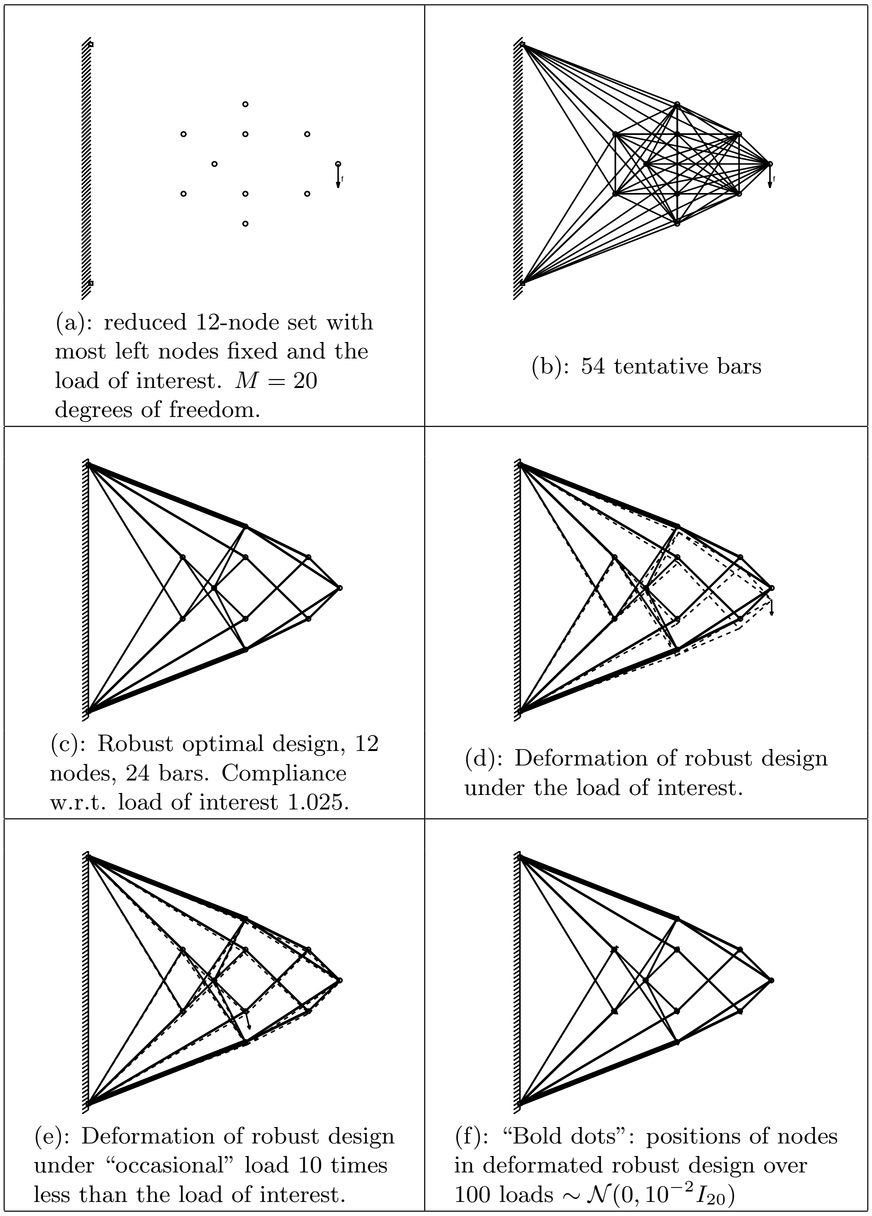
Lyapunov Stability Analysis. Consider a time-varying linear dynamical system "closed" by a linear output-based feedback:

$$\begin{array}{ll}(a) & \dot{x}(t) = A_t x(t) + B_t u(t) + R_t d_t \text{ [open loop system, or } \textit{plant}]\\ (b) & y(t) = C_t x(t) + D_t d_t \text{ [output]}\\ (c) & u(t) = K_t y(t) \text{ [output-based feedback]}\\ & \Downarrow\\ (d) & \dot{x}(t) = [A_t + B_t K_t C_t]x(t) + [R_t + B_t K_t D_t]d_t \text{ [closed loop system]}\end{array} \tag{3.4.22}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $d_t\in\mathbb{R}^p$, $y(t)\in\mathbb{R}^q$ are respectively, the state, the control, the external disturbance, and the output at time t, A_t, B_t, R_t, C_t, D_t are matrices of appropriate sizes specifying the dynamics of the system; and K_t is the feedback matrix. We assume that the dynamical system in question is *uncertain*, meaning that we do not know the dependencies of the matrices A_t,...,K_t on t; all we know is that the collection $M_t = (A_t, B_t, C_t, D_t, R_t, K_t)$ of all these matrices stays all the time within a given compact uncertainty set $\mathcal{M}$. For our further purposes, it makes sense to think that there exists an underlying time-invariant "nominal" system corresponding to known nominal values $A^{\rm n}$,...,$K^{\rm n}$ of the matrices $A_t, ..., K_t$, while the actual dynamics corresponds to the case when the matrices drift (perhaps, in a time-dependent fashion) around their nominal values.

An important desired property of a linear dynamical system is its *stability* — the fact that every state trajectory $x(t)$ of (every realization of) the closed loop system converges to 0 as $t\to\infty$, provided that the external disturbances d_t are identically zero. For a time-invariant linear system

$$\dot{x} = Q^{\rm n}x,$$



(a): reduced 12-node set with most left nodes fixed and the load of interest. $M = 20$ degrees of freedom.

(b): 54 tentative bars

(c): Robust optimal design, 12 nodes, 24 bars. Compliance w.r.t. load of interest 1.025.

(d): Deformation of robust design under the load of interest.

(e): Deformation of robust design under "occasional" load 10 times less than the load of interest.

(f): "Bold dots": positions of nodes in deformated robust design over 100 loads $\sim \mathcal{N}(0, 10^{-2} I_{20})$

Figure 3.4: Robust design.

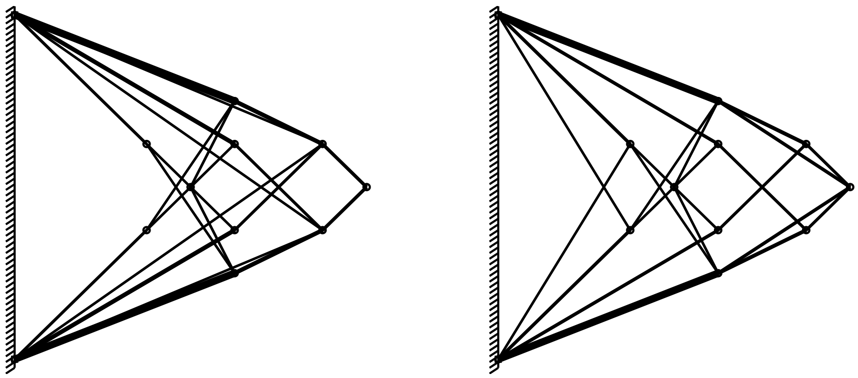


Figure 3.5: Nominal (left) and robust (right) designs.

the necessary and sufficient stability condition is that all eigenvalues of A have negative real parts or, equivalently, that there exists a *Lyapunov Stability Certificate* (LSC) — a positive definite symmetric matrix X such that

$$[Q^{\rm n}]^T X + XQ^{\rm n} \prec 0.$$

For uncertain system (3.4.22), a *sufficient* stability condition is that all matrices

$$Q \in \mathcal{Q} = \{Q = A^M + B^M K^M C^M : M \in \mathcal{M}\}$$

have a common LSC X, that is, there exists $X \succ 0$ such that

$$\begin{array}{ll} (a) & Q^T X + XQ^T \prec 0 \quad \forall Q \in \mathcal{Q} \\ & \Updownarrow \\ (b) & [A^M + B^M K^M C^M]^T X + X[A^M + B^M K^M C^M] \prec 0 \quad \forall M \in \mathcal{M}; \end{array} \tag{3.4.23}$$

here A^M,...,K^M are the components of a collection $M \in \mathcal{M}$.

The fact that the existence of a common LSC for all matrices $Q \in \mathcal{Q}$ is sufficient for the stability of the closed loop system is nearly evident. Indeed, since $\mathcal{M}$ is compact, for every feasible solution $X \succ 0$ of the semi-infinite LMI (3.4.23) one has

$$\forall M \in \mathcal{M} : [A^M + B^M K^M C^M]^T X + X[A^M + B^M K^M C^M] \prec -\alpha X \tag{$*$}$$

with appropriate $\alpha > 0$. Now let us look what happens with the quadratic form $x^T X x$ along a state trajectory $x(t)$ of (3.4.22). Setting $f(t) = x^T(t)Xx(t)$ and invoking (3.4.22.d), we have

$$\begin{array}{rl} f'(t) & = \dot{x}^T(t)Xx(t) + x(t)X\dot{x}(t) \\ & = x^T(t)\left[[A_t + B_t K_t C_t]^T X + X[A_t + B_t K_t C_t]\right] x(t) \\ & \le -\alpha f(t), \end{array}$$

where the concluding inequality is due to ($*$). From the resulting differential inequality

$$f'(t) \le -\alpha f(t)$$

it follows that

$$f(t) \le \exp\{-\alpha t\} f(0) \to 0, \ t \to \infty.$$

Recalling that $f(t) = x^T(t)Xx(t)$ and X is positive definite, we conclude that $x(t) \to 0$ as $t \to \infty$.

Observe that the set $\mathcal{Q}$ is compact along with $\mathcal{M}$. It follows that X is an LSC if and only if $X \succ 0$ and

$$\exists \beta > 0 : Q^T X + XQ \preceq -\beta I \quad \forall Q \in \mathcal{Q}$$

$$\Leftrightarrow \quad \exists \beta > 0 : Q^T X + XQ \preceq -\beta I \quad \forall Q \in \text{Conv}(\mathcal{Q}).$$

Multiplying such an X by an appropriate positive real, we can ensure that

$$X \succeq I \ \& \ Q^T X + XQ \preceq -I \quad \forall Q \in \text{Conv}(\mathcal{Q}). \tag{3.4.24}$$

Thus, we lose nothing when requiring from an LSC to satisfy the latter system of (semi-infinite) LMIs, and from now on LSCs in question will be exactly the solutions of this system.

Observe that (3.4.24) is nothing but the RC of the uncertain system of LMIs

$$X \succeq I \ \& \ Q^T X + XQ \preceq -I, \tag{3.4.25}$$

the uncertain data being Q and the uncertainty set being $\text{Conv}(\mathcal{Q})$. Thus, RCs arise naturally in the context of Robust Control.

Now let us apply the results on tractability of the RCs of uncertain LMI in order to understand when the question of existence of an LSC for a given uncertain system (3.4.22) can be posed in a computationally tractable form. There are, essentially, two such cases — *polytopic* and *unstructured norm-bounded* uncertainty.

Polytopic uncertainty. By definition, polytopic uncertainty means that the set $\text{Conv}(\mathcal{Q})$ is given as a convex hull of an explicit list of "scenarios" Q^i, $i = 1, ..., N$:

$$\text{Conv}(\mathcal{Q}) = \text{Conv}\{Q^1, ..., Q^N\}.$$

In our context this situation occurs when the components A^M, B^M, C^M, K^M of $M \in \mathcal{M}$ run, independently of each other, through convex hulls of respective scenarios

$$S_A = \text{Conv}\{A^1, ..., A^{N_A}\}, S_B = \text{Conv}\{B^1, ..., B^{N_B}\},$$
$$S_C = \text{Conv}\{C^1, ..., C^{N_C}\}, S_K = \text{Conv}\{K^1, ..., K^{N_K}\};$$

in this case, the set $\text{Conv}(\mathcal{Q})$ is nothing but the convex hull of $N = N_A N_B N_C N_K$ "scenarios" $Q^{ijk\ell} = A^i + B^j K^\ell C^k$, $1 \leq i \leq N_A$,...,$1 \leq \ell \leq N_K$.

Indeed, $\mathcal{Q}$ clearly contains all matrices $Q^{ijk\ell}$ and therefore $\text{Conv}(\mathcal{Q}) \supset \text{Conv}(\{Q^{ijk\ell}\})$. On the other hand, the mapping $(A, B, C, K) \mapsto A + BKC$ is polylinear, so that the image $\mathcal{Q}$ of the set $S_A \times S_B \times S_C \times S_K$ under this mapping is contained in the convex set $\text{Conv}(\{Q^{ijk\ell}\})$, whence $\text{Conv}(\{Q^{ijk\ell}\}) \supset \text{Conv}(\mathcal{Q})$.

In the case in question we are in the situation of scenario perturbations, so that (3.4.25) is equivalent to the explicit system of LMIs

$$X \succeq I, [Q^i]^T X + XQ^i \preceq -I, \ i = 1, ..., N.$$

Unstructured norm-bounded uncertainty. Here

$$\text{Conv}(\mathcal{Q}) = \{Q = Q^{\text{n}} + U\zeta V : \zeta \in \mathbb{R}^{p \times q}, \|\zeta\|_{2,2} \leq \rho\}.$$

In our context this situation occurs, e.g., when 3 of the 4 matrices A^M, B^M, C^M, K^M, $M \in \mathcal{M}$, are in fact certain, and the remaining matrix, say, A^M, runs through a set of the form $\{A^{\text{n}} + G\zeta H : \zeta \in \mathbb{R}^{p \times q}, \|\zeta\|_{2,2} \leq \rho\}$.

In the case of unstructured norm-bounded uncertainty, the semi-infinite LMI in (3.4.25) is of the form

$$Q^T X + XQ \preceq -I \quad \forall Q \in \mathrm{Conv}(\mathcal{Q})$$
$$\Updownarrow$$
$$\underbrace{-I - [Q^{\mathrm{n}}]^T X - XQ^{\mathrm{n}}}_{\mathcal{A}^{\mathrm{n}}(X)} + [\underbrace{-XU}_{L^T(X)} \zeta \underbrace{V}_{R} + R^T \zeta^T L(X)] \succeq 0$$
$$\forall (\zeta \in \mathbb{R}^{p\times q}, \|\zeta\|_{2,2} \leq \rho).$$

Invoking Theorem 3.12, (3.4.25) is equivalent to the explicit system of LMIs

$$X \succeq I, \left[\begin{array}{c|c} \lambda I_p & \rho U^T X \\ \hline \rho XU & -I - [Q^{\mathrm{n}}]^T X - XQ^{\mathrm{n}} - \lambda V^T V \end{array}\right] \succeq 0. \tag{3.4.26}$$

in variables X, λ.

Lyapunov Stability Synthesis. We have considered the *Stability Analysis* problem, where one, given an uncertain closed-loop dynamical system along with the associated uncertainty set $\mathcal{M}$, seeks to verify a sufficient stability condition. A more challenging problem is *Stability Synthesis*: given an uncertain open loop system (3.4.22.*a*–*b*) along with the associated compact uncertainty set $\widehat{\mathcal{M}}$ in the space of collections $\widehat{M} = (A, B, C, D, R)$, find a linear output-based feedback

$$u(t) = Ky(t)$$

and an LSC for the resulting closed loop system.

The Synthesis problem has a nice solution, due to [22], in the case of *state-based* feedback (that is, $C_t \equiv I$) and under the assumption that the feedback is implemented exactly, so that the state dynamics of the closed loop system is given by

$$\dot{x}(t) = [A_t + B_t K]x(t) + [R_t + B_t K D_t]d_t. \tag{3.4.27}$$

The pairs (K, X) of "feedback – LSC" that we are looking for are exactly the feasible solutions to the system of semi-infinite matrix inequalities in variables X, K:

$$X \succ 0 \ \& \ [A + BK]^T X + X[A + BK] \prec 0 \ \ \forall [A, B] \in \mathcal{AB}; \tag{3.4.28}$$

here $\mathcal{AB}$ is the projection of $\widehat{M}$ on the space of $[A, B]$ data. The difficulty is that the system is *nonlinear* in the variables. As a remedy, let us carry out the nonlinear substitution of variables $X = Y^{-1}$, $K = ZY^{-1}$. With this substitution, (3.4.28) becomes a system in the new variables Y, Z:

$$Y \succ 0 \ \& \ [A + BZY^{-1}]^T Y^{-1} + Y^{-1}[A + BZY^{-1}] \prec 0 \ \ \forall [A, B] \in \mathcal{AB};$$

multiplying both sides of the second matrix inequality from the left and from the right by Y, we convert the system to the equivalent form

$$Y \succ 0, \ \& \ AY + YA^T + BZ + Z^T B^T \prec 0 \quad \forall [A, B] \in \mathcal{AB}.$$

Since $\mathcal{AB}$ is compact along with $\widehat{\mathcal{M}}$, the solutions to the latter system are exactly the pairs (Y, Z) that can be obtained by scaling $(Y, Z) \mapsto (\lambda Y, \lambda Z)$, $\lambda > 0$, from the solutions to the system of semi-infinite LMIs

$$Y \succeq I \ \& \ AY + YA^T + BZ + Z^T B^T \preceq -I \quad \forall [A, B] \in \mathcal{AB} \tag{3.4.29}$$

in variables Y, Z. When the uncertainty $\mathcal{AB}$ can be represented either as a polytopic, or as unstructured norm-bounded, the system (3.4.29) of semi-infinite LMIs admits an equivalent tractable reformulation.

3.5 Approximating RCs of Uncertain Semidefinite Problems

3.5.1 Tight Tractable Approximations of RCs of Uncertain SDPs with Structured Norm-Bounded Uncertainty

We have seen that the possibility to reformulate the RC of an uncertain semidefinite program in a computationally tractable form is a "rare commodity," so that there are all reasons to be interested in the second best thing — in situations where the RC admits a tight tractable approximation. To the best of our knowledge, just one such case is known — the case of *structured norm-bounded uncertainty* we are about to consider in this section.

Uncertain LMI with Structured Norm-Bounded Perturbations

Consider an uncertain LMI

$$\mathcal{A}_\zeta(y) \succeq 0 \tag{3.4.6}$$

where the "body" $\mathcal{A}_\zeta(y)$ is bi-linear in the design vector y and the perturbation vector ζ. The definition of a structured norm-bounded perturbation follows the path we got acquainted with in section 3:

Definition 3.6 *We say that the uncertain constraint* (3.4.6) *is affected by structured norm-bounded uncertainty with uncertainty level ρ, if*

1. The perturbation set $\mathcal{Z}_\rho$ is of the form

$$\mathcal{Z}_\rho = \left\{ \zeta = (\zeta^1, ..., \zeta^L) : \begin{array}{l} \zeta^\ell \in \mathbb{R}, |\zeta^\ell| \leq \rho, \ \ell \in \mathcal{I}_\mathrm{s} \\ \zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell} : \|\zeta^\ell\|_{2,2} \leq \rho, \ \ell \notin \mathcal{I}_\mathrm{s} \end{array} \right\} \tag{3.5.1}$$

2. The body $\mathcal{A}_\zeta(y)$ of the constraint can be represented as

$$\begin{array}{rcl} \mathcal{A}_\zeta(y) & = & \mathcal{A}^\mathrm{n}(y) + \sum\limits_{\ell \in \mathcal{I}_\mathrm{S}} \zeta^\ell \mathcal{A}_\ell(y) \\ & & + \sum\limits_{\ell \notin \mathcal{I}_\mathrm{s}} \left[L_\ell^T(y) \zeta^\ell R_\ell + R_\ell^T [\zeta^\ell]^T L_\ell(y) \right], \end{array} \tag{3.5.2}$$

where $\mathcal{A}_\ell(y)$, $\ell \in \mathcal{I}_\mathrm{s}$, and $L_\ell(y)$, $\ell \notin \mathcal{I}_\mathrm{s}$, are affine in y, and R_ℓ, $\ell \notin \mathcal{I}_\mathrm{s}$, are nonzero.

Theorem 3.13 *Given uncertain LMI* (3.4.6) *with structured norm-bounded uncertainty* (3.5.1), (3.5.2)*, let us associate with it the following system of LMIs in variables Y_ℓ, $\ell = 1, ..., L$, λ_ℓ, $\ell \notin \mathcal{I}_\mathrm{s}$, y:*

$$\begin{array}{ll} (a) & Y_\ell \succeq \pm \mathcal{A}_\ell(y), \ \ell \in \mathcal{I}_\mathrm{s} \\ (b) & \left[\begin{array}{c|c} \lambda_\ell I_{p_\ell} & L_\ell(y) \\ \hline L_\ell^T(y) & Y_\ell - \lambda_\ell R_\ell^T R_\ell \end{array} \right] \succeq 0, \ \ell \notin \mathcal{I}_\mathrm{s} \\ (c) & \mathcal{A}^\mathrm{n}(y) - \rho \sum\limits_{\ell=1}^{L} Y_\ell \succeq 0 \end{array} \tag{3.5.3}$$

Then system (3.5.3) *is a safe tractable approximation of the RC*

$$\mathcal{A}_\zeta(y) \succeq 0 \ \forall \zeta \in \mathcal{Z}_\rho \tag{3.5.4}$$

of (3.4.6), (3.5.1), (3.5.2)*, and the tightness factor of this approximation does not exceed* $\vartheta(\mu)$*, where* μ *is the smallest integer* ≥ 2 *such that* $\mu \geq \max_y \mathrm{Rank}(\mathcal{A}_\ell(y))$ *for all* $\ell \in \mathcal{I}_\mathrm{s}$*, and* $\vartheta(\cdot)$ *is a universal function of* μ *such that*

$$\vartheta(2) = \frac{\pi}{2},\ \vartheta(4) = 2,\ \vartheta(\mu) \leq \pi\sqrt{\mu/2},\ \mu > 2.$$

The approximation is exact, if either $L = 1$*, or all perturbations are scalar, (i.e.,* $\mathcal{I}_\mathrm{s} = \{1, ..., L\}$*) and all* $\mathcal{A}_\ell(y)$ *are of ranks not exceeding 1.*

Proof. Let us fix y and observe that a collection $y, Y_1, ..., Y_L$ can be extended to a feasible solution of (3.5.3) if and only if

$$\forall \zeta \in \mathcal{Z}_\rho : \left\{ \begin{array}{l} -\rho Y_\ell \preceq \zeta^\ell \mathcal{A}_\ell(y),\ \ell \in \mathcal{I}_\mathrm{s}, \\ -\rho Y_\ell \preceq L_\ell^T(y)\zeta^\ell R_\ell + R_\ell^T[\zeta^\ell]^T L_\ell(y),\ \ell \not\in \mathcal{I}_\mathrm{s} \end{array} \right.$$

(see Theorem 3.12). It follows that if, in addition, Y_ℓ satisfy (3.5.3.c), then y is feasible for (3.5.4), so that (3.5.3) is a safe tractable approximation of (3.5.4). The fact that this approximation is tight within the factor $\vartheta(\mu)$ is readily given by the real case Matrix Cube Theorem (Theorem A.7). The fact that the approximation is exact when $L = 1$ is evident when $\mathcal{I}_\mathrm{s} = \{1\}$ and is readily given by Theorem 3.12 when $\mathcal{I}_\mathrm{s} = \emptyset$. The fact that the approximation is exact when all perturbations are scalar and all matrices $\mathcal{A}_\ell(y)$ are of ranks not exceeding 1 is evident. □

Application: Lyapunov Stability Analysis/Synthesis Revisited

We start with the Analysis problem. Consider the uncertain time-varying dynamical system (3.4.22) and assume that the uncertainty set $\mathrm{Conv}(\mathcal{Q}) = \mathrm{Conv}(\{A^M + B^M K^M C^M\} : M \in \mathcal{M}\})$ in (3.4.24) is an *interval* uncertainty, meaning that

$$\begin{array}{c} \mathrm{Conv}(\mathcal{Q}) = Q^\mathrm{n} + \rho\mathcal{Z},\ \mathcal{Z} = \{\sum_{\ell=1}^L \zeta_\ell U_\ell : \|\zeta\|_\infty \leq 1\}, \\ \mathrm{Rank}(U_\ell) \leq \mu,\ 1 \leq \ell \leq L. \end{array} \tag{3.5.5}$$

Such a situation (with $\mu = 1$) arises, e.g., when two of the 3 matrices B_t, C_t, K_t are certain, and the remaining one of these 3 matrices, say, K_t, and the matrix A_t are affected by entry-wise uncertainty:

$$\{(A^M, K^M) : M \in \mathcal{M}\} = \left\{(A, K) : \begin{array}{l} |A_{ij} - A^\mathrm{n}_{ij}| \leq \rho\alpha_{ij} \forall (i,j) \\ |K_{pq} - K^\mathrm{n}_{pq}| \leq \rho\kappa_{pq}\,\forall (p,q) \end{array} \right\},$$

In this case, denoting by B^n, C^n the (certain!) matrices B_t, C_t, we clearly have

$$\begin{array}{rl} \mathrm{Conv}(\mathcal{Q}) = & \underbrace{A^\mathrm{n} + B^\mathrm{n}K^\mathrm{n}C^\mathrm{n}}_{Q^\mathrm{n}} + \rho\Big\{ [\sum_{i,j} \xi_{ij}[\alpha_{ij} e_i e_j^T] \\ & + \sum_{p,q} \eta_{pq}[\kappa_{pq} B^\mathrm{n} f_p g_q^T C^\mathrm{n}]] : |\xi_{ij}| \leq 1, |\eta_{pq}| \leq 1 \Big\}, \end{array}$$

where e_i, f_p, g_q are the standard basic orths in the spaces $\mathbb{R}^{\dim x}$, $\mathbb{R}^{\dim u}$ and $\mathbb{R}^{\dim y}$, respectively. Note that the matrix coefficients at the "elementary perturbations" ξ_{ij}, η_{pq} are of rank 1, and these perturbations, independently of each other, run through $[-1, 1]$ — exactly as required in (3.5.5) for $\mu = 1$.

In the situation of (3.5.5), the semi-infinite Lyapunov LMI

$$Q^T X + XQ \preceq -I \ \forall Q \in \mathrm{Conv}(\mathcal{Q})$$

in (3.4.24) reads

$$\begin{array}{l}\underbrace{-I - [Q^{\rm n}]^T X - XQ^{\rm n}}_{\mathcal{A}^{\rm n}(X)} + \rho \sum\limits_{\ell=1}^{L} \zeta_\ell \underbrace{[-U_\ell^T X - XU_\ell]}_{\mathcal{A}_\ell(X)} \\ \succeq 0 \ \forall (\zeta : |\zeta_\ell| \leq 1, \, \ell = 1, ..., L).\end{array} \tag{3.5.6}$$

We are in the case of structured norm-bounded perturbations with $\mathcal{I}_{\rm s} = \{1, ..., L\}$. Noting that the ranks of all matrices $\mathcal{A}_\ell(X)$ never exceed 2μ (since all U_ℓ are of ranks $\leq \mu$), the safe tractable approximation of (3.5.6) given by Theorem 3.13 is tight within the factor $\vartheta(2\mu)$. It follows, in particular, that *in the case of* (3.5.5) *with* $\mu = 1$, *we can find efficiently a lower bound, tight within the factor* $\pi/2$, *on the Lyapunov Stability Radius of the uncertain system* (3.4.22) (that is, on the supremum of those ρ for which the stability of our uncertain dynamical system can be certified by an LSC). The lower bound in question is the supremum of those ρ for which the approximation is feasible, and this supremum can be easily approximated to whatever accuracy by bisection.

We can process in the same fashion the Lyapunov Stability Synthesis problem in the presence of interval uncertainty. Specifically, assume that $C_t \equiv I$ and the uncertainty set $\mathcal{AB} = \{[A^M, B^M] : M \in \mathcal{M}\}$ underlying the Synthesis problem is an interval uncertainty:

$$\mathcal{AB} = [A^{\rm n}, B^{\rm n}] + \rho\{\sum_{\ell=1}^{L} \zeta_\ell U_\ell : \|\zeta\|_\infty \leq 1\}, \quad \mathrm{Rank}(U_\ell) \leq \mu \, \forall \ell. \tag{3.5.7}$$

We arrive at the situation of (3.5.7) with $\mu = 1$, e.g., when $\mathcal{AB}$ corresponds to entry-wise uncertainty:

$$\mathcal{AB} = [A^{\rm n}, B^{\rm n}] + \rho\{H \equiv [\delta A, \delta B] : |H_{ij}| \leq h_{ij} \, \forall i, j\}.$$

In the case of (3.5.7) the semi-infinite LMI in (3.4.29) reads

$$\begin{array}{l}\underbrace{-I - [A^{\rm n}, B^{\rm n}][Y; Z] - [Y; Z]^T [A^{\rm n}, B^{\rm n}]^T}_{\mathcal{A}^{\rm n}(Y,Z)} \\ + \rho \sum\limits_{\ell=1}^{L} \zeta_\ell \underbrace{[-U_\ell [Y; Z] - [Y; Z]^T U_\ell^T]}_{\mathcal{A}_\ell(Y,Z)} \succeq 0 \, \forall (\zeta : |\zeta_\ell| \leq 1, \, \ell = 1, ..., L).\end{array} \tag{3.5.8}$$

We again reach a situation of structured norm-bounded uncertainty with $\mathcal{I}_{\rm s} = \{1, ..., L\}$ and all matrices $\mathcal{A}_\ell(\cdot)$, $\ell = 1, ..., L$, being of ranks at most 2μ. Thus, Theorem 3.13 provides us with a tight, within factor $\vartheta(2\mu)$, safe tractable approximation of the Lyapunov Stability Synthesis problem.

Illustration: Controlling a multiple pendulum. Consider a multiple pendulum ("a train") depicted in figure 3.6. Denoting by m_i, $i = 1, ..., 4$, the masses of the "engine" ($i = 1$) and the "cars" ($i = 2, 3, 4$, counting from right to left), Newton's laws for the dynamical system in question read

$$\begin{array}{rcl} m_1 \frac{d^2}{dt^2} x_1(t) &=& -\kappa_1 x_1(t) + \kappa_1 x_2(t) + u(t) \\ m_2 \frac{d^2}{dt^2} x_2(t) &=& \kappa_1 x_1(t) - (\kappa_1 + \kappa_2) x_2(t) + \kappa_2 x_3(t) \\ m_3 \frac{d^2}{dt^2} x_3(t) &=& \kappa_2 x_2(t) - (\kappa_2 + \kappa_3) x_3(t) + \kappa_3 x_4(t) \\ m_4 \frac{d^2}{dt^2} x_4(t) &=& \kappa_3 x_3(t) - \kappa_3 x_4(t), \end{array} \tag{3.5.9}$$

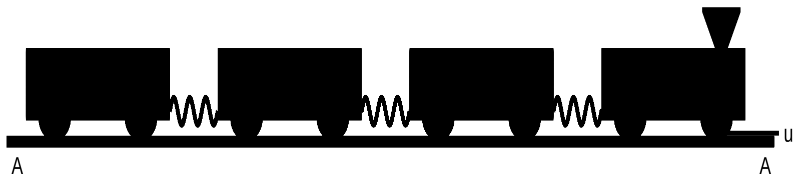


Figure 3.6: "Train": 4 masses (3 "cars" and "engine") linked by elastic springs and sliding without friction (aside of controlled force u) along "rail" AA.

where $x_i(t)$ are shifts of the engine and the cars from their respective positions in the state of rest (where nothing moves and the springs are neither shrunk nor expanded), and κ_i are the elasticity constants of the springs (counted from right to left). Passing from masses m_i to their reciprocals $\mu_i = 1/m_i$ and adding to the coordinates of the cars their velocities $v_i(t) = \dot{x}_i(t)$, we can rewrite (3.5.9) as the system of 8 linear differential equations:

$$\dot{x}(t) = \underbrace{\left[\begin{array}{cccc|cccc} &&&& 1 &&& \\ &&&& & 1 && \\ &&&& && 1 & \\ &&&& &&& 1 \\ \hline -\kappa_1\mu_1 & \kappa_1\mu_1 &&&&&& \\ \kappa_1\mu_2 & -[\kappa_1+\kappa_2]\mu_2 & \kappa_2\mu_2 &&&&& \\ & \kappa_2\mu_3 & -[\kappa_2+\kappa_3]\mu_3 & \kappa_3\mu_3 &&&& \\ && \kappa_3\mu_4 & -\kappa_3\mu_4 &&&& \end{array}\right]}_{A_\mu} x(t) + \underbrace{\left[\begin{array}{c} \\ \\ \\ \\ \hline \mu_1 \\ \\ \\ \\ \end{array}\right]}_{B_\mu} u(t) \tag{3.5.10}$$

where $x(t) = [x_1(t); x_2(t); x_3(t); x_4(t); v_1(t); v_2(t); v_3(t); v_4(t)]$. System (3.5.10) "as it is" (i.e., with trivial control $u(\cdot) \equiv 0$) is unstable; not only it has a solution that does not converge to 0 as $t \to \infty$, it has even an unbounded solution (specifically, one where $x_i(t) = vt$, $v_i(t) \equiv v$, which corresponds to uniform motion of the cars and the engine with no tensions in the springs). Let us look for a stabilizing state-based linear feedback controller

$$u(t) = Kx(t), \tag{3.5.11}$$

that is robust w.r.t. the masses of the cars and the engine when they vary in given segments Δ_i, $i = 1, ..., 4$. To this end we can apply the Lyapunov Stability Synthesis machinery. Observe that to say that the masses m_i run, independently of each other, through given segments is exactly the same as to say that their reciprocals μ_i run, independently of each other, through other given segments Δ_i'; thus, our goal is as follows:

> **Stabilization:** *Given elasticity constants κ_i and segments $\Delta_i' \subset \{\mu > 0\}$, $i = 1, ..., 4$, find a linear feedback* (3.5.11) *and a Lyapunov Stability Certificate X for the*

corresponding closed loop system (3.5.10), (3.5.11), *with the uncertainty set for the system being*

$$\mathcal{AB} = \{[A_\mu, B_\mu] : \mu_i \in \Delta_i', \, i = 1, ..., 4\}.$$

Note that in our context the Lyapunov Stability Synthesis approach is, so to speak, "doubly conservative." First, the existence of a common LSC for all matrices Q from a given compact set $\mathcal{Q}$ is only a *sufficient* condition for the stability of the uncertain dynamical system

$$\dot{x}(t) = Q_t x(t), \;\; Q_t \in \mathcal{Q} \, \forall t,$$

and as such this condition is conservative. Second, in our train example there are reasons to think of m_i as of uncertain data (in reality the loads of the cars and the mass of the engine could vary from trip to trip, and we would not like to re-adjust the controller as long as these changes are within a reasonable range), but there is absolutely no reason to think of these masses as varying in time. Indeed, we could perhaps imagine a mechanism that makes the masses m_i time-dependent, but with this mechanism our original model (3.5.9) becomes invalid — Newton's laws in the form of (3.5.9) are not applicable to systems with varying masses and at the very best they offer a reasonable approximation of the true model, provided that the changes in masses are slow. Thus, in our train example a common LSC for all matrices $Q = A + BK$, $[A, B] \in \mathcal{AB}$, would guarantee much more than required, namely, that all trajectories of the closed loop system "train plus feedback controller" converge to 0 as $t \to \infty$ even in the case when the parameters $\mu_i \in \Delta_i'$ vary in time at a high speed. This is much more than what we actually need — convergence to 0 of all trajectories in the case when $\mu_i \in \Delta_i'$ do not vary in time.

The system of semi-infinite LMIs we are about to process in the connection of the Lyapunov Stability Synthesis is

$$\begin{array}{ll} (a) & [A, B][Y; Z] + [Y; Z]^T [A, B]^T \preceq -\alpha Y, \;\forall [A, B] \in \mathcal{AB} \\ (b) & Y \succeq I \\ (c) & Y \leq \chi I, \end{array} \tag{3.5.12}$$

where $\alpha > 0$ and $\chi > 1$ are given. This system differs slightly from the "canonical" system (3.4.29), and the difference is twofold:

- [major] in (3.4.29), the semi-infinite Lyapunov LMI is written as

$$[A, B][Y; Z] + [Y; Z]^T [A, B]^T \preceq -I,$$

which is just a convenient way to express the relation

$$[A, B][Y; Z] + [Y; Z]^T [A, B]^T \prec 0, \forall [A, B] \in \mathcal{AB}.$$

Every feasible solution $[Y; Z]$ to this LMI with $Y \succ 0$ produces a stabilizing feedback $K = ZY^{-1}$ and the common LSC $X = Y^{-1}$ for all instances of the matrix $Q = A + BK$, $[A, B] \in \mathcal{AB}$, of the closed loop system, i.e.,

$$[A + BK]^T X + X[A + BK] \prec 0 \;\; \forall [A, B] \in \mathcal{AB}.$$

The latter condition, however, says nothing about the corresponding decay rate. In contrast, when $[Y; Z]$ is feasible for (3.5.12.a, b), the associated stabilizing feedback $K = ZY^{-1}$ and LSC $X = Y^{-1}$ satisfy the relation

$$[A + BK]^T X + X[A + BK] \prec -\alpha X \;\; \forall [A, B] \in \mathcal{AB},$$

and this relation, as we have seen when introducing the Lyapunov Stability Certificate, implies that

$$x^T(t)Xx(t) \le \exp\{-\alpha t\}x^T(0)Xx(0),\ t \ge 0,$$

which guarantees that the decay rate in the closed loop system is at least α. In our illustration (same as in real life), we prefer to deal with this "stronger" form of the Lyapunov Stability Synthesis requirement, in order to have a control over the decay rate associated with the would-be controller.

- [minor] In (3.5.12) we impose an upper bound on the condition number (ratio of the maximal and minimal eigenvalues) of the would-be LSC; with normalization of Y given by (3.5.12.b), this bound is ensured by (3.5.12.c) and is precisely χ. The only purpose of this bound is to avoid working with extremely ill-conditioned positive definite matrices, which can cause numerical problems.

Now let us use Theorem 3.13 to get a tight safe tractable approximation of the semi-infinite system of LMIs (3.5.12). Denoting by μ_i^{n} the midpoints of the segments Δ_i' and by δ_i the half-width of these segments, we have

$$\begin{array}{rcl}
\mathcal{AB} & \equiv & \{[A_\mu, B_\mu] : \mu_i \in \Delta_i', i = 1, ..., 4\} \\
 & = & \{[A_{\mu^{\mathrm{n}}}, B_{\mu^{\mathrm{n}}}] + \sum\limits_{\ell=1}^{4} \zeta_\ell U_\ell : |\zeta_\ell| \le 1,\ \ell = 1, ..., 4\}, \\
U_\ell & = & \delta_\ell p_\ell q_\ell^T,
\end{array}$$

where $p_\ell \in \mathbb{R}^8$ has the only nonzero entry, equal to 1, in the position $4+\ell$, and

$$\begin{bmatrix} q_1^T \\ q_2^T \\ q_3^T \\ q_4^T \end{bmatrix} = \left[\begin{array}{c|c|c|c|c|c|c|c}
-\kappa_1 & \kappa_1 & & & & & & 1 \\ \hline
\kappa_1 & -[\kappa_1+\kappa_2] & \kappa_2 & & & & & \\ \hline
 & \kappa_2 & -[\kappa_2+\kappa_3] & \kappa_3 & & & & \\ \hline
 & & \kappa_3 & -\kappa_3 & & & &
\end{array}\right]$$

Consequently, the analogy of (3.5.12) with uncertainty level ρ ((3.5.12) itself corresponds to $\rho = 1$) is the semi-infinite system of LMIs

$$\begin{array}{l}
\underbrace{-\alpha Y - [A_{\mu^{\mathrm{n}}}, B_{\mu^{\mathrm{n}}}][Y;Z] - [Y;Z]^T[A_{\mu^{\mathrm{n}}}, B_{\mu^{\mathrm{n}}}]^T}_{\mathcal{A}^{\mathrm{n}}(Y,Z)} \\
+\rho \sum\limits_{\ell=1}^{4} \zeta_\ell (\underbrace{-\delta_\ell[p_\ell q_\ell^T[Y;Z] + [Y;Z]^T q_\ell p_\ell^T]}_{\mathcal{A}_\ell(Y,Z)}) \succeq 0\ \forall(\zeta : |\zeta_\ell| \le 1,\ \ell = 1, ..., 4) \\
Y \succeq I_8,\ Y \preceq \chi I_8
\end{array} \tag{3.5.13}$$

in variables Y, Z (cf. (3.5.8)). The safe tractable approximation of this semi-infinite system of LMIs as given by Theorem 3.13 is the system of LMIs

$$\begin{array}{l}
Y_\ell \succeq \pm \mathcal{A}_\ell(Y,Z),\ \ell = 1, ..., 4 \\
\mathcal{A}^{\mathrm{n}}(Y,Z) - \rho \sum\limits_{\ell=1}^{4} Y_\ell \succeq 0 \\
Y \succeq I_8,\ Y \preceq \chi I_8
\end{array} \tag{3.5.14}$$

in variables $Y, Z, Y_1, ..., Y_4$. Since all U_ℓ are of rank 1 and therefore all $\mathcal{A}_\ell(Y,Z)$ are of rank ≤ 2, Theorem 3.13 states that this safe approximation is tight within the factor $\pi/2$.

Of course, in our toy example no approximation is needed — the set $\mathcal{AB}$ is a polytopic uncertainty with just $2^4 = 16$ vertices, and we can straightforwardly convert (3.5.13) into an exactly equivalent system of 18 LMIs

$$\begin{array}{l}\mathcal{A}^{\mathrm{n}}(Y,Z) \succeq \rho \sum\limits_{\ell=1}^{4} \epsilon_\ell \mathcal{A}_\ell(Y,Z),\ \epsilon_\ell = \pm 1,\ \ell = 1,...,4\\ Y \succeq I_8,\ Y \preceq \chi I_8\end{array}$$

in variables Y, Z. The situation would change dramatically if there were, say, 30 cars in our train rather than just 3. Indeed, in the latter case the precise "polytopic" approach would require solving a system of $2^{31} + 2 = 2,147,483,650$ LMIs of the size 62×62 in variables $Y \in \mathbf{S}^{62}, Z \in \mathbb{R}^{1\times 63}$, which is a bit too much... In contrast, the approximation (3.5.14) is a system of just $31 + 2 = 33$ LMIs of the size 62×62 in variables $\{Y_\ell \in \mathbf{S}^{62}\}_{\ell=1}^{31}$, $Y \in \mathbf{S}^{62}$, $Z \in \mathbb{R}^{1\times 63}$ (totally $(31+1)\frac{62\cdot 63}{2} + 63 = 60606$ scalar decision variables). One can argue that the latter problem still is too large from a practical perspective. But in fact it can be shown that in this problem, one can easily eliminate the matrices Y_ℓ (every one of them can be replaced with a *single* scalar decision variable, cf. Antenna example on p. 106), which reduces the design dimension of the approximation to $31 + \frac{62\cdot 63}{2} + 63 = 2047$. A convex problem of this size can be solved pretty routinely.

We are about to present numerical results related to stabilization of our toy 3-car train. The setup in our computations is as follows:

$$\begin{array}{c}\kappa_1 = \kappa_2 = \kappa_3 = 100.0; \alpha = 0.01; \chi = 10^8;\\ \Delta_1' = [0.5, 1.5], \Delta_2' = \Delta_3' = \Delta_4' = [1.5, 4.5],\end{array}$$

which corresponds to the mass of the engine varying in $[2/3, 2]$ and the masses of the cars varying in $[2/9, 2/3]$.

We computed, by a kind of bisection, the largest ρ for which the approximation (3.5.14) is feasible; the optimal feedback we have found is

$$\begin{array}{rl} u = & 10^7\big[-0.2892x_1 - 2.5115x_2 + 6.3622x_3 - 3.5621x_4\\ & \quad -0.0019v_1 - 0.0912v_2 - 0.0428v_3 + 0.1305v_4\big],\end{array}$$

and the (lower bound on the) Lyapunov Stability radius of the closed loop system as yielded by our approximation is $\widehat{\rho} = 1.05473$. This bound is > 1, meaning that our feedback stabilizes the train in the above ranges of the masses of the engine and the cars (and in fact, even in slightly larger ranges $0.65 \le m_1 \le 2.11$, $0.22 \le m_2, m_3, m_4 \le 0.71$). An interesting question is by how much the *lower bound* $\widehat{\rho}$ is less than the Lyapunov Stability radius ρ_* of the closed loop system. Theory guarantees that the ratio $\rho_*/\widehat{\rho}$ should be $\le \pi/2 = 1.570...$. In our small problem we can compute ρ_* by applying the polytopic uncertainty approach, that results in $\rho_* = 1.05624$. Thus, in reality $\rho_*/\widehat{\rho} \approx 1.0014$, much better than the theoretical bound 1.570.... In figure 3.7, we present sample trajectories of the closed loop system yielded by our design, the level of perturbations being 1.054 — pretty close to $\widehat{\rho} = 1.05473$.

3.6 Approximating Chance Constrained CQIs and LMIs

In the first reading this section can be skipped.

Below we develop safe tractable approximations of *chance constrained* randomly perturbed Conic Quadratic and Linear Matrix Inequalities. For omitted proofs, see [3].

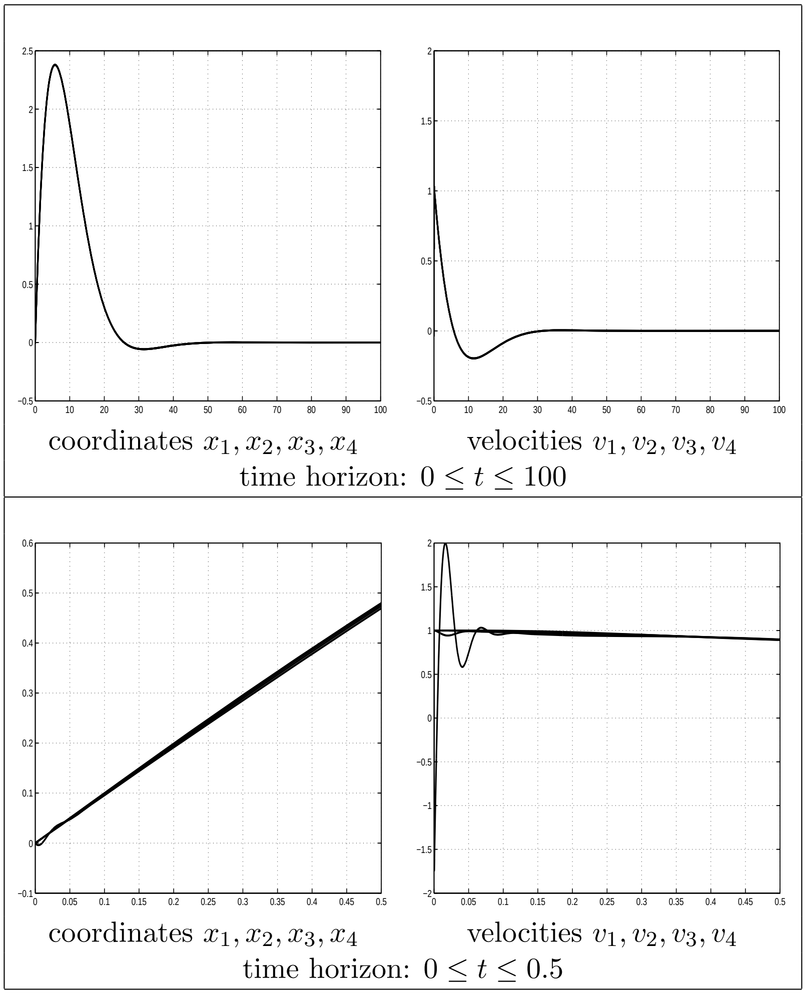


Figure 3.7: Sample trajectories of the 3-car train.

3.6.1 Chance Constrained LMIs

We have considered the Robust of uncertain conic quadratic and semidefinite programs. Now we intend to consider *randomly perturbed* CQPs and SDPs and to derive safe approximations of their chance constrained versions (cf. section 2.1). From this perspective, it is convenient to treat chance constrained CQPs as particular cases of chance constrained SDPs (such an option is given by Lemma 3.1), so that in the sequel we focus on chance constrained SDPs. Thus, we are interested in a randomly perturbed semidefinite program

$$\min_y \left\{ c^T y : \mathcal{A}^{\mathrm{n}}(y) + \rho \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell(y) \succeq 0, y \in \mathcal{Y} \right\}, \tag{3.6.1}$$

where $\mathcal{A}^{\mathrm{n}}(y)$ and all $\mathcal{A}^\ell(y)$ are affine in y, $\rho \geq 0$ is the "perturbation level," $\zeta = [\zeta_1; ...; \zeta_L]$ is a random perturbation, and $\mathcal{Y}$ is a semidefinite representable set. We associate with this problem its *chance constrained* version

$$\min_y \left\{ c^T y : \mathrm{Prob}\left\{ \mathcal{A}^{\mathrm{n}}(y) + \rho \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell(y) \succeq 0 \right\} \geq 1 - \epsilon, y \in \mathcal{Y} \right\} \tag{3.6.2}$$

where $\epsilon \ll 1$ is a given positive tolerance. Our goal is to build a computationally tractable safe approximation of (3.6.2). We start with assumptions on the random variables ζ_ℓ, which will be in force everywhere in the following:

> *Random variables ζ_ℓ, $\ell = 1, ..., L$, are independent with zero mean satisfying either*
> **A.I** ["bounded case"] $|\zeta_\ell| \leq 1$, $\ell = 1, ..., L$,
> *or*
> **A.II** ["Gaussian case"] $\zeta_\ell \sim \mathcal{N}(0,1), \ell = 1, ..., L$.

Note that most of the results to follow can be extended to the case when ζ_ℓ are independent with zero means and "light tail" distributions. We prefer to require more in order to avoid too many technicalities.

Approximating Chance Constrained LMIs: Preliminaries

The problem we are facing is basically as follows:

> (?) *Given symmetric matrices $A, A_1,...,A_L$, find a verifiable sufficient condition for the relation*
> $$\mathrm{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq A\} \geq 1 - \epsilon. \tag{3.6.3}$$

Since ζ is with zero mean, it is natural to require $A \succeq 0$ (this condition clearly is necessary when ζ is symmetrically distributed w.r.t. 0 and $\epsilon < 0.5$). Requiring a bit more, namely, $A \succ 0$, we can reduce the situation to the case when $A = I$, due to

$$\mathrm{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq A\} = \mathrm{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell \underbrace{A^{-1/2} A_\ell A^{-1/2}}_{B_\ell} \preceq I\}. \tag{3.6.4}$$

Now let us try to guess a verifiable sufficient condition for the relation

$$\mathrm{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell B_\ell \preceq I\} \geq 1 - \epsilon. \tag{3.6.5}$$

First of all, we do not lose much when strengthening the latter relation to

$$\mathrm{Prob}\{\| \sum_{\ell=1}^{L} \zeta_\ell B_\ell \| \leq 1\} \geq 1 - \epsilon \tag{3.6.6}$$

(here and in what follows, $\|\cdot\|$ stands for the standard matrix norm $\|\cdot\|_{2,2}$). Indeed, the latter condition is nothing but

$$\mathrm{Prob}\{-I \preceq \sum_{\ell=1}^{L} \zeta_\ell B_\ell \preceq I\} \geq 1 - \epsilon,$$

so that it implies (3.6.5). In the case of ζ symmetrically distributed w.r.t. the origin, we have a "nearly inverse" statement: the validity of (3.6.5) implies the validity of (3.6.6) with ϵ increased to 2ϵ.

The central observation is that whenever (3.6.6) holds true and the distribution of the random matrix

$$S = \sum_{\ell=1}^{L} \zeta_\ell B_\ell$$

is not pathological, we should have

$$\mathbf{E}\{\|S^2\|\} \leq O(1),$$

whence, by Jensen's Inequality,

$$\|\mathbf{E}\{S^2\}\| \leq O(1)$$

as well. Taking into account that $\mathbf{E}\{S^2\} = \sum\limits_{\ell=1}^{L} \mathbf{E}\{\zeta_\ell^2\}B_\ell^2$, we conclude that when all quantities $\mathbf{E}\{\zeta_\ell^2\}$ are of order of 1, we should have $\|\sum_{\ell=1}^{L} B_\ell^2\| \leq O(1)$, or, which is the same,

$$\sum_{\ell=1}^{L} B_\ell^2 \preceq O(1)I. \tag{3.6.7}$$

By the above reasoning, (3.6.7) is a kind of a necessary condition for the validity of the chance constraint (3.6.6), at least for random variables ζ_ℓ that are symmetrically distributed w.r.t. the origin and are "of order of 1." To some extent, this condition can be treated as nearly sufficient, as is shown by the following two theorems.

Theorem 3.14 *Let $B_1, ..., B_L \in \mathbf{S}^m$ be deterministic matrices such that*

$$\sum_{\ell=1}^{L} B_\ell^2 \preceq I \tag{3.6.8}$$

and $\Upsilon > 0$ be a deterministic real. Let, further, ζ_ℓ, $\ell = 1, ..., L$, be independent random variables taking values in $[-1, 1]$ such that

$$\chi \equiv \mathrm{Prob}\left\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| \leq \Upsilon\right\} > 0. \tag{3.6.9}$$

Then

$$\forall \Omega > \Upsilon : \mathrm{Prob}\left\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega\right\} \leq \frac{1}{\chi}\exp\{-(\Omega - \Upsilon)^2/16\}. \tag{3.6.10}$$

Proof. Let $Q = \{z \in \mathbb{R}^L : \|\sum\limits_\ell z_\ell B_\ell\| \leq 1\}$. Observe that

$$\|[\sum_\ell z_\ell B_\ell]u\|_2 \leq \sum_\ell |z_\ell|\|B_\ell u\|_2 \leq \left(\sum_\ell z_\ell^2\right)^{1/2}\left(\sum_\ell u^T B_\ell^2 u\right)^{1/2} \leq \|z\|_2\|u\|_2,$$

where the concluding relation is given by (3.6.8). It follows that $\|\sum_\ell z_\ell B_\ell\| \leq \|z\|_2$, whence Q contains the unit $\|\cdot\|_2$-ball B centered at the origin in $\mathbb{R}^L$. Besides this, Q is clearly closed, convex and symmetric w.r.t. the origin. Invoking the Talagrand Inequality (Theorem A.9), we have

$$\mathbf{E}\left\{\exp\{\mathrm{dist}^2_{\|\cdot\|_2}(\zeta, \Upsilon Q)/16\}\right\} \leq (\mathrm{Prob}\{\zeta \in \Upsilon Q\})^{-1} = \frac{1}{\chi}. \tag{3.6.11}$$

Now, when ζ is such that $\|\sum\limits_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega$, we have $\zeta \notin \Omega Q$, whence, due to symmetry and convexity of Q, the set $(\Omega - \Upsilon)Q + \zeta$ does not intersect the set ΥQ. Since Q contains B, the set $(\Omega - \Upsilon)Q + \zeta$ contains $\|\cdot\|_2$-ball, centered at ζ, of the radius $\Omega - \Upsilon$, and therefore this ball does not intersect ΥQ either, whence $\mathrm{dist}_{\|\cdot\|_2}(\zeta, \Upsilon Q) > \Omega - \Upsilon$. The resulting relation

$$\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega \Leftrightarrow \zeta \notin \Omega Q \Rightarrow \mathrm{dist}_{\|\cdot\|_2}(\zeta, \Upsilon Q) > \Omega - \Upsilon$$

combines with (3.6.11) and the Tschebyshev Inequality to imply that

$$\mathrm{Prob}\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega\} \leq \frac{1}{\chi}\exp\{-(\Omega - \Upsilon)^2/16\}. \qquad \square$$

Theorem 3.15 *Let $B_1, ..., B_L \in \mathbf{S}^m$ be deterministic matrices satisfying* (3.6.8) *and $\Upsilon > 0$ be a deterministic real. Let, further, ζ_ℓ, $\ell = 1, ..., L$, be independent $\mathcal{N}(0,1)$ random variables such that* (3.6.9) *holds true with $\chi > 1/2$.*

Then

$$\begin{array}{l} \forall \Omega \geq \Upsilon : \mathrm{Prob}\{\|\sum\limits_{\ell=1}^{L} \zeta_\ell B_\ell\| > \Omega\} \\ \quad \leq \mathrm{Erf}\left(\mathrm{ErfInv}(1-\chi) + (\Omega - \Upsilon)\max[1, \Upsilon^{-1}\mathrm{ErfInv}(1-\chi)]\right) \\ \quad \leq \exp\{-\frac{\Omega^2\Upsilon^{-2}\mathrm{ErfInv}^2(1-\chi)}{2}\}, \end{array} \tag{3.6.12}$$

where $\mathrm{Erf}(\cdot)$ *and* $\mathrm{ErfInv}(\cdot)$ *are the error and the inverse error functions, see* (2.2.6), (2.2.7).

Proof. Let $Q = \{z \in \mathbb{R}^L : \|\sum_\ell z_\ell B_\ell\| \leq \Upsilon\}$. By the same argument as in the beginning of the proof of Theorem 3.14, Q contains the centered at the origin $\|\cdot\|_2$-ball of the radius Υ. Besides this, by definition of Q we have $\mathrm{Prob}\{\zeta \in Q\} \geq \chi$. Invoking item (i) of Theorem A.10, Q contains the centered at the origin $\|\cdot\|_2$-ball of the radius $r = \max[\mathrm{ErfInv}(1-\chi), \Upsilon]$, whence, by item (ii) of this Theorem, (3.6.12) holds true. □

The last two results are stated next in a form that is better suited for our purposes.

Corollary 3.1 *Let $A, A_1, ..., A_L$ be deterministic matrices from $\mathbf{S}^m$ such that*

$$\exists \{Y_\ell\}_{\ell=1}^L : \left\{ \begin{array}{l} \left[\begin{array}{c|c} Y_\ell & A_\ell \\ \hline A_\ell & A \end{array}\right] \succeq 0,\ 1 \leq \ell \leq L \\ \sum\limits_{\ell=1}^{L} Y_\ell \preceq A \end{array} \right. , \tag{3.6.13}$$

let $\Upsilon > 0$, $\chi > 0$ be deterministic reals and $\zeta_1, ..., \zeta_L$ be independent random variables satisfying either **A.I***, or* **A.II***, and such that*

$$\mathrm{Prob}\{-\Upsilon A \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Upsilon A\} \geq \chi. \tag{3.6.14}$$

Then

(i) *When ζ_ℓ satisfy* **A.I***, we have*

$$\forall \Omega > \Upsilon : \mathrm{Prob}\{-\Omega A \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Omega A\} \geq 1 - \frac{1}{\chi}\exp\{-(\Omega - \Upsilon)^2/16\}; \tag{3.6.15}$$

(ii) *When ζ_ℓ satisfy* **A.II***, and, in addition, $\chi > 0.5$, we have*

$$\begin{array}{l} \forall \Omega > \Upsilon : \mathrm{Prob}\{-\Omega A \preceq \sum\limits_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Omega A\} \\ \quad \geq 1 - \mathrm{Erf}\left(\mathrm{ErfInv}(1-\chi) + (\Omega - \Upsilon)\max\left[1, \frac{\mathrm{ErfInv}(1-\chi)}{\Upsilon}\right]\right), \end{array} \tag{3.6.16}$$

with $\mathrm{Erf}(\cdot)$, $\mathrm{ErfInv}(\cdot)$ *given by* (2.2.6), (2.2.7).

Proof. Let us prove (i). Given positive δ, let us set $A^\delta = A + \delta I$. Observe that the premise in (3.6.14) clearly implies that $A \succeq 0$, whence $A^\delta \succ 0$. Now let Y_ℓ be such that the conclusion in (3.6.13) holds true. Then $\left[\begin{array}{c|c} Y_\ell & A_\ell \\ \hline A_\ell & A^\delta \end{array}\right] \succeq 0$, whence, by the Schur Complement Lemma, $Y_\ell \succeq A_\ell[A^\delta]^{-1}A_\ell$, so that

$$\sum_\ell A_\ell[A^\delta]^{-1}A_\ell \preceq \sum_\ell Y_\ell \preceq A \preceq A^\delta.$$

We see that

$$\sum_\ell \big[\underbrace{[A^\delta]^{-1/2}A_\ell[A^\delta]^{-1/2}}_{B_\ell^\delta}\big]^2 \preceq I.$$

Further, relation (3.6.14) clearly implies that

$$\mathrm{Prob}\{-\Upsilon A^\delta \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Upsilon A^\delta\} \geq \chi,$$

or, which is the same,

$$\mathrm{Prob}\{-\Upsilon I \preceq \sum_\ell \zeta_\ell B_\ell^\delta \preceq \Upsilon I\} \geq \chi.$$

Applying Theorem 3.14, we conclude that

$$\Omega > \Upsilon \Rightarrow \mathrm{Prob}\{-\Omega I \preceq \sum_\ell \zeta_\ell B_\ell^\delta \preceq \Omega I\} \geq 1 - \frac{1}{\chi}\exp\{-(\Omega-\Upsilon)^2/16\},$$

which in view of the structure of B_ℓ^δ is the same as

$$\Omega > \Upsilon \Rightarrow \mathrm{Prob}\{-\Omega A^\delta \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Omega A^\delta\} \geq 1 - \frac{1}{\chi}\exp\{-(\Omega-\Upsilon)^2/16\}. \tag{3.6.17}$$

For every $\Omega > \Upsilon$, the sets $\{\zeta : -\Omega A^{1/t} \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Omega A^{1/t}\}$, $t = 1, 2, ...$, shrink as t grows, and their intersection over $t = 1, 2, ...$ is the set $\{\zeta : -\Omega A \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Omega A\}$, so that (3.6.17) implies (3.6.15), and (i) is proved. The proof of (ii) is completely similar, with Theorem 3.15 in the role of Theorem 3.14. □

Comments. When $A \succ 0$, invoking the Schur Complement Lemma, the condition (3.6.13) is satisfied iff it is satisfied with $Y_\ell = A_\ell A^{-1} A_\ell$, which in turn is the case iff $\sum_\ell A_\ell A^{-1} A_\ell \preceq A$, or which is the same, iff $\sum_\ell [A^{-1/2} A_\ell A^{-1/2}]^2 \preceq I$. Thus, *condition* (3.6.4), (3.6.7) *introduced in connection with Problem* (?), *treated as a condition on the variable symmetric matrices* $A, A_1, ..., A_L$, *is LMI-representable,* (3.6.13) *being the representation.* Further, (3.6.13) can be written as the following explicit LMI on the matrices $A, A_1, ..., A_L$:

$$\mathrm{Arrow}(A, A_1, ..., A_L) \equiv \left[\begin{array}{c|ccc} A & A_1 & ... & A_L \\ \hline A_1 & A & & \\ \vdots & & \ddots & \\ A_L & & & A \end{array}\right] \succeq 0. \tag{3.6.18}$$

Indeed, when $A \succ 0$, the Schur Complement Lemma says that the matrix $\mathrm{Arrow}(A, A_1, ..., A_L)$ is $\succeq 0$ if and only if

$$\sum_\ell A_\ell A^{-1} A_\ell \preceq A,$$

and this is the case if and only if (3.6.13) holds. Thus, (3.6.13) and (3.6.18) are equivalent to each other when $A \succ 0$, which, by standard approximation argument, implies the equivalence of these two properties in the general case (that is, when $A \succeq 0$). It is worthy of noting that the set of matrices $(A, A_1, ..., A_L)$ satisfying (3.6.18) form a cone that can be considered as the matrix analogy of the Lorentz cone (look what happens when all the matrices are 1×1 ones).

3.6.2 The Approximation Scheme

To utilize the outlined observations and results in order to build a safe/"almost safe" tractable approximation of a chance constrained LMI in (3.6.2), we proceed as follows.

1) We introduce the following:

Conjecture 3.1 *Under assumptions* **A.I** *or* **A.II**, *condition* (3.6.13) *implies the validity of* (3.6.14) *with known in advance* $\chi > 1/2$ *and "a moderate" (also known in advance)* $\Upsilon > 0$.

With properly chosen χ and Υ, this Conjecture indeed is true, see below. We, however, prefer not to stick to the corresponding worst-case-oriented values of χ and Υ and consider $\chi > 1/2$, $\Upsilon > 0$ as somehow chosen parameters of the construction to follow, and we proceed *as if* we know in advance that our conjecture, with the chosen Υ, χ, is true. Eventually we shall explain how to justify this tactics.

2) Trusting in Conjecture 3.1, we have at our disposal constants $\Upsilon > 0$, $\chi \in (0.5, 1]$ such that (3.6.13) implies (3.6.14). We claim that *modulo Conjecture 3.1, the following systems of LMIs in variables* $y, U_1, ..., U_L$ *are safe tractable approximations of the chance constrained LMI in* (3.6.2):
In the case of **A.I**:

$$\begin{array}{ll}(a) & \left[\begin{array}{c|c} U_\ell & \mathcal{A}^\ell(y) \\ \hline \mathcal{A}^\ell(y) & \mathcal{A}^{\rm n}(y)\end{array}\right] \succeq 0,\ 1 \leq \ell \leq L \\ (b) & \rho^2 \sum\limits_{\ell=1}^L U_\ell \preceq \Omega^{-2}\mathcal{A}^{\rm n}(y),\ \ \Omega = \Upsilon + 4\sqrt{\ln(\chi^{-1}\epsilon^{-1})};\end{array} \tag{3.6.19}$$

In the case of **A.II**:

$$\begin{array}{ll}(a) & \left[\begin{array}{c|c} U_\ell & \mathcal{A}^\ell(y) \\ \hline \mathcal{A}^\ell(y) & \mathcal{A}^{\rm n}(y)\end{array}\right] \succeq 0,\ 1 \leq \ell \leq L \\ (b) & \rho^2 \sum\limits_{\ell=1}^L U_\ell \preceq \Omega^{-2}\mathcal{A}^{\rm n}(y),\ \ \Omega = \Upsilon + \frac{\max[\mathrm{ErfInv}(\epsilon) - \mathrm{ErfInv}(1-\chi), 0]}{\max[1, \Upsilon^{-1}\mathrm{ErfInv}(1-\chi)]} \\ & \leq \Upsilon + \max\left[\mathrm{ErfInv}(\epsilon) - \mathrm{ErfInv}(1-\chi), 0\right].\end{array} \tag{3.6.20}$$

Indeed, assume that y can be extended to a feasible solution $(y, U_1, ..., U_L)$ of (3.6.19). Let us set $A = \Omega^{-1}\mathcal{A}^{\rm n}(y)$, $A_\ell = \rho\mathcal{A}^\ell(y)$, $Y_\ell = \Omega\rho^2 U_\ell$. Then $\left[\begin{array}{c|c} Y_\ell & A_\ell \\ \hline A_\ell & A\end{array}\right] \succeq 0$ and $\sum\limits_\ell Y_\ell \preceq A$ by (3.6.19). Applying Conjecture 3.1 to the matrices $A, A_1, ..., A_L$, we conclude that (3.6.14) holds true as well. Applying Corollary 3.1.(i), we get

$$\begin{array}{l}\mathrm{Prob}\left\{\rho\sum\limits_\ell \zeta_\ell\mathcal{A}^\ell(y) \not\preceq \mathcal{A}^{\rm n}(y)\right\} = \mathrm{Prob}\left\{\sum\limits_\ell \zeta_\ell A_\ell \not\preceq \Omega A\right\} \\ \leq \chi^{-1}\exp\{-(\Omega-\Upsilon)^2/16\} = \epsilon,\end{array}$$

as claimed.

Relation (3.6.20) can be justified, modulo the validity of Conjecture 3.1, in the same fashion, with item (ii) of Corollary 3.1 in the role of item (i).

3) We replace the chance constrained LMI problem (3.6.2) with the outlined safe (modulo the validity of Conjecture 3.1) approximation, thus arriving at the approximating problem

$$\min_{y,\{U_\ell\}}\left\{c^Ty : \begin{array}{l}\left[\begin{array}{c|c} U_\ell & \mathcal{A}^\ell(y) \\ \hline \mathcal{A}^\ell(y) & \mathcal{A}^{\rm n}(y)\end{array}\right] \succeq 0,\ 1 \leq \ell \leq L \\ \rho^2\sum\limits_\ell U_\ell \preceq \Omega^{-2}\mathcal{A}^{\rm n}(y),\ y \in \mathcal{Y}\end{array}\right\}, \tag{3.6.21}$$

where Ω is given by the required tolerance *and our guesses for* Υ *and* χ according to (3.6.19) or (3.6.20), depending on whether we are in the case of a bounded random perturbation model (Assumption **A.I**) or a Gaussian one (Assumption **A.II**).

We solve the approximating SDO problem and obtain its optimal solution y_*. If (3.6.21) were indeed a safe approximation of (3.6.2), we would be done: y_* would be a *feasible* suboptimal solution to the chance constrained problem of interest. However, since we are not sure of the validity of Conjecture 3.1, we need an additional phase — *post-optimality analysis* — aimed at justifying the feasibility of y_* for the chance constrained problem. Note that *at this phase, we should not bother about the validity of Conjecture 3.1 in full generality — all we need is to justify the validity of the relation*

$$\mathrm{Prob}\{-\Upsilon A \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Upsilon A\} \geq \chi \tag{3.6.22}$$

for specific matrices

$$A = \Omega^{-1}\mathcal{A}^{\rm n}(y_*),\ A_\ell = \rho\mathcal{A}^\ell(y_*),\ \ell = 1, ..., L, \tag{3.6.23}$$

which we have in our disposal after y_* is found, and which indeed satisfy (3.6.13) (cf. "justification" of approximations (3.6.19), (3.6.20) in item 2)).

In principle, there are several ways to justify (3.6.22):

1. Under certain structural assumptions on the matrices A, A_ℓ and with properly chosen χ, Υ, our Conjecture 3.1 is provably true. Specifically, we shall see in section 3.6.4 that:
   (a) when A, A_ℓ are diagonal, (which corresponds to the semidefinite reformulation of a Linear Optimization problem), Conjecture 3.1 holds true with $\chi = 0.75$ and $\Upsilon = \sqrt{3\ln(8m)}$ (recall that m is the size of the matrices $A, A_1, ..., A_L$);
   (b) when A, A_ℓ are arrow matrices, (which corresponds to the semidefinite reformulation of a conic quadratic problem), Conjecture 3.1 holds true with $\chi = 0.75$ and $\Upsilon = 4\sqrt{2}$.
2. Utilizing deep results from Functional Analysis, it can be proved (see [3, Proposition B.5.2]) that Conjecture 3.1 is true *for all matrices* $A, A_1, ..., A_L$ when $\chi = 0.75$ and $\Upsilon = 4\sqrt{\ln\max[m, 3]}$. It should be added that in order for our Conjecture 3.1 to be true for all L and all $m \times m$ matrices $A, A_1, ..., A_L$ with χ not too small, Υ should be *at least* $O(1)\sqrt{\ln m}$ with appropriate positive absolute constant $O(1)$.

In view of the above facts, we could *in principle* avoid the necessity to rely on any conjecture. However, the "theoretically valid" values of Υ, χ are *by definition* worst-case oriented and can be too conservative for the particular matrices we are interested in. The situation is even worse: these theoretically valid values reflect not the worst case "as it is," but rather our abilities to analyze this worst case and therefore are conservative estimates of the "true" (and already conservative) Υ, χ. This is why we prefer to use a technique that is based on *guessing* Υ, χ and a subsequent "verification of the guess" by a *simulation-based* justification of (3.6.22).

Comments. Note that our proposed course of action is completely similar to what we did in section 2.2. The essence of the matter there was as follows: we were interested in building a safe approximation of the chance constraint

$$\sum_{\ell=1}^{L} \zeta_\ell a_\ell \le a \tag{3.6.24}$$

with deterministic $a, a_1, ..., a_L \in \mathbb{R}$ and random ζ_ℓ satisfying Assumption **A.I**. To this end, we used the *provable fact* expressed by Proposition 2.1:

> *Whenever random variables $\zeta_1, ..., \zeta_L$ satisfy* **A.I** *and deterministic reals $b, a_1, ..., a_L$ are such that*
>
> $$\sqrt{\sum_{\ell=1}^{L} a_\ell^2} \le b,$$
>
> *or, which is the same,*
>
> $$\mathrm{Arrow}(b, a_1, ..., a_L) \equiv \left[\begin{array}{c|ccc} b & a_1 & \dots & a_L \\ \hline a_1 & b & & \\ \vdots & & \ddots & \\ a_L & & & b \end{array}\right] \succeq 0,$$
>
> *one has*
>
> $$\forall \Omega > 0 : \mathrm{Prob}\left\{\sum_{\ell=1}^{L} \zeta_\ell a_\ell \le \Omega b\right\} \ge 1 - \psi(\Omega),$$
> $$\psi(\Omega) = \exp\{-\Omega^2/2\}.$$

As a result, the condition

$$\mathrm{Arrow}(\Omega^{-1}a, a_1, ..., a_L) \equiv \left[\begin{array}{c|ccc} \Omega^{-1}a & a_1 & ... & a_L \\ \hline a_1 & \Omega^{-1}a & & \\ \vdots & & \ddots & \\ a_L & & & \Omega^{-1}a \end{array}\right] \succeq 0$$

is sufficient for the validity of the chance constraint

$$\mathrm{Prob}\left\{\sum_\ell \zeta_\ell a_\ell \leq a\right\} \geq 1 - \psi(\Omega).$$

What we are doing under Assumption **A.I** now can be sketched as follows: we are interested in building a safe approximation of the chance constraint

$$\sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq A \tag{3.6.25}$$

with deterministic $A, A_1, ..., A_L \in \mathbf{S}^m$ and random ζ_ℓ satisfying Assumption **A.I**. To this end, we use the following *provable fact* expressed by Theorem 3.14:

> *Whenever random variables* $\zeta_1, ..., \zeta_L$ *satisfy* **A.I** *and deterministic symmetric matrices* $B, A_1, ..., A_L$ *are such that*
>
> $$\mathrm{Arrow}(B, A_1, ..., A_L) \equiv \left[\begin{array}{c|ccc} B & A_1 & ... & A_L \\ \hline A_1 & B & & \\ \vdots & & \ddots & \\ A_L & & & B \end{array}\right] \succeq 0, \tag{!}$$
>
> *and*
>
> $$\mathrm{Prob}\{-\Upsilon B \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Upsilon B\} \geq \chi \tag{$*$}$$
>
> *with certain* $\chi, \Upsilon > 0$, *one has*
>
> $$\forall \Omega > \Upsilon : \mathrm{Prob}\left\{\sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Omega B\right\} \geq 1 - \psi_{\Upsilon,\chi}(\Omega),$$
> $$\psi_{\Upsilon,\chi}(\Omega) = \chi^{-1}\exp\{-(\Omega-\Upsilon)^2/16\}.$$

As a result, the condition

$$\mathrm{Arrow}(\Omega^{-1}A, A_1, ..., A_L) \equiv \left[\begin{array}{c|ccc} \Omega^{-1}A & A_1 & ... & A_L \\ \hline A_1 & \Omega^{-1}A & & \\ \vdots & & \ddots & \\ A_L & & & \Omega^{-1}A \end{array}\right] \succeq 0$$

is a sufficient condition for the validity of the chance constraint

$$\mathrm{Prob}\left\{\sum_\ell \zeta_\ell A_\ell \preceq A\right\} \geq 1 - \psi_{\Upsilon,\chi}(\Omega),$$

provided that $\Omega > \Upsilon$ *and* $\chi > 0$, $\Upsilon > 0$ *are such that the matrices* $B, A_1, ..., A_L$ *satisfy* ($*$).

The constructions are pretty similar; the only difference is that in the matrix case we need an additional "provided that," which is absent in the scalar case. In fact, it is automatically present in the scalar case: from the Tschebyshev Inequality it follows that when $B, A_1, ..., A_L$ are scalars, condition (!) implies the validity of $(*)$ with, say, $\chi = 0.75$ and $\Upsilon = 2$. We now could apply the matrix-case result to recover the scalar-case, at the cost of replacing $\psi(\Omega)$ with $\psi_{2,0.75}(\Omega)$, which is not that big a loss.

Conjecture 3.1 *suggests* that in the matrix case we also should not bother much about "provided that" — it is automatically implied by (!), perhaps with a somehow worse value of Υ, but still not too large. As it was already mentioned, we can prove certain versions of the Conjecture, and we can also verify its validity, for *guessed* χ, Υ and matrices $B, A_1, ..., A_L$ that we are interested in, by simulation. The latter is the issue we consider next.

Simulation-Based Justification of (3.6.22)

Let us start with the following simple situation: there exists a random variable ξ taking value 1 with probability p and value 0 with probability $1-p$; we can simulate ξ, that is, for every sample size N, observe realizations $\xi^N = (\xi_1, ..., \xi_N)$ of N independent copies of ξ. We do not know p, and our goal is to infer a reliable lower bound on this quantity from simulations. The simplest way to do this is as follows: given "reliability tolerance" $\delta \in (0,1)$, a sample size N and an integer L, $0 \le L \le N$, let

$$\widehat{p}_{N,\delta}(L) = \min\left\{ q \in [0,1] : \sum_{k=L}^{N} \binom{N}{k} q^k (1-q)^{N-k} \ge \delta \right\}.$$

The interpretation of $\widehat{p}_{N,\delta}(L)$ is as follows: imagine we are flipping a coin, and let q be the probability to get heads. We restrict q to induce chances at least δ to get L or more heads when flipping the coin N times, and $\widehat{p}_{N,\delta}(L)$ is exactly the smallest of these probabilities q. Observe that

$$(L > 0, \widehat{p} = \widehat{p}_{N,\delta}(L)) \Rightarrow \sum_{k=L}^{N} \binom{N}{k} \widehat{p}^k (1-\widehat{p})^{N-k} = \delta \tag{3.6.26}$$

and that $\widehat{p}_{N,\delta}(0) = 0$.

An immediate observation is as follows:

Lemma 3.4 *For a fixed N, let $L(\xi^N)$ be the number of ones in a sample ξ^N, and let*

$$\widehat{p}(\xi^N) = \widehat{p}_{N,\delta}(L(\xi^N)).$$

Then

$$\mathrm{Prob}\{\widehat{p}(\xi^N) > p\} \le \delta. \tag{3.6.27}$$

Proof. Let

$$M(p) = \min\left\{ \mu \in \{0,1,...,N\} : \sum_{k=\mu+1}^{N} \binom{N}{k} p^k (1-p)^{N-k} \le \delta \right\}$$

(as always, a sum over empty set of indices is 0) and let Θ be the event $\{\xi^N : L(\xi^N) > M(p)\}$, so that by construction

$$\mathrm{Prob}\{\Theta\} \le \delta.$$

Now, the function

$$f(q) = \sum_{k=M(p)}^{N} \binom{N}{k} q^k (1-q)^{N-k}$$

is a nondecreasing function of $q \in [0,1]$, and by construction $f(p) > \delta$; it follows that if ξ^N is such that $\widehat{p} \equiv \widehat{p}(\xi^N) > p$, then $f(\widehat{p}) > \delta$ as well:

$$\sum_{k=M(p)}^{N} \binom{N}{k} \widehat{p}^k (1-\widehat{p})^{N-k} > \delta \tag{3.6.28}$$

and, besides this, $L(\xi^N) > 0$ (since otherwise $\widehat{p} = \widehat{p}_{N,\delta}(0) = 0 \le p$). Since $L(\xi^N) > 0$, we conclude from (3.6.26) that

$$\sum_{k=L(\xi^N)}^{N} \binom{N}{k} \widehat{p}^k (1-\widehat{p})^{N-k} = \delta,$$

which combines with (3.6.28) to imply that $L(\xi^N) > M(p)$, that is, ξ^N in question is such that the event Θ takes place. The bottom line is: the probability of the event $\widehat{p}(\xi^N) > p$ is at most the probability of Θ, and the latter, as we remember, is $\le \delta$. □

Lemma 3.4 says that the simulation-based (and thus random) quantity $\widehat{p}(\xi^N)$ is, with probability at least $1-\delta$, a *lower bound* for unknown probability $p \equiv \text{Prob}\{\xi = 1\}$. When p is not small, this bound is reasonably good already for moderate N, even when δ is extremely small, say, $\delta = 10^{-10}$. For example, here are simulation results for $p = 0.8$ and $\delta = 10^{-10}$:

N	10	100	1,000	10,000	100,000
$\widehat{p}$	0.06032	0.5211	0.6992	0.7814	0.7908

Coming back to our chance constrained problem (3.6.2), we can now use the outlined bounding scheme in order to carry out post-optimality analysis, namely, as follows:

> **Acceptance Test**: *Given a reliability tolerance* $\delta \in (0,1)$, *guessed* Υ, χ *and a solution* y_* *to the associated problem* (3.6.21), *build the matrices* (3.6.23). *Choose an integer* N, *generate a sample of* N *independent realizations* $\zeta^1, ..., \zeta^N$ *of the random vector* ζ, *compute the quantity*
>
> $$L = \text{Card}\{i : -\Upsilon A \preceq \sum_{\ell=1}^{L} \zeta_\ell^i A_\ell \preceq \Upsilon A\}$$
>
> *and set*
>
> $$\widehat{\chi} = \widehat{p}_{N,\delta}(L).$$
>
> *If* $\widehat{\chi} \ge \chi$, *accept* y_*, *that is, claim that* y_* *is a feasible solution to the chance constrained problem of interest* (3.6.2).

By the above analysis, the random quantity $\widehat{\chi}$ is, with probability $\ge 1-\delta$, a lower bound on $p \equiv \text{Prob}\{-\Upsilon A \preceq \sum_\ell \zeta_\ell A_\ell \preceq \Upsilon A\}$, so that the probability to accept y_* in the case when $p < \chi$ is at most δ. When this "rare event" does not occur, the relation (3.6.22) is satisfied, and therefore y_* is indeed feasible for the chance constrained problem. In other words, the probability to accept y_* when it is *not* a feasible solution to the problem of interest is at most δ.

The outlined scheme does not say what to do if y_* does *not* pass the Acceptance Test. A naive approach would be to check whether y_* satisfies the chance constraint by direct simulation. This approach indeed is workable when ϵ is not too small (say, $\epsilon \ge 0.001$); for small ϵ, however, it would require an unrealistically large simulation sample. A practical alternative is to resolve the approximating problem with Υ increased by a reasonable factor (say, 1.1 or 2), and to repeat this "trial and error" process until the Acceptance Test is passed.

A Modification

The outlined approach can be somehow streamlined when applied to a slightly modified problem (3.6.2), specifically, to the problem

$$\max_{\rho,y}\left\{\rho : \text{Prob}\left\{\mathcal{A}^{\rm n}(y) + \rho\sum_{\ell=1}^{L}\zeta_\ell \mathcal{A}^\ell(y) \succeq 0\right\} \ge 1-\epsilon, c^T y \le \tau_*, y \in \mathcal{Y}\right\} \tag{3.6.29}$$

where τ_* is a given upper bound on the original objective. Thus, now we want to maximize the level of random perturbations under the restrictions that $y \in \mathcal{Y}$ satisfies the chance constraint and is not too bad in terms of the original objective.

Approximating this problem by the method we have developed in the previous section, we end up with the problem

$$\min_{\beta,y,\{U_\ell\}} \left\{ \beta : \begin{array}{l} \left[\begin{array}{c|c} U_\ell & \mathcal{A}^\ell(y) \\ \hline \mathcal{A}^\ell(y) & \mathcal{A}^{\rm n}(y) \end{array}\right] \succeq 0,\ 1 \le \ell \le L \\ \\ \sum_\ell U_\ell \preceq \beta \mathcal{A}^{\rm n}(y),\ c^T y \le \tau_*,\ y \in \mathcal{Y} \end{array} \right\} \tag{3.6.30}$$

(cf. (3.6.21); in terms of the latter problem, $\beta = (\Omega\rho)^{-2}$, so that maximizing ρ is equivalent to minimizing β). Note that this problem remains the same whatever our guesses for Υ, χ. Further, (3.6.30) is a so called *GEVP* — Generalized Eigenvalue problem; while not being exactly a semidefinite program, it can be reduced to a "short sequence" of semidefinite programs via bisection in β and thus is efficiently solvable. Solving this problem, we arrive at a solution $\beta_*, y_*, \{U_\ell^*\}$; all we need is to understand what is the "feasibility radius" $\rho_*(y_*)$ of y_* — the largest ρ for which (y_*, ρ) satisfies the chance constraint in (3.6.29). As a matter of fact, we cannot compute this radius efficiently; what we will actually build is a reliable *lower bound* on the feasibility radius. This can be done by a suitable modification of the Acceptance Test. Let us set

$$A = \mathcal{A}^{\rm n}(y_*), A_\ell = \beta_*^{-1/2} \mathcal{A}^\ell(y_*),\ \ell = 1, ..., L; \tag{3.6.31}$$

note that these matrices satisfy (3.6.13). We apply to the matrices $A, A_1, ..., A_L$ the following procedure:

Randomized r-procedure:

Input: A collection of symmetric matrices $A, A_1, ..., A_L$ satisfying (3.6.13) and $\epsilon, \delta \in (0, 1)$.

Output: A random $r \ge 0$ such that with probability at least $1 - \delta$ one has

$$\text{Prob}\{\zeta : -A \preceq r \sum_{\ell=1}^L \zeta_\ell A_\ell \preceq A\} \ge 1 - \epsilon. \tag{3.6.32}$$

Description:

1. We choose a K-point grid $\Gamma = \{\omega_1 < \omega_2 < ... < \omega_K\}$ with $\omega_1 \ge 1$ and a reasonably large ω_K, e.g., the grid
$$\omega_k = 1.1^k$$
and choose K large enough to ensure that Conjecture 3.1 holds true with $\Upsilon = \omega_K$ and $\chi = 0.75$; note that $K = O(1)\ln(\ln m)$ will do;
2. We simulate N independent realizations $\zeta^1, ..., \zeta^N$ of ζ and compute the integers
$$L_k = \text{Card}\{i : -\omega_k A \preceq \sum_{\ell=1}^L \zeta_\ell^i A_\ell \preceq \omega_k A\}.$$
We then compute the quantities
$$\widehat{\chi}_k = \widehat{p}_{N,\delta/K}(L_k),\ k = 1, ..., K,$$
where $\delta \in (0, 1)$ is the chosen in advance "reliability tolerance."

Setting
$$\chi_k = \text{Prob}\{-\omega_k A \preceq \sum_{\ell=1}^L \zeta_\ell A_\ell \preceq \omega_k A\},$$
we infer from Lemma 3.4 that
$$\widehat{\chi}_k \le \chi_k,\ k = 1, ..., K \tag{3.6.33}$$
with probability at least $1 - \delta$.

3. We define a function $\psi(s)$, $s \geq 0$, as follows.
   In the bounded case (Assumption **A.I***)*, we set

$$\psi_k(s) = \begin{cases} 1, \; s \leq \omega_k \\ \min\left[1, \widehat{\chi}_k^{-1} \exp\{-(s-\omega_k)^2/16\}\right], \; s > \omega_k; \end{cases}$$

   In the Gaussian case (Assumption **A.II***)*, we set

$$\psi_k(s) = \begin{cases} 1, \;\text{ if } \widehat{\chi}_k \leq 1/2 \text{ or } s \leq \omega_k, \\ \mathrm{Erf}\big(\mathrm{ErfInv}(1-\widehat{\chi}_k) \\ \quad +(s-\omega_k)\max[1, \omega_k^{-1}\mathrm{ErfInv}(1-\widehat{\chi}_k)]\big), \\ \hfill \text{otherwise.} \end{cases}$$

   In both cases, we set

$$\psi(s) = \min_{1 \leq k \leq K} \psi_k(s).$$

   We claim that
   (!) *When* (3.6.33) *takes place* (recall that this happens with probability at least $1-\delta$), $\psi(s)$ *is, for all* $s \geq 0$, *an upper bound on* $1 - \mathrm{Prob}\{-sA \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq sA\}$.
   Indeed, in the case of (3.6.33), the matrices $A, A_1, ..., A_L$ (they from the very beginning are assumed to satisfy (3.6.13)) satisfy (3.6.14) with $\Upsilon = \omega_k$ and $\chi = \widehat{\chi}_k$; it remains to apply Corollary 3.1.

4. We set

$$s_* = \inf\{s \geq 0 : \psi(s) \leq \epsilon\}, \quad r = \frac{1}{s_*}$$

   and claim that with this r, (3.6.32) holds true.

Let us justify the outlined construction. Assume that (3.6.33) takes place. Then, by (!), we have

$$\mathrm{Prob}\{-sA \preceq \sum_\ell \zeta_\ell \preceq sA\} \geq 1 - \psi(s).$$

Now, the function $\psi(s)$ is clearly continuous; it follows that when s_* is finite, we have $\psi(s_*) \leq \epsilon$, and therefore (3.6.32) holds true with $r = 1/s_*$. If $s_* = +\infty$, then $r = 0$, and the validity of (3.6.32) follows from $A \succeq 0$ (the latter is due to the fact that $A, A_1, ..., A_L$ satisfy (3.6.13)).

When applying the Randomized r-procedure to matrices (3.6.31), we end up with $r = r_*$ satisfying, with probability at least $1-\delta$, the relation (3.6.32), and with our matrices $A, A_1, ..., A_L$ this relation reads

$$\mathrm{Prob}\{-\mathcal{A}^{\mathrm{n}}(y_*) \preceq r_* \beta_*^{-1/2} \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell(y_*) \preceq \mathcal{A}^{\mathrm{n}}(y_*)\} \geq 1 - \epsilon.$$

Thus, setting

$$\widehat{\rho} = \frac{r_*}{\sqrt{\beta_*}},$$

we get, with probability at least $1-\delta$, a valid lower bound on the feasibility radius $\rho_*(y_*)$ of y_*.

Illustration: Example 3.7 Revisited

Let us come back to the robust version of the Console Design problem (section 3.4.2, Example 3.7), where we were looking for a console capable (i) to withstand in a nearly optimal fashion a given load of interest, and (ii) to withstand equally well (that is, with the same or smaller compliance) every "occasional load"

g from the Euclidean ball $B_\rho = \{g : \|g\|_2 \le \rho\}$ of loads distributed along the 10 free nodes of the construction. Formally, our problem was

$$\max_{t,r}\left\{ r : \begin{array}{l} \left[\begin{array}{c|c} 2\tau_* & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \\ \left[\begin{array}{c|c} 2\tau_* & rh^T \\ \hline rh & A(t) \end{array}\right] \succeq 0 \;\forall (h : \|h\|_2 \le 1) \\ t \ge 0, \sum_{i=1}^N t_i \le 1 \end{array} \right\}, \tag{3.6.34}$$

where $\tau_* > 0$ and the load of interest f are given and $A(t) = \sum\limits_{i=1}^N t_i b_i b_i^T$ with $N = 54$ and known $(\mu = 20)$-dimensional vectors b_i. Note that what is now called r was called ρ in section 3.4.2.

Speaking about a console, it is reasonable to assume that in reality the "occasional load" vector is random $\sim \mathcal{N}(0, \rho^2 I_\mu)$ and to require that the construction should be capable of carrying such a load with the compliance $\le \tau_*$ with probability at least $1 - \epsilon$, with a very small value of ϵ, say, $\epsilon = 10^{-10}$. Let us now look for a console that satisfies these requirements with the largest possible value of ρ. The corresponding chance constrained problem is

$$\max_{t,\rho}\left\{ \rho : \begin{array}{l} \left[\begin{array}{c|c} 2\tau_* & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \\ \mathrm{Prob}_{h\sim\mathcal{N}(0,I_{20})}\left\{ \left[\begin{array}{c|c} 2\tau_* & \rho h^T \\ \hline \rho h & A(t) \end{array}\right] \succeq 0 \right\} \ge 1 - \epsilon \\ t \ge 0, \sum_{i=1}^N t_i \le 1 \end{array} \right\}, \tag{3.6.35}$$

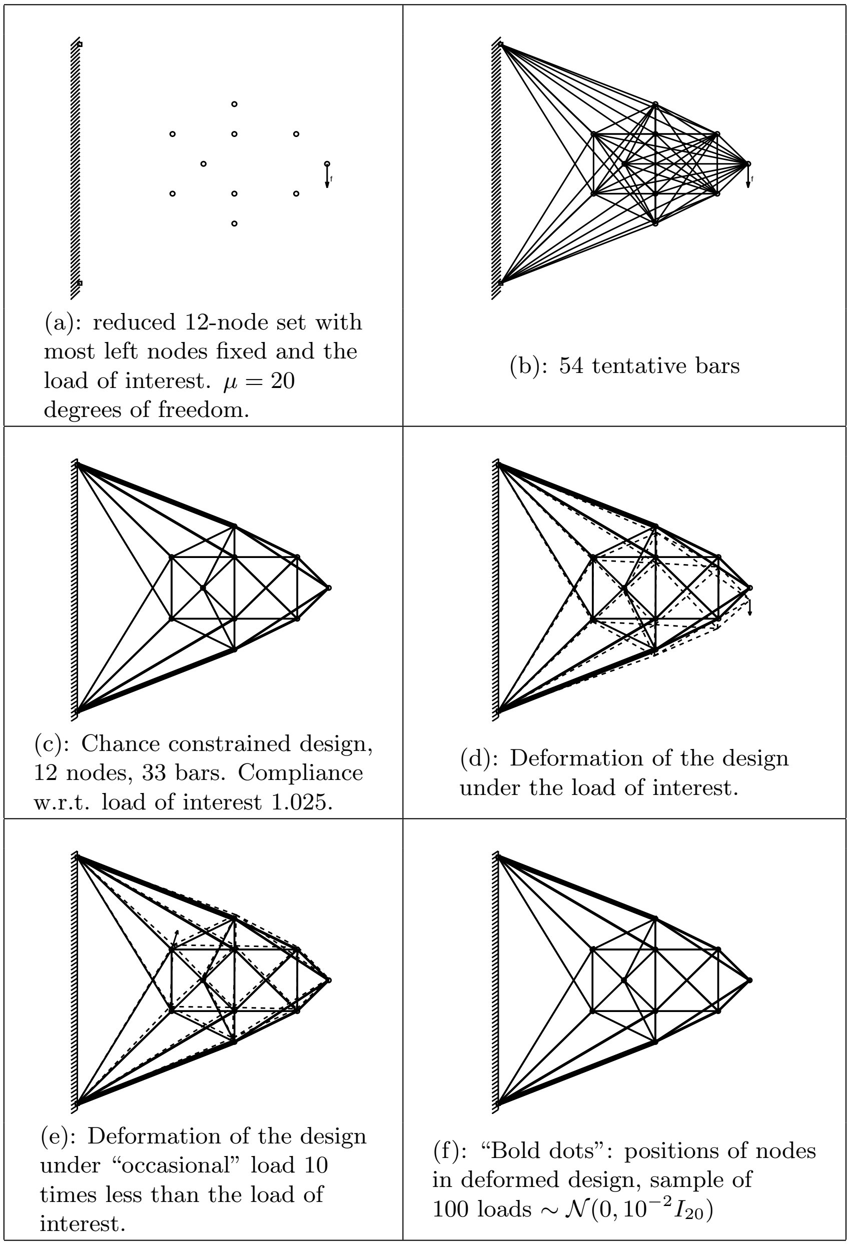
and its approximation (3.6.30) is

$$\min_{t,\beta,\{U_\ell\}_{\ell=1}^{20}}\left\{ \beta : \begin{array}{l} \left[\begin{array}{c|c} 2\tau_* & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \\ \left[\begin{array}{c|c} U_\ell & E_\ell \\ \hline E_\ell & Q(t) \end{array}\right] \succeq 0,\ 1 \le \ell \le \mu = 20 \\ \sum\limits_{\ell=1}^{\mu} U_\ell \preceq \beta Q(t),\ t \ge 0, \sum_{i=1}^N t_i \le 1 \end{array} \right\}, \tag{3.6.36}$$

where $E_\ell = e_0 e_\ell^T + e_\ell e_0^T$, $e_0, ..., e_\mu$ are the standard basic orths in $\mathbb{R}^{\mu+1} = \mathbb{R}^{21}$, and $Q(t)$ is the matrix $\mathrm{Diag}\{2\tau_*, A(t)\} \in \mathbf{S}^{\mu+1} = \mathbf{S}^{21}$.

Note that the matrices participating in this problem are simple enough to allow us to get without much difficulty a "nearly optimal" description of theoretically valid values of Υ, χ (see section 3.6.4). Indeed, here Conjecture 3.1 is valid with every $\chi \in (1/2, 1)$ provided that $\Upsilon \ge O(1)(1-\chi)^{-1/2}$. Thus, after the optimal solution $t_{\rm ch}$ to the approximating problem is found, we can avoid the simulation-based identification of a lower bound $\widehat{\rho}$ on $\rho_*(t_{\rm ch})$ (that is, on the largest ρ such that $(t_{\rm ch}, \rho)$ satisfies the chance constraint in (3.6.35)) and can get a 100%-reliable lower bound on this quantity, while the simulation-based technique is capable of providing no more than a $(1-\delta)$-reliable lower bound on $\rho_*(t_{\rm ch})$ with perhaps small, but positive δ. It turns out, however, that in our particular problem this 100%-reliable lower bound on $\rho_*(y_*)$ is significantly (by factor about 2) smaller than the $(1-\delta)$-reliable bound given by the outlined approach, even when δ is as small as 10^{-10}. This is why in the experiment we are about to discuss, we used the simulation-based lower bound on $\rho_*(t_{\rm ch})$.

The results of our experiment are as follows. The console given by the optimal solution to (3.6.36), let it be called the *chance constrained* design, is presented in figure 3.8 (cf. figures 3.3, 3.4 representing the nominal and the robust designs, respectively). The lower bounds on the feasibility radius for the chance constrained design associated with $\epsilon = \delta = 10^{-10}$ are presented in table 3.3; the plural ("bounds") comes from the fact that we worked with three different sample sizes N shown in table 3.3. Note that we can apply the outlined techniques to bound from below the feasibility radius of the *robust* design $t_{\rm rb}$ — the one given by the optimal solution to (3.6.34), see figure 3.4; the resulting bounds are presented in table 3.3.



(a): reduced 12-node set with most left nodes fixed and the load of interest. $\mu = 20$ degrees of freedom.

(b): 54 tentative bars

(c): Chance constrained design, 12 nodes, 33 bars. Compliance w.r.t. load of interest 1.025.

(d): Deformation of the design under the load of interest.

(e): Deformation of the design under "occasional" load 10 times less than the load of interest.

(f): "Bold dots": positions of nodes in deformed design, sample of 100 loads $\sim \mathcal{N}(0, 10^{-2} I_{20})$

Figure 3.8: Chance constrained design.

Design	Lower bound on feasibility radius		
	$N = 10,000$	$N = 100,000$	$N = 1,000,000$
chance constrained $t_{\rm ch}$	0.0354	0.0414	0.0431
robust $t_{\rm rb}$	0.0343	0.0380	0.0419

Table 3.3: $(1-10^{-10})$-confident lower bounds on feasibility radii for the chance constrained and the robust designs.

Finally, we note that we can exploit the specific structure of the particular problem in question to get alternative lower bounds on the feasibility radii of the chance constrained and the robust designs. Recall that the robust design ensures that the compliance of the corresponding console w.r.t. *any* load g of Euclidean norm $\leq r_*$ is at most τ_*; here $r_* \approx 0.362$ is the optimal value in (3.6.34). Now, if ρ is such that $\mathrm{Prob}_{h\sim\mathcal{N}(0,I_{20})}\{\rho\|h\|_2 > r_*\} \leq \epsilon = 10^{-10}$, then clearly ρ is a 100%-reliable lower bound on the feasibility radius of the robust design. We can easily compute the largest ρ satisfying the latter condition; it turns out to be 0.0381, 9% less than the best simulation-based lower bound. Similar reasoning can be applied to the chance constrained design $t_{\rm ch}$: we first find the largest $r = r_+$ for which $(t_{\rm ch}, r)$ is feasible for (3.6.34) (it turns out that $r_+ = 0.321$), and then find the largest ρ such that $\mathrm{Prob}_{h\sim\mathcal{N}(0,I_{20})}\{\rho\|h\|_2 > r_+\} \leq \epsilon = 10^{-10}$, ending up with the lower bound 0.0337 on the feasibility radius of the chance constrained design (25.5% worse than the best related bound in table 3.3).

3.6.3 Gaussian Majorization

Under favorable circumstances, we can apply the outlined approximation scheme to random perturbations that do not fit exactly neither Assumption **A.I**, nor Assumption **A.II.** As an instructive example, consider the case where the random perturbations ζ_ℓ, $\ell = 1,...,L$, in (3.6.1) are independent and symmetrically and *unimodally* distributed w.r.t. 0. Assume also that we can point out scaling factors $\sigma_\ell > 0$ such that the distribution of each ζ_ℓ is less diffuse than the Gaussian $\mathcal{N}(0,\sigma_\ell^2)$ distribution (see Definition 2.2). Note that in order to build a safe tractable approximation of the chance constrained LMI

$$\mathrm{Prob}\left\{\mathcal{A}^{\rm n}(y) + \sum_{\ell=1}^{L}\zeta_\ell\mathcal{A}_\ell(y) \succeq 0\right\} \geq 1-\epsilon, \tag{3.6.2}$$

or, which is the same, the constraint

$$\mathrm{Prob}\left\{\mathcal{A}^{\rm n}(y) + \sum_{\ell=1}^{L}\widetilde{\zeta}_\ell\widetilde{\mathcal{A}}^\ell(y) \succeq 0\right\} \geq 1-\epsilon \qquad \left[\begin{array}{c}\widetilde{\zeta}_\ell = \sigma_\ell^{-1}\zeta_\ell\\ \widetilde{\mathcal{A}}^\ell(y) = \sigma_\ell\mathcal{A}^\ell(y)\end{array}\right]$$

it suffices to build such an approximation for the symmetrized version

$$\mathrm{Prob}\{-\mathcal{A}^{\rm n}(y) \preceq \sum_{\ell=1}^{L}\widetilde{\zeta}_\ell\widetilde{\mathcal{A}}^\ell(y) \preceq \mathcal{A}^{\rm n}(y)\} \geq 1-\epsilon \tag{3.6.37}$$

of the constraint. Observe that the random variables $\widetilde{\zeta}_\ell$ are independent and possess symmetric and unimodal w.r.t. 0 distributions that are less diffuse than the $\mathcal{N}(0,1)$ distribution. Denoting by η_ℓ, $\ell = 1,...,L$, independent $\mathcal{N}(0,1)$ random variables and invoking the Majorization Theorem (Theorem 2.6), we see that the validity of the chance constraint

$$\mathrm{Prob}\{-\mathcal{A}^{\rm n}(y) \preceq \sum_{\ell=1}^{L}\eta_\ell\widetilde{\mathcal{A}}^\ell(y) \preceq \mathcal{A}^{\rm n}(y)\} \geq 1-\epsilon$$

— and this is the constraint we do know how to handle — is a sufficient condition for the validity of (3.6.37). Thus, in the case of unimodally and symmetrically distributed ζ_ℓ admitting "Gaussian majorants," we can act, essentially, as if we were in the Gaussian case **A.II**.

It is worth noticing that we can apply the outlined "Gaussian majorization" scheme even in the case when ζ_ℓ are symmetrically and unimodally distributed in $[-1,1]$ (a case that we know how to handle even without the unimodality assumption), and this could be profitable. Indeed, by Example 2.2 (section 2.7.2), in the case in question ζ_ℓ are less diffuse than the random variables $\eta_\ell \sim \mathcal{N}(0,2/\pi)$, and we can again reduce the situation to Gaussian. The advantage of this approach is that the absolute constant factor $\frac{1}{16}$ in the exponent in (3.6.15) is rather small. Therefore replacing (3.6.15) with (3.6.16), even after replacing our original variables ζ_ℓ with their less concentrated "Gaussian majorants" η_ℓ, can lead to better results. To illustrate this point, here is a report on a numerical experiment:

1) We generated $L = 100$ matrices $A_\ell \in \mathbf{S}^{40}$, $\ell = 1,...,L$, such that $\sum_\ell A_\ell^2 \preceq I$, (which clearly implies that $A = I, A_1,...,A_L$ satisfy (3.6.13));

2) We applied the bounded case version of the Randomized r procedure to the matrices $A, A_1,...,A_L$ and the independent random variables ζ_ℓ uniformly distributed on $[-1,1]$, setting δ and ϵ to 10^{-10};

3) We applied the Gaussian version of the same procedure, with the same ϵ, δ, to the matrices $A, A_1,...,A_L$ and independent $\mathcal{N}(0,2/\pi)$ random variables η_ℓ in the role of ζ_ℓ.

In both 2) and 3), we used the same grid $\omega_k = 0.01 \cdot 10^{0.1k}$, $0 \le k \le 40$.

By the above arguments, both in 2) and in 3) we get, with probability at least $1-10^{-10}$, lower bounds on the largest ρ such that

$$\mathrm{Prob}\{-I \preceq \rho \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq I\} \ge 1 - 10^{-10}.$$

Here are the bounds obtained:

Bounding scheme	Lower Bound	
	$N = 1000$	$N = 10000$
2)	0.0489	0.0489
3)	0.185	0.232

We see that while we can process the case of uniformly distributed ζ_ℓ "as it is," it is better to process it via Gaussian majorization.

To conclude this section, we present another "Gaussian Majorization" result. Its advantage is that it does not require the random variables ζ_ℓ to be symmetrically or unimodally distributed; what we need, essentially, is just independence plus zero means. We start with some definitions. Let $\mathcal{R}_n$ be the space of Borel probability distributions on $\mathbb{R}^n$ with zero mean. For a random variable η taking values in $\mathbb{R}^n$, we denote by P_η the corresponding distribution, and we write $\eta \in \mathcal{R}_n$ to express that $P_\eta \in \mathcal{R}_n$. Let also $\mathcal{CF}_n$ be the set of all *convex* functions f on $\mathbb{R}^n$ with linear growth, meaning that there exists $c_f < \infty$ such that $|f(u)| \le c_f(1 + \|u\|_2)$ for all u.

Definition 3.7 *Let $\xi, \eta \in \mathcal{R}_n$. We say that η dominates ξ (notation: $\xi \preceq_c \eta$, or $P_\xi \preceq_c P_\eta$, or $\eta \succeq_c \xi$, or $P_\eta \succeq_c P_\xi$) if*

$$\int f(u)dP_\xi(u) \le \int f(u)dP_\eta(u)$$

for every $f \in \mathcal{CF}_n$.

Note that in the literature the relation $\succeq_c$ is called "convex dominance." The properties of the relation $\succeq_c$ we need are summarized as follows:

Proposition 3.3

1. *$\preceq_c$ is a partial order on $\mathcal{R}_n$.*
2. *If $P_1,...,P_k,Q_1,...,Q_k \in \mathcal{R}_n$, and $P_i \preceq_c Q_i$ for every i, then $\sum_i \lambda_i P_i \preceq_c \sum_i \lambda_i Q_i$ for all nonnegative λ_i with unit sum.*

3. *If* $\xi \in \mathcal{R}_n$ *and* $t \geq 1$ *is deterministic, then* $t\xi \succeq_{\rm c} \xi$.

4. *Let* $P_1, Q_1 \in \mathcal{R}_r$, $P_2, Q_2 \in \mathcal{R}_s$ *be such that* $P_i \preceq_{\rm c} Q_i$, $i = 1,2$. *Then* $P_1 \times P_2 \preceq_{\rm c} Q_1 \times Q_2$. *In particular, if* $\xi_1, ..., \xi_n, \eta_1, ..., \eta_n \in \mathcal{R}_1$ *are independent and* $\xi_i \preceq_{\rm c} \eta_i$ *for every* i, *then* $[\xi_1; ...; \xi_n] \preceq_{\rm c} [\eta_1; ...; \eta_n]$.

5. *If* $\xi_1, ..., \xi_k, \eta_1, ..., \eta_k \in \mathcal{R}_n$ *are independent random variables,* $\xi_i \preceq_{\rm c} \eta_i$ *for every* i, *and* $S_i \in \mathbb{R}^{m\times n}$ *are deterministic matrices, then* $\sum_i S_i\xi_i \preceq_{\rm c} \sum_i S_i\eta_i$.

6. *Let* $\xi \in \mathcal{R}_1$ *be supported on* $[-1,1]$ *and* $\eta \sim \mathcal{N}(0, \pi/2)$. *Then* $\eta \succeq_{\rm c} \xi$.

7. *If* ξ, η *are symmetrically and unimodally distributed w.r.t. the origin scalar random variables with finite expectations and* $\eta \succeq_{\rm m} \xi$ *(see section 2.7.2), then* $\eta \succeq_{\rm c} \xi$ *as well. In particular, if* ξ *has unimodal w.r.t. 0 distribution and is supported on* $[-1,1]$ *and* $\eta \sim \mathcal{N}(0, 2/\pi)$, *then* $\eta \succeq_{\rm c} \xi$ *(cf. Example 2.2).*

8. *Assume that* $\xi \in \mathcal{R}_n$ *is supported in the unit cube* $\{u : \|u\|_\infty \leq 1\}$ *and is "absolutely symmetrically distributed," meaning that if* J *is a diagonal matrix with diagonal entries* ± 1, *then* $J\xi$ *has the same distribution as* ξ. *Let also* $\eta \sim \mathcal{N}(0, (\pi/2)I_n)$. *Then* $\xi \preceq_{\rm c} \eta$.

9. *Let* $\xi, \eta \in \mathcal{R}_r$, $\xi \sim \mathcal{N}(0, \Sigma)$, $\eta \sim \mathcal{N}(0, \Theta)$ *with* $\Sigma \preceq \Theta$. *Then* $\xi \preceq_{\rm c} \eta$.

The main result here is as follows.

Theorem 3.16 [Gaussian Majorization [3, Theorem 10.3.3]] *Let* $\eta \sim \mathcal{N}(0, I_L)$, *and let* $\zeta \in \mathcal{R}_L$ *be such that* $\zeta \preceq_{\rm c} \eta$. *Let, further,* $Q \subset \mathbb{R}^L$ *be a closed convex set such that*

$$\chi \equiv \text{Prob}\{\eta \in Q\} > 1/2.$$

Then for every $\gamma > 1$, *one has*

$$\begin{array}{rl} \text{Prob}\{\zeta \notin \gamma Q\} \leq & \inf\limits_{1\leq\beta<\gamma} \frac{1}{\gamma-\beta} \int\limits_\beta^\infty \text{Erf}(r\text{ErfInv}(1-\chi))dr \\ \leq & \inf\limits_{1\leq\beta<\gamma} \frac{1}{2(\gamma-\beta)} \int\limits_\beta^\infty \exp\{-r^2\text{ErfInv}^2(1-\chi)/2\}dr, \end{array} \quad (3.6.38)$$

where Erf(·), ErfInv(·) *are given by* (2.2.6), (2.2.7).

The assumption $\zeta \preceq_{\rm c} \eta$ *is valid, in particular, if* $\zeta = [\zeta_1; ...; \zeta_L]$ *with independent* ζ_ℓ *such that* $P_{\zeta_\ell} \in \mathcal{R}_1$ *and* $P_{\zeta_\ell} \preceq_{\rm c} \mathcal{N}(0,1)$.

3.6.4 Chance Constrained LMIs: Special Cases

We intend to consider two cases where it is easy to justify Conjecture 3.1. While the structural assumptions on the matrices $A, A_1, ..., A_L$ in these two cases seem to be highly restrictive, the results are nevertheless important: they cover the situations arising in randomly perturbed Linear and Conic Quadratic Optimization. We begin with a slight relaxation of Assumptions **A.I–II**:

> **Assumption A.III**: The random perturbations $\zeta_1, ..., \zeta_L$ are independent, zero mean and "of order of 1," meaning that
>
> $$\mathbf{E}\{\exp\{\zeta_\ell^2\}\} \leq \exp\{1\},\ \ell = 1, ..., L.$$

Note that Assumption **A.III** is implied by **A.I** and is "almost implied" by **A.II**; indeed, $\zeta_\ell \sim \mathcal{N}(0,1)$ implies that the random variable $\widetilde{\zeta}_\ell = \sqrt{(1-e^{-2})/2}\zeta_\ell$ satisfies $\mathbf{E}\{\exp\{\widetilde{\zeta}_\ell^2\}\} \leq \exp\{1\}$.

The Diagonal Case: Chance Constrained Linear Optimization

Theorem 3.17 *Let $A, A_1, ..., A_L \in \mathbf{S}^m$ be diagonal matrices satisfying* (3.6.13) *and let the random variables ζ_ℓ satisfy Assumption* **A.III**. *Then, for every $\chi \in (0,1)$, with $\Upsilon = \Upsilon(\chi) \equiv \sqrt{3\ln\left(\frac{2m}{1-\chi}\right)}$ one has*

$$\mathrm{Prob}\{-\Upsilon A \preceq \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq \Upsilon A\} \geq \chi \tag{3.6.39}$$

(cf. (3.6.14)*). In the case of $\zeta_\ell \sim \mathcal{N}(0,1)$, relation* (3.6.39) *holds true with $\Upsilon = \Upsilon(\chi) \equiv \sqrt{2\ln\left(\frac{m}{1-\chi}\right)}$.*

Proof. It is immediately seen that we lose nothing when assuming that $A \succ 0$ (cf. the proof of Corollary 3.1). With this assumption, passing from diagonal matrices A, A_ℓ to the diagonal matrices $B_\ell = A^{-1/2} A_\ell A^{-1/2}$, the statement to be proved reads as follows:

> *If $B_\ell \in \mathbf{S}^m$ are deterministic diagonal matrices such that $\sum_\ell B_\ell^2 \preceq I$ and ζ_ℓ satisfy* **A.III**, *then, for every $\chi \in (0,1)$, one has*
>
> $$\mathrm{Prob}\{\|\sum_{\ell=1}^{L} \zeta_\ell B_\ell\| \leq \underbrace{\sqrt{3\ln\left(\frac{2m}{1-\chi}\right)}}_{\Upsilon(\chi)}\} \geq \chi. \tag{3.6.40}$$
>
> *When $\zeta_\ell \sim \mathcal{N}(0,1)$, $\ell = 1, ..., L$, the relation remains true with $\Upsilon(\chi)$ reduced to $\sqrt{2\ln(m/(1-\chi))}$.*

The proof of the latter statement is based on the standard argument used in deriving results on large deviations of sums of "light-tail" independent random variables. First we need the following result.

Lemma 3.5 *Let β_ℓ, $\ell = 1, ..., L$, $\gamma > 0$ be deterministic reals such that $\sum_\ell \beta_\ell^2 \leq 1$. Then*

$$\forall \Upsilon > 0 : \mathrm{Prob}\left\{|\sum_{\ell=1}^{L} \beta_\ell \zeta_\ell| > \Upsilon\right\} \leq 2\exp\{-\Upsilon^2/3\}. \tag{3.6.41}$$

Proof of Lemma 3.5. Observe, first, that whenever ξ is a random variable with zero mean such that $\mathbf{E}\{\exp\{\xi^2\}\} \leq \exp\{1\}$, one has

$$\mathbf{E}\{\exp\{\gamma\xi\}\} \leq \exp\{3\gamma^2/4\}. \tag{3.6.42}$$

> Indeed, observe that by Holder Inequality the relation $\mathbf{E}\left\{\exp\{\xi^2\}\right\} \leq \exp\{1\}$ implies that $\mathbf{E}\left\{\exp\{s\xi^2\}\right\} \leq \exp\{s\}$ for all $s \in [0,1]$. It is immediately seen that $\exp\{x\} - x \leq \exp\{9x^2/16\}$ for all x. Assuming that $9\gamma^2/16 \leq 1$, we therefore have
>
> $$\begin{aligned}\mathbf{E}\left\{\exp\{\gamma\xi\}\right\} &= \mathbf{E}\left\{\exp\{\gamma\xi\} - \gamma\xi\right\} \ [\xi \text{ is with zero mean}] \\ &\leq \mathbf{E}\left\{\exp\{9\gamma^2\xi^2/16\}\right\} \\ &\leq \exp\{9\gamma^2/16\} \ [\text{since } 9\gamma^2/16 \leq 1] \\ &\leq \exp\{3\gamma^2/4\},\end{aligned}$$
>
> as required in (3.6.42). Now let $9\gamma^2/16 \geq 1$. For all γ we have $\gamma\xi \leq 3\gamma^2/8 + 2\xi^2/3$, whence
>
> $$\begin{aligned}\mathbf{E}\left\{\exp\{\gamma\xi\}\right\} \leq \ & \exp\{3\gamma^2/8\}\exp\{2\xi^2/3\} \leq \exp\{3\gamma^2/8 + 2/3\} \\ & \leq \exp\{3\gamma^2/4\} \ [\text{since } \gamma^2 \geq 16/9]\end{aligned}$$
>
> We see that (3.6.42) is valid for all γ.

We now have

$$\begin{array}{l}\mathbf{E}\left\{\exp\{\gamma\sum_{\ell=1}^L\beta_\ell\zeta_\ell\}\right\}=\prod\limits_{\ell=1}^L\mathbf{E}\left\{\exp\{\gamma\beta_\ell\zeta_\ell\}\right\}\ \ [\zeta_1,...,\zeta_L \text{ are independent}]\\ \leq \prod\limits_{\ell=1}^L \exp\{3\gamma^2\beta_\ell^2/4\}\ [\text{by Lemma}]\\ \leq \exp\{3\gamma^2/4\}\ [\text{since } \sum_\ell\beta_\ell^2\leq 1].\end{array}$$

We now have

$$\begin{array}{l}\text{Prob}\left\{\sum_{\ell=1}^L\beta_\ell\zeta_\ell>\Upsilon\right\}\\ \leq \min_{\gamma\geq 0}\exp\{-\Upsilon\gamma\}\mathbf{E}\left\{\exp\{\gamma\sum_\ell\beta_\ell\zeta_\ell\}\right\}\ \ [\text{Tschebyshev Inequality}]\\ \leq \min_{\gamma\geq 0}\exp\{-\Upsilon\gamma+3\gamma^2/4\}\ [\text{by (3.6.42)}]\\ =\exp\{-\Upsilon^2/3\}.\end{array}$$

Replacing ζ_ℓ with $-\zeta_\ell$, we get that $\text{Prob}\left\{\sum_\ell\beta_\ell\zeta_\ell<-\Upsilon\right\}\leq\exp\{-\Upsilon^2/3\}$ as well, and (3.6.41) follows. □

Proof of (3.6.39). Let s_i be the i-th diagonal entry in the random diagonal matrix $S=\sum\limits_{\ell=1}^L\zeta_\ell B_\ell$. Taking into account that B_ℓ are diagonal with $\sum\limits_\ell B_\ell^2\preceq I$, we can apply Lemma 3.5 to get the bound

$$\text{Prob}\{|s_i|>\Upsilon\}\leq 2\exp\{-\Upsilon^2/3\};$$

since $\|S\|=\max\limits_{1\leq i\leq m}|s_i|$, (3.6.40) follows.

Refinements in the case of $\zeta_\ell\sim\mathcal{N}(0,1)$ are evident: here the i-th diagonal entry s_i in the random diagonal matrix $S=\sum\limits_\ell\zeta_\ell B_\ell$ is $\sim\mathcal{N}(0,\sigma_i^2)$ with $\sigma_i\leq 1$, whence $\text{Prob}\{|s_i|>\Upsilon\}\leq\exp\{-\Upsilon^2/2\}$ and therefore $\text{Prob}\{\|S\|>\Upsilon\}\leq m\exp\{-\Upsilon^2/2\}$, so that $\Upsilon(\chi)$ in (3.6.40) can indeed be reduced to $\sqrt{2\ln(m/(1-\chi))}$. □

The case of chance constrained LMI with diagonal matrices $\mathcal{A}^{\rm n}(y)$, $\mathcal{A}^\ell(y)$ has an important application — Chance Constrained Linear Optimization. Indeed, consider a randomly perturbed Linear Optimization problem

$$\min_y\left\{c^Ty: A_\zeta y\geq b_\zeta\right\} \tag{3.6.43}$$

where A_ζ, b_ζ are affine in random perturbations ζ:

$$[A_\zeta,b_\zeta]=[A^{\rm n},b^{\rm n}]+\sum_{\ell=1}^L\zeta_\ell[A^\ell,b^\ell];$$

as usual, we have assumed w.l.o.g. that the objective is certain. The chance constrained version of this problem is

$$\min_y\left\{c^Ty:\text{Prob}\{A_\zeta y\geq b_\zeta\}\geq 1-\epsilon\right\}. \tag{3.6.44}$$

Setting $\mathcal{A}^{\rm n}(y)=\text{Diag}\{A^{\rm n}y-b^{\rm n}\}$, $\mathcal{A}^\ell(y)=\text{Diag}\{A^\ell y-b^\ell\}$, $\ell=1,...,L$, we can rewrite (3.6.44) equivalently as the chance constrained semidefinite problem

$$\min_y\left\{c^Ty:\text{Prob}\{\mathcal{A}_\zeta(y)\succeq 0\}\geq 1-\epsilon\right\},\ \mathcal{A}_\zeta(y)=\mathcal{A}^{\rm n}(y)+\sum_\ell\zeta_\ell\mathcal{A}^\ell(y), \tag{3.6.45}$$

and process this problem via the outlined approximation scheme. Note the essential difference between what we are doing now and what was done in lecture 2. There we focused on safe approximation of chance constrained *scalar* linear inequality, here we are speaking about approximating a chance constrained coordinate-wise *vector* inequality. Besides this, our approximation scheme is, in general, "semi-analytic" — it involves simulation and as a result produces a solution that is feasible for the chance constrained problem with probability close to 1, but not with probability 1.

Of course, the safe approximations of chance constraints developed in lecture 2 can be used to process coordinate-wise vector inequalities as well. The natural way to do it is to replace the chance constrained vector inequality in (3.6.44) with a bunch of chance constrained scalar inequalities

$$\mathrm{Prob}\left\{(A_\zeta y - b_\zeta)_i \geq 0\right\} \geq 1-\epsilon_i,\ i=1,...,m \equiv \dim b_\zeta, \tag{3.6.46}$$

where the tolerances $\epsilon_i \geq 0$ satisfy the relation $\sum\limits_i \epsilon_i = \epsilon$. The validity of (3.6.46) clearly is a sufficient condition for the validity of the chance constraint in (3.6.44), so that replacing these constraints with their safe tractable approximations from lecture 2, we end up with a safe tractable approximation of the chance constrained LO problem (3.6.44). A drawback of this approach is in the necessity to "guess" the quantities ϵ_i. The ideal solution would be to treat them as additional decision variables and to optimize the safe approximation in both y and ϵ_i. Unfortunately, all approximation schemes for scalar chance constraints presented in lecture 2 result in approximations that are *not* jointly convex in $y, \{\epsilon_i\}$. As a result, joint optimization in y, ϵ_i is more wishful thinking than a computationally solid strategy. Seemingly the only simple way to resolve this difficulty is to set all ϵ_i equal to ϵ/m.

It is instructive to compare the "constraint-by-constraint" safe approximation of a chance constrained LO (3.6.44) given by the results of lecture 2 with our present approximation scheme. To this end, let us focus on the following version of the chance constrained problem:

$$\max_{\rho,y}\left\{\rho : c^T y \leq \tau_*, \mathrm{Prob}\left\{A_{\rho\zeta} y \geq b_{\rho\zeta}\right\} \geq 1-\epsilon\right\} \tag{3.6.47}$$

(cf. (3.6.29)). To make things as simple as possible, we assume also that $\zeta_\ell \sim \mathcal{N}(0,1)$, $\ell = 1,...,L$.

The "constraint-by-constraint" safe approximation of (3.6.47) is the chance constrained problem

$$\max_{\rho,y}\left\{\rho : c^T y \leq \tau_*, \mathrm{Prob}\left\{(A_{\rho\zeta} y - b_{\rho\zeta})_i \geq 0\right\} \geq 1-\epsilon/m\right\},$$

where m is the number of rows in A_ζ. A chance constraint

$$\mathrm{Prob}\left\{(A_{\rho\zeta} y - b_{\rho\zeta})_i \geq 0\right\} \geq 1-\epsilon/m$$

can be rewritten equivalently as

$$\mathrm{Prob}\{[b^{\mathrm{n}} - A^{\mathrm{n}} y]_i + \rho\sum_{\ell=1}^{L}[b^\ell - A^\ell y]_i \zeta_\ell > 0\} \leq \epsilon/m.$$

Since $\zeta_\ell \sim \mathcal{N}(0,1)$ are independent, this scalar chance constraint is exactly equivalent to

$$[b^{\mathrm{n}} - A^{\mathrm{n}} y]_i + \rho \mathrm{ErfInv}(\epsilon/m)\sqrt{\sum_\ell [b^\ell - A^\ell y]_i^2} \leq 0.$$

The associated safe tractable approximation of the problem of interest (3.6.47) is the conic quadratic program

$$\max_{\rho,y}\left\{\rho : c^T y \leq \tau_*, \mathrm{ErfInv}(\epsilon/m)\sqrt{\sum_\ell [b^\ell - A^\ell y]_i^2} \leq \frac{[A^{\mathrm{n}} y - b^{\mathrm{n}}]_i}{\rho}, 1 \leq i \leq m\right\}. \tag{3.6.48}$$

Now let us apply our new approximation scheme, which treats the chance constrained vector inequality in (3.6.44) "as a whole." To this end, we should solve the problem

$$\min_{\nu,y,\{U_\ell\}}\left\{\nu : \begin{array}{l} c^T y \leq \tau_*, \left[\begin{array}{c|c} U_\ell & \mathrm{Diag}\{A^\ell y - b^\ell\} \\ \hline \mathrm{Diag}\{A^\ell y - b^\ell\} & \mathrm{Diag}\{A^{\mathrm{n}} y - b^{\mathrm{n}}\}\end{array}\right] \succeq 0, \\ \hfill 1 \leq \ell \leq L \\ \sum\limits_\ell U_\ell \preceq \nu \mathrm{Diag}\{A^{\mathrm{n}} y - b^{\mathrm{n}}\},\ c^T y \leq \tau_* \end{array}\right\}, \tag{3.6.49}$$

treat its optimal solution y_* as the y component of the optimal solution to the approximation and then bound from below the feasibility radius $\rho_*(y_*)$ of this solution, (e.g., by applying to y_* the Randomized r procedure). Observe that problem (3.6.49) is nothing but the problem

$$\min_{\nu,y}\left\{\nu : \begin{array}{l}\sum\limits_{\ell=1}^{L}[A^\ell y-b]_i^2/[A^{\rm n}y-b^{\rm n}]_i\leq\nu[A^{\rm n}y-b^{\rm n}]_i, 1\leq i\leq m,\\ A^{\rm n}y-b^{\rm n}\geq 0,\ c^Ty\leq\tau_*\end{array}\right\},$$

where $a^2/0$ is 0 for $a=0$ and is $+\infty$ otherwise. Comparing the latter problem with (3.6.48), we see that

> *Problems* (3.6.49) *and* (3.6.48) *are equivalent to each other, the optimal values being related as*
> $$\mathrm{Opt}(3.6.48)=\frac{1}{\mathrm{ErfInv}(\epsilon/m)\sqrt{\mathrm{Opt}(3.6.49)}}.$$
> *Thus, the approaches we are comparing result in the same vector of decision variables y_*, the only difference being the resulting value of a lower bound on the feasibility radius of y_*. With the "constraint-by-constraint" approach originating from lecture 2, this value is the optimal value in* (3.6.48), *while with our new approach, which treats the vector inequality $Ax\geq b$ "as a whole," the feasibility radius is bounded from below via the provable version of Conjecture 3.1 given by Theorem 3.17, or by the Randomized r procedure.*

A natural question is, which one of these approaches results in a less conservative lower bound on the feasibility radius of y_*. On the theoretical side of this question, it is easily seen that when the second approach utilizes Theorem 3.17, it results in the same (within an absolute constant factor) value of ρ as the first approach. From the practical perspective, however, it is much more interesting to consider the case where the second approach exploits the Randomized r procedure, since experiments demonstrate that this version is less conservative than the "100%-reliable" one based on Theorem 3.17. Thus, let us focus on comparing the "constraint-by-constraint" safe approximation of (3.6.44), let it be called Approximation I, with Approximation II based on the Randomized r procedure. Numerical experiments show that no one of these two approximations "generically dominates" the other one, so that the best thing is to choose the best — the largest — of the two respective lower bounds.

The Arrow Case: Chance Constrained Conic Quadratic Optimization

We are about to justify Conjecture 3.1 in the *arrow-type case*, that is, when the matrices $A_\ell\in\mathbf{S}^m$, $\ell=1,...,L$, are of the form

$$A_\ell=[ef_\ell^T+f_\ell e^T]+\lambda_\ell G, \tag{3.6.50}$$

where $e,f_\ell\in\mathbb{R}^m$, $\lambda_\ell\in\mathbb{R}$ and $G\in\mathbf{S}^m$. We encounter this case in the Chance Constrained Conic Quadratic Optimization. Indeed, a Chance Constrained CQI

$$\mathrm{Prob}\{\|A(y)\zeta+b(y)\|_2\leq c^T(y)\zeta+d(y)\}\geq 1-\epsilon, \qquad [A(\cdot):p\times q]$$

can be reformulated equivalently as the chance constrained LMI

$$\mathrm{Prob}\{\left[\begin{array}{c|c}c^T(y)\zeta+d(y) & \zeta^TA^T(y)+b^T(y)\\ \hline A(y)\zeta+b(y) & (c^T(y)\zeta+d(y))I\end{array}\right]\succeq 0\}\geq 1-\epsilon \tag{3.6.51}$$

(see Lemma 3.1). In the notation of (3.6.1), for this LMI we have

$$\mathcal{A}^{\rm n}(y)=\left[\begin{array}{c|c}d(y) & b^T(y)\\ \hline b(y) & d(y)I\end{array}\right],\ \mathcal{A}^\ell(y)=\left[\begin{array}{c|c}c_\ell(y) & a_\ell^T(y)\\ \hline a_\ell(y) & c_\ell(y)I\end{array}\right],$$

where $a_\ell(y)$ in (3.6.50) is ℓ-th column of $A(y)$. We see that the matrices $\mathcal{A}^\ell(y)$ are arrow-type $(p+1)\times(p+1)$ matrices where e in (3.6.50) is the first basic orth in $\mathbb{R}^{p+1}$, $f_\ell=[0;a_\ell(y)]$ and $G=I_{p+1}$.

Another example is the one arising in the chance constrained Truss Topology Design problem, see section 3.6.2.

The justification of Conjecture 3.1 in the arrow-type case is given by the following

Theorem 3.18 *Let $m\times m$ matrices $A_1,...,A_L$ of the form* (3.6.50) *along with a matrix $A\in\mathbf{S}^m$ satisfy the relation* (3.6.13)*, and ζ_ℓ be independent with zero means and such that $\mathbf{E}\{\zeta_\ell^2\}\leq\sigma^2$, $\ell=1,...,L$ (under Assumption* **A.III***, one can take $\sigma=\sqrt{\exp\{1\}-1}$). Then, for every $\chi\in(0,1)$, with $\Upsilon=\Upsilon(\chi)\equiv\frac{2\sqrt{2}\sigma}{\sqrt{1-\chi}}$ one has*

$$\mathrm{Prob}\{-\Upsilon A\preceq\sum_{\ell=1}^L\zeta_\ell A_\ell\preceq\Upsilon A\}\geq\chi \tag{3.6.52}$$

(cf. (3.6.14)*). When ζ satisfies Assumption* **A.I***, or ζ satisfies Assumption* **A.II** *and $\chi\geq\frac{6}{7}$, relation* (3.6.52) *is satisfied with $\Upsilon=\Upsilon_{\mathrm{I}}(\chi)\equiv 2+4\sqrt{3\ln\frac{4}{1-\chi}}$ and with $\Upsilon=\Upsilon_{\mathrm{II}}(\chi)\equiv\sqrt{3\left(1+3\ln\frac{1}{1-\chi}\right)}$, respectively.*

Proof. First of all, when ζ_ℓ, $\ell=1,...,L$, satisfy Assumption **A.III**, we indeed have $\mathbf{E}\{\zeta_\ell^2\}\leq\exp\{1\}-1$ due to $t^2\leq\exp\{t^2\}-1$ for all t. Further, same as in the proof of Theorem 3.17, it suffices to consider the case when $A\succ 0$ and to prove the following statement:

> *Let A_ℓ be of the form of* (3.6.50) *and such that the matrices $B_\ell=A^{-1/2}A_\ell A^{-1/2}$ satisfy $\sum\limits_\ell B_\ell^2\preceq I$. Let, further, ζ_ℓ satisfy the premise in Theorem 3.18. Then, for every $\chi\in(0,1)$, one has*
>
> $$\mathrm{Prob}\{\|\sum_{\ell=1}^L\zeta_\ell B_\ell\|\leq\frac{2\sqrt{2}\sigma}{\sqrt{1-\chi}}\}\geq\chi. \tag{3.6.53}$$

Observe that B_ℓ are also of the arrow-type form (3.6.50):

$$B_\ell=[gh_\ell^T+h_\ell g^T]+\lambda_\ell H \qquad [g=A^{-1/2}e,h_\ell=A^{-1/2}f_\ell,H=A^{-1/2}GA^{-1/2}]$$

Note that w.l.o.g. we can assume that $\|g\|_2=1$ and then rotate the coordinates to make g the first basic orth. In this situation, the matrices B_ℓ become

$$B_\ell=\left[\begin{array}{c|c}q_\ell & r_\ell^T\\\hline r_\ell & \lambda_\ell Q\end{array}\right]; \tag{3.6.54}$$

by appropriate scaling of λ_ℓ, we can ensure that $\|Q\|=1$. We have

$$B_\ell^2=\left[\begin{array}{c|c}q_\ell^2+r_\ell^Tr_\ell & q_\ell r_\ell^T+\lambda_\ell r_\ell^TQ\\\hline q_\ell r_\ell+\lambda_\ell Qr_\ell & r_\ell r_\ell^T+\lambda_\ell^2Q^2\end{array}\right].$$

We conclude that $\sum\limits_{\ell=1}^L B_\ell^2\preceq I_m$ implies that $\sum\limits_\ell(q_\ell^2+r_\ell^Tr_\ell)\leq 1$ and $[\sum\limits_\ell\lambda_\ell^2]Q^2\preceq I_{m-1}$; since $\|Q^2\|=1$, we arrive at the relations

$$\begin{array}{ll}(a) & \sum_\ell\lambda_\ell^2\leq 1,\\ (b) & \sum_\ell(q_\ell^2+r_\ell^Tr_\ell)\leq 1.\end{array} \tag{3.6.55}$$

Now let $p_\ell=[0;r_\ell]\in\mathbb{R}^m$. We have

$$\begin{array}{rl} & S[\zeta]\equiv\sum_\ell\zeta_\ell B_\ell=[g^T(\underbrace{\sum\nolimits_\ell\zeta_\ell p_\ell}_{\xi})+\xi^Tg]+\mathrm{Diag}\{\underbrace{\sum\nolimits_\ell\zeta_\ell q_\ell}_{\theta},(\underbrace{\sum\nolimits_\ell\zeta_\ell\lambda_\ell}_{\eta})Q\}\\ \Rightarrow & \|S[\zeta]\|\leq\|g\xi^T+\xi g^T\|+\max[|\theta|,|\eta|\|Q\|]=\|\xi\|_2+\max[|\theta|,|\eta|].\end{array}$$

Setting

$$\alpha=\sum_\ell r_\ell^Tr_\ell,\ \beta=\sum_\ell q_\ell^2,$$

we have $\alpha+\beta\leq 1$ by (3.6.55.b). Besides this,

$$\begin{array}{ll}
\mathbf{E}\{\xi^T\xi\} = \sum_{\ell,\ell'}\mathbf{E}\{\zeta_\ell\zeta_{\ell'}\}p_\ell^Tp_{\ell'} = \sum_\ell \mathbf{E}\{\zeta_\ell^2\}r_\ell^Tr_\ell & [\zeta_\ell \text{ are independent zero mean}]\\
\leq \sigma^2\sum_\ell r_\ell^Tr_\ell = \sigma^2\alpha & \\
\quad\Rightarrow \quad \mathrm{Prob}\{\|\xi\|_2>t\}\leq \frac{\sigma^2\alpha}{t^2}\ \forall t>0 & [\text{Tschebyshev Inequality}]\\
\mathbf{E}\{\eta^2\} = \sum_\ell \mathbf{E}\{\zeta_\ell^2\}\lambda_\ell^2 \leq \sigma^2\sum_\ell\lambda_\ell^2\leq\sigma^2 & [\ (3.6.55.a)]\\
\quad\Rightarrow \quad \mathrm{Prob}\{|\eta|>t\}\leq \frac{\sigma^2}{t^2}\ \forall t>0 & [\text{Tschebyshev Inequality}]\\
\mathbf{E}\{\theta^2\} = \sum_\ell \mathbf{E}\{\zeta_\ell^2\}q_\ell^2\leq\sigma^2\beta & \\
\quad\Rightarrow \quad \mathrm{Prob}\{|\theta|>t\}\leq \frac{\sigma^2\beta}{t^2}\ \forall t>0 & [\text{Tschebyshev Inequality}].
\end{array}$$

Thus, for every $\Upsilon>0$ and all $\lambda\in(0,1)$ we have

$$\begin{array}{rcl}
\mathrm{Prob}\{\|S[\zeta]\|>\Upsilon\} & \leq & \mathrm{Prob}\{\|\xi\|_2+\max[|\theta|,|\eta|]>\Upsilon\}\leq \mathrm{Prob}\{\|\xi\|_2>\lambda\Upsilon\}\\
& & +\mathrm{Prob}\{|\theta|>(1-\lambda)\Upsilon\}+\mathrm{Prob}\{|\eta|>(1-\lambda)\Upsilon\}\\
& \leq & \frac{\sigma^2}{\Upsilon^2}\left[\frac{\alpha}{\lambda^2}+\frac{\beta+1}{(1-\lambda)^2}\right],
\end{array}$$

whence, due to $\alpha+\beta\leq 1$,

$$\mathrm{Prob}\{\|S[\zeta]\|>\Upsilon\}\leq\frac{\sigma^2}{\Upsilon^2}\max_{\alpha\in[0,1]}\min_{\lambda\in(0,1)}\left[\frac{\alpha}{\lambda^2}+\frac{2-\alpha}{(1-\lambda)^2}\right]=\frac{8\sigma^2}{\Upsilon^2};$$

with $\Upsilon=\Upsilon(\chi)$, this relation implies (3.6.52).

Assume now that ζ_ℓ satisfy Assumption **A.I**. We should prove that here the relation (3.6.52) holds true with $\Upsilon=\Upsilon_{\rm I}(\chi)$, or, which is the same,

$$\mathrm{Prob}\left\{\|S[\zeta]\|>\Upsilon\right\}\leq 1-\chi,\ S[\zeta]=\sum_\ell\zeta_\ell B_\ell=\left[\begin{array}{c|c}\sum_\ell\zeta_\ell q_\ell & \sum_\ell\zeta_\ell r_\ell^T\\ \hline \sum_\ell\zeta_\ell r_\ell & (\sum_\ell\zeta_\ell\lambda_\ell)Q\end{array}\right]. \tag{3.6.56}$$

Observe that for a symmetric block-matrix $P=\left[\begin{array}{c|c}A & B^T\\ \hline B & C\end{array}\right]$ we have $\|P\|\leq\|\left[\begin{array}{c|c}\|A\| & \|B\|\\ \hline \|B\| & \|C\|\end{array}\right]\|$, and that the norm of a symmetric matrix does not exceed its Frobenius norm, whence

$$\|S[\zeta]\|^2\leq|\sum_\ell\zeta_\ell q_\ell|^2+2\|\sum_\ell\zeta_\ell r_\ell\|_2^2+|\sum_\ell\zeta_\ell\lambda_\ell|^2\equiv\alpha[\zeta] \tag{3.6.57}$$

(recall that $\|Q\|=1$). Let E_ρ be the ellipsoid $E_\rho=\{z:\alpha[z]\leq\rho^2\}$. Observe that E_ρ contains the centered at the origin Euclidean ball of radius $\rho/\sqrt{3}$. Indeed, applying the Cauchy Inequality, we have

$$\alpha[z]\leq\left(\sum_\ell z_\ell^2\right)\left[\sum_\ell q_\ell^2+2\sum_\ell\|r_\ell\|_2^2+\sum_\ell\lambda_\ell^2\right]\leq 3\sum_\ell z_\ell^2$$

(we have used (3.6.55)). Further, ζ_ℓ are independent with zero mean and $\mathbf{E}\{\zeta_\ell^2\}\leq 1$ for every ℓ; applying the same (3.6.55), we therefore get $\mathbf{E}\{\alpha[\zeta]\}\leq 3$. By the Tschebyshev Inequality, we have

$$\mathrm{Prob}\{\zeta\in E_\rho\}\equiv\mathrm{Prob}\{\alpha[\zeta]\leq\rho^2\}\geq 1-\frac{3}{\rho^2}.$$

Invoking the Talagrand Inequality (Theorem A.9), we have

$$\rho^2>3\Rightarrow\mathbf{E}\left\{\exp\{\frac{\mathrm{dist}^2_{\|\cdot\|_2}(\zeta,E_\rho)}{16}\}\right\}\leq\frac{1}{\mathrm{Prob}\{\zeta\in E_\rho\}}\leq\frac{\rho^2}{\rho^2-3}.$$

On the other hand, if $r > \rho$ and $\alpha[\zeta] > r^2$, then $\zeta \notin (r/\rho)E_\rho$ and therefore $\mathrm{dist}_{\|\cdot\|_2}(\zeta, E_\rho) \geq (r/\rho - 1)\rho/\sqrt{3} = (r-\rho)/\sqrt{3}$ (recall that E_ρ contains the centered at the origin $\|\cdot\|_2$-ball of radius $\rho/\sqrt{3}$). Applying the Tschebyshev Inequality, we get

$$\begin{array}{l} r^2 > \rho^2 > 3 \Rightarrow \mathrm{Prob}\{\alpha[\zeta] > r^2\} \leq \mathbf{E}\left\{\exp\{\frac{\mathrm{dist}^2_{\|\cdot\|_2}(\zeta,E_\rho)}{16}\}\right\}\exp\{-\frac{(r-\rho)^2}{48}\} \\ \leq \frac{\rho^2\exp\{-\frac{(r-\rho)^2}{48}\}}{\rho^2-3}. \end{array}$$

With $\rho = 2$, $r = \Upsilon_{\mathrm{I}}(\chi) = 2 + 4\sqrt{3\ln\frac{4}{1-\chi}}$ this bound implies $\mathrm{Prob}\{\alpha[\zeta] > r^2\} \leq 1-\chi$; recalling that $\sqrt{\alpha[\zeta]}$ is an upper bound on $\|S[\zeta]\|$, we see that (3.6.52) indeed holds true with $\Upsilon = \Upsilon_{\mathrm{I}}(\chi)$.

Now consider the case when $\zeta \sim \mathcal{N}(0, I_L)$. Observe that $\alpha[\zeta]$ is a homogeneous quadratic form of ζ: $\alpha[\zeta] = \zeta^T A\zeta$, $A_{ij} = q_i q_j + 2r_i^T r_j + \lambda_i\lambda_j$. We see that the matrix A is positive semidefinite, and $\mathrm{Tr}(A) = \sum_i (q_i^2 + \lambda_i^2 + 2\|r_i\|_2^2) \leq 3$. Denoting by μ_ℓ the eigenvalues of A, we have $\zeta^T A\zeta = \sum_{\ell=1}^L \mu_\ell\xi_\ell^2$, where $\xi \sim \mathcal{N}(0, I_L)$ is an appropriate rotation of ζ. Now we can use the Bernstein scheme to bound from above $\mathrm{Prob}\{\alpha[\zeta] > \rho^2\}$:

$$\begin{array}{l} \forall(\gamma \geq 0, \max_\ell \gamma\mu_\ell < 1/2): \\ \ln\left(\mathrm{Prob}\{\alpha[\zeta] > \rho^2\}\right) \leq \ln\left(\mathbf{E}\left\{\exp\{\gamma\zeta^T A\zeta\}\right\}\exp\{-\gamma\rho^2\}\right) \\ = \ln\left(\mathbf{E}\left\{\exp\{\gamma\sum_\ell \mu_\ell\xi_\ell^2\}\right\}\right) - \gamma\rho^2 = \sum_\ell \ln\left(\mathbf{E}\left\{\exp\{\gamma\mu_\ell\xi_\ell^2\}\right\}\right) - \gamma\rho^2 \\ = -\frac{1}{2}\sum_\ell \ln(1-2\mu_\ell\gamma) - \gamma\rho^2. \end{array}$$

The concluding expression is a convex and monotone function of μ's running through the box $\{0 \leq \mu_\ell < \frac{1}{2\gamma}\}$. It follows that when $\gamma < 1/6$, the maximum of the expression over the set $\{\mu_1, ..., \mu_L \geq 0, \sum_\ell \mu_\ell \leq 3\}$ is $-\frac{1}{2}\ln(1-6\gamma) - \gamma\rho^2$. We get

$$0 \leq \gamma < \frac{1}{6} \Rightarrow \ln\left(\mathrm{Prob}\{\alpha[\zeta] > \rho^2\}\right) \leq -\frac{1}{2}\ln(1-6\gamma) - \gamma\rho^2.$$

Optimizing this bound in γ and setting $\rho^2 = 3(1+\Delta)$, $\Delta \geq 0$, we get $\mathrm{Prob}\{\alpha[\zeta] > 3(1+\Delta)\} \leq \exp\{-\frac{1}{2}[\Delta - \ln(1+\Delta)]\}$. It follows that if $\chi \in (0,1)$ and $\Delta = \Delta(\chi) \geq 0$ is such that $\Delta - \ln(1+\Delta) = 2\ln\frac{1}{1-\chi}$, then

$$\mathrm{Prob}\{\|S[\zeta]\| > \sqrt{3(1+\Delta)}\} \leq \mathrm{Prob}\{\alpha[\zeta] > 3(1+\Delta)\} \leq 1-\chi.$$

It is easily seen that when $1-\chi \leq \frac{1}{7}$, one has $\Delta(\chi) \leq 3\ln\frac{1}{1-\chi}$, that is, $\mathrm{Prob}\{\|S[\zeta]\| > \sqrt{3\left(1+3\ln\frac{1}{1-\chi}\right)}\} \leq 1-\chi$, which is exactly what was claimed in the case of Gaussian ζ. □

Application: Recovering Signal from Indirect Noisy Observations

Consider the situation as follows (cf. section 3.2.6): we observe in noise a linear transformation

$$u = As + \rho\xi \tag{3.6.58}$$

of a random signal $s \in \mathbb{R}^n$; here A is a given $m\times n$ matrix, $\xi \sim \mathcal{N}(0, I_m)$ is the noise, (which is independent of s), and $\rho \geq 0$ is a (deterministic) noise level. Our goal is to find a linear estimator

$$\widehat{s}(u) = Gu \equiv GAs + \rho G\xi \tag{3.6.59}$$

such that

$$\mathrm{Prob}\{\|\widehat{s}(u) - s\|_2 \leq \tau_*\} \geq 1-\epsilon, \tag{3.6.60}$$

where $\tau_* > 0$ and $\epsilon \ll 1$ are given. Note that the probability in (3.6.60) is taken w.r.t. the joint distribution of s and ξ. We assume below that $s \sim \mathcal{N}(0, C)$ with known covariance matrix $C \succ 0$.

Besides this, we assume that $m \geq n$ and A is of rank n. When there is no observation noise, we can recover s from u in a linear fashion without any error; it follows that when $\rho > 0$ is small enough, there exists G that makes (3.6.60) valid. Let us find the largest such ρ, that is, let us solve the optimization problem

$$\max_{G,\rho} \{\rho : \text{Prob}\{\|(GA - I_n)s + \rho G\xi\|_2 \leq \tau_*\} \geq 1 - \epsilon\}. \tag{3.6.61}$$

Setting $S = C^{1/2}$ and introducing a random vector $\theta \sim \mathcal{N}(0, I_n)$ independent of ξ (so that the random vector $[S^{-1}s; \xi]$ has exactly the same $\mathcal{N}(0, I_{n+m})$ distribution as the vector $\zeta = [\theta; \xi]$), we can rewrite our problem equivalently as

$$\max_{G,\rho} \{\rho : \text{Prob}\{\|H_\rho(G)\zeta\|_2 \leq \tau_*\} \geq 1 - \epsilon\}, \;\; H_\rho(G) = [(GA - I_n)S, \rho G]. \tag{3.6.62}$$

Let $h^\ell_\rho(G)$ be the ℓ-th column in the matrix $H_\rho(G)$, $\ell = 1, ..., L = m + n$. Invoking Lemma 3.1, our problem is nothing but the chance constrained program

$$\begin{array}{c} \max\limits_{G,\rho} \left\{ \rho : \text{Prob}\left\{ \sum\limits_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell_\rho(G) \preceq \tau_* \mathcal{A}^{\rm n} \equiv \tau_* I_{n+1} \right\} \geq 1 - \epsilon \right\} \\ \mathcal{A}^\ell_\rho(G) = \left[\begin{array}{c|c} & [h^\ell_\rho(G)]^T \\ \hline h^\ell_\rho(G) & \end{array} \right]. \end{array} \tag{3.6.63}$$

We intend to process the latter problem as follows:

A) We use our "Conjecture-related" approximation scheme to build a nondecreasing continuous function $\Gamma(\rho) \to 0, \rho \to +0$, and matrix-valued function G_ρ (both functions are efficiently computable) such that

$$\text{Prob}\{\|(GA - I_n)s + \rho G\xi\|_2 > \tau_*\} = \text{Prob}\{\sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell_\rho(G_\rho) \not\preceq \tau_* I_{n+1}\} \leq \Gamma(\rho). \tag{3.6.64}$$

B) We then solve the approximating problem

$$\max_\rho \{\rho : \Gamma(\rho) \leq \epsilon\}. \tag{3.6.65}$$

Clearly, a feasible solution ρ to the latter problem, along with the associated matrix G_ρ, form a feasible solution to the problem of interest (3.6.63). On the other hand, the approximating problem is efficiently solvable: $\Gamma(\rho)$ is nondecreasing, efficiently computable and $\Gamma(\rho) \to 0$ as $\rho \to +0$, so that the approximating problem can be solved efficiently by bisection. We find a feasible nearly optimal solution $\widehat{\rho}$ to the approximating problem and treat $(\widehat{\rho}, G_{\widehat{\rho}})$ as a suboptimal solution to the problem of interest. By our analysis, this solution is feasible for the latter problem.

Remark 3.2 In fact, the constraint in (3.6.62) is simpler than a general-type chance constrained conic quadratic inequality — it is a chance constrained Least Squares inequality (the right hand side is affected neither by the decision variables, nor by the noise), and as such it admits a Bernstein-type approximation described in section 2.5.3, see Corollary 2.1. Of course, in the outlined scheme one can use the Bernstein approximation as an alternative to the Conjecture-related approximation.

Now let us look at steps A, B in more details.
Step A). We solve the semidefinite program

$$\nu_*(\rho) = \min_{\nu,G} \left\{ \nu : \sum_{\ell=1}^{L} (\mathcal{A}^\ell_\rho(G))^2 \preceq \nu I_{n+1} \right\}; \tag{3.6.66}$$

whenever $\rho > 0$, this problem clearly is solvable. Due to the fact that part of the matrices $\mathcal{A}^\ell_\rho(G)$ are independent of ρ, and the remaining ones are proportional to ρ, the optimal value is a positive continuous and nondecreasing function of $\rho > 0$. Finally, $\nu_*(\rho) \to +0$ as $\rho \to +0$ (look what happens at the point G satisfying the relation $GA = I_n$).

Let G_ρ be an optimal solution to (3.6.66). Setting $A_\ell = \mathcal{A}^\ell_\rho(G_\rho)\nu_*^{-\frac{1}{2}}(\rho)$, $A = I_{n+1}$, the arrow-type matrices $A, A_1, ..., A_L$ satisfy (3.6.13); invoking Theorem 3.18, we conclude that

$$\begin{array}{l}\chi \in [\frac{6}{7}, 1) \Rightarrow \mathrm{Prob}\{-\Upsilon(\chi)\nu_*^{\frac{1}{2}}(\rho)I_{n+1} \preceq \sum\limits_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell_\rho(G_\rho) \preceq \Upsilon(\chi)\nu_*^{\frac{1}{2}}(\rho)I_{n+1}\} \\ \geq \chi,\ \Upsilon(\chi) = \sqrt{3\left(1 + 3\ln\frac{1}{1-\chi}\right)}.\end{array}$$

Now let χ and ρ be such that $\chi \in [6/7, 1)$ and $\Upsilon(\chi)\sqrt{\nu_*(\rho)} \leq \tau_*$. Setting

$$Q = \{z : \|\sum_{\ell=1}^{L} z_\ell \mathcal{A}^\ell_\rho(G_\rho)\| \leq \Upsilon(\chi)\sqrt{\nu_*(\rho)}\},$$

we get a closed convex set such that the random vector $\zeta \sim \mathcal{N}(0, I_{n+m})$ takes its values in Q with probability $\geq \chi > 1/2$. Invoking Theorem A.10 (where we set $\alpha = \tau_*/(\Upsilon(\chi)\sqrt{\nu_*(\rho)})$), we get

$$\begin{array}{l}\mathrm{Prob}\left\{\sum\limits_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell_\rho(G_\rho) \not\preceq \tau_* I_{n+1}\right\} \leq \mathrm{Erf}\left(\frac{\tau_* \mathrm{ErfInv}(1-\chi)}{\sqrt{\nu_*(\rho)}\Upsilon(\chi)}\right) \\ = \mathrm{Erf}\left(\frac{\tau_* \mathrm{ErfInv}(1-\chi)}{\sqrt{3\nu_*(\rho)\left[1+3\ln\frac{1}{1-\chi}\right]}}\right).\end{array}$$

Setting

$$\Gamma(\rho) = \inf_\chi \left\{ \mathrm{Erf}\left(\frac{\tau_* \mathrm{ErfInv}(1-\chi)}{\sqrt{3\nu_*(\rho)\left[1 + 3\ln\frac{1}{1-\chi}\right]}}\right) : \begin{array}{l}\chi \in [6/7, 1), \\ 3\nu_*(\rho)\left[1 + 3\ln\frac{1}{1-\chi}\right] \leq \tau_*^2\end{array} \right\} \tag{3.6.67}$$

(if the feasible set of the right hand side optimization problem is empty, then, by definition, $\Gamma(\rho) = 1$), we ensure (3.6.64). Taking into account that $\nu_*(\rho)$ is a nondecreasing continuous function of $\rho > 0$ that tends to 0 as $\rho \to +0$, it is immediately seen that $\Gamma(\rho)$ possesses these properties as well.

Solving (3.6.66). Good news is that problem (3.6.66) has a closed form solution. To see this, note that the matrices $\mathcal{A}^\ell_\rho(G)$ are pretty special arrow type matrices: their diagonal entries are zero, so that these $(n+1)\times(n+1)$ matrices are of the form $\left[\begin{array}{c|c} & [h^\ell_\rho(G)]^T \\ \hline h^\ell_\rho(G) & \end{array}\right]$ with n-dimensional vectors $h^\ell_\rho(G)$ affinely depending on G. Now let us make the following observation:

Lemma 3.6 *Let $f_\ell \in \mathbb{R}^n$, $\ell = 1, ..., L$, and $\nu \geq 0$. Then*

$$\sum_{\ell=1}^{L} \left[\begin{array}{c|c} & f_\ell^T \\ \hline f_\ell & \end{array}\right]^2 \preceq \nu I_{n+1} \tag{$*$}$$

if and only if $\sum\limits_\ell f_\ell^T f_\ell \leq \nu$.

Proof. Relation $(*)$ is nothing but

$$\sum_\ell \left[\begin{array}{c|c} f_\ell^T f_\ell & \\ \hline & f_\ell f_\ell^T \end{array}\right] \preceq \nu I_{n+1},$$

so it definitely implies that $\sum\limits_\ell f_\ell^T f_\ell \leq \nu$. To prove the inverse implication, it suffices to verify that the relation $\sum\limits_\ell f_\ell^T f_\ell \leq \nu$ implies that $\sum\limits_\ell f_\ell f_\ell^T \preceq \nu I_n$. This is immediate due to $\mathrm{Tr}(\sum\limits_\ell f_\ell f_\ell^T) = \sum\limits_\ell f_\ell^T f_\ell \leq \nu$, (note that the matrix $\sum_\ell f_\ell f_\ell^T$ is positive semidefinite, and therefore its maximal eigenvalue does not exceed its trace). □

In view of Lemma 3.6, the optimal solution and the optimal value in (3.6.66) are exactly the same as their counterparts in the minimization problem

$$\nu = \min_G \sum_\ell [h^\ell_\rho(G)]^T h^\ell_\rho(G).$$

Thus, (3.6.66) is nothing but the problem

$$\nu_*(\rho) = \min_G \left\{ \mathrm{Tr}((GA - I_n)C(GA - I)^T) + \rho^2 \mathrm{Tr}(GG^T) \right\}. \tag{3.6.68}$$

The objective in this unconstrained problem has a very transparent interpretation: it is the mean squared error of the linear estimator $\widehat{s} = Gu$, the noise intensity being ρ. The matrix G minimizing this objective is called the *Wiener filter*; a straightforward computation yields

$$\begin{array}{rcl} G_\rho & = & CA^T(ACA^T + \rho^2 I_m)^{-1}, \\ \nu_*(\rho) & = & \mathrm{Tr}\left((G_\rho A - I_n)C(G_\rho A - I_n)^T + \rho^2 G_\rho G_\rho^T\right). \end{array} \tag{3.6.69}$$

Remark 3.3 The Wiener filter is one of the oldest and the most basic tools in Signal Processing; it is good news that our approximation scheme recovers this tool, albeit from a different perspective: we were seeking a linear filter that ensures that with probability $1-\epsilon$ the recovering error does not exceed a given threshold (a problem that seemingly does not admit a closed form solution); it turned out that the *sub*optimal solution yielded by our approximation scheme is the precise solution to a simple classical problem.

Refinements. The pair $(\widehat{\rho}, G_{\mathrm{W}} = G_{\widehat{\rho}})$ ("W" stands for "Wiener") obtained via the outlined approximation scheme is feasible for the problem of interest (3.6.63). However, we have all reason to expect that our provably 100%-reliable approach is conservative — exactly because of its 100% reliability. In particular, it is very likely that $\widehat{\rho}$ is a too conservative lower bound on the actual feasibility radius $\rho_*(G_{\mathrm{W}})$ — the largest ρ such that (ρ, G_{W}) is feasible for the chance constrained problem of interest. We can try to improve this lower bound by the Randomized r procedure, e.g., as follows:

Given a confidence parameter $\delta \in (0,1)$, we run $\nu = 10$ steps of bisection on the segment $\Delta = [\widehat{\rho}, 100\widehat{\rho}]$. At a step t of this process, given the previous localizer Δ_{t-1} (a segment contained in Δ, with $\Delta_0 = \Delta$), we take as the current trial value ρ_t of ρ the midpoint of Δ_{t-1} and apply the Randomized r procedure in order to check whether (ρ_t, G_{W}) is feasible for (3.6.63). Specifically, we

- compute the $L = m + n$ vectors $h^\ell_{\rho_t}(G_{\mathrm{W}})$ and the quantity $\mu_t = \sqrt{\sum_{\ell=1}^{m+n} \|h^\ell_{\rho_t}(G_{\mathrm{W}})\|_2^2}$. By Lemma 3.6, we have

  $$\sum_{\ell=1}^{L} \left[\mathcal{A}^\ell_{\rho_t}(G_{\mathrm{W}})\right]^2 \preceq \mu_t^2 I_{n+1},$$

  so that the matrices $A = I_{n+1}$, $A_\ell = \mu_t^{-1}\mathcal{A}^\ell_{\rho_t}(G_{\mathrm{W}})$ satisfy (3.6.13);

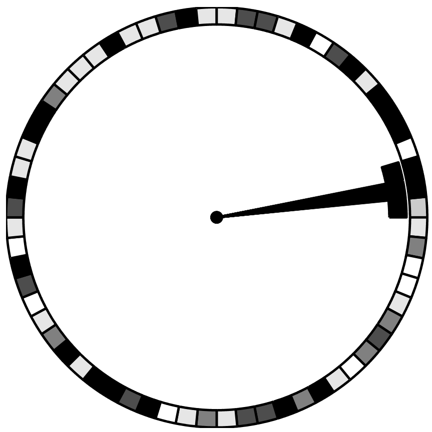
- apply to the matrices $A, A_1, ..., A_L$ the Randomized r procedure with parameters $\epsilon, \delta/\nu$, thus ending up with a random quantity r_t such that "up to probability of bad sampling $\leq \delta/\nu$," one has

  $$\mathrm{Prob}\{\zeta : -I_{n+1} \preceq r_t \sum_{\ell=1}^{L} \zeta_\ell A_\ell \preceq I_{n+1}\} \geq 1 - \epsilon,$$

  or, which is the same,

  $$\mathrm{Prob}\{\zeta : -\frac{\mu_t}{r_t} I_{n+1} \preceq \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}^\ell_\rho(G_W) \preceq \frac{\mu_t}{r_t} I_{n+1}\} \geq 1 - \epsilon. \tag{3.6.70}$$

  Note that when the latter relation is satisfied and $\frac{\mu_t}{r_t} \leq \tau_*$, the pair (ρ_t, G_{W}) is feasible for (3.6.63);



$(K * s)_i = 0.2494 s_{i-1} + 0.5012 s_i + 0.2494 s_{i+1}$

Figure 3.9: A scanner.

- finally, complete the bisection step, namely, check whether $\mu_t/r_t \leq \tau_*$. If it is the case, we take as our new localizer Δ_t the part of Δ_{t-1} to the right of ρ_t, otherwise Δ_t is the part of Δ_{t-1} to the left of ρ_t.

After ν bisection steps are completed, we claim that the left endpoint $\widetilde{\rho}$ of the last localizer Δ_ν is a lower bound on $\rho_*(G_{\rm W})$. Observe that this claim is valid, provided that all ν inequalities (3.6.70) take place, which happens with probability at least $1-\delta$.

Illustration: Deconvolution. A rotating scanning head reads random signal s as shown in figure 3.9. The signal registered when the head observes bin i, $0 \leq i < n$, is

$$u_i = (As)_i + \rho\xi_i \equiv \sum_{j=-d}^{d} K_j s_{(i-j) \bmod n} + \rho\xi_i,\ 0 \leq i < n,$$

where $r = p \bmod n$, $0 \leq r < n$, is the remainder when dividing p by n. The signal s is assumed to be Gaussian with zero mean and known covariance $C_{ij} = \mathbf{E}\{s_i s_j\}$ depending on $(i-j) \bmod n$ only ("stationary periodic discrete-time Gaussian process"). The goal is to find a linear recovery $\widehat{s} = Gu$ and the largest ρ such that

$$\mathrm{Prob}_{[s;\xi]}\left\{\|G(As+\rho\xi) - s\|_2 \leq \tau_*\right\} \geq 1-\epsilon.$$

We intend to process this problem via the outlined approach using two safe approximations of the chance constraint of interest — the Conjecture-related and the Bernstein (see Remark 3.2). The recovery matrices and critical levels of noise as given by these two approximations will be denoted G_W, ρ_W ("W" for "Wiener") and G_B, ρ_B ("B" for "Bernstein"), respectively.

Note that in the case in question one can immediately verify that the matrices A^TA and C commute. Whenever this is the case, the computational burden to compute G_W and G_B reduces dramatically. Indeed, after appropriate rotations of x and y we arrive at the situation where both A and C are diagonal, in which case in both our approximation schemes one loses nothing by restricting G to be diagonal. This significantly reduces the dimensions of the convex problems we need to solve.

In the experiment we use

$$n = 64,\ d = 1,\ \tau_* = 0.1\sqrt{n} = 0.8,\ \epsilon = 1.\text{e-}4;$$

C was set to the unit matrix, (meaning that $s \sim \mathcal{N}(0, I_{64})$), and the convolution kernel K is the one shown in figure 3.9. After (G_W, ρ_w) and (G_B, ρ_B) were computed, we used the Randomized r procedure

Admissible noise level	Bernstein approximation	Conjecture-related approximation
Before refinement	1.92e-4	1.50e-4
After refinement (δ = 1.e-6)	3.56e-4	3.62e-4

Table 3.4: Results of deconvolution experiment.

Noise level	$\mathrm{Prob}\{\|\widehat{s}-s\|_2 > \tau_*\}$	
	$G = G_B$	$G = G_W$
3.6e-4	0	0
7.2e-4	6.7e-3	6.7e-3
1.0e-3	7.4e-2	7.5e-2

Table 3.5: Empirical value of $\mathrm{Prob}\{\|\widehat{s}-s\|_2 > 0.8\}$ based on 10,000 simulations.

with δ = 1.e-6 to refine the critical values of noise for G_W and G_B; the refined values of ρ are denoted $\widehat{\rho}_W$ and $\widehat{\rho}_B$, respectively.

The results of the experiments are presented in table 3.4. While G_B and G_W turned out to be close, although not identical, the critical noise levels as yielded by the Conjecture-related and the Bernstein approximations differ by $\approx 30\%$. The refinement increases these critical levels by a factor ≈ 2 and makes them nearly equal. The resulting critical noise level 3.6e-4 is not too conservative: the simulation results shown in table 3.5 demonstrate that at a twice larger noise level, the probability for the chance constraint to be violated is by far larger than the required 1.e-4.

Modifications. We have addressed the Signal Recovery problem (3.6.58), (3.6.59), (3.6.60) in the case when $s \sim \mathcal{N}(0, C)$ is random, the noise is independent of s and the probability in (3.6.60) is taken w.r.t. the joint distribution of ξ and s. Next we want to investigate two other versions of the problem.

Recovering a uniformly distributed signal. Assume that the signal s is

(a) uniformly distributed in the unit box $\{s \in \mathbb{R}^n : \|s\|_\infty \leq 1\}$,

or

(b) uniformly distributed on the vertices of the unit box

and is independent of ξ. Same as above, our goal is to ensure the validity of (3.6.60) with as large ρ as possible. To this end, let us use Gaussian Majorization. Specifically, in the case of (a), let $\widetilde{s} \sim \mathcal{N}(0, (2/\pi)I)$. As it was explained in section 3.6.3, the condition

$$\mathrm{Prob}\{\|(GA - I)\widetilde{s} + \rho G\xi\|_2 \leq \tau_*\} \geq 1 - \epsilon$$

is sufficient for the validity of (3.6.60). Thus, we can use the Gaussian case procedure presented in section 3.6.4 with the matrix $(2/\pi)I$ in the role of C; an estimator that is good in this case will be at least as good in the case of the signal s.

In case of (b), we can act similarly, utilizing Theorem 3.16. Specifically, let $\widetilde{s} \sim \mathcal{N}(0, (\pi/2)I)$ be independent of ξ. Consider the parametric problem

$$\nu(\rho) \equiv \min_G \left\{ \frac{\pi}{2} \mathrm{Tr}\left((GA - I)(GA - I)^T\right) + \rho^2 \mathrm{Tr}(GG^T) \right\}, \tag{3.6.71}$$

$\rho \geq 0$ being the parameter (cf. (3.6.68) and take into account that the latter problem is equivalent to (3.6.66)), and let G_ρ be an optimal solution to this problem. The same reasoning as on p. 168 shows that

$$\begin{array}{l} 6/7 \leq \chi < 1 \Rightarrow \mathrm{Prob}\{(\widetilde{s}, \xi) : \|(G_\rho A - I)\widetilde{s} + \rho G_\rho \xi\|_2 \leq \Upsilon(\chi)\nu_*^{1/2}(\rho)\} \geq \chi, \\ \Upsilon(\chi) = \sqrt{3\left(1 + 3\ln\frac{1}{1-\chi}\right)}. \end{array}$$

Applying Theorem 3.16 to the convex set $Q = \{(z,x) : \|(G_\rho A - I)z + \rho G_\rho x\|_2 \leq \Upsilon(\chi)\nu_*^{1/2}(\rho)$ and the random vectors $[s;\xi]$, $[\widetilde{s};\xi]$, we conclude that

$$\forall \left(\begin{smallmatrix}\chi\in[6/7,1)\\ \gamma>1\end{smallmatrix}\right) : \mathrm{Prob}\{(s,\xi) : \|(G_\rho A - I)s + \rho G_\rho \xi\|_2 > \gamma\Upsilon(\chi)\nu_*^{1/2}(\rho)\}$$
$$\leq \min_{\beta\in[1,\gamma)} \tfrac{1}{\gamma-\beta}\int_\beta^\infty \mathrm{Erf}(r\mathrm{ErfInv}(1-\chi))dr.$$

We conclude that setting

$$\widetilde{\Gamma}(\rho) = \inf_{\chi,\gamma,\beta}\left\{ \tfrac{1}{\gamma-\beta}\int_\beta^\infty \mathrm{Erf}(r\mathrm{ErfInv}(1-\chi))dr : \begin{array}{l} 6/7 \leq \chi < 1, \gamma > 1 \\ 1 \leq \beta < \gamma \\ \gamma\Upsilon(\chi)\nu_*^{1/2}(\rho) \leq \tau_* \end{array}\right\}$$
$$\left[\Upsilon(\chi) = \sqrt{3\left(1 + 3\ln\tfrac{1}{1-\chi}\right)}\right]$$

($\widetilde{\Gamma}(\rho) = 1$ when the right hand side problem is infeasible), one has

$$\mathrm{Prob}\{(s,\xi) : \|(G_\rho A - I)s + \rho G_\rho\xi\|_2 > \tau_*\} \leq \widetilde{\Gamma}(\rho)$$

(cf. p. 168). It is easily seen that $\widetilde{\Gamma}(\cdot)$ is a continuous nondecreasing function of $\rho > 0$ such that $\widetilde{\Gamma}(\rho) \to 0$ as $\rho \to +0$, and we end up with the following safe approximation of the Signal Recovery problem:

$$\max_\rho \left\{\rho : \widetilde{\Gamma}(\rho) \leq \epsilon\right\}$$

(cf. (3.6.65)).

Note that in the above "Gaussian majorization" scheme we could use the Bernstein approximation, based on Corollary 2.1, of the chance constraint $\mathrm{Prob}\{\|(GA-I)\widetilde{s} + \rho G\xi\|_2 \leq \tau_*\} \geq 1-\epsilon$ instead of the Conjecture-related approximation.

The case of deterministic uncertain signal. Up to now, signal s was considered as random and independent of ξ, and the probability in (3.6.60) was taken w.r.t. the joint distribution of s and ξ; as a result, certain "rare" realizations of the signal can be recovered very poorly. Our current goal is to understand what happens when we replace the specification (3.6.60) with

$$\begin{array}{l}\forall (s\in\mathcal{S}) : \\ \mathrm{Prob}\{\xi : \|Gu - s\|_2 \leq \tau_*\} \equiv \mathrm{Prob}\{\xi : \|(GA-I)s + \rho G\xi\|_2 \leq \tau_*\} \geq 1-\epsilon,\end{array} \tag{3.6.72}$$

where $\mathcal{S} \subset \mathbb{R}^n$ is a given compact set.

Our starting point is the following observation:

Lemma 3.7 *Let G, $\rho \geq 0$ be such that*

$$\Theta \equiv \frac{\tau_*^2}{\max\limits_{s\in\mathcal{S}} s^T(GA-I)^T(GA-I)s + \rho^2\mathrm{Tr}(G^TG)} \geq 1. \tag{3.6.73}$$

Then for every $s \in \mathcal{S}$ one has

$$\mathrm{Prob}_{\zeta\sim\mathcal{N}(0,I)}\{\|(GA-I)s + \rho G\zeta\|_2 > \tau_*\} \leq \exp\left\{-\tfrac{(\Theta-1)^2}{4(\Theta+1)}\right\}. \tag{3.6.74}$$

Proof. There is nothing to prove when $\Theta = 1$, so that let $\Theta > 1$. Let us fix $s \in \mathcal{S}$ and let $g = (GA-I)s$, $W = \rho^2 G^TG$, $w = \rho G^Tg$. We have

$$\begin{array}{l}\mathrm{Prob}\{\|(GA-I)s + \rho G\zeta\|_2 > \tau_*\} = \mathrm{Prob}\left\{\|g + \rho G\zeta\|_2^2 > \tau_*^2\right\} \\ = \mathrm{Prob}\left\{\zeta^T[\rho^2G^TG]\zeta + 2\zeta^T\rho G^Tg > \tau_*^2 - g^Tg\right\} \\ = \mathrm{Prob}\left\{\zeta^TW\zeta + 2\zeta^Tw > \tau_*^2 - g^Tg\right\}.\end{array} \tag{3.6.75}$$

Denoting by λ the vector of eigenvalues of W, we can assume w.l.o.g. that $\lambda \neq 0$, since otherwise $W = 0$, $w = 0$ and thus the left hand side in (3.6.75) is 0 (note that $\tau_*^2 - g^T g > 0$ due to (3.6.73) and since $s \in \mathcal{S}$), and thus (3.6.74) is trivially true. Setting

$$\Omega = \frac{\tau_*^2 - g^T g}{\sqrt{\lambda^T\lambda + w^T w}}$$

and invoking Proposition 2.3, we arrive at

$$\begin{array}{l}
\mathrm{Prob}\{\|(GA-I)s + \rho G\zeta\|_2 > \tau_*\} \leq \exp\left\{-\frac{\Omega^2\sqrt{\lambda^T\lambda+w^Tw}}{4\left[2\sqrt{\lambda^T\lambda+w^Tw}+\|\lambda\|_\infty\Omega\right]}\right\} \\
= \exp\left\{-\frac{[\tau_*^2-g^Tg]^2}{4[2[\lambda^T\lambda+w^Tw]+\|\lambda\|_\infty[\tau_*^2-g^Tg]]}\right\} \\
= \exp\left\{-\frac{[\tau_*^2-g^Tg]^2}{4[2[\lambda^T\lambda+g^T[\rho^2GG^T]g]+\|\lambda\|_\infty[\tau_*^2-g^Tg]]}\right\} \\
\leq \exp\left\{-\frac{[\tau_*^2-g^Tg]^2}{4\|\lambda\|_\infty[2[\|\lambda\|_1+g^Tg]+[\tau_*^2-g^Tg]]}\right\},
\end{array} \tag{3.6.76}$$

where the concluding inequality is due to $\rho^2 GG^T \preceq \|\lambda\|_\infty I$ and $\lambda^T\lambda \leq \|\lambda\|_\infty\|\lambda\|_1$. Further, setting $\alpha = g^T g$, $\beta = \mathrm{Tr}(\rho^2 G^T G)$ and $\gamma = \alpha + \beta$, observe that $\beta = \|\lambda\|_1 \geq \|\lambda\|_\infty$ and $\tau_*^2 \geq \Theta\gamma \geq \gamma$ by (3.6.73). It follows that

$$\frac{[\tau_*^2 - g^Tg]^2}{4\|\lambda\|_\infty\,[2[\|\lambda\|_1 + g^Tg] + [\tau_*^2 - g^Tg]]} \geq \frac{(\tau_*^2 - \gamma + \beta)^2}{4\beta(\tau_*^2 + \gamma + \beta)} \geq \frac{(\tau_*^2-\gamma)^2}{4\gamma(\tau_*^2+\gamma)},$$

where the concluding inequality is readily given by the relations $\tau_*^2 \geq \gamma \geq \beta > 0$. Thus, (3.6.76) implies that

$$\mathrm{Prob}\{\|(GA-I)s + \rho G\zeta\|_2 > \tau_*\} \leq \exp\left\{-\frac{(\tau_*^2-\gamma)^2}{4\gamma(\tau_*^2+\gamma)}\right\} \leq \exp\left\{-\frac{(\Theta-1)^2}{4(\Theta+1)}\right\}. \qquad \square$$

Lemma 3.7 suggests a safe approximation of the problem of interest as follows. Let $\Theta(\epsilon) > 1$ be given by

$$\exp\{-\frac{(\Theta-1)^2}{4(\Theta+1)}\} = \epsilon \qquad [\Rightarrow \Theta(\epsilon) = (4+o(1))\ln(1/\epsilon) \text{ as } \epsilon \to +0]$$

and let

$$\phi(G) = \max_{s\in\mathcal{S}} s^T(GA-I)^T(GA-I)s, \tag{3.6.77}$$

(this function clearly is convex). By Lemma 3.7, the optimization problem

$$\max_{\rho,G}\left\{\rho : \phi(G) + \rho^2\mathrm{Tr}(G^TG) \leq \gamma_* \equiv \Theta^{-1}(\epsilon)\tau_*^2\right\} \tag{3.6.78}$$

is a safe approximation of the problem of interest. Applying bisection in ρ, we can reduce this problem to a "short series" of convex feasibility problems of the form

$$\text{find } G\text{: } \phi(G) + \rho^2\mathrm{Tr}(G^TG) \leq \gamma_*. \tag{3.6.79}$$

Whether the latter problems are or are not computationally tractable depends on whether the function $\phi(G)$ is so, which happens if and only if we can efficiently optimize positive semidefinite quadratic forms s^TQs over $\mathcal{S}$.

Example 3.8 Let $\mathcal{S}$ be an ellipsoid centered at the origin:

$$\mathcal{S} = \{s = Hv : v^Tv \leq 1\}$$

In this case, it is easy to compute $\phi(G)$ — this function is semidefinite representable:

$$\begin{array}{l}\phi(G) \le t \Leftrightarrow \max\limits_{s\in\mathcal{S}} s^T(GA-I)^T(GA-I)s \le t \\ \Leftrightarrow \max\limits_{v:\|v\|_2\le 1} v^T(H^T(GA-I)^T(GA-I)Hv \le t \\ \Leftrightarrow \lambda_{\max}(H^T(GA-I)^T(GA-I)H) \le t \\ \Leftrightarrow tI - H^T(GA-I)^T(GA-I)H \succeq 0 \Leftrightarrow \left[\begin{array}{c|c} tI & H^T(GA-I)^T \\ \hline (GA-I)H & I \end{array}\right] \succeq 0,\end{array}$$

where the concluding $\Leftrightarrow$ is given by the Schur Complement Lemma. Consequently, (3.6.79) is the efficiently solvable convex feasibility problem

$$\text{Find } G, t:\quad t + \rho^2\mathrm{Tr}(G^TG) \le \gamma_*,\ \left[\begin{array}{c|c} tI & H^T(GA-I)^T \\ \hline (GA-I)H & I \end{array}\right] \succeq 0.$$

Example 3.8 allows us to see the dramatic difference between the case where we are interested in "highly reliable with high probability" recovery of a *random* signal and "highly reliable" recovery of *every* realization of uncertain signal. Specifically, assume that G, ρ are such that (3.6.60) is satisfied with $s \sim \mathcal{N}(0, I_n)$. Note that when n is large, s is nearly uniformly distributed over the sphere $\mathcal{S}$ of radius $\sqrt{n}$ (indeed, $s^Ts = \sum\limits_i s_i^2$, and by the Law of Large Numbers, for $\delta > 0$ the probability of the event $\{\|s\|_2 \notin [(1-\delta)\sqrt{n}, (1+\delta)\sqrt{n}]\}$ goes to 0 as $n \to \infty$, in fact exponentially fast. Also, the direction $s/\|s\|_2$ of s is uniformly distributed on the unit sphere). Thus, the recovery in question is, essentially, a highly reliable recovery of random signal uniformly distributed over the above sphere $\mathcal{S}$. Could we expect the recovery to "nearly satisfy" (3.6.72), that is, to be reasonably good in the worst case over the signals from $\mathcal{S}$? The answer is negative when n is large. Indeed, a *sufficient* condition for (3.6.60) to be satisfied is

$$\mathrm{Tr}((GA-I)^T(GA-I)) + \rho^2\mathrm{Tr}(G^TG) \le \frac{\tau_*^2}{O(1)\ln(1/\epsilon)} \qquad (*)$$

with appropriately chosen absolute constant $O(1)$. A *necessary* condition for (3.6.72) to be satisfied is

$$n\lambda_{\max}((GA-I)^T(GA-I)) + \rho^2\mathrm{Tr}(G^TG) \le O(1)\tau_*^2. \qquad (**)$$

Since the trace of the $n \times n$ matrix $Q = (GA-I)^T(GA-I)$ can be nearly n times less than $n\lambda_{\max}(Q)$, the validity of $(*)$ *by far* does not imply the validity of $(**)$. To be more rigorous, consider the case when $\rho = 0$ and $GA - I = \mathrm{Diag}\{1, 0, ..., 0\}$. In this case, the $\|\cdot\|_2$-norm of the recovering error, in the case of $s \sim \mathcal{N}(0, I_n)$, is just $|s_1|$, and $\mathrm{Prob}\{|s_1| > \tau_*\} \le \epsilon$ provided that $\tau_* \ge \sqrt{2\ln(2/\epsilon)}$, in particular, when $\tau_* = \sqrt{2\ln(2/\epsilon)}$. At the same time, when $s = \sqrt{n}[1; 0; ...; 0] \in \mathcal{S}$, the norm of the recovering error is $\sqrt{n}$, which, for large n, is incomparably larger than the above τ_*.

Example 3.9 Here we consider the case where $\phi(G)$ cannot be computed efficiently, specifically, the case where $\mathcal{S}$ is the unit box $B_n = \{s \in \mathbb{R}^n : \|s\|_\infty \le 1\}$ (or the set V_n of vertices of this box). Indeed, it is known that for a general-type positive definite quadratic form s^TQs, computing its maximum over the unit box is NP-hard, even when instead of the precise value of the maximum its 4%-accurate approximation is sought. In situations like this we could replace $\phi(G)$ in the above scheme by its efficiently computable upper bound $\widehat{\phi}(G)$. To get such a bound in the case when $\mathcal{S}$ is the unit box, we can use the following wonderful result:

Nesterov's $\frac{\pi}{2}$ Theorem [76] *Let $A \in \mathbf{S}^n_+$. Then the efficiently computable quantity*

$$\mathrm{SDP}(A) = \min_{\lambda\in\mathbb{R}^n}\left\{\sum_i \lambda_i : \mathrm{Diag}\{\lambda\} \succeq A\right\}$$

is an upper bound, tight within the factor $\frac{\pi}{2}$, on the quantity

$$\mathrm{Opt}(A) = \max_{s\in B_n} s^TAs.$$

Assuming that $\mathcal{S}$ is B_n (or V_n), Nesterov's $\frac{\pi}{2}$ Theorem provides us with an efficiently computable and tight, within the factor $\frac{\pi}{2}$, upper bound

$$\widehat{\phi}(G) = \min_{\lambda}\left\{\sum_i \lambda_i : \left[\begin{array}{c|c} \mathrm{Diag}(\lambda) & (GA-I)^T \\ \hline GA-I & I \end{array}\right] \succeq 0\right\}$$

on $\phi(G)$. Replacing $\phi(\cdot)$ by its upper bound, we pass from the intractable problems (3.6.79) to their tractable approximations

$$\text{find } G, \lambda: \quad \sum_i \lambda_i + \rho^2 \mathrm{Tr}(G^T G) \leq \gamma_*, \left[\begin{array}{c|c} \mathrm{Diag}(\lambda) & (GA-I)^T \\ \hline GA-I & I \end{array}\right] \succeq 0; \qquad (3.6.80)$$

we then apply bisection in ρ to rapidly approximate the largest $\rho = \rho_*$, along with the associated $G = G_*$, for which problems (3.6.80) are solvable, thus getting a feasible solution to the problem of interest.

3.7 Exercises

Exercise 3.1 Consider a semi-infinite conic constraint

$$\forall(\zeta \in \rho\mathcal{Z}) : a_0[x] + \sum_{\ell=1}^{L} \zeta_i a_\ell[x] \in \mathbf{Q} \qquad (C_{\mathcal{Z}}[\rho])$$

Assume that for certain ϑ and some closed convex set $\mathcal{Z}_*$, $0 \in \mathcal{Z}_*$, the constraint $(C_{\mathcal{Z}_*}[\cdot])$ admits a safe tractable approximation tight within the factor ϑ. Now let $\mathcal{Z}$ be a closed convex set that can be approximated, up to a factor λ, by $\mathcal{Z}_*$, meaning that for certain $\gamma > 0$ we have

$$\gamma\mathcal{Z}_* \subset \mathcal{Z} \subset (\lambda\gamma)\mathcal{Z}_*.$$

Prove that $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation, tight within the factor $\lambda\vartheta$.

Exercise 3.2 Let $\vartheta \geq 1$ be given, and consider the semi-infinite conic constraint $(C_{\mathcal{Z}}[\cdot])$ "as a function of $\mathcal{Z}$," meaning that $a_\ell[\cdot]$, $0 \leq \ell \leq L$, and $\mathbf{Q}$ are once and forever fixed. In what follows, $\mathcal{Z}$ always is a solid (convex compact set with a nonempty interior) symmetric w.r.t. 0.

Assume that whenever $\mathcal{Z}$ is an ellipsoid centered at the origin, $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation tight within factor ϑ (as it is the case for $\vartheta = 1$ when $\mathbf{Q}$ is the Lorentz cone, see section 3.2.5).

1. Prove that when $\mathcal{Z}$ is the intersection of M centered at the origin ellipsoids:

$$\mathcal{Z} = \{\zeta : \zeta^T Q_i \zeta \leq 1,\ i = 1, ..., M\} \qquad [Q_i \succeq 0, \textstyle\sum_i Q_i \succ 0]$$

   $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation tight within the factor $\sqrt{M}\vartheta$.

2. Prove that if $\mathcal{Z} = \{\zeta : \|\zeta\|_\infty \leq 1\}$, then $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation tight within the factor $\vartheta\sqrt{\dim \zeta}$.

3. Assume that $\mathcal{Z}$ is the intersection of M ellipsoids not necessarily centered at the origin. Prove that then $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation tight within a factor $\sqrt{2M}\vartheta$.

Exercise 3.3 Consider the situation as follows (cf. section 3.2.6). We are given an observation

$$y = Ax + b \in \mathbb{R}^m$$

of unknown signal $x \in \mathbb{R}^n$. The matrix $B \equiv [A; b]$ is not known exactly; all we know is that $B \in \mathcal{B} = \{B = B_{\rm n} + L^T \Delta R : \Delta \in \mathbb{R}^{p\times q}, \|\Delta\|_{2,2} \leq \rho\}$. Build an estimate v of the vector Qx, where Q is a given $k \times n$ matrix, that minimizes the worst-case, over all possible true values of x, $\|\cdot\|_2$ estimation error.

Exercise 3.4 Consider an uncertain Least Squares inequality

$$\|A(\eta)x + b(\eta)\|_2 \leq \tau, \;\; \eta \in \rho\mathcal{Z}$$

where $\mathcal{Z}$, $0 \in \mathrm{int}\mathcal{Z}$, is a symmetric w.r.t. the origin convex compact set that is the intersection of $J > 1$ ellipsoids not necessarily centered at the origin:

$$\mathcal{Z} = \{\eta : (\eta - a_j)^T Q_j (\eta - a_j) \leq 1, \, 1 \leq j \leq J\} \qquad [Q_j \succeq 0, \textstyle\sum_j Q_j \succ 0]$$

Prove that the RC of the uncertain inequality in question admits a safe tractable approximation tight within the factor $O(1)\sqrt{\ln J}$ (cf. Theorem 3.9).

Exercise 3.5 [Robust Linear Estimation, see [44]] Let a signal $v \in \mathbb{R}^n$ be observed according to

$$y = Av + \xi,$$

where A is an $m \times n$ matrix, known up to "unstructured norm-bounded perturbation":

$$A \in \mathcal{A} = \{A = A_{\rm n} + L^T \Delta R : \Delta \in \mathbb{R}^{p\times q}, \|\Delta\|_{2,2} \leq \rho\},$$

and ξ is a zero mean random noise with a known covariance matrix Σ. Our a priori information on v is that

$$v \in V = \{v : v^T Q v \leq 1\},$$

where $Q \succ 0$. We are looking for a linear estimate

$$\widehat{v} = Gy$$

with the smallest possible worst-case mean squared error

$$\mathrm{EstErr} = \sup_{v\in V, A\in\mathcal{A}} \left(\mathbf{E}\left\{\|G[Av+\xi] - v\|_2^2\right\}\right)^{1/2}$$

(cf. section 3.2.6).

1) Reformulate the problem of building the optimal estimate equivalently as the RC of uncertain semidefinite program with unstructured norm-bounded uncertainty and reduce this RC to an explicit semidefinite program.

2) Assume that $m = n$, $\Sigma = \sigma^2 I_n$, and the matrices $A_{\rm n}^T A_{\rm n}$ and Q commute, so that $A_{\rm n} = V\mathrm{Diag}\{a\}U^T$ and $Q = U\mathrm{Diag}\{q\}U^T$ for certain orthogonal matrices U, V and certain vectors $a \geq 0$, $q > 0$. Let, further, $\mathcal{A} = \{A_{\rm n} + \Delta : \|\Delta\|_{2,2} \leq \rho\}$. Prove that in the situation in question we lose nothing when looking for G in the form of

$$G = U\mathrm{Diag}\{g\}V^T,$$

and build an explicit convex optimization program with just two variables specifying the optimal choice of G.

Exercise 3.6

1) Let $p, q \in \mathbb{R}^n$ and $\lambda > 0$. Prove that $\lambda pp^T + \frac{1}{\lambda} qq^T \succeq \pm[pq^T + qp^T]$.

2) Let p, q be as in 1) with $p, q \neq 0$, and let $Y \in \mathbf{S}^n$ be such that $Y \succeq \pm[pq^T + qp^T]$. Prove that there exists $\lambda > 0$ such that $Y \succeq \lambda pp^T + \frac{1}{\lambda} qq^T$.

3) Consider the semi-infinite LMI of the following specific form:

$$\forall (\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1) : \mathcal{A}_\mathrm{n}(x) + \rho \sum_{\ell=1}^{L} \zeta_\ell \left[L_\ell^T(x) R_\ell + R_\ell^T L_\ell(x) \right] \succeq 0, \tag{3.7.1}$$

where $L_\ell^T(x), R_\ell^T \in \mathbb{R}^n$, $R_\ell \neq 0$ and $L_\ell(x)$ are affine in x, as is the case in Lyapunov Stability Analysis/Synthesis under interval uncertainty (3.5.7) with $\mu = 1$.

Prove that the safe tractable approximation, tight within the factor $\pi/2$, of (3.7.1), that is, the system of LMIs

$$\begin{array}{l} Y_\ell \succeq \pm \left[L_\ell^T(x) R_\ell + R_\ell^T L_\ell(x) \right], \; 1 \leq \ell \leq L \\ \mathcal{A}_\mathrm{n}(x) - \rho \sum_{\ell=1}^{L} Y_\ell \succeq 0 \end{array} \tag{3.7.2}$$

in x and in matrix variables $Y_1, ..., Y_L$ is equivalent to the LMI

$$\left[\begin{array}{c|cccc} \mathcal{A}_\mathrm{n}(x) - \rho \sum_{\ell=1}^{L} \lambda_\ell R_\ell^T R_\ell & L_1^T(x) & L_2^T(x) & \cdots & L_L^T(x) \\ \hline L_1(x) & \lambda_1/\rho & & & \\ L_2(x) & & \lambda_2/\rho & & \\ \vdots & & & \ddots & \\ L_L(x) & & & & \lambda_L/\rho \end{array} \right] \succeq 0 \tag{3.7.3}$$

in x and real variables $\lambda_1 ..., \lambda_L$. Here the equivalence means that x can be extended to a feasible solution of (3.7.2) if and only if it can be extended to a feasible solution of (3.7.3).

Exercise 3.7 Consider the Signal Processing problem as follows. We are given uncertainty-affected observations

$$y = Av + \xi$$

of a signal v known to belong to a set V. Uncertainty "sits" in the "measurement error" ξ, known to belong to a given set Ξ, and in A — all we know is that $A \in \mathcal{A}$. We assume that V and Ξ are intersections of ellipsoids centered at the origin:

$$V = \{v \in \mathbb{R}^n : v^T P_i v \leq 1, \; 1 \leq i \leq I\}, \; [P_i \succeq 0, \textstyle\sum_i P_i \succ 0]$$

$$\Xi = \{\xi \in \mathbb{R}^m : \xi^T Q_j \xi \leq \rho_\xi^2, \; 1 \leq j \leq J\}, \; [Q_j \succeq 0, \textstyle\sum_j Q_j \succeq 0]$$

and $\mathcal{A}$ is given by structured norm-bounded perturbations:

$$\mathcal{A} = \{A = A_\mathrm{n} + \sum_{\ell=1}^{L} L_\ell^T \Delta_\ell R_\ell, \Delta_\ell \in \mathbb{R}^{p_\ell \times q_\ell}, \|\Delta_\ell\|_{2,2} \leq \rho_A\}.$$

We are interested to build a linear estimate $\widehat{v} = Gy$ of v via y. The $\|\cdot\|_2$ error of such an estimate at a particular v is

$$\|Gy - v\|_2 = \|G[Av + \xi] - v\|_2 = \|(GA - I)v + G\xi\|_2,$$

and we want to build G that minimizes the worst, over all v, A, ξ compatible with our a priori information, estimation error

$$\max_{\xi \in \Xi, v \in V, A \in \mathcal{A}} \|(GA - I)v + G\xi\|_2.$$

Build a safe tractable approximation of this problem that seems reasonably tight when ρ_ξ and ρ_A are small.

Lecture 4

Globalized Robust Counterparts of Uncertain Linear and Conic Problems

In this lecture we extend the concept of Robust Counterpart in order to gain certain control on what happens when the actual data perturbations run out of the postulated perturbation set.

4.1 Globalized Robust Counterparts — Motivation and Definition

Let us come back to Assumptions A.1 – A.3 underlying the concept of Robust Counterpart and concentrate on **A.3**. This assumption is not a "universal truth" — in reality, there are indeed constraints that cannot be violated (e.g., you cannot order a negative supply), but also constraints whose violations, while undesirable, can be tolerated to some degree, (e.g., sometimes you can tolerate a shortage of a certain resource by implementing an "emergency measure" like purchasing it on the market, employing sub-contractors, taking out loans, etc.). Immunizing such "soft" constraints against data uncertainty should perhaps be done in a more flexible fashion than in the usual Robust Counterpart. In the latter, we ensure a constraint's validity for all realizations of the data from a given uncertainty set and do not care what happens when the data are outside of this set. For a soft constraint, we can take care of what happens in this latter case as well, namely, by ensuring *controlled deterioration* of the constraint when the data runs away from the uncertainty set. We are about to build a mathematically convenient model capturing the above requirements.

4.1.1 The Case of Uncertain Linear Optimization

Consider an uncertain linear constraint in variable x

$$[a^0 + \sum_{\ell=1}^{L} \zeta_\ell a^\ell]^T x \leq [b^0 + \sum_{\ell=1}^{L} \zeta_\ell b^\ell] \tag{4.1.1}$$

where ζ is the perturbation vector (cf. (1.3.4), (1.3.5)). Let $\mathcal{Z}_+$ be the set of all "physically possible" perturbations, and $\mathcal{Z} \subset \mathcal{Z}_+$ be the "normal range" of the perturbations — the one for which we insist on the constraint to be satisfied. With the usual RC approach, we treat $\mathcal{Z}$ as the only set of perturbations and require a candidate solution x to satisfy the constraint for all $\zeta \in \mathcal{Z}$. With our new approach, we add the requirement that *the violation of constraint in the case when* $\zeta \in \mathcal{Z}_+ \backslash \mathcal{Z}$ (that is a "physically possible" perturbation that is outside of the normal range) *should be bounded by a constant times the distance from* ζ *to* $\mathcal{Z}$. Both requirements — the validity of the constraint for $\zeta \in \mathcal{Z}$ and the bound on the constraint's violation when $\zeta \in \mathcal{Z}_+ \backslash \mathcal{Z}$ can be expressed by a single requirement

$$[a^0 + \sum_{\ell=1}^{L} \zeta_\ell a^\ell]^T x - [b^0 + \sum_{\ell=1}^{L} \zeta_\ell b^\ell] \leq \alpha \mathrm{dist}(\zeta, \mathcal{Z}) \quad \forall \zeta \in \mathcal{Z}_+,$$

where $\alpha \geq 0$ is a given "global sensitivity."

In order to make the latter requirement tractable, we add some structure to our setup. Specifically, let us assume that:

(G.a) The normal range $\mathcal{Z}$ of the perturbation vector ζ is a nonempty closed convex set;

(G.b) The set $\mathcal{Z}_+$ of all "physically possible" perturbations is the sum of $\mathcal{Z}$ and a closed convex cone $\mathcal{L}$:

$$\mathcal{Z}_+ = \mathcal{Z} + \mathcal{L} = \{\zeta = \zeta' + \zeta'' : \zeta' \in \mathcal{Z}, \zeta'' \in \mathcal{L}\}; \tag{4.1.2}$$

(G.c) We measure the distance from a point $\zeta \in \mathcal{Z}_+$ to the normal range $\mathcal{Z}$ of the perturbations in a way that is consistent with the structure (4.1.2) of $\mathcal{Z}_+$, specifically, by

$$\mathrm{dist}(\zeta, \mathcal{Z} \mid \mathcal{L}) = \inf_{\zeta'} \left\{ \|\zeta - \zeta'\| : \zeta' \in \mathcal{Z}, \zeta - \zeta' \in \mathcal{L} \right\}, \tag{4.1.3}$$

where $\|\cdot\|$ is a fixed norm on $\mathbb{R}^L$.

In what follows, we refer to a triple $(\mathcal{Z}, \mathcal{L}, \|\cdot\|)$ arising in (G.a–c) as a *perturbation structure* for the uncertain constraint (4.1.1).

Definition 4.1 *Given $\alpha \geq 0$ and a perturbation structure $(\mathcal{Z}, \mathcal{L}, \|\cdot\|)$, we say that a vector x is a globally robust feasible solution to uncertain linear constraint* (4.1.1) *with global sensitivity α, if x satisfies the semi-infinite constraint*

$$[a^0 + \sum_{\ell=1}^{L} \zeta_\ell a^\ell]^T x - [b^0 + \sum_{\ell=1}^{L} \zeta_\ell b^\ell] \leq \alpha \, \mathrm{dist}(\zeta, \mathcal{Z} | \mathcal{L}) \;\; \forall \zeta \in \mathcal{Z}_+ = \mathcal{Z} + \mathcal{L}. \tag{4.1.4}$$

We refer to the semi-infinite constraint (4.1.4) *as the Globalized Robust Counterpart (GRC) of the uncertain constraint* (4.1.1).

Note that global sensitivity $\alpha = 0$ corresponds to the most conservative attitude where the constraint must be satisfied for all physically possible perturbations; with $\alpha = 0$, the GRC becomes the usual RC of the uncertain constraint with $\mathcal{Z}_+$ in the role of the perturbation set. The larger α, the less conservative the GRC.

Now, given an uncertain Linear Optimization program with affinely perturbed data

$$\left\{ \min_x \left\{ c^T x : Ax \leq b \right\} : [A, b] = [A^0, b^0] + \sum_{\ell=1}^{L} \zeta_\ell [A^\ell, b^\ell] \right\} \tag{4.1.5}$$

(w.l.o.g., we assume that the objective is certain) and a perturbation structure $(\mathcal{Z}, \mathcal{L}, \|\cdot\|)$, we can replace every one of the constraints with its Globalized Robust Counterpart, thus ending up with the GRC of (4.1.5). In this construction, we can associate different sensitivity parameters α to different constraints. Moreover, we can treat these sensitivities as design variables rather than fixed parameters, add linear constraints on these variables, and optimize both in x *and* α an objective function that is a mixture of the original objective and a weighted sum of the sensitivities.

4.1.2 The Case of Uncertain Conic Optimization

Consider an uncertain conic problem (3.1.2), (3.1.3):

$$\min_x \left\{c^T x + d : A_i x - b_i \in \mathbf{Q}_i,\ 1 \leq i \leq m\right\}, \tag{4.1.6}$$

where $\mathbf{Q}_i \subset \mathbb{R}^{k_i}$ are nonempty closed convex sets given by finite lists of conic inclusions:

$$\mathbf{Q}_i = \{u \in \mathbb{R}^{k_i} : Q_{i\ell} u - q_{i\ell} \in \mathbf{K}_{i\ell},\ \ell = 1, ..., L_i\}, \tag{4.1.7}$$

with closed convex pointed cones $\mathbf{K}_{i\ell}$, and let the data be affinely parameterized by the perturbation vector ζ:

$$(c, d, \{A_i, b_i\}_{i=1}^m) = (c^0, d^0, \{A_i^0, b_i^0\}_{i=1}^m) + \sum_{\ell=1}^L \zeta_\ell (c^\ell, d^\ell, \{A_i^\ell, b_i^\ell\}_{i=1}^m). \tag{4.1.8}$$

When extending the notion of Globalized Robust Counterpart from the case of Linear Optimization to the case of Conic one, we need a small modification. Assuming, same as in the former case, that the set $\mathcal{Z}_+$ of all "physically possible" realizations of the perturbation vector ζ is of the form $\mathcal{Z}_+ = \mathcal{Z} + \mathcal{L}$, where $\mathcal{Z}$ is the closed convex normal range of ζ and $\mathcal{L}$ is a closed convex cone, observe that in the conic case, as compared to the LO one, the left hand side of our uncertain constraint (4.1.6) is vector rather than scalar, so that a straightforward analogy of (4.1.4) does not make sense. Note, however, that when rewriting (4.1.1) in our present "inclusion form"

$$[a^0 + \sum_{\ell=1}^L \zeta_\ell a^\ell]^T x - [b^0 + \sum_{\ell=1}^L \zeta_\ell b^\ell] \in \mathbf{Q} \equiv \mathbb{R}_-, \tag{$*$}$$

relation (4.1.4) says exactly that the distance from the left hand side of $(*)$ to $\mathbf{Q}$ does not exceed $\alpha \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L})$ for all $\zeta \in \mathcal{Z} + \mathcal{L}$. In this form, the notion of global sensitivity admits the following multi-dimensional extension:

Definition 4.2 *Consider an uncertain convex constraint*

$$[P_0 + \sum_{\ell=1}^L \zeta_\ell P_\ell] y - [p^0 + \sum_{\ell=1}^L \zeta_\ell p^\ell] \in \mathbf{Q}, \tag{4.1.9}$$

where $\mathbf{Q}$ is a nonempty closed convex subset in $\mathbb{R}^k$. Let $\|\cdot\|_Q$ be a norm on $\mathbb{R}^k$, $\|\cdot\|_Z$ be a norm on $\mathbb{R}^L$, $\mathcal{Z} \subset \mathbb{R}^L$ be a nonempty closed convex normal range of perturbation ζ, and $\mathcal{L} \subset\in \mathbb{R}^L$ be a *closed convex cone. We say that a candidate solution y is robust feasible, with global sensitivity α, for* (4.1.9)*, under the perturbation structure* $(\|\cdot\|_Q, \|\cdot\|_Z, \mathcal{Z}, \mathcal{L})$*, if*

$$\begin{array}{l}\mathrm{dist}([P_0+\sum_{\ell=1}^L \zeta_\ell P_\ell]y-[p^0+\sum_{\ell=1}^L\zeta_\ell p^\ell],\mathbf{Q})\leq \alpha \mathrm{dist}(\zeta,\mathcal{Z}|\mathcal{L})\\ \qquad\qquad\qquad\qquad\forall \zeta\in\mathcal{Z}_+=\mathcal{Z}+\mathcal{L}\\ \left[\begin{array}{rcl}\mathrm{dist}(u,\mathbf{Q}) &=& \min\limits_v\{\|u-v\|_Q : v\in\mathbf{Q}\}\\ \mathrm{dist}(\zeta,\mathcal{Z}|\mathcal{L}) &=& \min\limits_v\{\|\zeta-v\|_Z : v\in\mathcal{Z}, \zeta-v\in\mathcal{L}\}\end{array}\right].\end{array} \tag{4.1.10}$$

Sometimes it is necessary to add some structure to the latter definition. Specifically, assume that the space $\mathbb{R}^L$ where ζ lives is given as a direct product:

$$\mathbb{R}^L = \mathbb{R}^{L_1}\times ...\times \mathbb{R}^{L_S}$$

and let $\mathcal{Z}^s\subset\mathbb{R}^{L_s}$, $\mathcal{L}^s\subset\mathbb{R}^{L_s}$, $\|\cdot\|_s$ be, respectively, closed nonempty convex set, closed convex cone and a norm on $\mathbb{R}^{L_s}$, $s=1,...,S$. For $\zeta\in\mathbb{R}^L$, let ζ^s, $s=1,...,S$, be the projections of ζ onto the direct factors $\mathbb{R}^{L_s}$ of $\mathbb{R}^L$. The "structured version" of Definition 4.2 is as follows:

Definition 4.3 *A candidate solution y to the uncertain constraint* (4.1.9) *is robust feasible with global sensitivities α_s, $1\leq s\leq S$, under the perturbation structure* $(\|\cdot\|_Q, \{\mathcal{Z}^s,\mathcal{L}^s,\|\cdot\|_s\}_{s=1}^S)$*, if*

$$\begin{array}{l}\mathrm{dist}([P_0+\sum_{\ell=1}^L \zeta_\ell P_\ell]y-[p^0+\sum_{\ell=1}^L\zeta_\ell p^\ell],\mathbf{Q})\leq \sum_{s=1}^S\alpha_s \mathrm{dist}(\zeta^s,\mathcal{Z}^s|\mathcal{L}^s)\\ \qquad\qquad\qquad\qquad\forall \zeta\in\mathcal{Z}_+=\underbrace{(\mathcal{Z}^1\times...\times\mathcal{Z}^S)}_{\mathcal{Z}}+\underbrace{\mathcal{L}^1\times...\times\mathcal{L}^S}_{\mathcal{L}}\\ \left[\begin{array}{rcl}\mathrm{dist}(u,\mathbf{Q}) &=& \min\limits_v\{\|u-v\|_Q : v\in\mathbf{Q}\}\\ \mathrm{dist}(\zeta^s,\mathcal{Z}^s|\mathcal{L}^s) &=& \min\limits_{v^s}\{\|\zeta^s-v^s\|_s : v^s\in\mathcal{Z}^s, \zeta^s-v^s\in\mathcal{L}^s\}.\end{array}\right]\end{array} \tag{4.1.11}$$

Note that Definition 4.2 can be obtained from Definition 4.3 by setting $S=1$. We refer to the semi-infinite constraints (4.1.10), (4.1.11) as to *Globalized Robust Counterparts* of the uncertain constraint (4.1.9) w.r.t. the perturbations structure in question. When building the GRC of uncertain problem (4.1.6), (4.1.8), we first rewrite it as an uncertain problem

$$\min_{y=(x,t)}\left\{t:\begin{array}{l}c^Tx+d-t\equiv\overbrace{[c^0+\sum_{\ell=1}^L\zeta_\ell c^\ell]^Tx+[d^0+\sum_{\ell=1}^L\zeta_\ell d^\ell]-t}^{[P_{00}+\sum_{\ell=1}^L\zeta_\ell P_{0\ell}]y-[p_0^0+\sum_{\ell=1}^L\zeta_\ell p_0^\ell]}\in\mathbf{Q}_0\equiv\mathbb{R}_-\\ A_ix-b_i\equiv\underbrace{[A_i^0+\sum_{\ell=1}^L\zeta_\ell A_i^\ell]x-[b_i^0+\sum_{\ell=1}^L\zeta_\ell b_i^\ell]}_{[P_{i0}+\sum_{\ell=1}^L\zeta_\ell P_{i\ell}]y-[p_i^0+\sum_{\ell=1}^L\zeta_\ell p_i^\ell]}\in\mathbf{Q}_i,\ 1\leq i\leq m\end{array}\right\}$$

with certain objective, and then replace the constraints with their Globalized RCs. The underlying perturbation structures and global sensitivities may vary from constraint to constraint.

4.1.3 Safe Tractable Approximations of GRCs

A Globalized RC, the same as the plain one, can be computationally intractable, in which case we can look for the second best thing — a safe tractable approximation of the GRC. This notion is defined as follows (cf. Definition 3.2):

Definition 4.4 *Consider the uncertain convex constraint* (4.1.9) *along with its GRC* (4.1.11). *We say that a system $\mathcal{S}$ of convex constraints in variables y, $\alpha = (\alpha_1, ..., \alpha_S) \geq 0$, and, perhaps, additional variables u, is a safe approximation of the GRC, if the projection of the feasible set of $\mathcal{S}$ on the space of (y, α) variables is contained in the feasible set of the GRC:*

$$\begin{array}{l}\forall(\alpha = (\alpha_1, ..., \alpha_S) \geq 0, y) :\\ (\exists u :\ (y, \alpha, u) \text{ satisfies } \mathcal{S}) \Rightarrow (y, \alpha) \text{ satisfies } (4.1.11).\end{array}$$

This approximation is called tractable, provided that $\mathcal{S}$ is so, (e.g., $\mathcal{S}$ is an explicit system of CQIs/LMIs of, more general, the constraints in $\mathcal{S}$ are efficiently computable).

When quantifying the tightness of an approximation, we, as in the case of RC, assume that the normal range $\mathcal{Z} = \mathcal{Z}^1 \times ... \times \mathcal{Z}^S$ of the perturbations contains the origin and is included in the single-parametric family of normal ranges:

$$\mathcal{Z}_\rho = \rho\mathcal{Z},\ \ \rho > 0.$$

As a result, the GRC (4.1.11) of (4.1.9) becomes a member, corresponding to $\rho = 1$, of the single-parametric family of constraints

$$\begin{array}{c}\mathrm{dist}([P_0 + \sum_{\ell=1}^{L} \zeta_\ell P_\ell]y - [p^0 + \sum_{\ell=1}^{L} \zeta_\ell p^\ell], \mathbf{Q}) \leq \sum_{s=1}^{S} \alpha_s \mathrm{dist}(\zeta^s, \mathcal{Z}^s | \mathcal{L}^s)\\ \forall \zeta \in \mathcal{Z}_+^\rho = \underbrace{\rho(\mathcal{Z}^1 \times ... \times \mathcal{Z}^S)}_{\mathcal{Z}_\rho} + \underbrace{\mathcal{L}^1 \times ... \times \mathcal{L}^S}_{\mathcal{L}}\end{array} \qquad (\mathrm{GRC}_\rho)$$

in variables y, α. We define the tightness factor of a safe tractable approximation of the GRC as follows (cf. Definition 3.3):

Definition 4.5 *Assume that we are given an approximation scheme that associates with* (GRC_ρ) *a finite system $\mathcal{S}_\rho$ of efficiently computable convex constraints on variables y, α and, perhaps, additional variables u, depending on $\rho > 0$ as a parameter. We say that this approximation scheme is a safe tractable approximation of the GRC tight, within tightness factor $\vartheta \geq 1$, if*

(i) *For every $\rho > 0$, $\mathcal{S}_\rho$ is a safe tractable approximation of* (GRC_ρ)*: whenever $(y, \alpha \geq 0)$ can be extended to a feasible solution of $\mathcal{S}_\rho$, (y, α) satisfies* (GRC_ρ)*;*

(ii) *Whenever $\rho > 0$ and $(y, \alpha \geq 0)$ are such that (y, α) cannot be extended to a feasible solution of $\mathcal{S}_\rho$, the pair $(y, \vartheta^{-1}\alpha)$ is not feasible for* $(\mathrm{GRC}_{\vartheta\rho})$.

4.2 Tractability of GRC in the Case of Linear Optimization

As in the case of the usual Robust Counterpart, the central question of computational tractability of the Globalized RC of an uncertain LO reduces to a similar question for the GRC (4.1.4) of a single uncertain linear constraint (4.1.1). The latter question is resolved to a large extent by the following observation:

Proposition 4.1 *A vector x satisfies the semi-infinite constraint* (4.1.4) *if and only if x satisfies the following pair of semi-infinite constraints:*

$$\begin{array}{ll}(a) & \left[a^0+\sum_{\ell=1}^L \zeta_\ell a^\ell\right]^T x \leq \left[b^0+\sum_{\ell=1}^L \zeta_\ell b^\ell\right] \ \forall \zeta \in \mathcal{Z} \\ (b) & \left[\sum_{\ell=1}^L \Delta_\ell a^\ell\right]^T x \leq \left[\sum_{\ell=1}^L \Delta_\ell b^\ell\right] + \alpha \ \forall \Delta \in \widetilde{\mathcal{Z}} \equiv \{\Delta \in \mathcal{L} : \|\Delta\| \leq 1\}.\end{array} \tag{4.2.1}$$

Remark 4.1 Proposition 4.1 implies that the GRC of an uncertain linear inequality is *equivalent* to a pair of semi-infinite linear inequalities of the type arising in the usual RC. Consequently, we can invoke the representation results of section 1.3 to show that *under mild assumptions on the perturbation structure, the GRC* (4.1.4) *can be represented by a "short" system of explicit convex constraints.*

Proof of Proposition 4.1. Let x satisfy (4.1.4). Then x satisfies (4.2.1.a) due to $\mathrm{dist}(\zeta,\mathcal{Z}|\mathcal{L}) = 0$ for $\zeta \in \mathcal{Z}$. In order to demonstrate that x satisfies (4.2.1.b) as well, let $\bar\zeta \in \mathcal{Z}$ and $\Delta \in \mathcal{L}$ with $\|\Delta\| \leq 1$. By (4.1.4) and since $\mathcal{L}$ is a cone, for every $t > 0$ we have $\zeta_t := \bar\zeta + t\Delta \in \mathcal{Z} + \mathcal{L}$ and $\mathrm{dist}(\zeta_t, \mathcal{Z}|\mathcal{L}) \leq \|t\Delta\| \leq t$; applying (4.1.4) to $\zeta = \zeta_t$, we therefore get

$$\left[a^0+\sum_{\ell=1}^L \bar\zeta_\ell a^\ell\right]^T x + t\left[\sum_{\ell=1}^L \Delta_\ell a^\ell\right]^T x \leq \left[b^0+\sum_{\ell=1}^L \bar\zeta_\ell b^\ell\right] + t\left[\sum_{\ell=1}^L \Delta_\ell b^\ell\right] + \alpha t.$$

Dividing both sides in this inequality by t and passing to limit as $t \to \infty$, we see that the inequality in (4.2.1.b) is valid at our Δ. Since $\Delta \in \widetilde{\mathcal{Z}}$ is arbitrary, x satisfies (4.2.1.b), as claimed.

It remains to prove that if x satisfies (4.2.1), then x satisfies (4.1.4). Indeed, let x satisfy (4.2.1). Given $\zeta \in \mathcal{Z} + \mathcal{L}$ and taking into account that $\mathcal{Z}$ and $\mathcal{L}$ are closed, we can find $\bar\zeta \in \mathcal{Z}$ and $\Delta \in \mathcal{L}$ such that $\bar\zeta + \Delta = \zeta$ and $t := \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L}) = \|\Delta\|$. Representing $\Delta = te$ with $e \in \mathcal{L}$, $\|e\| \leq 1$, we have

$$\begin{array}{l}\left[a^0+\sum_{\ell=1}^L \zeta_\ell a^\ell\right]^T x - \left[b^0+\sum_{\ell=1}^L \zeta_\ell b^\ell\right] \\ = \underbrace{\left[a^0+\sum_{\ell=1}^L \bar\zeta_\ell a^\ell\right]^T x - \left[b^0+\sum_{\ell=1}^L \bar\zeta_\ell b^\ell\right]}_{\leq 0 \text{ by } (4.2.1.a)} + \underbrace{\left[\sum_{\ell=1}^L \Delta_\ell a^\ell\right]^T x - \left[\sum_{\ell=1}^L \Delta_\ell b^\ell\right]}_{\substack{= t\left[\left[\sum_{\ell=1}^L e_\ell a^\ell\right]^T x - \left[\sum_{\ell=1}^L e_\ell b^\ell\right]\right] \\ \leq t\alpha \text{ by } (4.2.1.b)}} \\ \leq t\alpha = \alpha \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L}).\end{array}$$

Since $\zeta \in \mathcal{Z} + \mathcal{L}$ is arbitrary, x satisfies (4.1.4). □

Example 4.1 Consider the following 3 perturbation structures $(\mathcal{Z}, \mathcal{L}, \|\cdot\|)$:
(a) $\mathcal{Z}$ is a box $\{\zeta : |\zeta_\ell| \leq \sigma_\ell, 1 \leq \ell \leq L\}$, $\mathcal{L} = \mathbb{R}^L$ and $\|\cdot\| = \|\cdot\|_1$;
(b) $\mathcal{Z}$ is an ellipsoid $\{\zeta : \sum_{\ell=1}^L \zeta_\ell^2/\sigma_\ell^2 \leq \Omega^2\}$, $\mathcal{L} = \mathbb{R}^L_+$ and $\|\cdot\| = \|\cdot\|_2$;
(c) $\mathcal{Z}$ is the intersection of a box and an ellipsoid: $\mathcal{Z} = \{\zeta : |\zeta_\ell| \leq \sigma_\ell, 1 \leq \ell \leq L, \sum_{\ell=1}^L \zeta_\ell^2/\sigma_\ell^2 \leq \Omega^2\}$, $\mathcal{L} = \mathbb{R}^L$, $\|\cdot\| = \|\cdot\|_\infty$.

In these cases the GRC of (4.1.1) is equivalent to the finite systems of explicit convex inequalities as follows:
Case (a):

$$\begin{array}{ll} (a) & [a^0]^T x + \sum_{\ell=1}^{L} \sigma_\ell |[a^\ell]^T x - b^\ell| \leq b^0 \\ (b) & |[a^\ell]^T x - b^\ell| \leq \alpha,\ \ell = 1, ..., L \end{array}$$

Here (a) represents the constraint $(4.2.1.a)$ (cf. Example 1.4), and (b) represents the constraint $(4.2.1.b)$ (why?)
Case (b):

$$\begin{array}{ll} (a) & [a^0]^T x + \Omega \left(\sum_{\ell=1}^{L} \sigma_\ell^2 ([a^\ell]^T x - b^\ell)^2 \right)^{1/2} \leq b^0 \\ (b) & \left(\sum_{\ell=1}^{L} \max^2 [[a^\ell]^T x - b^\ell, 0] \right)^{1/2} \leq \alpha. \end{array}$$

Here (a) represents the constraint $(4.2.1.a)$ (cf. Example 1.5), and (b) represents the constraint $(4.2.1.b)$.
Case (c):

$$\begin{array}{ll} (a.1) & [a^0]^T x + \sum_{\ell=1}^{L} \sigma_\ell |z_\ell| + \Omega \left(\sum_{\ell=1}^{L} \sigma_\ell^2 w_\ell^2 \right)^{1/2} \leq b^0 \\ (a.2) & z_\ell + w_\ell = [a^\ell]^T x - b^\ell,\ \ell = 1, ..., L \\ (b) & \sum_{\ell=1}^{L} |[a^\ell]^T x - b^\ell| \leq \alpha. \end{array}$$

Here $(a.1–2)$ represent the constraint $(4.2.1.a)$ (cf. Example 1.6), and (b) represents the constraint $(4.2.1.b)$.

4.2.1 Illustration: Antenna Design

We are about to illustrate the GRC approach by applying it to the Antenna Design problem (Example 1.1), where we are interested in the uncertain LP problem of the form

$$\left\{ \min_{x,\tau} \{ \tau : \|d - D(I + \mathrm{Diag}\{\zeta\})x\|_\infty \leq \tau \} : \|\zeta\|_\infty \leq \rho \right\}; \tag{4.2.2}$$

here D is a given $m \times L$ matrix and $d \in \mathbb{R}^m$ is a given vector. The RC of the problem is equivalent to

$$\mathrm{Opt}(\rho) = \min_x \left\{ F_x(\rho) := \max_{\|\zeta\|_\infty \leq \rho} \|d - D(I + \mathrm{Diag}\{\zeta\})x\|_\infty \right\} \tag{RC$_\rho$}$$

For a candidate design x, the function $F_x(\rho)$ of the uncertainty level ρ has a very transparent interpretation: it is the worst-case, over perturbations ζ with $\|\zeta\|_\infty \leq \rho$, loss (deviation of the synthesized diagram from the target one) of this design. This clearly is a convex and nondecreasing function of ρ.

Let us fix a "reference uncertainty level" $\bar{\rho} \geq 0$ and equip our uncertain problem with the perturbation structure

$$\mathcal{Z} = \{\zeta : \|\zeta\|_\infty \leq \bar{\rho}\},\ \mathcal{L} = \mathbb{R}^L,\ \|\cdot\| = \|\cdot\|_\infty. \tag{4.2.3}$$

With this perturbation structure, it can be immediately derived from Proposition 4.1 (do it!) that a pair (τ, x) is a robust feasible solution to the GRC with global sensitivity α if and only if

$$\tau \geq F_x(\bar{\rho}) \ \&\ \alpha \geq \alpha(x) = \lim_{\rho \to \infty} \frac{d}{d\rho} F_\rho(x) = \max_{i \leq m} \sum_{j=1}^{L} |D_{ij}||x_j|.$$

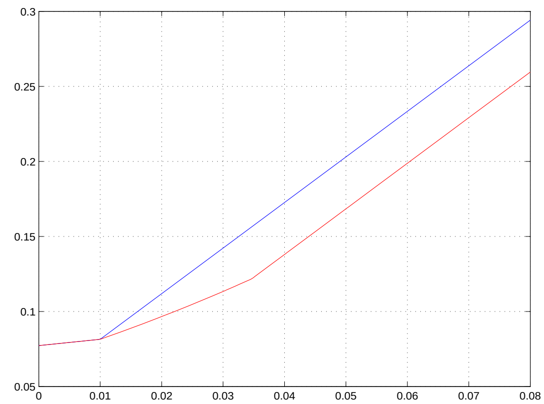


Figure 4.1: Bound (4.2.5) with $\rho_0 = 0.01$ (blue) on the loss $F_x(\rho)$ (red) associated with the optimal solution x to ($\mathrm{RC}_{0.01}$). The common values of the bound and the loss at $\rho_0 = 0.01$ is the optimal value of ($\mathrm{RC}_{0.01}$).

Note that the best (the smallest) value $\alpha(x)$ of global sensitivity which, for appropriately chosen τ, makes (τ, x) feasible for the GRC depends solely on x; we shall refer to this quantity as to the *global sensitivity of* x. Due to its origin, and since $F_x(\rho)$ is convex and nondecreasing, $\alpha(x)$ and a value of $F_x(\cdot)$ at a particular $\rho = \rho_0 \geq 0$ imply a piecewise linear upper bound on $F_x(\cdot)$:

$$\forall \rho \geq 0 : F_x(\rho) \leq \begin{cases} F_x(\rho_0), & \rho < \rho_0 \\ F_x(\rho_0) + \alpha(x)[\rho - \rho_0], & \rho \geq \rho_0 \end{cases}. \tag{4.2.4}$$

Taking into account also the value of F_x at 0, we can improve this bound to

$$\forall \rho \geq 0 : F_x(\rho) \leq \begin{cases} \frac{\rho_0 - \rho}{\rho_0} F_x(0) + \frac{\rho}{\rho_0} F_x(\rho_0), & \rho < \rho_0 \\ F_x(\rho_0) + \alpha(x)[\rho - \rho_0], & \rho \geq \rho_0 \end{cases}. \tag{4.2.5}$$

In figure 4.1, we plot the latter bound and the true $F_x(\cdot)$ for the robust design built in section 1.4.1 (that is, the optimal solution to ($\mathrm{RC}_{0.01}$)), choosing $\rho_0 = 0.01$. When designing a robust antenna, our "ideal goal" would be to choose the design x which makes the loss $F_x(\rho)$ as small as possible *for all values of* ρ; of course, this goal usually cannot be achieved. With the usual RC approach, we fix the uncertainty level at $\bar{\rho}$ and minimize over x the loss at this particular value of ρ, with no care of how rapidly this loss grows when the true uncertainty level ρ exceeds our guess $\bar{\rho}$. This makes sense when we understand well what is the uncertainty level at which our system should work, which sometimes is not the case. With the GRC approach, we can, to some extent, take care of both the value of F_x at $\rho = \bar{\rho}$ and of the rate at which the loss grows with ρ, thus making our design better suited to the situations when it should be used in a wide range of uncertainty levels. For example, we can act as follows:

- We first solve ($\mathrm{RC}_{\bar{\rho}}$), thus getting the "reference" design $\bar{x}$ with the loss at the uncertainty level $\bar{\rho}$ as small as possible, so that $F_{\bar{x}}(\bar{\rho}) = \mathrm{Opt}(\mathrm{RC}_{\bar{\rho}})$;
- We then increase the resulting loss by certain percentage δ (say, $\delta = 0.1$) and choose, as

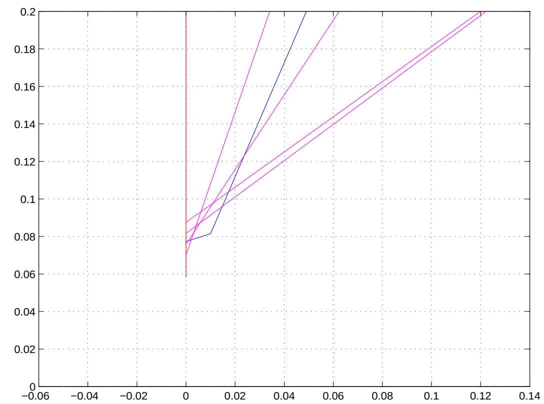


Figure 4.2: Red and magenta: bounds (4.2.4) on the losses for optimal solutions to (4.2.6) for the values of δ listed in table 4.1; the bounds correspond to $\rho_0 = 0$. Blue: bound (4.2.5) with $\rho_0 = 0.01$ on the loss $F_x(\rho)$ associated with the optimal solution x to $(\mathrm{RC}_{0.01})$.

δ	0	0.2	0.3	0.4	0.5
$\beta_*(\delta)$	2.6×10^5	3.8134	1.9916	0.9681	0.9379

Table 4.1: Tolerances δ and quantities $\beta_*(\delta)$ (the global sensitivities of the optimal solutions to (4.2.6)). Pay attention to how huge is the global sensitivity $\beta_*(0)$ of the nominal optimal design. For comparison: the global sensitivity of the robust design built in section 1.4.1 is just 3.0379.

the actual design, the solution to the minimization problem

$$\min_x \left\{\alpha(x) : F_x(\bar{\rho}) \leq (1+\delta)\mathrm{Opt}(\mathrm{RC}_{\bar{\rho}})\right\}.$$

In other words, we allow for a controlled sacrifice in the loss at the "nominal" uncertainty level $\bar{\rho}$ in order to get as good as possible upper bound (4.2.4) on the loss in the range $\rho \geq \bar{\rho}$.

In figure 4.2 and in table 4.1, we illustrate the latter approach in the special case when $\bar{\rho} = 0$. In this case, we want from our design to perform nearly as well as (namely, within percentage δ of) the nominally optimal design in the ideal case of no actuation errors, and optimize under this restriction the global sensitivity of the design w.r.t. the magnitude of actuation errors. Mathematically, this reduces to solving a simple LP problem

$$\beta_*(\delta) = \min_x \left\{ \max_{i \leq m} \sum_{j=1}^{L} |D_{ij}||x_j| : \|d - Dx\|_\infty \leq (1+\delta)\min_u \|d - Du\|_\infty \right\}. \tag{4.2.6}$$

4.3 Tractability of GRC in the Case of Conic Optimization

4.3.1 Decomposition

Preliminaries

Recall the notion of the *recessive cone* of a closed and nonempty convex set $\mathbf{Q}$:

Definition 4.6 *Let $\mathbf{Q} \subset \mathbb{R}^k$ be a nonempty closed convex set and $\bar{x} \in \mathbf{Q}$. The recessive cone $\mathrm{Rec}(\mathbf{Q})$ of Q is comprised of all rays emanating from $\bar{x}$ and contained in $\mathbf{Q}$:*

$$\mathrm{Rec}(\mathbf{Q}) = \{h \in \mathbb{R}^k : \bar{x} + th \in \mathbf{Q} \forall t \geq 0\}.$$

(Due to closedness and convexity of $\mathbf{Q}$, the right hand side set in this formula is independent of the choice of $\bar{x} \in \mathbf{Q}$ and is a nonempty closed convex cone in $\mathbb{R}^k$.)

Example 4.2
(i) The recessive cone of a nonempty bounded and closed convex set $\mathbf{Q}$ is trivial: $\mathrm{Rec}(\mathbf{Q}) = \{0\}$;
(ii) The recessive cone of a closed convex cone $\mathbf{Q}$ is $\mathbf{Q}$ itself;
(iii) The recessive cone of the set $\mathbf{Q} = \{x : Ax - b \in \mathbf{K}\}$, where $\mathbf{K}$ is a closed convex cone, is the set $\{h : Ah \in \mathbf{K}\}$;
(iv.a) Let $\mathbf{Q}$ be a closed convex set and $e_i \to e$, $i \to \infty$, $t_i \geq 0$, $t_i \to \infty$, $i \to \infty$, be sequences of vectors and reals such that $t_i e_i \in \mathbf{Q}$ for all i. Then $e \in \mathrm{Rec}(\mathbf{Q})$.
(iv.b) Vice versa: every $e \in \mathrm{Rec}(\mathbf{Q})$ can be represented in the form of $e = \lim_{i\to\infty} e_i$ with vectors e_i such that $ie_i \in \mathbf{Q}$.
Proof. (iv.a): Let $\bar{x} \in \mathbf{Q}$. With our e_i and t_i, for every $t > 0$ we have $\bar{x} + te_i - t/t_i\bar{x} = (t/t_i)(t_ie_i) + (1 - t/t_i)\bar{x}$. For all but finitely many values of i, the right hand side in this equality is a convex combination of two vectors from $\mathbf{Q}$ and therefore belongs to $\mathbf{Q}$; for $i \to \infty$, the left hand side converges to $\bar{x} + te$. Since $\mathbf{Q}$ is closed, we conclude that $\bar{x} + te \in \mathbf{Q}$; since $t > 0$ is arbitrary, we get $e \in \mathrm{Rec}(\mathbf{Q})$.
(iv.b): Let $e \in \mathrm{Rec}(\mathbf{Q})$ and $\bar{x} \in \mathbf{Q}$. Setting $e_i = i^{-1}(\bar{x} + ie)$, we have $ie_i \in \mathbf{Q}$ and $e_i \to e$ as $i \to \infty$. □

The Main Result

The following statement is the "multi-dimensional" extension of Proposition 4.1:

Proposition 4.2 *A candidate solution y is feasible for the GRC* (4.1.11) *of the uncertain constraint* (4.1.9) *if and only if x satisfies the following system of semi-infinite constraints:*

$$\begin{array}{ll}(a) & \overbrace{[P_0 + \sum_{\ell=1}^L \zeta_\ell P_\ell]^T y - [p^0 + \sum_{\ell=1}^L \zeta_\ell p^\ell]}^{P(y,\zeta)} \in \mathbf{Q} \\ & \qquad\qquad \forall \zeta \in \mathcal{Z} \equiv \mathcal{Z}^1 \times ... \times \mathcal{Z}^S \\ (b_s) & \mathrm{dist}(\overbrace{\sum_{\ell=1}^L [P_\ell y - p^\ell](E_s\zeta^s)_\ell}^{\Phi(y)E_s\zeta^s}, \mathrm{Rec}(\mathbf{Q})) \leq \alpha_s \\ & \qquad\qquad \forall \zeta^s \in \mathcal{L}^s_{\|\cdot\|_s} \equiv \{\zeta^s \in \mathcal{L}^s : \|\zeta^s\|_s \leq 1\}, \quad s = 1, ..., S,\end{array} \tag{4.3.1}$$

where E_s is the natural embedding of $\mathbb{R}^{L_s}$ into $\mathbb{R}^L = \mathbb{R}^{L_1} \times ... \times \mathbb{R}^{L_S}$ and $\mathrm{dist}(u, \mathrm{Rec}(\mathbf{Q})) = \min\limits_{v \in \mathrm{Rec}(\mathbf{Q})} \|u - v\|_Q$.

Proof. Assume that y satisfies (4.1.11), and let us verify that y satisfies (4.3.1). Relation (4.3.1.a) is evident. Let us fix $s \leq S$ and verify that y satisfies (4.3.1.b_s). Indeed, let $\bar\zeta \in \mathcal{Z}$ and $\zeta^s \in \mathcal{L}^s_{\|\cdot\|_s}$. For $i = 1, 2, ...$, let ζ_i be given by $\zeta_i^r = \bar\zeta^r$, $r \neq s$, and $\zeta_i^s = \bar\zeta^s + i\zeta^s$, so that $\mathrm{dist}(\zeta_i^r, \mathcal{Z}^r|\mathcal{L}^r)$ is 0 for $r \neq s$ and is $\leq i$ for $r = s$. Since y is feasible for (4.1.11), we have

$$\mathrm{dist}(\underbrace{[P_0 + \sum_{\ell=1}^{L}(\zeta_i)_\ell P_\ell]y - [p^0 + \sum_{\ell=1}^{L}(\zeta_i)_\ell p^\ell]}_{P(y,\zeta_i)=P(y,\bar\zeta)+i\Phi(y)E_s\zeta^s}, \mathbf{Q}) \leq \alpha_s i,$$

that is, there exists $q_i \in \mathbf{Q}$ such that

$$\|P(y,\bar\zeta) + i\Phi(y)E_s\zeta^s - q_i\|_Q \leq \alpha_s i.$$

From this inequality it follows that $\|q_i\|_Q/i$ remains bounded when $i \to \infty$; setting $q_i = ie_i$ and passing to a subsequence $\{i_\nu\}$ of indices i, we may assume that $e_{i_\nu} \to e$ as $\nu \to \infty$; by item (iv.a) of Example 4.2, we have $e \in \mathrm{Rec}(\mathbf{Q})$. We further have

$$\begin{array}{rcl}\|\Phi(y)E_s\zeta^s - e_{i_\nu}\|_Q &=& i_\nu^{-1}\|i_\nu\Phi(y)E_s\zeta - q_{i_\nu}\|_Q \\ &\leq& i_\nu^{-1}\left[\|P(y,\bar\zeta) + i_\nu\Phi(y)E_s\zeta^s - q_{i_\nu}\|_Q + i_\nu^{-1}\|P(y,\bar\zeta)\|_Q\right] \\ &\leq& \alpha_s + i_\nu^{-1}\|P(y,\bar\zeta)\|_Q,\end{array}$$

whence, passing to limit as $\nu \to \infty$, $\|\Phi(y)E_s\zeta^s - e\|_Q \leq \alpha_s$, whence, due to $e \in \mathrm{Rec}(\mathbf{Q})$, we have $\mathrm{dist}(\Phi(y)E_s\zeta^s, \mathrm{Rec}(\mathbf{Q})) \leq \alpha_s$. Since $\zeta^s \in \mathcal{L}^s_{\|\cdot\|_s}$ is arbitrary, (4.3.1.b_s) holds true.

Now assume that y satisfies (4.3.1), and let us prove that y satisfies (4.1.11). Indeed, given $\zeta \in \mathcal{Z} + \mathcal{L}$, we can find $\bar\zeta^s \in \mathcal{Z}^s$ and $\delta^s \in \mathcal{L}^s$ in such a way that $\zeta^s = \bar\zeta^s + \delta^s$ and $\|\delta^s\|_s = \mathrm{dist}(\zeta^s, \mathcal{Z}^s|\mathcal{L}^s)$. Setting $\bar\zeta = (\bar\zeta^1, ..., \bar\zeta^S)$ and invoking (4.3.1.a), the vector $\bar u = P(y,\bar\zeta)$ belongs to $\mathbf{Q}$. Further, for every s, by (4.3.1.b_s), there exists $\delta u^s \in \mathrm{Rec}(\mathbf{Q})$ such that $\|\Phi(y)E_s\delta^s - \delta u^s\|_Q \leq \alpha_s\|\delta^s\|_s = \alpha_s \mathrm{dist}(\zeta^s, \mathcal{Z}^s|\mathcal{L}^s)$. Since $P(y,\zeta) = P(y,\bar\zeta) + \sum_s \Phi(y)E_s\delta^s$, we have

$$\|P(y,\zeta) - \underbrace{[\bar u + \sum_s \delta u^s]}_{v}\|_Q \leq \|\underbrace{P(y,\bar\zeta) - \bar u}_{=0}\|_Q + \sum_s \underbrace{\|\Phi(y)E_s\delta^s - \delta u^s\|_Q}_{\leq \alpha_s \mathrm{dist}(\zeta^s,\mathcal{Z}^s|\mathcal{L}^s)};$$

since $\bar u \in \mathbf{Q}$ and $\delta u^s \in \mathrm{Rec}(\mathbf{Q})$ for all s, we have $v \in \mathbf{Q}$, so that the inequality implies that

$$\mathrm{dist}(P(y,\zeta), \mathbf{Q}) \leq \sum_s \alpha_s \mathrm{dist}(\zeta^s, \mathcal{Z}^s|\mathcal{L}^s).$$

Since $\zeta \in \mathcal{Z} + \mathcal{L}$ is arbitrary, y satisfies (4.1.11). □

Consequences of Main Result

Proposition 4.2 demonstrates that the GRC of an uncertain constraint (4.1.9) is equivalent to the explicit system of semi-infinite constraints (4.3.1). We are well acquainted with the constraint (4.3.1.a) — it is nothing but the RC of the uncertain constraint (4.1.9) with the normal range $\mathcal{Z}$ of the perturbations in the role of the uncertainty set. As a result, we have certain knowledge of how to convert this semi-infinite constraint into a tractable form or how to build its tractable safe approximation. What is new is the constraint (4.3.1.b), which is of the following generic form:

We are given

• an Euclidean space E with inner product $\langle\cdot,\cdot\rangle_E$, a norm (not necessarily the Euclidean one) $\|\cdot\|_E$, and a closed convex cone K^E in E;

• an Euclidean space F with inner product $\langle\cdot,\cdot\rangle_F$, norm $\|\cdot\|_F$ and a closed convex cone K^F in F.

These data define a function on the space $\mathcal{L}(E,F)$ of linear mappings $\mathcal{M}$ from E to F, specifically, the function

$$\begin{array}{c}\Psi(\mathcal{M}) = \max\limits_{e}\left\{\mathrm{dist}_{\|\cdot\|_F}(\mathcal{M}e, K^F) : e\in K^E, \|e\|_E\leq 1\right\},\\ \mathrm{dist}_{\|\cdot\|_F}(f,K^F) = \min\limits_{g\in K^F}\|f-g\|_F.\end{array} \tag{4.3.2}$$

Note that $\Psi(\mathcal{M})$ is a kind of a norm: it is nonnegative, satisfies the requirement $\Psi(\lambda\mathcal{M}) = \lambda\Psi(\mathcal{M})$ when $\lambda\geq 0$, and satisfies the triangle inequality $\Psi(\mathcal{M}+\mathcal{N})\leq\Psi(\mathcal{M})+\Psi(\mathcal{N})$. The properties of a norm that are missing are symmetry (in general, $\Psi(-\mathcal{M})\neq\Psi(\mathcal{M})$) and strict positivity (it may happen that $\Psi(\mathcal{M}) = 0$ for $\mathcal{M}\neq 0$). Note also that in the case when $K^F=\{0\}$, $K^E = E$, $\Psi(\mathcal{M}) = \max\limits_{e:\|e\|_E\leq 1}\|\mathcal{M}e\|_F$ becomes the usual norm of a linear mapping induced by given norms in the origin and the destination spaces.

The above setting gives rise to a convex inequality

$$\Psi(\mathcal{M})\leq\alpha \tag{4.3.3}$$

in variables $\mathcal{M},\alpha$. Note that every one of the constraints (4.3.1.b) is obtained from a convex inequality of the form (4.3.3) by affine substitution

$$\mathcal{M}\leftarrow H_s(y),\ \alpha\leftarrow\alpha_s$$

where $H_s(y)\in\mathcal{L}(E_s,F_s)$ is affine in y. Indeed, (4.3.1.b_s) is obtained in this fashion when specifying

- $(E,\langle\cdot,\cdot\rangle_E)$ as the Euclidean space where $\mathcal{Z}^s,\mathcal{L}^s$ live, and $\|\cdot\|_E$ as $\|\cdot\|_s$;
- $(F,\langle\cdot,\cdot\rangle_F)$ as the Euclidean space where $\mathbf{Q}$ lives, and $\|\cdot\|_F$ as $\|\cdot\|_Q$;
- K^E as the cone $\mathcal{L}^s$, and K^F as the cone $\mathrm{Rec}(\mathbf{Q})$;
- $H(y)$ as the linear map $\zeta^s\mapsto\Phi(y)E_s\zeta^s$.

It follows that efficient processing of constraints (4.3.1.b) reduces to a similar task for the associated constraints

$$\Psi_s(\mathcal{M}_s)\leq\alpha_s \tag{$\mathcal{C}_s$}$$

of the form (4.3.3). Assume, e.g., that we are smart enough to build, for certain $\vartheta\geq 1$,

(i) a ϑ-tight safe tractable approximation of the semi-infinite constraint (4.3.1.a) with $\mathcal{Z}_\rho = \rho\mathcal{Z}_1$ in the role of the perturbation set. Let this approximation be a system $\mathcal{S}^a_\rho$ of explicit convex constraints in variables y and additional variables u;

(ii) for every $s=1,...,S$ a ϑ-tight efficiently computable upper bound on the function $\Psi_s(\mathcal{M}_s)$, that is, a system $\mathcal{S}^s$ of efficiently computable convex constraints on matrix variable $\mathcal{M}_s$, real variable τ_s and, perhaps, additional variables u^s such that

(a) whenever $(\mathcal{M}_s,\tau_s)$ can be extended to a feasible solution of $\mathcal{S}^s$, we have $\Psi_s(\mathcal{M}_s)\leq\tau_s$,

(b) whenever $(\mathcal{M}_s, \tau_s)$ cannot be extended to a feasible solution of $\mathcal{S}^s$, we have $\vartheta\Psi_s(\mathcal{M}_s) > \tau_s$.

In this situation, we can point out a safe tractable approximation, tight within the factor ϑ (see Definition 4.5), of the GRC in question. To this end, consider the system of constraints in variables $y, \alpha_1, ..., \alpha_S, u, u^1, ..., u^S$ as follows:

$$(y,u) \text{ satisfies } \mathcal{S}^a_\rho \text{ and } \{(H_s(y), \alpha_s, u^s) \text{ satisfies } \mathcal{S}^s,\ s = 1, ..., S\}, \tag{$\mathcal{S}_\rho$}$$

and let us verify that this is a ϑ-tight safe computationally tractable approximation of the GRC. Indeed, $\mathcal{S}_\rho$ is an explicit system of efficiently computable convex constraints and as such is computationally tractable. Further, $\mathcal{S}_\rho$ is a safe approximation of the (GRC_ρ). Indeed, if (y, α) can be extended to a feasible solution of $\mathcal{S}_\rho$, then y satisfies $(4.3.1.a)$ with $\mathcal{Z}_\rho$ in the role of $\mathcal{Z}$ (since (y, u) satisfies $\mathcal{S}^a_\rho$) and (y, α_s) satisfies $(4.3.1.b_s)$ due to (ii.a) (recall that $(4.3.1.b_s)$ is equivalent to $\Psi_s(H_s(y)) \leq \alpha_s$). Finally, assume that (y, α) cannot be extended to a feasible solution of $(\mathcal{S}_\rho)$, and let us prove that then $(y, \vartheta^{-1}\alpha)$ is not feasible for $(\mathrm{GRC}_{\vartheta\rho})$. Indeed, if (y, α) cannot be extended to a feasible solution to $\mathcal{S}_\rho$, then either y cannot be extended to a feasible solution of $\mathcal{S}^a_\rho$, or for certain s (y, α_s) cannot be extended to a feasible solution of $\mathcal{S}^s$. In the first case, y does not satisfy $(4.3.1.a)$ with $\mathcal{Z}_{\vartheta\rho}$ in the role of $\mathcal{Z}$ by (i); in the second case, $\vartheta^{-1}\alpha_s < \Psi_s(H_s(y))$ by (ii.b), so that in both cases the pair $(y, \vartheta^{-1}\alpha)$ is not feasible for $(\mathrm{GRC}_{\vartheta\rho})$.

We have reduced the tractability issues related to Globalized RCs to similar issues for RCs (which we have already investigated in the CO case) and to the issue of efficient bounding of $\Psi(\cdot)$. The rest of this section is devoted to investigating this latter issue.

4.3.2 Efficient Bounding of $\Psi(\cdot)$

Symmetry

We start with observing that the problem of efficient computation of (a tight upper bound on) $\Psi(\cdot)$ possesses a kind of symmetry. Indeed, consider a setup

$$\Xi = (E, \langle\cdot,\cdot\rangle_E, \|\cdot\|_E, K^E; F, \langle\cdot,\cdot\rangle_F, \|\cdot\|_F, K^F)$$

specifying Ψ, and let us associate with Ξ its *dual* setup

$$\Xi_* = (F, \langle\cdot,\cdot\rangle_F, \|\cdot\|_F^*, K_*^F; E, \langle\cdot,\cdot\rangle_E, \|\cdot\|_E^*, K_*^E),$$

where

- for a norm $\|\cdot\|$ on a Euclidean space $(G, \langle\cdot,\cdot\rangle_G)$, its *conjugate* norm $\|\cdot\|^*$ is defined as

$$\|u\|^* = \max_v \{\langle u, v\rangle_G : \|v\| \leq 1\};$$

- For a closed convex cone K in a Euclidean space $(G, \langle\cdot,\cdot\rangle_G)$, its *dual* cone is defined as

$$K_* = \{y : \langle y, h\rangle_G \geq 0 \ \ \forall h \in K\}.$$

.

Recall that the *conjugate* to a linear map $\mathcal{M} \in \mathcal{L}(E,F)$ from Euclidean space E to Euclidean space F is the linear map $\mathcal{M}^* \in \mathcal{L}(F,E)$ uniquely defined by the identity

$$\langle \mathcal{M}e, f\rangle_F = \langle e, \mathcal{M}^* f\rangle_E \quad \forall (e \in E, f \in F);$$

representing linear maps by their matrices in a fixed pair of orthonormal bases in E, F, the matrix representing $\mathcal{M}^*$ is the transpose of the matrix representing $\mathcal{M}$. Note that twice taken dual/conjugate of an entity recovers the original entity: $(K_*)_* = K$, $(\|\cdot\|^*)^* = \|\cdot\|$, $(\mathcal{M}^*)^* = \mathcal{M}$, $(\Xi_*)_* = \Xi$.

Recall that the functions $\Psi(\cdot)$ are given by setups Ξ of the outlined type according to

$$\Psi(\mathcal{M}) \equiv \Psi_\Xi(\mathcal{M}) = \max_{e\in E} \left\{ \mathrm{dist}_{\|\cdot\|_F}(\mathcal{M}e, K^F) : e \in K^E, \|e\|_E \le 1 \right\}.$$

The aforementioned symmetry is nothing but the following simple statement:

Proposition 4.3 *For every setup $\Xi = (E, ..., K^F)$ and every $\mathcal{M} \in \mathcal{L}(E,F)$ one has*

$$\Psi_\Xi(\mathcal{M}) = \Psi_{\Xi_*}(\mathcal{M}^*).$$

Proof. Let $H, \langle\cdot,\cdot\rangle_H$ be a Euclidean space. Recall that the *polar* of a closed convex set $X \subset H$, $0 \in X$, is the set $X^o = \{y \in H : \langle y, x\rangle_H \le 1 \quad \forall x \in X\}$. We need the following facts:

(a) If $X \subset H$ is closed, convex and $0 \in X$, then so is X^o, and $(X^o)^o = X$ [87];

(b) If $X \subset H$ is convex compact, $0 \in X$, and $K^H \subset H$ is closed convex cone, then $X + K^H$ is closed and

$$(X + K^H)^o = X^o \cap (-K^H_*).$$

Indeed, the arithmetic sum of a compact and a closed set is closed, so that $X + K^H$ is closed, convex, and contains 0. We have

$$f \in (X + K^H)^o \Leftrightarrow 1 \ge \sup_{x\in X, h\in K^H} \langle f, x+h\rangle_H = \sup_{x\in X}\langle f, x\rangle_H + \sup_{h\in K^H}\langle f, h\rangle_H;$$

since K^H is a cone, the concluding inequality is possible iff $f \in X^o$ and $f \in -K^H_*$.

(c) Let $\|\cdot\|$ be a norm in H. Then for every $\alpha > 0$ one has $(\{x : \|x\| \le \alpha\})^o = \{x : \|x\|^* \le 1/\alpha\}$ (evident).

When $\alpha > 0$, we have

$$\begin{array}{lll}
& \Psi_\Xi(\mathcal{M}) \le \alpha & \\
\Leftrightarrow & \left\{ \begin{array}{l} \forall e \in K^E \cap \{e : \|e\|_E \le 1\} : \\ \mathcal{M}e \in \{f : \|f\|_F \le \alpha\} + K^F \end{array} \right. & \text{[by definition]} \\
\Leftrightarrow & \left\{ \begin{array}{l} \forall e \in K^E \cap \{e : \|e\|_E \le 1\} : \\ \mathcal{M}e \in \left[[\{f : \|f\|_F \le \alpha\} + K^F]^o \right]^o \end{array} \right. & \text{[by (a)]} \\
\Leftrightarrow & \left\{ \begin{array}{l} \forall e \in K^E \cap \{e : \|e\|_E \le 1\} : \\ \langle \mathcal{M}e, f\rangle_F \le 1 \; \forall f \in \underbrace{[\{f : \|f\|_F \le \alpha\} + K^F]^o}_{=\{f : \|f\|^*_F \le \alpha^{-1}\} \cap (-K^F_*)} \\ \end{array} \right. & \text{[by (b), (c)]}
\end{array}$$

$$
\begin{array}{ll}
\Leftrightarrow & \left\{\begin{array}{l}\forall e \in K^E \cap \{e : \|e\|_E \le 1\} : \\ \langle e, \mathcal{M}^* f\rangle_E \le 1 \;\forall f \in \{f : \|f\|_F^* \le \alpha^{-1}\} \cap (-K_*^F)\end{array}\right. & \\
\Leftrightarrow & \left\{\begin{array}{l}\forall e \in K^E \cap \{e : \|e\|_E \le \alpha^{-1}\} : \\ \langle e, \mathcal{M}^* f\rangle_E \le 1 \;\forall f \in \{f : \|f\|_F^* \le 1\} \cap (-K_*^F)\end{array}\right. & \text{[evident]} \\
\Leftrightarrow & \left\{\begin{array}{l}\forall e \in [-(-K_*^E)_*] \cap [\{e : \|e\|_E^* \le \alpha\}^o] : \\ \langle e, \mathcal{M}^* f\rangle_E \le 1 \;\forall f \in \{f : \|f\|_F^* \le 1\} \cap (-K_*^F)\end{array}\right. & \text{[by (c)]} \\
\Leftrightarrow & \left\{\begin{array}{l}\forall e \in \left[(-K_*^E) + \{e : \|e\|_E^* \le \alpha\}\right]^o : \\ \langle \mathcal{M}^* f, e\rangle_E \le 1 \;\forall f \in \{f : \|f\|_F^* \le 1\} \cap (-K_*^F)\end{array}\right. & \text{[by (b)]} \\
\Leftrightarrow & \left\{\begin{array}{l}\forall f \in \{f : \|f\|_F^* \le 1\} \cap (-K_*^F) : \\ \langle \mathcal{M}^* f, e\rangle_E \le 1 \,\forall e \in [(-K_*^E) + \{e : \|e\|_E^* \le \alpha\}]^o\end{array}\right. & \\
\Leftrightarrow & \left\{\begin{array}{l}\forall f \in \{f : \|f\|_F^* \le 1\} \cap (-K_*^F) : \\ \mathcal{M}^* f \in (-K_*^E) + \{e : \|e\|_E^* \le \alpha\}\end{array}\right. & \text{[by (a)]} \\
\Leftrightarrow & \left\{\begin{array}{l}\forall f \in K_F^* \cap \{f : \|f\|_F^* \le 1\} : \\ \mathcal{M}^* f \in K_*^E + \{e : \|e\|_E^* \le \alpha\}\end{array}\right. & \\
\Leftrightarrow & \Psi_{\Xi_*}(\mathcal{M}^*) \le \alpha. & \square
\end{array}
$$

Good GRC setups

Proposition 4.3 says that "good" setups Ξ — those for which $\Psi_\Xi(\cdot)$ is efficiently computable or admits a tight, within certain factor ϑ, efficiently computable upper bound — always come in symmetric pairs: if Ξ is good, so is Ξ_*, and vice versa. In what follows, we refer to members of such a symmetric pair as to *counterparts* of each other. We are about to list a number of good pairs. From now on, we assume that all components of a setup in question are "computationally tractable," specifically, that the cones K^E, K^F and the epigraphs of the norms $\|\cdot\|_E$, $\|\cdot\|_F$ are given by LMI representations (or, more general, by systems of efficiently computable convex constraints). Below, we denote by B_E and B_F the unit balls of the norms $\|\cdot\|_E$, $\|\cdot\|_F$, respectively. Here are several good GRC setups:

A: $K^E = \{0\}$. The counterpart is

A*: $K^F = F$.

These cases are trivial: $\Psi_\Xi(\mathcal{M}) \equiv 0$.

B: $K^E = E$, $B_E = \mathrm{Conv}\{e^1, ..., e^N\}$, *the list* $\{e^i\}_{i=1}^N$ *is available.* The counterpart is the case

B*: $K^F = \{0\}$, $B_F = \{f : \langle f^i, f\rangle_F \le 1, i = 1, ..., N\}$, *the list* $\{f^i\}_{i=1}^N$ *is available.*

Standard example for **B** is $E = \mathbb{R}^n$ with the standard inner product, $K^E = E$, $\|e\| = \|e\|_1 \equiv \sum_j |e_j|$. Standard example for **B*** is $F = \mathbb{R}^m$ with the standard inner product, $\|f\|_F = \|f\|_\infty = \max_j |f_j|$.

The cases in question are easy. Indeed, in the case of **B** we clearly have

$$\Psi(\mathcal{M}) = \max_{1 \le j \le N} \mathrm{dist}_{\|\cdot\|_F}(\mathcal{M} e_i, K^F),$$

and thus $\Psi(\mathcal{M})$ is efficiently computable (as the maximum of a finite family of efficiently computable quantities $\mathrm{dist}_{\|\cdot\|_F}(\mathcal{M} e_i, K^F)$). Assuming, e.g., that E, F are, respectively, $\mathbb{R}^m$ and $\mathbb{R}^n$ with the standard inner products, and that K^F, $\|\cdot\|_F$ are given by strictly feasible conic representations:

$$
\begin{array}{l}
K^F = \{f : \exists u : Pf + Qu \in \mathbf{K}^1\}, \\
\{t \ge \|f\|_F\} \Leftrightarrow \left\{\exists v : Rf + tr + Sv \in \mathbf{K}^2\right\}
\end{array}
$$

the relation

$$\Psi(\mathcal{M}) \le \alpha$$

can be represented equivalently by the following explicit system of conic constraints

$$\begin{array}{ll}(a) & Pf^i + Qu^i \in \mathbf{K}^1,\ i = 1, ..., N\\ (b) & R(\mathcal{M}e^i - f^i) + \alpha r + Sv^i \in \mathbf{K}^2,\ i = 1, ..., N\end{array}$$

in variables $\mathcal{M}, \alpha, u^i, f^i, v^i$. Indeed, relations (a) equivalently express the requirement $f^i \in K^F$, while relations (b) say that $\|\mathcal{M}e^i - f^i\|_F \leq \alpha$.

C: $K^E = E$, $K^F = \{0\}$. The counterpart case is exactly the same.

In the case of **C**, $\Psi(\cdot)$ is the norm of a linear map from E to F induced by given norms on the origin and the destination spaces:

$$\Psi(\mathcal{M}) = \max_e \{\|\mathcal{M}e\|_F : \|e\|_E \leq 1\}.$$

Aside of situations covered by **B**, **B***, there is only one generic situation where computing the norm of a linear map is easy — this is the situation where both $\|\cdot\|_E$ and $\|\cdot\|_F$ are Euclidean norms. In this case, we lose nothing by assuming that $E = \ell_2^n$ (that is, E is $\mathbb{R}^n$ with the standard inner product and the standard norm $\|e\|_2 = \sqrt{\sum_i e_i^2}$), $F = \ell_2^m$, and let M be the $m \times n$ matrix representing the map $\mathcal{M}$ in the standard bases of E and F. In this case, $\Psi(\mathcal{M}) = \|M\|_{2,2}$ is the maximal singular value of M and as such is efficiently computable. A semidefinite representation of the constraint $\|M\|_{2,2} \leq \alpha$ is

$$\left[\begin{array}{c|c}\alpha I_n & M^T\\ \hline M & \alpha I_m\end{array}\right] \succeq 0.$$

Now consider the case when $E = \ell_p^n$ (that is, E is $\mathbb{R}^n$ with the standard inner product and the norm

$$\|e\|_p = \begin{cases}\left(\sum_j |e_j|^p\right)^{1/p} & , 1 \leq p < \infty\\ \max_j |e_j| & , p = \infty\end{cases},$$

and $F = \ell_r^m$, $1 \leq r, p \leq \infty$. Here again we can naturally identify $\mathcal{L}(E, F)$ with the space $\mathbb{R}^{m\times n}$ of real $m \times n$ matrices, and the problem of interest is to compute

$$\|M\|_{p,r} = \max_e \{\|Me\|_r : \|e\|_p \leq 1\}.$$

The case of $p = r = 2$ is the just considered "purely Euclidean" situation; the cases of $p = 1$ and of $r = \infty$ are covered by **B**, **B***. These are the only 3 cases when computing $\|\cdot\|_{p,r}$ is known to be easy. It is also known that it is NP-hard to compute the matrix norm in question when $p > r$. However, in the case of $p \geq 2 \geq r$ there exists a tight efficiently computable upper bound on $\|M\|_{p,r}$ due to Nesterov [97, Theorem 13.2.4]. Specifically, Nesterov shows that when $\infty \geq p \geq 2 \geq r \geq 1$, the explicitly computable quantity

$$\Psi_{p,r}(M) = \frac{1}{2} \min_{\substack{\mu\in\mathbb{R}^n\\ \nu\in\mathbb{R}^m}} \left\{\|\mu\|_{\frac{p}{p-2}} + \|\nu\|_{\frac{r}{2-r}} : \left[\begin{array}{c|c}\mathrm{Diag}\{\mu\} & M^T\\ \hline M & \mathrm{Diag}\{\nu\}\end{array}\right] \succeq 0\right\}$$

is an upper bound on $\|M\|_{p,r}$, and this bound is tight within the factor $\vartheta = \left[\frac{2\sqrt{3}}{\pi} - \frac{2}{3}\right]^{-1} \approx 2.2936$:

$$\|M\|_{p,r} \leq \Psi_{p,r}(M) \leq \left[\frac{2\sqrt{3}}{\pi} - \frac{2}{3}\right]^{-1} \|M\|_{p,r}$$

(depending on values of p, r, the tightness factor can be improved; e.g., when $p = \infty, r = 2$, it is just $\sqrt{\pi/2} \approx 1.2533...$).

It follows that the explicit system of efficiently computable convex constraints

$$\left[\begin{array}{c|c} \mathrm{Diag}\{\mu\} & M^T \\ \hline M & \mathrm{Diag}\{\nu\} \end{array}\right] \succeq 0, \ \frac{1}{2}\left[\|\mu\|_{\frac{p}{p-2}} + \|\nu\|_{\frac{r}{2-r}}\right] \le \alpha \tag{4.3.4}$$

in variables M, α, μ, ν is a safe tractable approximation of the constraint

$$\|M\|_{p,r} \le \alpha,$$

which is tight within the factor ϑ. In some cases the value of the tightness factor can be improved; e.g., when $p = \infty$, $r = 2$ and when $p = 2$, $r = 1$, the tightness factor does not exceed $\sqrt{\pi/2}$.

Most of the tractable (or nearly so) cases considered so far deal with the case when $K^F = \{0\}$ (the only exception is the case $\mathbf{B}^*$ that, however, imposes severe restrictions on $\|\cdot\|_E$). In the GRC context, that means that we know nearly nothing about what to do when the recessive cone of the right hand side set $\mathbf{Q}$ in (4.1.6) is nontrivial, or, which is the same, $\mathbf{Q}$ is unbounded. This is not that disastrous — in many cases, boundedness of the right hand side set is not a severe restriction. However, it is highly desirable, at least from the academic viewpoint, to know something about the case when K^F is nontrivial, in particular, when K^F is a nonnegative orthant, or a Lorentz, or a Semidefinite cone (the two latter cases mean that (4.1.6) is an uncertain CQI, respectively, uncertain LMI). We are about to consider several such cases.

D: $F = \ell_\infty^m$, K^F *is a "sign" cone,* meaning that $K^F = \{u \in \ell_\infty^m : u_i \ge 0, i \in I_+, u_i \le 0, i \in I_-, u_i = 0, i \in I_0\}$, *where* I_+, I_-, I_0 *are given non-intersecting subsets of the index set* $i = \{1, ..., m\}$.

The counterpart is

D*: $E = \ell_1^m$, $K^E = \{v \in \ell_1^m : v_j \ge 0, j \in J_+, v_j \le 0, j \in J_-, v_j = 0, j \in J_0\}$, *where* J_+, J_-, J_0 *are given non-overlapping subsets of the index set* $\{1, ..., m\}$.

In the case of $\mathbf{D}^*$, assuming, for the sake of notational convenience, that $J_+ = \{1, ..., p\}$, $J_- = \{p+1, ..., q\}$, $J_0 = \{r+1, ..., m\}$ and denoting by e^j the standard basic orths in ℓ_1, we have

$$\begin{array}{rl} B \equiv & \{v \in K^E : \|v\|_E \le 1\} = \mathrm{Conv}\{e^1, ..., e^p, -e^{p+1}, ..., -e^q, \pm e^{q+1}, ..., \pm e^r\} \\ \equiv & \mathrm{Conv}\{g^1, ..., g^s\}, s = 2r - q. \end{array}$$

Consequently,

$$\Psi(\mathcal{M}) = \max_{1 \le j \le s} \mathrm{dist}_{\|\cdot\|_F}(\mathcal{M}g^j, K^F)$$

is efficiently computable (cf. case $\mathbf{B}$).

E: $F = \ell_2^m$, $K^F = \mathbf{L}^m \equiv \{f \in \ell_2^m : f_m \ge \sqrt{\sum_{i=1}^{m-1} f_i^2}\}$, $E = \ell_2^n$, $K^E = E$.

The counterpart is

E*: $F = \ell_2^n$, $K^F = \{0\}$, $E = \ell_2^m$, $K^E = \mathbf{L}^m$.

In the case of $\mathbf{E}^*$, let $D = \{e \in K^E : \|e\|_2 \le 1\}$, and let

$$B = \{e \in E : e_1^2 + ... + e_{m-1}^2 + 2e_m^2 \le 1\}.$$

Let us represent a linear map $\mathcal{M} : \ell_2^m \to \ell_2^n$ by its matrix M in the standard bases of the origin and the destination spaces. Observe that

$$B \subset D_s \equiv \mathrm{Conv}\{D \cup (-D)\} \subset \sqrt{3/2}B \tag{4.3.5}$$

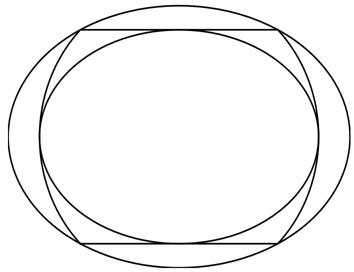


Figure 4.3: 2-D cross-sections of the solids B, $\sqrt{3/2}B$ (ellipses) and D_s by a 2-D plane passing through the common symmetry axis $e_1 = ... = e_{m-1} = 0$ of the solids.

(see figure 4.3). Now, let B_F be the unit Euclidean ball, centered at the origin, in $F = \ell_2^m$. By definition of $\Psi(\cdot)$ and due to $K^F = \{0\}$, we have

$$\Psi(\mathcal{M}) \leq \alpha \Leftrightarrow \mathcal{M}D \subset \alpha B_F \Leftrightarrow (\mathcal{M}D \cup (-\mathcal{M}D)) \subset \alpha B_F \Leftrightarrow \mathcal{M}D_s \subset \alpha B_F.$$

Since $D_s \subset \sqrt{3/2}B$, the inclusion $\mathcal{M}(\sqrt{3/2}B) \subset \alpha B_F$ is a sufficient condition for the validity of the inequality $\Psi(\mathcal{M}) \leq \alpha$, and since $B \subset D_s$, this condition is tight within the factor $\sqrt{3/2}$. (Indeed, if $\mathcal{M}(\sqrt{3/2}B) \not\subset \alpha B_F$, then $\mathcal{M}B \not\subset \sqrt{2/3}\alpha B_F$, meaning that $\Psi(\mathcal{M}) > \sqrt{2/3}\alpha$.) Noting that $\mathcal{M}(\sqrt{3/2}B) \leq \alpha$ if and only if $\|M\Delta\|_{2,2} \leq \alpha$, where $\Delta = \mathrm{Diag}\{\sqrt{3/2}, ..., \sqrt{3/2}, \sqrt{3/4}\}$, we conclude that the efficiently verifiable convex inequality

$$\|M\Delta\|_{2,2} \leq \alpha$$

is a safe tractable approximation, tight within the factor $\sqrt{3/2}$, of the constraint $\Psi(\mathcal{M}) \leq \alpha$.

F: $F = \mathbf{S}^m$, $\|\cdot\|_F = \|\cdot\|_{2,2}$, $K^F = \mathbf{S}^m_+$, $E = \ell_\infty^n$, $K^E = E$.

The counterpart is

F*: $F = \ell_1^n$, $K^F = \{0\}$, $E = \mathbf{S}^m$, $\|e\|_E = \sum_{i=1}^m |\lambda_i(e)|$, *where* $\lambda_1(e) \geq \lambda_2(e) \geq ... \geq \lambda_m(e)$ *are the eigenvalues of* e, $K^E = \mathbf{S}^m_+$.

In the case of **F**, given $\mathcal{M} \in \mathcal{L}(\ell_\infty^n, \mathbf{S}^m)$, let $e^1, ..., e^n$ be the standard basic orths of ℓ_∞^n, and let $B_E = \{v \in \ell_\infty^n : \|u\|_\infty \leq 1\}$. We have

$$\begin{array}{l}\{\Psi(\mathcal{M}) \leq \alpha\} \Leftrightarrow \left\{\forall v \in B_E\, \exists V \succeq 0 : \max_i |\lambda_i(\mathcal{M}v - V)| \leq \alpha\right\} \\ \Leftrightarrow \{\forall v \in B_E : \mathcal{M}v + \alpha I_m \succeq 0\}.\end{array}$$

Thus, the constraint

$$\Psi(\mathcal{M}) \leq \alpha \tag{$*$}$$

is equivalent to

$$\alpha I + \sum_{i=1}^n v_i(\mathcal{M}e^i) \succeq 0 \;\forall (v : \|v\|_\infty \leq 1).$$

It follows that the explicit system of LMIs

$$\begin{array}{l} Y_i \succeq \pm \mathcal{M}e^i,\, i = 1, ..., n \\ \alpha I_m \succeq \sum_{i=1}^n Y_i \end{array} \tag{4.3.6}$$

in variables $\mathcal{M}$, α, $Y_1, ..., Y_n$ is a safe tractable approximation of the constraint $(*)$. Now let

$$\Theta(\mathcal{M}) = \vartheta(\mu(\mathcal{M})),\ \mu(\mathcal{M}) = \max_{1\leq i\leq n} \mathrm{Rank}(\mathcal{M}e^i),$$

where $\vartheta(\mu)$ is the function defined in the Real Case Matrix Cube Theorem, so that $\vartheta(1) = 1$, $\vartheta(2) = \pi/2$, $\vartheta(4) = 2$, and $\vartheta(\mu) \leq \pi\sqrt{\mu/2}$ for $\mu \geq 1$. Invoking this Theorem (see the proof of Theorem 3.4), we conclude that the *local* tightness factor of our approximation does not exceed $\Theta(\mathcal{M})$, meaning that if $(\mathcal{M}, \alpha)$ cannot be extended to a feasible solution of (4.3.6), then

$$\Theta(\mathcal{M})\Psi(\mathcal{M}) > \alpha.$$

4.3.3 Illustration: Robust Least Squares Antenna Design

We are about to illustrate our findings by applying a GRC-based approach to the Least Squares Antenna Design problem (see p. 104 and Example 1.1). Our motivation and course of actions here are completely similar to those we used in the case of $\|\cdot\|_\infty$ design considered in section 4.2.1. At present, we are interested in the uncertain conic problem

$$\left\{\min_{x,\tau} \left\{\tau : [h - H(I + \mathrm{Diag}\{\zeta\})x; \tau] \in \mathbf{L}^{m+1}\right\} : \|\zeta\|_\infty \leq \rho\right\}$$

where $H = WD \in \mathbb{R}^{m\times L}$ and $h = Wd \in \mathbb{R}^m$ are given matrix and vector, see p. 104. We equip the problem with the same perturbation structure (4.2.3) as in the case of $\|\cdot\|_\infty$-design:

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq \bar{\rho}\},\ \mathcal{L} = \mathbb{R}^L,\ \|\zeta\| \equiv \|\zeta\|_\infty$$

and augment this structure with the $\|\cdot\|_2$-norm on the embedding space $\mathbb{R}^{m+1}$ of the right hand side $\mathbf{Q} := \mathbf{L}^{m+1}$ of the conic constraint in question. Now the robust properties of a candidate design x are fully characterized by the worst-case loss

$$F_x(\rho) = \max_{\|\zeta\|_\infty\leq\rho} \|H(I + \mathrm{Diag}\{\zeta\})x - h\|_2,$$

which is a convex and nondecreasing function of $\rho \geq 0$. Invoking Proposition 4.2, it is easily seen that a par (τ, x) is feasible for the GRC of our uncertain problem with global sensitivity α if and only if

$$\begin{array}{c} F_{\bar{\rho}}(x) \leq \tau \ \&\ \ \alpha \geq \alpha(x) := \max_\zeta \left\{\mathrm{dist}_{\|\cdot\|_2}([D[x]\zeta; 0], \mathbf{L}^{m+1}) : \|\zeta\|_\infty \leq 1\right\} \\ D[x] = H\mathrm{Diag}\{x\} \in \mathbb{R}^{m\times L}. \end{array}$$

Similarly to the case of $\|\cdot\|_\infty$-synthesis, the minimal possible value $\alpha(x)$ of α depends solely on x; we call it *global sensitivity* of a candidate design x. Note that $F_x(\cdot)$ and $\alpha(x)$ are linked by the relation similar, although not identical to, the relation $\alpha(x) = \lim\limits_{\rho\to\infty} \frac{d}{d\rho}F_x(\rho)$ we had in the case of $\|\cdot\|_\infty$-design; now this relation modifies to

$$\alpha(x) = 2^{-1/2} \lim_{\rho\to+\infty} \frac{d}{d\rho}F_x(\rho).$$

Indeed, denoting $\beta(x) = \lim\limits_{\rho\to+\infty} \frac{d}{d\rho}F_x(\rho)$ and taking into account the fact that $F_x(\cdot)$ is a nondecreasing convex function, we have $\beta(x) = \lim\limits_{\rho\to\infty} F_x(\rho)/\rho$. Now, from the structure of F it immediately follows that $\lim\limits_{\rho\to\infty} F_x(\rho)/\rho = \max\limits_{\zeta} \{\|D[x]\zeta\|_2 : \|\zeta\|_\infty \leq 1\}$. Observing that the $\|\cdot\|_2$-distance from a vector $[u;0] \in \mathbb{R}^{m+1}$ to the Lorentz cone $\mathbf{L}^{m+1}$ clearly is $2^{-1/2}\|u\|_2$ and looking at the definition of $\alpha(x)$, we conclude that $\alpha(x) = 2^{-1/2}\beta(x)$, as claimed.

Now, in the case of $\|\cdot\|_\infty$-synthesis, the quantities $F_x(\rho)$ and $\alpha(x)$ were easy to compute, which is not the case now. However, we can built tight within the factor $\sqrt{\pi/2}$ tractable *upper bounds* on these quantities, namely, as follows.

We have

$$\begin{aligned} F_x(\rho) &= \max_{\zeta:\|\zeta\|_\infty\leq\rho} \|h - H(1+\mathrm{Diag}\{\zeta\})x\|_2 = \max_{\zeta:\|\zeta\|_\infty\leq\rho} \|h - Hx + D[x]\zeta\|_2 \\ &= \max_{\|[t;\eta]\|_\infty\leq 1} \|t[h-Hx] + \rho D[x]\eta\|_2, \end{aligned}$$

and the concluding quantity is nothing but the norm of the linear map $[t:\zeta] \mapsto [h - Hx; \rho D[x]][t;\zeta]$ induced by the norm $\|\cdot\|_\infty$ in the origin and the norm $\|\cdot\|_2$ in the destination spaces. By Nesterov's theorem, see p. 194, the efficiently computable quantity

$$\widehat{F}_x(\rho) = \min_{\mu,\nu}\left\{\frac{\|\mu\|_1+\nu}{2} : \left[\begin{array}{c|c} \nu I_m & [h-Hx, \rho D[x]] \\ \hline [h-Hx,\rho D[x]]^T & \mathrm{Diag}\{\mu\}\end{array}\right] \succeq 0\right\} \tag{4.3.7}$$

is a tight within the factor $\sqrt{\pi/2}$ upper bound on $F_x(\rho)$; note that this bound, same as $F_x(\cdot)$ itself, is a convex and nondecreasing function of ρ.

Further, we have

$$\alpha(x) = \max_{\|\zeta\|_\infty\leq 1} \mathrm{dist}_{\|\cdot\|_2}([D[x]\zeta;0], \mathbf{L}^{m+1}) = 2^{-1/2} \max_{\|\zeta\|_\infty\leq 1} \|D[x]\zeta\|_2,$$

that is, $\alpha(x)$ is proportional to the norm of the linear mapping $\zeta \mapsto D[x]\zeta$ induced by the $\|\cdot\|_\infty$-norm in the origin and by the $\|\cdot\|_2$-norm in the destination spaces. Same as above, we conclude that the efficiently computable quantity

$$\widehat{\alpha}(x) = \min_{\mu,\nu}\left\{\frac{\|\mu\|_1+\nu}{2\sqrt{2}} : \left[\begin{array}{c|c} \nu I_m & D[x] \\ \hline D^T[x] & \mathrm{Diag}\{\mu\}\end{array}\right] \succeq 0\right\} \tag{4.3.8}$$

is a tight within the factor $\sqrt{\pi/2}$ upper bound on $\alpha(x)$. Note that $\widehat{F}_x(\cdot)$ and $\widehat{\alpha}(x)$ are linked by exactly the same relation as $F_x(\rho)$ and $\alpha(x)$, namely,

$$\widehat{\alpha}(x) = 2^{-1/2} \lim_{\rho\to\infty} \frac{d}{d\rho}\widehat{F}_x(\rho)$$

(why?).

Similarly to the situation of section 4.2.1, the function $\widehat{F}_x(\cdot)$ (and thus the true loss $F_x(\rho)$) admits the piecewise linear upper bound

$$\left\{\begin{array}{ll} \frac{\rho_0-\rho}{\rho_0}\widehat{F}_x(0) + \frac{\rho}{\rho_0}\widehat{F}_x(\rho_0), & 0 \leq \rho < \rho_0, \\ \widehat{F}_x(\rho_0) + 2^{1/2}\widehat{\alpha}(x)[\rho-\rho_0], & \rho \geq \rho_0, \end{array}\right. , \tag{4.3.9}$$

$\rho_0 \geq 0$ being the parameter of the bound.

We have carried out numerical experiments completely similar to those reported in section 4.2.1, that is, built solutions to the optimization problems

$$\widehat{\beta}_*(\delta) = \min_x \left\{\widehat{\alpha}(x) : F_x(0) \leq (1+\delta)\min_u \|Hu - h\|_2\right\}; \tag{4.3.10}$$

the results are presented in figure 4.4 and table 4.2.

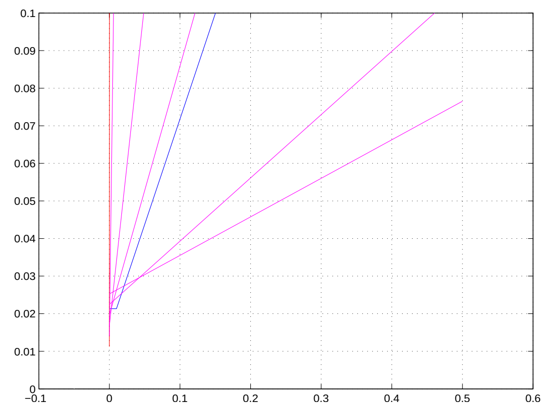


Figure 4.4: Red and magenta: bounds (4.3.9) on the losses for optimal solutions to (4.3.10) for the values of δ listed in table 4.2; the bounds correspond to $\rho_0 = 0$. Blue: bound (4.2.5) with $\rho_0 = 0.01$ on the loss $F_x(\rho)$ associated with the robust design built in section 3.3.1.

δ	0	0.25	0.50	0.75	1.00	1.25
$\widehat{\beta}_*(\delta)$	9441.4	14.883	1.7165	0.6626	0.1684	0.1025

Table 4.2: Tolerances δ and quantities $\widehat{\beta}_*(\delta)$ (tight within the factor $\pi/2$ upper bounds on global sensitivities of the optimal solutions to (4.3.10)). Pay attention to how huge is the global sensitivity ($\geq 2\widehat{\beta}_*(0)/\pi$) of the nominal Least Squares-optimal design. For comparison: the global sensitivity of the robust design $x_{\rm r}$ built in section 3.3.1 is $\leq \widehat{\alpha}(x_{\rm r}) = 0.3962$.

4.4 Illustration: Robust Analysis of Nonexpansive Dynamical Systems

We are about to illustrate the techniques we have developed by applying them to the problem of *robust nonexpansiveness analysis* coming from Robust Control; in many aspects, this problem resembles the Robust Lyapunov Stability Analysis problem we have considered in sections 3.4.2 and 3.5.1.

4.4.1 Preliminaries: Nonexpansive Linear Dynamical Systems

Consider an uncertain time-varying linear dynamical system (cf. (3.4.22)):

$$\begin{array}{rcl} \dot{x}(t) & = & A_t x(t) + B_t u(t) \\ y(t) & = & C_t x(t) + D_t u(t) \end{array} \tag{4.4.1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output and $u \in \mathbb{R}^q$ is the control. The system is assumed to be uncertain, meaning that all we know about the matrix $\Sigma_t = \left[\begin{array}{c|c} A_t & B_t \\ \hline C_t & D_t \end{array}\right]$ is that at every time instant t it belongs to a given *uncertainty set* $\mathcal{U}$.

System (4.4.1) is called *nonexpansive* (more precisely, *robustly nonexpansive w.r.t. uncertainty set* $\mathcal{U}$), if

$$\int_0^t y^T(s)y(s)ds \leq \int_0^t u^T(s)u(s)ds$$

for all $t \geq 0$ and for all trajectories of (all realizations of) the system such that $z(0) = 0$. In what follows, we focus on the simplest case of a system with $y(t) \equiv x(t)$, that is, on the case of $C_t \equiv I$, $D_t \equiv 0$. Thus, from now on the system of interest is

$$\begin{array}{rcl} \dot{x}(t) & = & A_t x(t) + B_t u(t) \\ & & [A_t, B_t] \in \mathcal{AB} \subset \mathbb{R}^{n \times m} \ \forall t, \\ & & m = n + q = \dim x + \dim u. \end{array} \tag{4.4.2}$$

Robust nonexpansiveness now reads

$$\int_0^t x^T(s)x(s)ds \leq \int_0^t u^T(s)u(s)ds \tag{4.4.3}$$

for all $t \geq 0$ and all trajectories $x(\cdot)$, $x(0) = 0$, of all realizations of (4.4.2).

Similarly to robust stability, robust nonexpansiveness admits a *certificate* that is a matrix $X \in \mathbf{S}^n_+$. Specifically, such a certificate is a solution of the following system of LMIs in matrix variable $X \in \mathbf{S}^m$:

$$\begin{array}{ll} (a) & X \succeq 0 \\ (b) & \forall [A, B] \in \mathcal{AB} : \\ & \mathcal{A}(A, B; X) \equiv \left[\begin{array}{c|c} -I_n - A^T X - XA & -XB \\ \hline -B^T X & I_q \end{array}\right] \succeq 0. \end{array} \tag{4.4.4}$$

The fact that solvability of (4.4.4) is a *sufficient* condition for robust nonexpansiveness of (4.4.2) is immediate: if X solves (4.4.4), $x(\cdot)$, $u(\cdot)$ satisfy (4.4.2) and $x(0) = 0$, then

$$\begin{array}{l} u^T(s)u(s) - x^T(s)x(s) - \frac{d}{ds}\left[x^T(s)Xx(s)\right] = u^T(s)u(s) - x^T(s)x(s) \\ -[\dot{x}^T(s)Xx(s) + x^T(s)X\dot{x}(s)] = u^T(s)u(s) - x^T(s)x(s) \\ -[A_s x(s) + B_s u(s)]^T X x(s) - x^T(s)X[A_s x(s) + B_s u(s)] \\ = \left[x^T(s), u^T(s)\right] \mathcal{A}(A_s, B_s; X) \left[\begin{array}{c} x(s) \\ u(s) \end{array}\right] \geq 0, \end{array}$$

whence

$$\begin{array}{l} t > 0 \Rightarrow \int\limits_0^t [u^T(s)u(s) - x^T(s)x(s)]ds \geq x^T(t)Xx(t) - x^T(0)Xx(0) \\ = x^T(t)Xx(t) \geq 0. \end{array}$$

It should be added that when (4.4.2) is time-invariant, (i.e., $\mathcal{AB}$ is a singleton) and satisfies mild regularity conditions, the existence of the outlined certificate, (i.e., the solvability of (4.4.4)), is sufficient *and necessary* for nonexpansiveness.

Now, (4.4.4) is nothing but the RC of the system of LMIs in matrix variable $X \in \mathbf{S}^n$:

$$\begin{array}{ll} (a) & X \succeq 0 \\ (b) & \mathcal{A}(A, B; X) \in \mathbf{S}^m_+, \end{array} \tag{4.4.5}$$

the uncertain data being $[A, B]$ and the uncertainty set being $\mathcal{AB}$. From now on we focus on the *interval uncertainty*, where the uncertain data $[A, B]$ in (4.4.5) is parameterized by perturbation $\zeta \in \mathbb{R}^L$ according to

$$[A, B] = [A_\zeta, B_\zeta] \equiv [A^{\rm n}, B^{\rm n}] + \sum_{\ell=1}^{L} \zeta_\ell e_\ell f_\ell^T; \tag{4.4.6}$$

here $[A^{\rm n}, B^{\rm n}]$ is the nominal data and $e_\ell \in \mathbb{R}^n$, $f_\ell \in \mathbb{R}^m$ are given vectors.

Imagine, e.g., that the entries in the uncertain matrix $[A, B]$ drift, independently of each other, around their nominal values. This is a particular case of (4.4.6) where $L = nm$, $\ell = (i, j)$, $1 \leq i \leq n$, $1 \leq j \leq m$, and the vectors e_ℓ and f_ℓ associated with $\ell = (i, j)$ are, respectively, the i-th standard basic orth in $\mathbb{R}^n$ multiplied by a given deterministic real δ_ℓ ("typical variability" of the data entry in question) and the j-th standard basic orth in $\mathbb{R}^m$.

4.4.2 Robust Nonexpansiveness: Analysis via GRC

The GRC setup and its interpretation

We are about to consider the GRC of the uncertain system of LMIs (4.4.5) affected by interval uncertainty (4.4.6). Our "GRC setup" will be as follows:

1. We equip the space $\mathbb{R}^L$ where the perturbation ζ lives with the uniform norm $\|\zeta\|_\infty = \max_\ell |\zeta_\ell|$, and specify the normal range of ζ as the box

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq r\} \tag{4.4.7}$$

with a given $r > 0$.

2. We specify the cone $\mathcal{L}$ as the entire $E = \mathbb{R}^L$, so that all perturbations are "physically possible."

3. The only uncertainty-affected LMI in our situation is (4.4.5.b); the right hand side in this LMI is the positive semidefinite cone $\mathbf{S}^{n+m}_+$ that lives in the space $\mathbf{S}^m$ of symmetric $m \times m$ matrices equipped with the Frobenius Euclidean structure. We equip this space with the standard spectral norm $\|\cdot\| = \|\cdot\|_{2,2}$.

Note that our setup belongs to what was called "case **F**" on p. 196.

Before processing the GRC of (4.4.5), it makes sense to understand what does it actually mean that X is a feasible solution to the GRC with global sensitivity α. By definition, this means three things:

A. $X \succeq 0$;

B. X is a robust feasible solution to (4.4.5.b), the uncertainty set being

$$\mathcal{AB}_r \equiv \{[A_\zeta, b_\zeta] : \|\zeta\|_\infty \le r\},$$

see (4.4.6); this combines with **A** to imply that if the perturbation $\zeta = \zeta^t$ underlying $[A_t, B_t]$ all the time remains in its normal range $\mathcal{Z} = \{\zeta : \|\zeta\|_\infty \le r\}$, the uncertain dynamical system (4.4.2) is robustly nonexpansive.

C. When $\rho > r$, we have

$$\forall(\zeta, \|\zeta\|_\infty \le \rho) : \mathrm{dist}(\mathcal{A}(A_\zeta, B_\zeta; X), \mathbf{S}^m_+) \le \alpha \mathrm{dist}(\zeta, \mathcal{Z}|\mathcal{L}) = \alpha(\rho - r),$$

or, recalling what is the norm on $\mathbf{S}^m$,

$$\forall(\zeta, \|\zeta\|_\infty \le \rho) : \mathcal{A}(A_\zeta, B_\zeta; X) \succeq -\alpha(\rho - r) I_m. \tag{4.4.8}$$

Now, repeating word for word the reasoning we used to demonstrate that (4.4.4) is sufficient for robust nonexpansiveness of (4.4.2), one can extract from (4.4.8) the following conclusion:

> (!) *Whenever in uncertain dynamical system* (4.4.2) *one has* $[A_t, B_t] = [A_{\zeta^t}, B_{\zeta^t}]$ *and the perturbation* ζ^t *remains all the time in the range* $\|\zeta^t\|_\infty \le \rho$, *one has*
>
> $$(1 - \alpha(\rho - r)) \int_0^t x^T(s) x(s) ds \le (1 + \alpha(\rho - r)) \int_0^t u^T(s) u(s) ds \tag{4.4.9}$$
>
> *for all* $t \ge 0$ *and all trajectories of the dynamical system such that* $x(0) = 0$.

We see that global sensitivity α indeed controls "deterioration of nonexpansiveness" as the perturbations run out of their normal range $\mathcal{Z}$: when the $\|\cdot\|_\infty$ distance from ζ^t to $\mathcal{Z}$ all the time remains bounded by $\rho - r \in [0, \frac{1}{\alpha})$, relation (4.4.9) guarantees that the L_2 norm of the state trajectory on every time horizon can be bounded by constant times the L_2 norm of the control on the this time horizon. The corresponding constant $\left(\frac{1+\alpha(\rho-r)}{1-\alpha(\rho-r)}\right)^{1/2}$ is equal to 1 when $\rho = r$ and grows with ρ, blowing up to $+\infty$ as $\rho - r$ approaches the critical value α^{-1}, and the larger α, the smaller is this critical value.

Processing the GRC

Observe that (4.4.4) and (4.4.6) imply that

$$\begin{array}{c} \mathcal{A}(A_\zeta, B_\zeta; X) = \mathcal{A}(A^{\rm n}, B^{\rm n}; X) - \sum_{\ell=1}^L \zeta_\ell \left[L_\ell^T(X) R_\ell + R_\ell^T L_\ell(X)\right], \\ L_\ell^T(X) = [X e_\ell; 0_{m-n,1}],\ R_\ell^T = f_\ell. \end{array} \tag{4.4.10}$$

Invoking Proposition 4.2, the GRC in question is equivalent to the following system of LMIs in variables X and α:

$$\begin{array}{ll} (a) & X \succeq 0 \\ (b) & \forall(\zeta, \|\zeta\|_\infty \le r): \\ & \qquad \mathcal{A}(A^{\rm n}, B^{\rm n}; X) + \sum_{\ell=1}^L \zeta_\ell \left[L_\ell^T(X) R_\ell + R_\ell^T L_\ell(X)\right] \succeq 0 \\ (c) & \forall(\zeta, \|\zeta\|_\infty \le 1): \sum_{\ell=1}^L \zeta_\ell \left[L_\ell^T(X) R_\ell + R_\ell^T L_\ell(X)\right] \succeq -\alpha I_m. \end{array} \tag{4.4.11}$$

Note that the semi-infinite LMIs (4.4.11.b, c) are affected by structured norm-bounded uncertainty with 1×1 scalar perturbation blocks (see section 3.5.1). Invoking Theorem 3.13, the system of LMIs

$$\begin{array}{ll} (a) & X \succeq 0 \\ (b.1) & Y_\ell \succeq \pm\left[L_\ell^T(X) R_\ell + R_\ell^T L_\ell(X)\right],\ 1 \le \ell \le L \\ (b.2) & \mathcal{A}(A^{\rm n}, B^{\rm n}; X) - r \sum_{\ell=1}^L Y_\ell \succeq 0 \\ (c.1) & Z_\ell \succeq \pm\left[L_\ell^T(X) R_\ell + R_\ell^T L_\ell(X)\right],\ 1 \le \ell \le L \\ (c.2) & \alpha I_m - \sum_{\ell=1}^L Z_\ell \succeq 0 \end{array}$$

in matrix variables $X, \{Y_\ell, Z_\ell\}_{\ell=1}^L$ and in scalar variable α is a safe tractable approximation of the GRC, tight within the factor $\frac{\pi}{2}$. Invoking the result stated in (!) on p. 106, we can reduce the design dimension of this approximation; the equivalent reformulation of the approximation is the SDO program

$$\min \alpha$$

s.t.

$$\begin{array}{l} X \succeq 0 \\ \left[\begin{array}{c|ccc} \mathcal{A}(A^{\rm n}, B^n; X) - r\sum_{\ell=1}^L \lambda_\ell R_\ell^T R_\ell & L_1^T(X) & \cdots & L_L^T(X) \\ \hline L_1(X) & \lambda_1/r & & \\ \vdots & & \ddots & \\ L_L(X) & & & \lambda_L/r \end{array}\right] \succeq 0 \\ \left[\begin{array}{c|ccc} \alpha I_m - \sum_{\ell=1}^L \mu_\ell R_\ell^T R_\ell & L_1^T(X) & \cdots & L_L^T(X) \\ \hline L_1(X) & \mu_1 & & \\ \vdots & & \ddots & \\ L_L(X) & & & \mu_L \end{array}\right] \succeq 0 \end{array} \tag{4.4.12}$$

in variable $X \in \mathbf{S}^m$ and scalar variables α, $\{\lambda_\ell, \mu_\ell\}_{\ell=1}^L$. Note that we have equipped our (approximate) GRC with the objective to minimize the global sensitivity of X; of course, other choices of the objective are possible as well.

Numerical illustration

The data. In the illustration we are about to present, the state dimension is $n = 5$, and the control dimension is $q = 2$, so that $m = \dim x + \dim u = 7$. The nominal data (chosen at random) are as follows:

$$[A^{\rm n}, B^{\rm n}] = M := \left[\begin{array}{ccccc|cc} -1.089 & -0.079 & -0.031 & -0.575 & -0.387 & 0.145 & 0.241 \\ -0.124 & -2.362 & -2.637 & 0.428 & 1.454 & -0.311 & 0.150 \\ -0.627 & 1.157 & -1.910 & -0.425 & -0.967 & 0.022 & 0.183 \\ -0.325 & 0.206 & 0.500 & -1.475 & 0.192 & 0.209 & -0.282 \\ 0.238 & -0.680 & -0.955 & -0.558 & -1.809 & 0.079 & 0.132 \end{array}\right].$$

The interval uncertainty (4.4.6) is specified as

$$[A_\zeta, b_\zeta] = M + \sum_{i=1}^{5}\sum_{j=1}^{7} \zeta_{ij} \underbrace{|M_{ij}| g_i}_{e_i} f_j^T,$$

where g_i, f_j are the standard basic orths in $\mathbb{R}^5$ and $\mathbb{R}^7$, respectively; in other words, every entry in $[A, B]$ is affected by its own perturbation, and the variability of an entry is the magnitude of its nominal value.

Normal range of perturbations. Next we should decide how to specify the normal range $\mathcal{Z}$ of the perturbations, i.e., the quantity r in (4.4.7). "In reality" this choice could come from the nature of the dynamical system in question and the nature of its environment. In our illustration there is no "nature and environment," and we specify r as follows. Let r_* be the largest r for which the robust nonexpansiveness of the system at the perturbation level r, (i.e., the perturbation set being the box $B_r = \{\zeta : \|\zeta\|_\infty \leq r\}$) admits a certificate. It would be quite reasonable to choose, as the normal range of perturbations $\mathcal{Z}$, the box B_{r_*}, so that the normal range of perturbations is the largest one where the robust nonexpansiveness still can be certified. Unfortunately, precise checking the existence of a certificate for a given box in the role of the perturbation set means to check the feasibility status of the system of LMIs

$$\begin{array}{ll} (a) & X \succeq 0 \\ (b) & \forall (\zeta, \|\zeta\|_\infty \leq r) : \mathcal{A}(A_\zeta, B_\zeta; X) \succeq 0 \end{array}$$

in matrix variable X, with $\mathcal{A}(\cdot, \cdot; \cdot)$ given in (4.4.4). This task seems to be intractable, so that we are forced to replace this system with its safe tractable approximation, tight within the factor $\pi/2$, specifically, with the system

$$\begin{array}{l} X \succeq 0 \\ \left[\begin{array}{c|ccc} \mathcal{A}(A^{\rm n}, B^n; X) - r\sum_{\ell=1}^{L} \lambda_\ell R_\ell^T R_\ell & L_1^T(X) & \cdots & L_L^T(X) \\ \hline L_1(X) & \lambda_1/r & & \\ \vdots & & \ddots & \\ L_L(X) & & & \lambda_L/r \end{array}\right] \succeq 0 \end{array} \tag{4.4.13}$$

in matrix variable X and scalar variables λ_ℓ (cf. (4.4.12)), with $R_\ell(X)$ and L_ℓ given by (4.4.10). The largest value r_1 of r for which the latter system is solvable (this quantity can be easily found by bisection) is a lower bound, tight within the factor $pi/2$, on r_*, and this is the quantity we use in the role of r when specifying the normal range of perturbations according to (4.4.7).

Applying this approach to the outlined data, we end up with

$$r = r_1 = 0.0346.$$

The results. With the outlined nominal and perturbation data and r, the optimal value in (4.4.12) turns out to be

$$\alpha_{\rm GRC} = 27.231.$$

It is instructive to compare this quantity with the global sensitivity of the RC-certificate $X_{\rm RC}$ of robust nonexpansiveness; by definition, $X_{\rm RC}$ is the X component of a feasible solution to (4.4.13) where r is set to r_1. This X clearly can be extended to a feasible solution to our safe tractable approximation (4.4.12) of the GRC; the smallest, over all these extensions, value of the global sensitivity α is

$$\alpha_{\rm RC} = 49.636,$$

which is by a factor 1.82 larger than $\alpha_{\rm GRC}$. It follows that the GRC-based analysis of the robust nonexpansiveness properties of the uncertain dynamical system in question provides us with essentially more optimistic results than the RC-based analysis. Indeed, a feasible solution $(\alpha, ...)$ to (4.4.12) provides us with the *upper bound*

$$C_*(\rho) \leq C_\alpha(\rho) \equiv \begin{cases} 1, & 0 \leq \rho \leq r \\ \frac{1+\alpha(\rho-r)}{1-\alpha(\rho-r)}, & r \leq \rho < r + \alpha^{-1} \end{cases} \tag{4.4.14}$$

(cf. (4.4.9)) on the "existing in the nature, but difficult to compute" quantity

$$C_*(\rho) = \inf \Big\{ C : \int_0^t x^T(s)x(s)ds \leq C \int_0^t u^T(s)u(s)ds \,\forall (t \geq 0, x(\cdot), u(\cdot)) : \\ x(0) = 0,\, \dot{x}(s) = A_{\zeta^s}x(s) + B_{\zeta^s}u(s), \|\zeta^s\|_\infty \leq \rho \,\forall s \Big\}$$

responsible for the robust nonexpansiveness properties of the dynamical system. The upper bounds (4.4.14) corresponding to $\alpha_{\rm RC}$ and $\alpha_{\rm GRC}$ are depicted on the left plot in figure 4.5 where we see that the GRC-based bound is much better than the RC-based bound.

Of course, both the bounds in question are conservative, and their "level of conservatism" is difficult to access theoretically: while we do understand how conservative our tractable approximations to intractable RC/GRC are, we have no idea how conservative the sufficient condition (4.4.4) for robust nonexpansiveness is (in this respect, the situation is completely similar to the one in Lyapunov Stability Analysis, see section 3.5.1). We can, however, run a brute force simulation to bound $C_*(\rho)$ from below. Specifically, generating a sample of perturbations of a given magnitude and checking the associated matrices $[A_\zeta, B_\zeta]$ for nonexpansiveness, we can build an upper bound $\overline{\rho}_1$ on the largest ρ for which every matrix $[A_\zeta, B_\zeta]$ with $\|\zeta\|_\infty \leq \rho$ generates a nonexpansive time-invariant dynamical system; $\overline{\rho}_1$ is, of course, greater than or equal to the largest $\rho = \rho_1$ for which $C_*(\rho) \leq 1$. Similarly, testing matrices A_ζ for stability, we can build an upper bound $\overline{\rho}_\infty$ on the largest $\rho = \rho_\infty$ for which all matrices A_ζ, $\|\zeta\|_\infty \leq \rho$, have all their eigenvalues in the closed left hand side plane; it is immediately seen that $C_*(\rho) = \infty$ when $\rho > \rho_\infty$. For our nominal and perturbation data, simulation yields

$$\overline{\rho}_1 = 0.310, \quad \overline{\rho}_\infty = 0.7854.$$

These quantities should be compared, respectively, to $r_1 = 0.0346$, (which clearly is a *lower* bound on the range ρ_1 of ρ's where $C_*(\rho) \leq 1$) and $r_\infty = r_1 + \alpha_{\rm GRC}^{-1}$ (this is the range of values of ρ where the GRC-originating upper bound (4.4.14) on $C_*(\rho)$ is finite; as such, r_∞ is a *lower* bound on ρ_∞). We see that in our numerical example the conservatism of our approach is "within one order of magnitude": $\overline{\rho}_1/r_1 \approx 8.95$ and $\overline{\rho}_\infty/r_\infty \approx 11.01$.

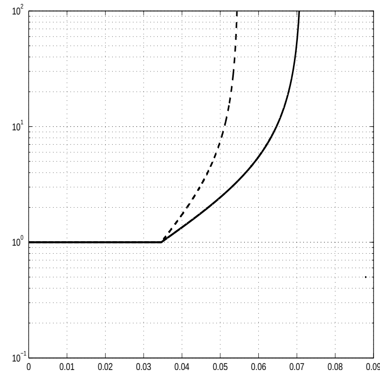


Figure 4.5: RC/GRC-based analysis: bounds (4.4.14) vs. ρ for $\alpha = \alpha_{\text{GRC}}$ (solid) and $\alpha = \alpha_{\text{RC}}$ (dashed).

4.5 Exercises

Exercise 4.1 Consider a situation as follows. A factory consumes n types of raw materials, coming from n different suppliers, to be decomposed into m pure components. The per unit content of component i in raw material j is $p_{ij} \geq 0$, and the necessary per month amount of component i is a given quantity $b_i \geq 0$. You need to make a long-term arrangement on the amounts of raw materials x_j coming every month from each of the suppliers, and these amounts should satisfy the system of linear constraints

$$Px \geq b, \; P = [p_{ij}].$$

The current per unit price of product j is c_j; this price, however, can vary in time, and from the history you know the volatilities $v_j \geq 0$ of the prices. How to choose x_j's in order to minimize the total cost of supply at the current prices, given an upper bound α on the sensitivity of the cost to possible future drifts in prices?

Test your model on the following data:

$$\begin{array}{c} n = 32, m = 8, p_{ij} \equiv 1/m, b_i \equiv 1.e3, \\ c_j = 0.8 + 0.2\sqrt{((j-1)/(n-1))}, v_j = 0.1(1.2 - c_j), \end{array}$$

and build the tradeoff curve "supply cost with current prices vs. sensitivity."

Exercise 4.2 Consider the Structural Design problem (p. 122 in section 3.4.2).

Recall that the nominal problem is

$$\min_{t,\tau} \left\{ \tau : t \in \mathcal{T}, \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array} \right] \succeq 0 \, \forall f \in \mathcal{F} \right\} \tag{SD}$$

where $\mathcal{T}$ is a given compact convex set of admissible designs, $\mathcal{F} \subset \mathbb{R}^m$ is a finite set of loads of interest and $A(t)$ is an $m \times m$ symmetric stiffness matrix affinely depending on t. Assuming that the load f "in reality" can run out of the set of scenarios $\mathcal{F}$ and thus can be treated as uncertain data element, we can look for a robust design. For the time being, we have already considered two robust versions of the problem. In the first, we extended the set $\mathcal{F}$ of actual loads of interest was extended to its union with an ellipsoid E centered at the origin (and thus we were interested to minimize the worst, with respect to loads of interest *and all "occasional" loads from* E, compliance of the construction. In the second (p. 154, section 3.6.2), we imposed an upper bound τ_* on the compliance w.r.t. the loads of interest and chance constrained the compliance w.r.t. Gaussian occasional loads. Along with these settings, it might make sense to control both the compliance w.r.t. loads of interest *and* its deterioration when these loads are perturbed. This is what we intend to consider now.

Consider the following "GRC-type" robust reformulation of the problem: we want to find a construction such that its compliance w.r.t. a load $f + \rho g$, where $f \in \mathcal{F}$, g is an occasional load from a given ellipsoid E centered at the origin, and $\rho \geq 0$ is a perturbation level, never exceeds a prescribed function $\phi(\rho)$. The corresponding robust problem is the semi-infinite program

(!) Given $\phi(\cdot)$, find $t \in \mathcal{T}$ such that

$$\left[\begin{array}{c|c} 2\phi(\rho) & [f+\rho g]^T \\ \hline f+\rho g & A(t) \end{array}\right] \succeq 0 \ \forall (f \in \mathcal{F}, g \in E).$$

1. Let us choose $\phi(\rho) = \tau + \alpha\rho$ – the choice straightforwardly inspired by the GRC methodology. Does the corresponding problem (!) make sense? If not, why the GRC methodology does not work in our case?

2. Let us set $\phi(\rho) = (\sqrt{\tau} + \rho\sqrt{\alpha})^2$. Does the corresponding problem (!) make sense? If yes, does (!) admit a computationally tractable reformulation?

Exercise 4.3 Consider the situation as follows:

Unknown signal z known to belong to a given ball $B = \{z \in \mathbb{R}^n : z^T z \leq 1\}$, $Q \succ 0$, is observed according to the relation

$$y = Az + \xi,$$

where y is the observation, A is a given $m \times n$ sensing matrix, and ξ is an observation error. Given y, we want to recover a linear form $f^T z$ of z; here f is a given vector. The normal range of the observation error is $\Xi = \{\xi \in \mathbb{R}^m : \|\xi\|_2 \leq 1\}$. We are seeking for a linear estimate $\widehat{f}(y) = g^T y$.

1. Formulate the problem of building the best, in the minimax sense (i.e., with the minimal worst case, w.r.t. $x \in B$ and $\xi \in \Xi$, recovering error), linear estimate as the RC of an uncertain LO problem and build tractable reformulation of this RC.

2. Formulate the problem of building a linear estimate with the worst case, over signals z with $\|z\|_2 \leq 1 + \rho_z$ and observation errors ξ with $\|\xi\|_2 \leq 1 + \rho_\xi$, risk for all $\rho_z, \rho_\xi \geq 0$, risk admitting the bound $\tau + \alpha_z\rho_z + \alpha_\xi\rho_\xi$ with given $\tau, \alpha_z, \alpha_x i$; thus, we want "desired performance" τ of the estimate in the normal range $B \times \Xi$ of $[z; \xi]$ and "controlled deterioration of this performance" when z and/or ξ run out of their normal ranges. Build a tractable reformulation of this problem.

Exercise 4.4 Consider situation completely similar to the one in Exercise 4.3, with the only difference that now we want to build a linear estimate Gy of the vector Cz, C being a given matrix, rather than to estimate a linear form of g.

1. Formulate the problem of building the best, in the minimax sense (i.e., with the minimal worst case, w.r.t. $z \in B$ and $\xi \in \Xi$, $\|\cdot\|_2$-recovering error), linear estimate of Cz as the RC of an uncertain Least Squares inequality and build a tight safe tractable approximation of this RC.

2. Formulate the problem of building a linear estimate of Cz with the worst case, over signals z with $\|z\|_2 \leq 1 + \rho_z$ and observation errors ξ with $\|\xi\|_2 \leq 1 + \rho_\xi$, risk admitting for all $\rho_z, \rho_\xi \geq 0$ the bound $\tau + \alpha_z \rho_z + \alpha_\xi \rho_\xi$ with given $\tau, \alpha_z, \alpha_x i$. Find a tight safe tractable approximation of this problem.

Lecture 5

Adjustable Robust Multistage Optimization

In this lecture we intend to investigate robust *multi-stage* linear and conic optimization.

5.1 Adjustable Robust Optimization: Motivation

Consider a general-type uncertain optimization problem — a collection

$$\mathcal{P} = \left\{ \min_x \{ f(x,\zeta) : F(x,\zeta) \in \mathbf{K} \} : \zeta \in \mathcal{Z} \right\} \tag{5.1.1}$$

of *instances* — optimization problems of the form

$$\min_x \{ f(x,\zeta) : F(x,\zeta) \in \mathbf{K} \},$$

where $x \in \mathbb{R}^n$ is the decision vector, $\zeta \in \mathbb{R}^L$ represents the uncertain data or data perturbation, the real-valued function $f(x,\zeta)$ is the objective, and the vector-valued function $F(x,\zeta)$ taking values in $\mathbb{R}^m$ along with a set $\mathbf{K} \subset \mathbb{R}^m$ specify the constraints; finally, $\mathcal{Z} \subset \mathbb{R}^L$ is the uncertainty set where the uncertain data is restricted to reside.

> Format (5.1.1) covers all uncertain optimization problems considered so far; moreover, in these latter problems the objective f and the right hand side F of the constraints always were *bi-affine* in x, ζ, (that is, affine in x when ζ is fixed, and affine in ζ, x being fixed), and $\mathbf{K}$ was a "simple" convex cone (a direct product of nonnegative rays/Lorentz cones/Semidefinite cones, depending on whether we were speaking about uncertain Linear, Conic Quadratic or Semidefinite Optimization). We shall come back to this "well-structured" case later; for our immediate purposes the specific conic structure of instances plays no role, and we can focus on "general" uncertain problems in the form of (5.1.1).

The Robust Counterpart of uncertain problem (5.1.1) is defined as the semi-infinite optimization problem

$$\min_{x,t} \{ t : \forall \zeta \in \mathcal{Z} : f(x,\zeta) \le t, F(x,\zeta) \in \mathbf{K} \}; \tag{5.1.2}$$

this is exactly what was so far called the RC of an uncertain problem.

Recall that our interpretation of the RC (5.1.2) as the natural source of robust/robust optimal solutions to the uncertain problem (5.1.1) is not self-evident, and its "informal justification" relies upon the specific assumptions A.1–3 on our "decision environment," see page 8. We have already relaxed somehow the last of these assumptions, thus arriving at the notion of Globalized Robust Counterpart, lecture 4. What is on our agenda now is to revise the first assumption, which reads

> A.1. All decision variables in (5.1.1) represent "here and now" decisions; they should get specific numerical values as a result of solving the problem *before* the actual data "reveals itself" and as such should be independent of the actual values of the data.

We have considered numerous examples of situations where this assumption is valid. At the same time, there are situations when it is too restrictive, since "in reality" some of the decision variables can adjust themselves, to some extent, to the actual values of the data. One can point out at least two sources of such adjustability: presence of *analysis variables* and *wait-and-see decisions.*

Analysis variables. Not always all decision variables x_j in (5.1.1) represent actual decisions; in many cases, some of x_j are slack, or analysis, variables introduced in order to convert the instances into a desired form, e.g., the one of Linear Optimization programs. It is very natural to allow for the analysis variables to depend on the true values of the data — why not?

Example 5.1 [cf. Example 1.3] Consider an "ℓ_1 constraint"

$$\sum_{k=1}^{K} |a_k^T x - b_k| \leq \tau; \tag{5.1.3}$$

you may think, e.g., about the Antenna Design problem (Example 1.1) where the "fit" between the actual diagram of the would-be antenna array and the target diagram is quantified by the $\|\cdot\|_1$ distance. Assuming that the data and x are real, (5.1.3) can be represented equivalently by the system of linear inequalities

$$-y_k \leq a_k^T x - b_k \leq y_k, \ \sum_k y_k \leq \tau$$

in variables x, y, τ. Now, when the data a_k, b_k are uncertain and the components of x do represent "here and now" decisions and should be independent of the actual values of the data, there is absolutely no reason to impose the latter requirement on the slack variables y_k as well: they do not represent decisions at all and just certify the fact that the actual decisions x, τ meet the requirement (5.1.3). While we can, of course, impose this requirement "by force," this perhaps will lead to a too conservative model. It seems to be completely natural to allow for the certificates y_k to depend on actual values of the data — it may well happen that then we shall be able to certify robust feasibility for (5.1.3) for a larger set of pairs (x, τ).

Wait-and-see decisions. This source of adjustability comes from the fact that some of the variables x_j represent decisions that are not "here and now" decisions, i.e., those that should be made before the true data "reveals itself." In multi-stage decision making processes, some x_j can represent "wait and see" decisions, which could be made after the controlled system "starts to live," at time instants when part (or all) of the true data is revealed. It is fully legitimate to allow for these decisions to depend on the part of the data that indeed "reveals itself" before the decision should be made.

Example 5.2 Consider a multi-stage inventory system affected by uncertain demand. The most interesting of the associated decisions — the *replenishment orders* — are made one at a time, and the replenishment order of "day" t is made when we already know the actual demands in the preceding days. It is completely natural to allow for the orders of day t to depend on the preceding demands.

5.2 Adjustable Robust Counterpart

A natural way to model adjustability of variables is as follows: for every $j \leq n$, we allow for x_j to depend on a prescribed "portion" $P_j\zeta$ of the true data ζ:

$$x_j = X_j(P_j\zeta), \tag{5.2.1}$$

where $P_1, ..., P_n$ are given in advance matrices specifying the "information base" of the decisions x_j, and $X_j(\cdot)$ are *decision rules* to be chosen; these rules can in principle be arbitrary functions on the corresponding vector spaces. For a given j, specifying P_j as the zero matrix, we force x_j to be completely independent of ζ, that is, to be a "here and now" decision; specifying P_j as the unit matrix, we allow for x_j to depend on the entire data (this is how we would like to describe the analysis variables). And the "in-between" situations, choosing P_j with $1 \leq \text{Rank}(P_j) < L$ enables one to model the situation where x_j is allowed to depend on a "proper portion" of the true data.

We can now replace in the usual RC (5.1.2) of the uncertain problem (5.1.1) the independent of ζ decision variables x_j with functions $X_j(P_j\zeta)$, thus arriving at the problem

$$\begin{array}{c} \min\limits_{t,\{X_j(\cdot)\}_{j=1}^n} \{t : \forall \zeta \in \mathcal{Z} : f(X(\zeta),\zeta) \leq t, F(X(\zeta),\zeta) \in \mathbf{K}\}, \\ X(\zeta) = [X_1(P_1\zeta); ...; X_n(P_n\zeta)]. \end{array} \tag{5.2.2}$$

The resulting optimization problem is called the *Adjustable Robust Counterpart* (ARC) of the uncertain problem (5.1.1), and the (collections of) decision rules $X(\zeta)$, which along with certain t are feasible for the ARC, are called *robust feasible decision rules*. The ARC is then the problem of specifying a collection of decision rules with prescribed information base that is feasible for as small t as possible. The *robust optimal decision rules* now replace the *constant* (non-adjustable, data-independent) robust optimal decisions that are yielded by the usual Robust Counterpart (5.1.2) of our uncertain problem. Note that the ARC is an extension of the RC; the latter is a "trivial" particular case of the former corresponding to the case of trivial information base in which all matrices P_j are zero.

5.2.1 Examples

We are about to present two instructive examples of uncertain optimization problems with adjustable variables.

Information base induced by time precedences. In many cases, decisions are made subsequently in time; whenever this is the case, a natural information base of the decision to be made at instant t ($t = 1, ..., N$) is the part of the true data that becomes known at time t. As an instructive example, consider a simple Multi-Period Inventory model mentioned in Example 5.2:

Example 5.2 continued. Consider an inventory system where d products share common warehouse capacity, the time horizon is comprised of N periods, and the goal is to minimize the total inventory management cost. Allowing for backlogged demand, the simplest model of such an inventory looks as follows:

$$
\begin{array}{lll}
\text{minimize} & C & \text{[inventory management cost]} \\
\text{s.t.} & & \\
(a) & C \geq \sum_{t=1}^{N} \left[c_{\mathrm{h},t}^T y_t + c_{\mathrm{b},t}^T z_t + c_{\mathrm{o},t}^T w_t \right] & \text{[cost description]} \\
(b) & x_t = x_{t-1} + w_t - \zeta_t,\ 1 \leq t \leq N & \text{[state equations]} \\
(c) & y_t \geq 0, y_t \geq x_t,\ 1 \leq t \leq N & \\
(d) & z_t \geq 0, z_t \geq -x_t,\ 1 \leq t \leq N & \\
(e) & \underline{w}_t \leq w_t \leq \overline{w}_t,\ 1 \leq t \leq N & \\
(f) & q^T y_t \leq r &
\end{array} \tag{5.2.3}
$$

The variables in this problem are:

- $C \in \mathbb{R}$ — (upper bound on) the total inventory management cost;
- $x_t \in \mathbb{R}^d$, $t = 1, ..., N$ — states. i-th coordinate x_t^i of vector x_t is the amount of product of type i that is present in the inventory at the time instant t (end of time interval # t). This amount can be nonnegative, meaning that the inventory at this time has x_t^i units of free product # i; it may be also negative, meaning that the inventory at the moment in question owes the customers $|x_t^i|$ units of the product i ("backlogged demand"). The initial state x_0 of the inventory is part of the data, and not part of the decision vector;
- $y_t \in \mathbb{R}^d$ are upper bounds on the positive parts of the states x_t, that is, (upper bounds on) the "physical" amounts of products stored in the inventory at time t, and the quantity $c_{\mathrm{h},t}^T y_t$ is the (upper bound on the) holding cost in the period t; here $c_{\mathrm{h},t} \in \mathbb{R}_+^d$ is a given vector of the holding costs per unit of the product. Similarly, the quantity $q^T y_t$ is (an upper bound on) the warehouse capacity used by the products that are "physically present" in the inventory at time t, $q \in \mathbb{R}_+^d$ being a given vector of the warehouse capacities per units of the products;
- $z_t \in \mathbb{R}^d$ are (upper bounds on) the backlogged demands at time t, and the quantities $c_{\mathrm{b},t}^T z_t$ are (upper bounds on) the penalties for these backlogged demands. Here $c_{\mathrm{b,t}} \in \mathbb{R}_+^d$ are given vectors of the penalties per units of the backlogged demands;
- $w_t \in \mathbb{R}^d$ is the vector of replenishment orders executed in period t, and the quantities $c_{\mathrm{o},t}^T w_t$ are the costs of executing these orders. Here $c_{\mathrm{o},t} \in \mathbb{R}_+^d$ are given vectors of per unit ordering costs.

With these explanations, the constraints become self-evident:

- (a) is the "cost description": it says that the total inventory management cost is comprised of total holding and ordering costs and of the total penalty for the backlogged demand;
- (b) are state equations: "what will be in the inventory at the end of period t (x_t) is what was there at the end of preceding period (x_{t-1}) plus the replenishment orders of the period (w_t) minus the demand of the period (ζ_t);
- (c), (d) are self-evident;
- (e) represents the upper and lower bounds on replenishment orders, and (f) expresses the requirement that (an upper bound on) the total warehouse capacity $q^T y_t$ utilized by products that are "physically present" in the inventory at time t should not be greater than the warehouse capacity r.

In our simple example, we assume that out of model's parameters

$$x_0, \{c_{\mathrm{h},t}, c_{\mathrm{b},t}, c_{\mathrm{o,t}}, \underline{w}_t, \overline{w}_t\}_{t=1}^N, q, r, \{\zeta_t\}_{t=1}^N$$

the only uncertain element is the *demand trajectory* $\zeta = [\zeta_1; ...; \zeta_N] \in \mathbb{R}^{dN}$, and that this trajectory is known to belong to a given uncertainty set $\mathcal{Z}$. The resulting uncertain Linear Optimization problem is comprised of instances (5.2.3) parameterized by the uncertain data — demand trajectory ζ — running through a given set $\mathcal{Z}$.

As far as the adjustability is concerned, all variables in our problem, except for the replenishment orders w_t, are analysis variables. As for the orders, the simplest assumption is that w_t should get numerical value at time t, and that at this time we already know the past demands $\zeta^{t-1} = [\zeta_1; ...; \zeta_{t-1}]$. Thus, the information base for w_t is $\zeta^{t-1} = P_t\zeta$ (with the convention that $\zeta^s = 0$ when $s < 0$). For the remaining analysis variables the information base is the entire demand trajectory ζ. Note that we can easily adjust this model to the case when there are lags in demand acquisition, so that w_t should depend on a prescribed initial segment $\zeta^{\tau(t)-1}$, $\tau(t) \leq t$, of ζ^{t-1} rather than on the entire ζ^{t-1}. We can equally easily account for the possibility, if any, to observe the demand "on line," by allowing w_t to depend on ζ^t rather than on ζ^{t-1}. Note that in all these cases the information base of the decisions is readily given by the natural time precedences between the "actual decisions" augmented by a specific demand acquisition protocol.

Example 5.3 Project management. Figure 5.1 is a simple *PERT diagram* — a graph representing a Project Management problem. This is an acyclic directed graph with nodes corresponding to *events*, and arcs corresponding to *activities*. Among the nodes there is a *start node* S with no incoming arcs and an *end node* F with no outgoing arcs, interpreted as "start of the project" and "completion of the project," respectively. The remaining nodes correspond to the events "a specific stage of the project is completed, and one can pass to another stage". For example, the diagram could represent creating a factory, with A, B, C being, respectively, the events "equipment to be installed is acquired and delivered," "facility #1 is built and equipped," "facility # 2 is built and equipped." The activities are jobs comprising the project. In our example, these jobs could be as follows:

a: acquiring and delivering the equipment for facilities ## 1,2
b: building facility # 1
c: building facility # 2
d: installing equipment in facility # 1
e: installing equipment in facility # 2
f: training personnel and preparing production at facility # 1
g: training personnel and preparing production at facility # 2

The topology of a PERT diagram represents *logical precedences* between the activities and events: a particular activity, say g, can start only after the event C occurs, and the latter event happens when both activities c and e are completed.

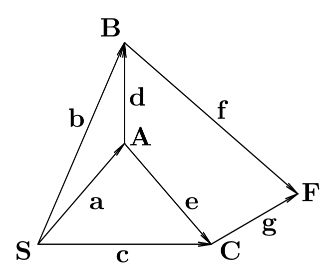


Figure 5.1: A PERT diagram.

In PERT models it is assumed that activities γ have nonnegative durations τ_γ (perhaps depending on control parameters), and are executed without interruptions, with possible idle periods between the moment when the start of an activity is allowed by the logical precedences and the moment when it is actually started. With these assumptions, one can write down a system of constraints on the time instants t_ν when events ν can take place. Denoting by $\Gamma = \{\gamma = (\mu_\gamma, \nu_\gamma)\}$ the set of arcs in a PERT diagram (μ_γ is the start- and ν_γ is the end-node of an arc γ), this system reads

$$t_{\mu_\gamma} - t_{\nu_\gamma} \geq \tau_\gamma \ \forall \gamma \in \Gamma. \tag{5.2.4}$$

"Normalizing" this system by the requirement

$$t_S = 0,$$

the values of t_F, which can be obtained from feasible solutions to the system, are achievable durations of the entire project. In a typical Project Management problem, one imposes an upper bound on t_F and minimizes, under this restriction, coupled with the system of constraints (5.2.4), some objective function.

As an example, consider the situation where the "normal" durations τ_γ of activities can be reduced at certain price ("in reality" this can correspond to investing into an activity extra manpower, machines, etc.). The corresponding model becomes

$$\tau_\gamma = \zeta_\gamma - x_\gamma, \ c_\gamma = f_\gamma(x_\gamma),$$

where ζ_γ is the "normal duration" of the activity, x_γ ("crush") is a nonnegative decision variable, and $c_\gamma = f_\gamma(x_\gamma)$ is the cost of the crush; here $f_\gamma(\cdot)$ is a given function. The associated optimization model might be, e.g., the problem of minimizing the total cost of the crushes under a given upper bound T on project's duration:

$$\min_{\substack{x=\{x_\gamma:\gamma\in\Gamma\} \\ \{t_\nu\}}} \left\{ \sum_\gamma f_\gamma(x_\gamma) : \begin{array}{l} t_{\mu_\gamma} - t_{\nu_\gamma} \geq \zeta_\gamma - x_\gamma \\ 0 \leq x_\gamma \leq \overline{x}_\gamma \end{array} \right\} \forall \gamma \in \Gamma, t_S = 0, t_F \leq T \right\}, \tag{5.2.5}$$

where $\overline{x}_\gamma$ are given upper bounds on crushes. Note that when $f_\gamma(\cdot)$ are convex functions, (5.2.5) is an explicit convex problem, and when, in addition to convexity, $f_\gamma(\cdot)$ are piecewise linear, (which is usually the case in reality and which we assume from now on), (5.2.5) can be straightforwardly converted to a Linear Optimization program.

Usually part of the data of a PERT problem are uncertain. Consider the simplest case when the only uncertain elements of the data in (5.2.5) are the normal durations ζ_γ of the activities (their uncertainty may come from varying weather conditions, inaccuracies in estimating the forthcoming effort, etc.). Let us assume that these durations are random variables, say, independent of each other, distributed in given segments $\Delta_\gamma = [\underline{\zeta}_\gamma, \overline{\zeta}_\gamma]$. To avoid pathologies, assume also that $\underline{\zeta}_\gamma \geq \overline{x}_\gamma$ for every γ ("you cannot make the duration negative"). Now (5.2.5) becomes an uncertain LO program with uncertainties affecting only the right hand sides of the constraints. A natural way to "immunize" the solutions to the problem against data uncertainty is to pass to the usual RC of the problem — to think of both t_γ and x_γ as of variables with values to be chosen in advance in such a way that the constraints in (5.2.4) are satisfied for all values of the data ζ_γ from the uncertainty set. With our model of the latter set the RC is nothing but the "worst instance" of our uncertain problem, the one where ζ_γ are set to their maximum possible values $\overline{\zeta}_\gamma$. For large PERT graphs, such an approach is very conservative: why should we care about the highly improbable case where all the normal durations — independent random variables! — are simultaneously at their worst-case values? Note that even taking into account that the normal durations are random and replacing the uncertain constraints in (5.2.5) by their chance constrained versions, we essentially do not reduce the conservatism. Indeed, every one of randomly perturbed constraints in (5.2.5) contains a *single* random perturbation, so that we cannot hope that random perturbations of a constraint will to some extent cancel each other. As a result, to require the validity of every uncertain constraint with probability 0.9 or 0.99 is the same as to require its validity "in the worst case" with just slightly reduced maximal normal durations of the activities.

A much more promising approach is to try to adjust our decisions "on line." Indeed, we are speaking about a process that evolves in time, with "actual decisions" represented by variables x_γ and t_ν's being the analysis variables. Assuming that the decision on x_γ can be postponed till the event μ_γ (the earliest time when the activity γ can be started) takes place, at that time we already know the actual durations of the activities terminated before the event μ_γ, we could then adjust our decision on x_γ in accordance with this information. The difficulty is that *we do not know in advance what will be the actual time precedences between the events — these precedences depend on our decisions and on the actual values of the uncertain data.* For example, in the situation described by figure 5.1, we, in general, cannot know in advance which one of the events B, C will precede the other one in time. As a result, in our present situation, in sharp contrast to the situation of Example 5.2, an attempt to fully utilize the possibilities to adjust the decisions to the actual values of the data results in an extremely complicated problem, where not only the decisions themselves, but the very information base of the decisions become dependent on the uncertain data and our policy. However, we could stick to something in-between "no adjustability at all" and "as much adjustability as possible." Specifically, we definitely know that if a pair of activities γ', γ are linked by a logical precedence, so that there exists an oriented route in the graph that starts with γ' and ends with γ, then the actual duration of γ' will be known before γ can start. Consequently, we can take, as the information base of an activity γ, the collection $\zeta^\gamma = \{\zeta_{\gamma'} : \gamma' \in \Gamma_-(\gamma)\}$, where $\Gamma_-(\gamma)$ is the set of all activities that logically precede the activity γ. In favorable circumstances, such an approach could reduce significantly the price of robustness as compared to the non-adjustable RC. Indeed, when plugging into the randomly perturbed constraints of (5.2.5) instead of constants x_γ functions $X_\gamma(\zeta^\gamma)$, and requiring from the resulting inequalities to be valid with probability $1-\epsilon$, we end up with a system of chance constraints such that some of them (in good cases, even most of them) involve many independent random perturbations each. When the functions $X_\gamma(\zeta^\gamma)$ are regular enough, (e.g., are affine), we can hope that the numerous independent perturbations affecting a chance constraint will to some extent cancel each other, and consequently, the resulting system of chance constraints will be significantly less conservative than the one corresponding to non-adjustable decisions.

5.2.2 Good News on the ARC

Passing from a trivial information base to a nontrivial one — passing from robust optimal *data-independent decisions* to robust optimal *data-based decision rules* can indeed dramatically reduce the associated robust optimal value.

Example 5.4 Consider the toy uncertain LO problem

$$\left\{\min_x \left\{ x_1 : \begin{array}{ll} x_2 \geq \frac{1}{2}\zeta x_1 + 1 & (a_\zeta) \\ x_1 \geq (2-\zeta)x_2 & (b_\zeta) \\ x_1, x_2 \geq 0 & (c_\zeta) \end{array} \right\} : 0 \leq \zeta \leq \rho \right\},$$

where $\rho \in (0,1)$ is a parameter (uncertainty level). Let us compare the optimal value of its non-adjustable RC (where both x_1 and x_2 must be independent of ζ) with the optimal value of the ARC where x_1 still is assumed to be independent of ζ ($P_1\zeta \equiv 0$) but x_2 is allowed to depend on ζ ($P_2\zeta \equiv \zeta$).

A feasible solution (x_1, x_2) of the RC should remain feasible for the constraint (a_ζ) when $\zeta = \rho$, meaning that $x_2 \geq \frac{\rho}{2}x_1 + 1$, and should remain feasible for the constraint (b_ζ) when $\zeta = 0$, meaning that $x_1 \geq 2x_2$. The two resulting inequalities imply that $x_1 \geq \rho x_1 + 2$, whence $x_1 \geq \frac{2}{1-\rho}$. Thus, Opt(RC)$\geq \frac{2}{1-\rho}$, whence Opt(RC)$\to \infty$ as $\rho \to 1 - 0$.

Now let us solve the ARC. Given $x_1 \geq 0$ and $\zeta \in [0, \rho]$, it is immediately seen that x_1 can be extended, by properly chosen x_2, to a feasible solution of (a_ζ) through (c_ζ) if and only if the pair $(x_1, x_2 = \frac{1}{2}\zeta x_1 + 1)$ is feasible for (a_ζ) through (c_ζ), that is, if and only if $x_1 \geq (2-\zeta)\left[\frac{1}{2}\zeta x_1 + 1\right]$ whenever $1 \leq \zeta \leq \rho$. The latter relation holds true when $x_1 = 4$ and $\rho \leq 1$ (since $(2-\zeta)\zeta \leq 1$ for $0 \leq \zeta \leq 2$). Thus, Opt(ARC)≤ 4, and the difference between Opt(RC) and Opt(ARC) and the ratio Opt(RC)/Opt(ARC) go to ∞ as $\rho \to 1 - 0$.

5.2.3 Bad News on the ARC

Unfortunately, from the computational viewpoint the ARC of an uncertain problem more often than not is wishful thinking rather than an actual tool. The reason comes from the fact that *the ARC is typically severely computationally intractable.* Indeed, (5.2.2) is an *infinite-dimensional problem*, where one wants to optimize over *functions* — decision rules — rather than vectors, and these functions, in general, depend on many real variables. It is unclear even how to *represent* a general-type candidate decision rule — a general-type multivariate function — in a computer. Seemingly the only option here is sticking to a chosen in advance *parametric family* of decision rules, like piece-wise constant/linear/quadratic functions of $P_j\zeta$ with simple domains of the pieces (say, boxes). With this approach, a candidate decision rule is identified by the vector of values of the associated parameters, and the ARC becomes a finite-dimensional problem, the parameters being our new decision variables. This approach is indeed possible and in fact will be the focus of what follows. However, it should be clear from the very beginning that if the parametric family in question is "rich enough" to allow for good approximation of "truly optimal" decision rules (think of polynomial splines of high degree as approximations to "not too rapidly varying" general-type multivariate functions), the number of parameters involved should be astronomically large, unless the dimension of ζ is really small, like 1 — 3 (think of how many coefficients there are in a single algebraic polynomial of degree 10 with 20 variables). Thus, aside of "really low dimensional" cases, "rich" general-purpose parametric families of decision rules are for all practical purposes as intractable as non-parametric families. In other words, when the dimension L of ζ is not too small, tractability of parametric families of decision rules is something opposite to their "approximation abilities," and sticking to tractable parametric families, we lose control of how far the optimal value of the "parametric" ARC is away from the optimal value of the "true" infinite-dimensional ARC. The only exception here seems to be the case when we are smart enough to utilize our knowledge of the structure of instances of the uncertain problem in question in order to identify the optimal decision rules up to a moderate number of parameters. *If* we indeed are that smart and *if* the parameters in question can be further identified numerically in a computationally efficient fashion, we indeed can end up with an optimal solution to the "true" ARC. Unfortunately, the two "if's" in the previous sentence are big if's indeed — to the best of our knowledge, the only *generic* situation when these conditions are satisfied is one treated the Dynamic Programming techniques. It seems that these techniques form the only component in the existing "optimization toolbox" that could be used to process the ARC numerically, at least when approximations of a provably high quality are sought. Unfortunately, the Dynamic Programming techniques are very "fragile" — they require instances of a very specific structure, suffer from "curse of dimensionality," etc. The bottom line, in our opinion, is that *aside of situations where Dynamic Programming is computationally efficient*, (which is an exception rather than a rule), *the only hopefully computationally tractable approach to optimizing over decision rules is to stick to their simple parametric families*, even at the price of giving up full control over the losses in optimality that can be incurred by such a simplification.

Before moving to an in-depth investigation of (a version of) the just outlined "simple approximation" approach to adjustable robust decision-making, it is worth pointing out two situations when no simple approximations are necessary, since the situations in question are very simple from the very beginning.

Simple case I: fixed recourse and scenario-generated uncertainty set

Consider an uncertain *conic* problem

$$\mathcal{P} = \left\{ \min_x \left\{ c_\zeta^T x + d_\zeta : A_\zeta x + b_\zeta \in \mathbf{K} \right\} : \zeta \in \mathcal{Z} \right\} \tag{5.2.6}$$

($A_\zeta, b_\zeta, c_\zeta, d_\zeta$ are affine in ζ, $\mathbf{K}$ is a computationally tractable convex cone) and assume that

1. $\mathcal{Z}$ is a scenario-generated uncertainty set, that is, a set given as a convex hull of finitely many "scenarios" ζ^s, $1 \le s \le S$;
2. The information base ensures that every variable x_j either is non-adjustable ($P_j = 0$), or is fully adjustable ($P_j = I$);
3. We are in the situation of *fixed recourse*, that is, for every adjustable variable x_j (one with $P_j \neq 0$), all its coefficients in the objective and the left hand side of the constraint are certain, (i.e., are independent of ζ).

W.l.o.g. we can assume that $x = [u; v]$, where the u variables are non-adjustable, and the v variables are fully adjustable; under fixed recourse, our uncertain problem can be written down as

$$\mathcal{P} = \left\{ \min_{u,v} \left\{ p_\zeta^T u + q^T v + d_\zeta : P_\zeta u + Qv + r_\zeta \in \mathbf{K} \right\} : \zeta \in \mathrm{Conv}\{\zeta^1, ..., \zeta^S\} \right\}$$

($p_\zeta, d_\zeta, P_\zeta, r_\zeta$ are affine in ζ). An immediate observation is that:

Theorem 5.1 *Under assumptions 1 – 3, the ARC of the uncertain problem $\mathcal{P}$ is equivalent to the computationally tractable conic problem*

$$Opt = \min_{t,u,\{v^s\}_{s=1}^S} \left\{ t : p_{\zeta^s} u + q^T v^s + d_{\zeta^s} \le t,\ P_{\zeta^s} u + Qv^s + r_{\zeta^s} \in \mathbf{K} \right\}. \tag{5.2.7}$$

Specifically, the optimal values in the latter problem and in the ARC of $\mathcal{P}$ are equal. Moreover, if $\bar{t}, \bar{u}, \{\bar{v}^s\}_{s=1}^S$ is a feasible solution to (5.2.7)*, then the pair $\bar{t}, \bar{u}$ augmented by the decision rule for the adjustable variables:*

$$v = \bar{V}(\zeta) = \sum_{s=1}^S \lambda_s(\zeta) \bar{v}^s$$

form a feasible solution to the ARC. Here $\lambda(\zeta)$ is an arbitrary nonnegative vector with the unit sum of entries such that

$$\zeta = \sum_{s=1}^S \lambda_s(\zeta) \zeta^S. \tag{5.2.8}$$

Proof. Observe first that $\lambda(\zeta)$ is well-defined for every $\zeta \in \mathcal{Z}$ due to $\mathcal{Z} = \text{Conv}\{\zeta^1, ..., \zeta^S\}$. Further, if $\bar{t}, \bar{u}, \{\bar{v}^s\}$ is a feasible solution of (5.2.7) and $\bar{V}(\zeta)$ is as defined above, then for every $\zeta \in \mathcal{Z}$ the following implications hold true:

$$\begin{array}{l}\bar{t} \geq p_{\zeta^s}\bar{u} + q^T\bar{v}^s + d_{\zeta^s}\,\forall s \Rightarrow \bar{t} \geq \sum_s \lambda_s(\zeta)\left[p_{\zeta^s}^T\bar{u} + q^T\bar{v}^s + d_{\zeta^s}\right]\\ = p_\zeta^T\bar{u} + q^T\bar{V}(\zeta) + d_\zeta,\\ \mathbf{K} \ni P_{\zeta^s}\bar{u} + Q\bar{v}^s + r_{\zeta^s}\,\forall s \Rightarrow \mathbf{K} \ni \sum_s \lambda_s(\zeta)\left[P_{\zeta^s}\bar{u} + Q\bar{v}^s + r_{\zeta^s}\right]\\ = P_\zeta\bar{u} + Q\bar{V}(\zeta) + r_\zeta\end{array}$$

(recall that $p_\zeta, ..., r_\zeta$ are affine in ζ). We see that $(\bar{t}, \bar{u}, \bar{V}(\cdot))$ is indeed a feasible solution to the ARC

$$\min_{t,u,V(\cdot)} \left\{t : p_\zeta^T u + q^T V(\zeta) + d_\zeta \leq t, P_\zeta u + QV(\zeta) + r_\zeta \in \mathbf{K}\,\forall \zeta \in \mathcal{Z}\right\}$$

of $\mathcal{P}$. As a result, the optimal value of the latter problem is $\leq$ Opt. It remains to verify that the optimal value of the ARC and Opt are equal. We already know that the first quantity is $\leq$ the second one. To prove the opposite inequality, note that if $(t, u, V(\cdot))$ is feasible for the ARC, then clearly $(t, u, \{v^s = V(\zeta^s)\})$ is feasible for (5.2.7). $\square$

The outlined result shares the same shortcoming as Theorem 3.1 from section 3.2.1: scenario-generated uncertainty sets are usually too "small" to be of much interest, unless the number L of scenarios is impractically large. It is also worth noticing that the assumption of fixed recourse is essential: it is easy to show (see [14]) that without it, the ARC may become intractable.

Simple case II: uncertain LO with constraint-wise uncertainty

Consider an uncertain LO problem

$$\mathcal{P} = \left\{\min_x \left\{c_\zeta^T x + d_\zeta : a_{i\zeta}^T x \leq b_{i\zeta},\, i = 1, ..., m\right\} : \zeta \in \mathcal{Z}\right\}, \tag{5.2.9}$$

where, as always, $c_\zeta, d_\zeta, a_{i\zeta}, b_{i\zeta}$ are affine in ζ. Assume that

1. The uncertainty is constraint-wise: ζ can be split into blocks $\zeta = [\zeta^0; ...; \zeta^m]$ in such a way that the data of the objective depend solely on ζ^0, the data of the i-th constraint depend solely on ζ^i, and the uncertainty set $\mathcal{Z}$ is the direct product of convex compact sets $\mathcal{Z}_0, \mathcal{Z}_1, ..., \mathcal{Z}_m$ in the spaces of $\zeta^0, ..., \zeta^m$;

2. One can point out a convex compact set $\mathcal{X}$ in the space of x variables such that whenever $\zeta \in \mathcal{Z}$ and x is feasible for the instance of $\mathcal{P}$ with the data ζ, one has $x \in \mathcal{X}$.
   The validity of the latter, purely technical, assumption can be guaranteed, e.g., when the constraints of the uncertain problem contain (certain) finite upper and lower bounds on every one of the decision variables. The latter assumption, for all practical purposes, is non-restrictive.

Our goal is to prove the following

Theorem 5.2 *Under the just outlined assumptions i) and ii), the ARC of* (5.2.9) *is equivalent to its usual RC (no adjustable variables): both ARC and RC have equal optimal values.*

Proof. All we need is to prove that the optimal value in the ARC is $\geq$ the one of the RC. When achieving this goal, we can assume w.l.o.g. that all decision variables are *fully adjustable* — are allowed to depend on the entire vector ζ. The "fully adjustable" ARC of (5.2.9) reads

$$\begin{array}{rcl}\mathrm{Opt(ARC)} & = & \min\limits_{X(\cdot),t}\left\{t: \begin{array}{l} c_{\zeta^0}^T X(\zeta) + d_{\zeta^0} - t \leq 0 \\ a_{i\zeta^i}^T X(\zeta) - b_{i\zeta^i} \leq 0, 1 \leq i \leq m \end{array} \right. \\ & & \qquad\qquad\qquad \left. \forall(\zeta \in \mathcal{Z}_0 \times ... \times \mathcal{Z}_m)\right\} \\ & = & \inf\left\{t : \forall(\zeta \in \mathcal{Z}_0 \times ... \times \mathcal{Z}_m)\exists x \in \mathcal{X} : \right. \\ & & \qquad \left. \alpha_{i,\zeta^i}^T x - \beta_i t + \gamma_{i,\zeta^i} \leq 0,\ 0 \leq i \leq m\right\},\end{array} \tag{5.2.10}$$

(the restriction $x \in \mathcal{X}$ can be added due to assumption i)), while the RC is the problem

$$\mathrm{Opt(RC)} = \inf\left\{t : \exists x \in \mathcal{X} : \alpha_{i\zeta^i}^T x - \beta_i t + \gamma_{i\zeta^i} \leq 0 \forall(\zeta \in \mathcal{Z}_0 \times ... \times \mathcal{Z}_m)\right\}; \tag{5.2.11}$$

here $\alpha_{i\zeta^i}$, $\gamma_{i\zeta^i}$ are affine in ζ^i and $\beta_i \geq 0$.

In order to prove that Opt(ARC) $\geq$ Opt(RC), it suffices to consider the case when Opt(ARC) $< \infty$ and to show that whenever a real $\bar{t}$ is $>$ Opt(ARC), we have $\bar{t} \geq$ Opt(RC). Looking at (5.2.11), we see that to this end it suffices to lead to a contradiction the statement that for some $\bar{t} >$ Opt(ARC) one has

$$\forall x \in \mathcal{X} \exists (i = i_x \in \{0, 1, ..., m\}, \zeta^i = \zeta_x^{i_x} \in \mathcal{Z}_{i_x}) : \alpha_{i_x\zeta_x^{i_x}}^T x - \beta_{i_x}\bar{t} + \gamma_{i_x\zeta_x^{i_x}} > 0. \tag{5.2.12}$$

Assume that $\bar{t} >$ Opt(ARC) and that (5.2.12) holds. For every $x \in \mathcal{X}$, the inequality

$$\alpha_{i_x\zeta_x^{i_x}}^T y - \beta_{i_x}\bar{t} + \gamma_{i_x\zeta_x^{i_x}} > 0$$

is valid when $y = x$; therefore, for every $x \in \mathcal{X}$ there exist $\epsilon_x > 0$ and a neighborhood U_x of x such that

$$\forall y \in U_x : \alpha_{i_x\zeta_x^{i_x}}^T y - \beta_{i_x}\bar{t} + \gamma_{i_x\zeta_x^{i_x}} \geq \epsilon_x.$$

Since $\mathcal{X}$ is a compact set, we can find finitely many points $x^1, ..., x^N$ such that $\mathcal{X} \subset \bigcup\limits_{j=1}^{N} U_{x^j}$. Setting $\epsilon = \min_j \epsilon_{x^j}$, $i[j] = i_{x^j}$, $\zeta[j] = \zeta_{x^j}^{i_{x^j}} \in \mathcal{Z}_{i[j]}$, and

$$f_j(y) = \alpha_{i[j],\zeta[j]}^T y - \beta_{i[j]}\bar{t} + \gamma_{i[j],\zeta[j]},$$

we end up with N affine functions of y such that

$$\max_{1\leq j\leq N} f_j(y) \geq \epsilon > 0\ \forall y \in \mathcal{X}.$$

Since $\mathcal{X}$ is a convex compact set and $f_j(\cdot)$ are affine (and thus convex and continuous) functions, the latter relation, by well-known facts from Convex Analysis (namely, the von Niemann Lemma), implies that there exists a collection of nonnegative weights λ_j with $\sum_j \lambda_j = 1$ such that

$$f(y) \equiv \sum_{j=1}^{N} \lambda_j f_j(y) \geq \epsilon\, \forall y \in \mathcal{X}. \tag{5.2.13}$$

Now let

$$\begin{array}{rcl} \omega_i & = & \sum_{j:i[j]=i} \lambda_j,\ i = 0, 1, ..., m; \\ \bar{\zeta}^i & = & \left\{ \begin{array}{ll} \sum_{j:i[j]=i} \frac{\lambda_j}{\omega_i} \zeta[j], & \omega_i > 0 \\ \text{a point from } \mathcal{Z}_i, & \omega_i = 0 \end{array} \right., \\ \bar{\zeta} & = & [\bar{\zeta}^0; ...; \bar{\zeta}^m]. \end{array}$$

Due to its origin, every one of the vectors $\bar{\zeta}^i$ is a convex combination of points from $\mathcal{Z}_i$ and as such belongs to $\mathcal{Z}_i$, since the latter set is convex. Since the uncertainty is constraint-wise, we conclude that $\bar{\zeta} \in \mathcal{Z}$. Since $\bar{t} > \mathrm{Opt(ARC)}$, we conclude from (5.2.10) that there exists $\bar{x} \in \mathcal{X}$ such that the inequalities

$$\alpha_{i\bar{\zeta}^i}^T \bar{x} - \beta_i \bar{t} + \gamma_{i\bar{\zeta}^i} \leq 0$$

hold true for every i, $0 \leq i \leq m$. Taking a weighted sum of these inequalities, the weights being ω_i, we get

$$\sum_{i:\omega_i>0} \omega_i [\alpha_{i\bar{\zeta}^i}^T \bar{x} - \beta_i \bar{t} + \gamma_{i\bar{\zeta}^i}] \leq 0. \tag{5.2.14}$$

At the same time, by construction of $\bar{\zeta}^i$ and due to the fact that $\alpha_{i\zeta^i}$, $\gamma_{i\zeta^i}$ are affine in ζ^i, for every i with $\omega_i > 0$ we have

$$[\alpha_{i\bar{\zeta}^i}^T \bar{x} - \beta_i \bar{t} + \gamma_{i\bar{\zeta}^i}] = \sum_{j:i[j]=i} \frac{\lambda_j}{\omega_i} f_j(\bar{x}),$$

so that (5.2.14) reads

$$\sum_{j=1}^{N} \lambda_j f_j(\bar{x}) \leq 0,$$

which is impossible due to (5.2.13) and to $\bar{x} \in \mathcal{X}$. We have arrived at the desired contradiction. □

5.3 Affinely Adjustable Robust Counterparts

We are about to investigate in-depth a specific version of the "parametric decision rules" approach we have outlined previously. At this point, we prefer to come back from general-type uncertain problem (5.1.1) to affinely perturbed uncertain conic problem

$$\mathcal{C} = \left\{ \min_{x \in \mathbb{R}^n} \left\{ c_\zeta^T x + d_\zeta : A_\zeta x + b_\zeta \in \mathbf{K} \right\} : \zeta \in \mathcal{Z} \right\}, \tag{5.3.1}$$

where $c_\zeta, d_\zeta, A_\zeta, b_\zeta$ are affine in ζ, $\mathbf{K}$ is a "nice" cone (direct product of nonnegative rays/Lorentz cones/semidefinite cones, corresponding to uncertain LP/CQP/SDP, respectively), and $\mathcal{Z}$ is a convex compact uncertainty set given by a strictly feasible SDP representation

$$\mathcal{Z} = \left\{ \zeta \in \mathbb{R}^L : \exists u : \mathcal{P}(\zeta, u) \succeq 0 \right\},$$

where $\mathcal{P}$ is affine in $[\zeta; u]$. Assume that along with the problem, we are given an information base $\{P_j\}_{j=1}^n$ for it; here P_j are $m_j \times n$ matrices. To save words (and without risk of ambiguity), we shall call such a pair "uncertain problem $\mathcal{C}$, information base" merely an *uncertain conic* *problem.* Our course of action is to restrict the ARC of the problem to a specific parametric family of decision rules, namely, the *affine* ones:

$$x_j = X_j(P_j\zeta) = p_j + q_j^T P_j\zeta,\ j = 1, ..., n. \tag{5.3.2}$$

The resulting restricted version of the ARC of (5.3.1), which we call the *Affinely Adjustable Robust Counterpart* (AARC), is the semi-infinite optimization program

$$\min_{t,\{p_j,q_j\}_{j=1}^n} \left\{ t : \begin{array}{l} \sum_{j=1}^n c_\zeta^j [p_j + q_j^T P_j\zeta] + d_\zeta - t \le 0 \\ \sum_{j=1}^n A_\zeta^j [p_j + q_j^T P_j\zeta] + b_\zeta \in \mathbf{K} \end{array} \right\} \forall \zeta \in \mathcal{Z} \right\}, \tag{5.3.3}$$

where c_ζ^j is j-th entry in c_ζ, and A_ζ^j is j-th column of A_ζ. Note that the variables in this problem are t and the coefficients p_j, q_j of the affine decision rules (5.3.2). As such, these variables do *not* specify uniquely the actual decisions x_j; these decisions are uniquely defined by these coefficients *and* the corresponding portions $P_j\zeta$ of the true data once the latter become known.

5.3.1 Tractability of the AARC

The rationale for focusing on affine decision rules rather than on other parametric families is that *there exists at least one important case when the AARC of an uncertain conic problem is, essentially, as tractable as the RC of the problem.* The "important case" in question is the one of *fixed recourse* and is defined as follows:

Definition 5.1 *Consider an uncertain conic problem* (5.3.1) *augmented by an information base* $\{P_j\}_{j=1}^n$. *We say that this pair is with fixed recourse, if the coefficients of every adjustable, (i.e., with* $P_j \neq 0$*), variable* x_j *are certain:*

$$\forall (j : P_j \neq 0) : \text{ both } c_\zeta^j \text{ and } A_\zeta^j \text{ are independent of } \zeta.$$

For example, both Examples 5.1 (Inventory) and 5.2 (Project Management) are uncertain problems with fixed recourse.

An immediate observation is as follows:

> **(!)** *In the case of fixed recourse, the AARC, similarly to the RC, is a semi-infinite conic problem — it is the problem*
>
> $$\min_{t,y=\{p_j,q_j\}} \left\{ t : \begin{array}{l} \widehat{c}_\zeta^T y + d_\zeta \le t \\ \widehat{A}_\zeta y + b_\zeta \in \mathbf{K} \end{array} \right\} \forall \zeta \in \mathcal{Z} \right\}, \tag{5.3.4}$$
>
> *with* $\widehat{c}_\zeta, d_\zeta, \widehat{A}_\zeta, b_\zeta$ *affine in* ζ*:*
>
> $$\begin{array}{rcl} \widehat{c}_\zeta^T y &=& \sum_j c_\zeta^j [p_j + q_j^T P_j\zeta] \\ \widehat{A}_\zeta y &=& \sum_j A_\zeta^j [p_j + q_j^T P_j\zeta]. \end{array} \qquad [y = \{[p_j, q_j]\}_{j=1}^n]$$

Note that it is exactly fixed recourse that makes $\widehat{c}_\zeta, \widehat{A}_\zeta$ affine in ζ; without this assumption, these entities are quadratic in ζ.

As far as the tractability issues are concerned, observation (!) is *the* main argument in favor of affine decision rules, *provided we are in the situation of fixed recourse.* Indeed, in the latter situation the AARC is a semi-infinite conic problem, and we can apply to it all the results of previous lectures related to tractable reformulations/tight safe tractable approximations of semi-infinite conic problems. Note that many of these results, while imposing certain restrictions on the geometries of the uncertainty set and the cone **K**, require from the objective (if it is uncertain) and the left hand sides of the uncertain constraints nothing more than bi-affinity in the decision variables and in the uncertain data. *Whenever this is the case, the "tractability status" of the AARC is not worse than the one of the usual RC.* In particular, *in the case of fixed recourse* we can:

1. Convert the AARC of an uncertain LO problem into an explicit efficiently solvable "well-structured" convex program (see Theorem 1.1).

2. Process efficiently the AARC of an uncertain conic quadratic problem with (common to all uncertain constraints) simple ellipsoidal uncertainty (see section 3.2.5).

3. Use a tight safe tractable approximation of an uncertain problem with linear objective and convex quadratic constraints with (common for all uncertain constraints) $\cap$-ellipsoidal uncertainty (see section 3.3.2): whenever $\mathcal{Z}$ is the intersection of M ellipsoids centered at the origin, the problem admits a safe tractable approximation tight within the factor $O(1)\sqrt{\ln(M)}$ (see Theorem 3.11).

The reader should be aware, however, that the AARC, in contrast to the usual RC, is *not* a constraint-wise construction, since when passing to the coefficients of affine decision rules as our new decision variables, the portion of the uncertain data affecting a particular constraint can change when allowing the original decision variables entering the constraint to depend on the uncertain data not affecting the constraint directly. This is where the words "common" in the second and the third of the above statements comes from. For example, the RC of an uncertain conic quadratic problem with the constraints of the form

$$\|A^i_\zeta x + b^i_\zeta\|_2 \leq x^T c^i_\zeta + d^i_\zeta,\ i = 1, ..., m,$$

is computationally tractable, provided that the projection $\mathcal{Z}_i$ of the overall uncertainty set $\mathcal{Z}$ onto the subspace of data perturbations of i-th constraint is an ellipsoid (section 3.2.5). To get a similar result for the AARC, we need *the overall uncertainty set $\mathcal{Z}$ itself* to be an ellipsoid, since otherwise the projection of $\mathcal{Z}$ on the data of the "AARC counterparts" of original uncertain constraints can be different from ellipsoids. The bottom line is that the claim that with fixed recourse, the AARC of an uncertain problem is "as tractable" as its RC should be understood with some caution. This, however, is not a big deal, since the "recipe" is already here: *Under the assumption of fixed recourse, the AARC is a semi-infinite conic problem, and in order to process it computationally, we can use all the machinery developed in the previous lectures. If this machinery allows for tractable reformulation/tight safe tractable approximation of the problem, fine, otherwise too bad for us."* Recall that there exists at least one really important case when everything is fine — this is the case of uncertain LO problem with fixed recourse.

It should be added that when processing the AARC in the case of fixed recourse, we can enjoy all the results on safe tractable approximations of chance constrained affinely perturbed scalar, conic quadratic and linear matrix inequalities developed in previous lectures. Recall that these results imposed certain restrictions on the distribution of ζ (like independence of $\zeta_1, ..., \zeta_L$), but never required more than affinity of the bodies of the constraints w.r.t. ζ, so that these results work equally well in the cases of RC and AARC.

Last, but not least, the concept of an Affinely Adjustable Robust Counterpart can be straightforwardly "upgraded" to the one of Affinely Adjustable *Globalized* Robust Counterpart. We have no doubts that a reader can carry out such an "upgrade" on his/her own and understands that in the case of fixed recourse, the above "recipe" is equally applicable to the AARC and the AAGRC.

5.3.2 Is Affinity an Actual Restriction?

Passing from *arbitrary* decision rules to *affine* ones seems to be a dramatic simplification. On a closer inspection, the simplification is not as severe as it looks, or, better said, the "dramatics" is not exactly where it is seen at first glance. Indeed, assume that we would like to use decision rules that are quadratic in $P_j\zeta$ rather than linear. Are we supposed to introduce a special notion of a "Quadratically Adjustable Robust Counterpart"? The answer is negative. All we need is to augment the data vector $\zeta = [\zeta_1; ...; \zeta_L]$ by extra entries — the pairwise products $\zeta_i\zeta_j$ of the original entries — and to treat the resulting "extended" vector $\widehat{\zeta} = \widehat{\zeta}[\zeta]$ as our new uncertain data. With this, the decision rules that are quadratic in $P_j\zeta$ become *affine* in $\widehat{P}_j\widehat{\zeta}[\zeta]$, where $\widehat{P}_j$ is a matrix readily given by P_j. More generally, assume that we want to use decision rules of the form

$$X_j(\zeta) = p_j + q_j^T \widehat{P}_j \widehat{\zeta}[\zeta], \tag{5.3.5}$$

where $p_j \in \mathbb{R}, q_j \in \mathbb{R}^{m_j}$ are "free parameters," (which can be restricted to reside in a given convex set), $\widehat{P}_j$ are given $m_j \times D$ matrices and

$$\zeta \mapsto \widehat{\zeta}[\zeta] : \mathbb{R}^L \to \mathbb{R}^D$$

is a given, possibly nonlinear, mapping. Here again we can pass from the original data vector ζ to the data vector $\widehat{\zeta}[\zeta]$, thus making the desired decision rules (5.3.5) merely affine in the "portions" $\widehat{P}_j\widehat{\zeta}$ of the new data vector. We see that when allowing for a seemingly harmless redefinition of the data vector, affine decision rules become as powerful as arbitrary affinely parameterized parametric families of decision rules. This latter class is really huge and, for all practical purposes, is as rich as the class of *all* decision rules. Does it mean that the concept of AARC is basically as flexible as the one of ARC? Unfortunately, the answer is negative, and the reason for the negative answer comes not from potential difficulties with extremely complicated nonlinear transformations $\zeta \mapsto \widehat{\zeta}[\zeta]$ and/or "astronomically large" dimension D of the transformed data vector. The difficulty arises already when the transformation is pretty simple, as is the case, e.g., when the coordinates in $\widehat{\zeta}[\zeta]$ are just the entries of ζ and the pairwise products of these entries. Here is where the difficulty arises. Assume that we are speaking about a single uncertain affinely perturbed scalar linear constraint, allow for quadratic dependence of the original decision variables on the data and pass to the associated adjustable robust counterpart of the constraint. As it was just explained, this counterpart is nothing but a semi-infinite scalar inequality

$$\forall (\widehat{\zeta} \in \mathcal{U}) : a_{0,\widehat{\zeta}} + \sum_{j=1}^{J} a_{j,\widehat{\zeta}} y_j \leq 0$$

where $a_{j,\widehat{\zeta}}$ are affine in $\widehat{\zeta}$, the entries in $\widehat{\zeta} = \widehat{\zeta}[\zeta]$ are the entries in ζ and their pairwise products, $\mathcal{U}$ is the image of the "true" uncertainty set $\mathcal{Z}$ under the *nonlinear* mapping $\zeta \to \widehat{\zeta}[\zeta]$, and y_j are our new decision variables (the coefficients of the quadratic decision rules). While the body of the constraint in question is bi-affine in y and in $\widehat{\zeta}$, this semi-infinite constraint can well be intractable, since the *uncertainty set $\mathcal{U}$ may happen to be intractable, even when $\mathcal{Z}$ is tractable.* Indeed, the tractability of a semi-infinite bi-affine scalar constraint

$$\forall (u \in \mathcal{U}) : f(y,u) \leq 0$$

heavily depends on whether the underlying uncertainty set $\mathcal{U}$ is convex and computationally tractable. When it is the case, we can, modulo minor technical assumptions, solve efficiently the *Analysis problem* of checking whether a given candidate solution y is feasible for the constraint — to this end, it suffices to maximize the affine function $f(y,\cdot)$ over the computationally tractable convex set $\mathcal{U}$. This, under minor technical assumptions, can be done efficiently. The latter fact, in turn, implies (again modulo minor technical assumptions) that we can optimize efficiently linear/convex objectives under the constraints with the above features, and this is basically all we need. The situation changes dramatically when the uncertainty set $\mathcal{U}$ is *not* a convex computationally tractable set. By itself, the convexity of $\mathcal{U}$ costs nothing: since f is bi-affine, the feasible set of the semi-infinite constraint in question remains intact when we replace $\mathcal{U}$ with its convex hull $\widehat{\mathcal{Z}}$. The actual difficulty is that the convex hull $\widehat{\mathcal{Z}}$ of the set $\mathcal{U}$ can be computationally intractable. In the situation we are interested in — the one where $\widehat{\mathcal{Z}} = \mathrm{Conv}\mathcal{U}$ and $\mathcal{U}$ is the image of a computationally tractable convex set $\mathcal{Z}$ under a *nonlinear* transformation $\zeta \mapsto \widehat{\zeta}[\zeta]$, $\widehat{\mathcal{Z}}$ can be computationally intractable already for pretty simple $\mathcal{Z}$ and nonlinear mappings $\zeta \mapsto \widehat{\zeta}[\zeta]$. It happens, e.g., when $\mathcal{Z}$ is the unit box $\|\zeta\|_\infty \leq 1$ and $\widehat{\zeta}[\zeta]$ is comprised of the entries in ζ and their pairwise products. In other words, the "Quadratically Adjustable Robust Counterpart" of an uncertain linear inequality with interval uncertainty is, in general, computationally intractable.

In spite of the just explained fact that "global lumbarization" of nonlinear decision rules via nonlinear transformation of the data vector not necessarily leads to tractable adjustable RCs, one should keep in mind this option, since it is important methodologically. Indeed, "global lumbarization" allows one to "split" the problem of processing the ARC, restricted to decision rules (5.3.5), into two subproblems:

(a) Building a tractable representation (or a tight tractable approximation) of the convex hull $\widehat{\mathcal{Z}}$ of the image $\mathcal{U}$ of the original uncertainty set $\mathcal{Z}$ under the nonlinear mapping $\zeta \mapsto \widehat{\zeta}[\zeta]$ associated with (5.3.5). Note that this problem by itself has nothing to do with adjustable robust counterparts and the like;

(b) Developing a tractable reformulation (or a tight safe tractable approximation) of the *Affinely Adjustable* Robust Counterpart of the uncertain problem in question, with $\widehat{\zeta}$ in the role of the data vector, the tractable convex set, yielded by (a), in the role of the uncertainty set, and the information base given by the matrices $\widehat{P}_j$.

Of course, the resulting two problems are not completely independent: the tractable convex set $\widehat{\mathcal{Z}}$ with which we, upon success, end up when solving (a) should be simple enough to allow for successful processing of (b). Note, however, that this "coupling of problems (a) and (b)" is of no importance when the uncertain problem in question is an LO problem with fixed recourse. Indeed, in this case the AARC of the problem is computationally tractable whatever the uncertainty set as long as it is tractable, therefore every tractable set $\widehat{\mathcal{Z}}$ yielded by processing of problem (a) will do.

Example 5.5 Assume that we want to process an uncertain LO problem

$$\begin{array}{c}\mathcal{C} = \left\{ \min_x \left\{ c_\zeta^T x + d_\zeta : A_\zeta x \geq b_\zeta \right\} : \zeta \in \mathcal{Z} \right\} \\ {[c_\zeta, d_\zeta, A_\zeta, b_\zeta : \text{ affine in } \zeta]}\end{array} \tag{5.3.6}$$

with fixed recourse and a tractable convex compact uncertainty set $\mathcal{Z}$, and consider a number of affinely parameterized families of decision rules.

A. *"Genuine" affine decision rules: x_j is affine in $P_j\zeta$.* As we have already seen, the associated ARC — the usual AARC of $\mathcal{C}$ — is computationally tractable.
B. *Piece-wise linear decision rules with fixed breakpoints.* Assume that the mapping $\zeta \mapsto \widehat{\zeta}[\zeta]$ augments the entries of ζ with finitely many entries of the form $\phi_i(\zeta) = \max\left[r_i, s_i^T\zeta\right]$, and the decision rules we intend to use should be affine in $\widehat{P}_j\widehat{\zeta}$, where $\widehat{P}_j$ are given matrices. In order to process the associated ARC in a computationally efficient fashion, all we need is to build a tractable representation of the set $\widehat{\mathcal{Z}} = \mathrm{Conv}\{\widehat{\zeta}[\zeta] : \zeta \in \mathcal{Z}\}$. While this could be difficult in general, there are useful cases when the problem is easy, e.g., the case where

$$\begin{array}{c}\mathcal{Z} = \{\zeta \in \mathbb{R}^L : f_k(\zeta) \leq 1, 1 \leq k \leq K\},\\ \widehat{\zeta}[\zeta] = [\zeta; (\zeta)_+; (\zeta)_-], \text{ with } (\zeta)_- = \max[\zeta, 0_{L\times 1}], (\zeta)_+ = \max[-\zeta, 0_{L\times 1}].\end{array}$$

Here, for vectors u, v, $\max[u, v]$ is taken coordinate-wise, and $f_k(\cdot)$ are lower semicontinuous and *absolutely symmetric* convex functions on $\mathbb{R}^L$, absolute symmetry meaning that $f_k(\zeta) \equiv f_k(\mathrm{abs}(\zeta))$ (abs acts coordinate-wise). (Think about the case when $f_k(\zeta) = \|[\alpha_{k1}\zeta_1; ...; \alpha_{kL}\zeta_L]\|_{p_k}$ with $p_k \in [1, \infty]$.) It is easily seen that if $\mathcal{Z}$ is bounded, then

$$\widehat{\mathcal{Z}} = \left\{\widehat{\zeta} = [\zeta; \zeta^+; \zeta^-] : \begin{array}{ll}(a) & f_k(\zeta^+ + \zeta^-) \leq 1,\ 1 \leq k \leq K\\ (b) & \zeta = \zeta^+ - \zeta^-\\ (c) & \zeta^{\pm} \geq 0\end{array}\right\}.$$

Indeed, (a) through (c) is a system of convex constraints on vector $\widehat{\zeta} = [\zeta; \zeta^+; \zeta^-]$, and since f_k are lower semicontinuous, the feasible set C of this system is convex and closed; besides, for $[\zeta; \zeta^+; \zeta^-] \in C$ we have $\zeta^+ + \zeta^- \in \mathcal{Z}$; since the latter set is bounded by assumption, the sum $\zeta^+ + \zeta^-$ is bounded uniformly in $\widehat{\zeta} \in C$, whence, by (a) through (c), C is bounded. Thus, C is a closed and bounded convex set. The image $\mathcal{U}$ of the set $\mathcal{Z}$ under the mapping $\zeta \mapsto [\zeta; (\zeta)_+; (\zeta)_-]$ clearly is contained in C, so that the convex hull $\widehat{\mathcal{Z}}$ of $\mathcal{U}$ is contained in C as well. To prove the inverse inclusion, note that since C is a (nonempty) convex compact set, it is the convex hull of the set of its extreme points, and therefore in order to prove that $\widehat{\mathcal{Z}} \supset C$ it suffices to verify that every extreme point $[\zeta; \zeta^+, \zeta^-]$ of C belongs to $\mathcal{U}$. But this is immediate: in an extreme point of C we should have $\min[\zeta_\ell^+, \zeta_\ell^-] = 0$ for every ℓ, since if the opposite were true for some $\ell = \bar{\ell}$, then C would contain a nontrivial segment centered at the point, namely, points obtained from the given one by the "3-entry perturbation" $\zeta_{\bar{\ell}}^+ \mapsto \zeta_{\bar{\ell}}^+ + \delta$, $\zeta_{\bar{\ell}}^- \mapsto \zeta_{\bar{\ell}}^+ - \delta$, $\zeta_{\bar{\ell}} \mapsto \zeta_{\bar{\ell}} + 2\delta$ with small enough $|\delta|$. Thus, every extreme point of C has $\min[\zeta^+, \zeta^-] = 0$, $\zeta = \zeta^+ - \zeta^-$, and a point of this type satisfying (a) clearly belongs to $\mathcal{U}$. □

C. *Separable decision rules.* Assume that $\mathcal{Z}$ is a box: $\mathcal{Z} = \{\zeta : \underline{a} \leq \zeta \leq \overline{a}\}$, and we are seeking for *separable* decision rules with a prescribed "information base," that is, for the decision rules of the form

$$x_j = \xi_j + \sum_{\ell \in I_j} f_\ell^j(\zeta_\ell),\ j = 1, ..., n, \tag{5.3.7}$$

where the only restriction on functions f_ℓ^j is to belong to given finite-dimensional linear spaces $\mathcal{F}_\ell$ of univariate functions. The sets I_j specify the information base of our decision rules. Some of these sets may be empty, meaning that the associated x_j are non-adjustable decision variables, in full accordance with the standard convention that a sum over an empty set of indices is 0. We consider two specific choices of the spaces $\mathcal{F}_\ell$:

C.1: $\mathcal{F}_\ell$ is comprised of all piecewise linear functions on the real axis with fixed breakpoints $a_{\ell 1} < ... < a_{\ell m}$ (w.l.o.g., assume that $\underline{a}_\ell < a_{\ell 1}$, $a_{\ell m} < \overline{a}_\ell$);

C.2: $\mathcal{F}_\ell$ is comprised of all algebraic polynomials on the axis of degree $\leq \kappa$.

Note that what follows works when m in **C.1** and κ in **C.2** depend on ℓ; in order to simplify notation, we do not consider this case explicitly.

C.1: Let us augment every entry ζ_ℓ of ζ with the reals $\zeta_{\ell i}[\zeta_\ell] = \max[\zeta_\ell, a_{\ell i}]$, $i = 1, ..., m$, and let us set $\zeta_{\ell 0}[\zeta_\ell] = \zeta_\ell$. In the case of **C.1**, decision rules (5.3.7) are exactly the rules where x_j is affine in $\{\zeta_{\ell i}[\zeta] : \ell \in I_j\}$; thus, all we need in order to process efficiently the ARC of (5.3.6) restricted to the decision rules in question is a tractable representation of the convex hull of the image $\mathcal{U}$ of $\mathcal{Z}$ under the mapping $\zeta \mapsto \{\zeta_{\ell i}[\zeta]\}_{\ell,i}$. Due to the direct product structure of $\mathcal{Z}$, the set $\mathcal{U}$ is the direct product, over $\ell = 1, ..., d$, of the sets

$$\mathcal{U}_\ell = \{[\zeta_{\ell 0}[\zeta_\ell]; \zeta_{\ell 1}[\zeta_\ell]; ...; \zeta_{\ell m}[\zeta_\ell]] : \underline{a}_\ell \leq \zeta_\ell \leq \overline{a}_\ell\},$$

so that all we need are tractable representations of the convex hulls of the sets $\mathcal{U}_\ell$. The bottom line is, that all we need is a tractable description of a set C of the form

$$C_m = \mathrm{Conv} S_m,\ S_m = \{[s_0; \max[s_0, a_1]; ...; \max[s_0, a_m]] : a_0 \leq s_0 \leq a_{m+1}\},$$

where $a_0 < a_1 < a_2 < ... < a_m < a_{m+1}$ are given. An explicit polyhedral description of the set C_m is given by the following

Lemma 5.1 [3, Lemma 14.3.3] *The convex hull C_m of the set S_m is*

$$C_m = \left\{[s_0; s_1; ...; s_m] : \left\{ \begin{array}{l} a_0 \leq s_0 \leq a_{m+1} \\ 0 \leq \frac{s_1 - s_0}{a_1 - a_0} \leq \frac{s_2 - s_1}{a_2 - a_1} \leq ... \leq \frac{s_{m+1} - s_m}{a_{m+1} - a_m} \leq 1 \end{array} \right. \right\}, \tag{5.3.8}$$

where $s_{m+1} = a_{m+1}$.

C.2: Similar to the case of **C.1**, in the case of **C.2** all we need in order to process efficiently the ARC of (5.3.6), restricted to decision rules (5.3.7), is a tractable representation of the set

$$C = \mathrm{Conv} S,\ S = \{\widehat{s} = [s; s^2; ...; s^\kappa] : |s| \leq 1\}.$$

(We have assumed w.l.o.g. that $\underline{a}_\ell = -1$, $\overline{a}_\ell = 1$.) Here is the description (originating from [75]):

Lemma 5.2 *The set $C = \mathrm{Conv} S$ admits the explicit semidefinite representation*

$$C = \left\{\widehat{s} \in \mathbb{R}^\kappa : \exists \lambda = [\lambda_0; ...; \lambda_{2\kappa}] \in \mathbb{R}^{2\kappa+1} : [1; \widehat{s}] = Q^T \lambda, [\lambda_{i+j}]_{i,j=0}^{\kappa} \succeq 0\right\}, \tag{5.3.9}$$

where the $(2\kappa + 1) \times (\kappa + 1)$ matrix Q is defined as follows: take a polynomial $p(t) = p_0 + p_1 t + ... + p_\kappa t^\kappa$ and convert it into the polynomial $\widehat{p}(t) = (1 + t^2)^\kappa p(2t/(1 + t^2))$. The vector of coefficients of $\widehat{p}$ clearly depends linearly on the vector of coefficients of p, and Q is exactly the matrix of this linear transformation.

Proof. 1^0. Let $P \subset \mathbb{R}^{\kappa+1}$ be the cone of vectors p of coefficients of polynomials $p(t) = p_0 + p_1 t + p_2 t^2 + ... + p_\kappa t^\kappa$ that are nonnegative on $[-1, 1]$, and P_* be the cone dual to P. We claim that

$$C = \{\widehat{s} \in \mathbb{R}^\kappa : [1; \widehat{s}] \in P_*\}. \tag{5.3.10}$$

Indeed, let C' be the right hand side set in (5.3.10). If $\widehat{s} = [s; s^2; ...; s^\kappa] \in S$, then $|s| \leq 1$, so that for every $p \in P$ we have $p^T[1; \widehat{s}] = p(s) \geq 0$. Thus, $[1; \widehat{s}] \in P_*$ and therefore $\widehat{s} \in C'$. Since C' is convex, we arrive at $C \equiv \mathrm{Conv} S \subset C'$. To prove the inverse inclusion, assume that there exists $\widehat{s} \notin C$ such that $z = [1; \widehat{s}] \in P_*$, and let us lead this assumption to a contradiction. Since $\widehat{s}$ is not in C and C is a closed convex set and clearly contains the origin, we can find a vector $q \in \mathbb{R}^\kappa$ such that $q^T \widehat{s} = 1$ and $\max_{r \in C} q^T r \equiv \alpha < 1$, or, which is the same due to $C = \mathrm{Conv} S$, $q^T[s; s^2; ...; s^\kappa] \leq \alpha < 1$ whenever $|s| \leq 1$. Setting $p = [\alpha; -q]$, we see that $p(s) \geq 0$ whenever $|s| \leq 1$, so that $p \in P$ and therefore $\alpha - q^T\widehat{s} = p^T[1;\widehat{s}] \geq 0$, whence $1 = q^T\widehat{s} \leq \alpha < 1$, which is a desired contradiction.

2^0. It remains to verify that the right hand side in (5.3.10) indeed admits representation (5.3.9). We start by deriving a semidefinite representation of the cone P_+ of (vectors of coefficients of) all polynomials $p(s)$ of degree not exceeding 2κ that are nonnegative on the entire axis. The representation is as follows. A $(\kappa+1)\times(\kappa+1)$ symmetric matrix W can be associated with the polynomial of degree $\leq 2\kappa$ given by $p_W(t) = [1;t;t^2;...;t^\kappa]^T W[1;t;t^2;...;t^\kappa]$, and the mapping $\mathcal{A} : W \mapsto p_W$ clearly is linear: $\left(\mathcal{A}[w_{ij}]_{i,j=0}^{\kappa}\right)_\nu = \sum_{0\leq i\leq\nu} w_{i,\nu-i}$, $0 \leq \nu \leq 2\kappa$. A dyadic matrix $W = ee^T$ "produces" in this way a polynomial that is the square of another polynomial: $\mathcal{A}ee^T = e^2(t)$ and as such is ≥ 0 on the entire axis. Since every matrix $W \succeq 0$ is a sum of dyadic matrices, we conclude that $\mathcal{A}W \in P_+$ whenever $W \succeq 0$. Vice versa, it is well known that every polynomial $p \in P_+$ is the sum of squares of polynomials of degrees $\leq \kappa$, meaning that every $p \in P_+$ is $\mathcal{A}W$ for certain W that is the sum of dyadic matrices and as such is $\succeq 0$. Thus,

$$P_+ = \{p = \mathcal{A}W : W \in \mathbf{S}_+^{\kappa+1}\}.$$

Now, the mapping $t \mapsto 2t/(1+t^2) : \mathbb{R} \to \mathbb{R}$ maps $\mathbb{R}$ onto the segment $[-1,1]$. It follows that *a polynomial p of degree $\leq \kappa$ is ≥ 0 on $[-1,1]$ if and only if the polynomial $\widehat{p}(t) = (1+t^2)^\kappa p(2t/(1+t^2))$ of degree $\leq 2\kappa$ is ≥ 0 on the entire axis*, or, which is the same, $p \in P$ if and only if $Qp \in P_+$. Thus,

$$P = \{p \in \mathbb{R}^{\kappa+1} : \exists W \in \mathbf{S}^{\kappa+1} : W \succeq 0, \mathcal{A}W = Qp\}.$$

Given this semidefinite representation of P, we can immediately obtain a semidefinite representation of P_*. Indeed,

$$\begin{array}{l} q \in P_* \Leftrightarrow 0 \leq \min_{p\in P}\{q^Tp\} \Leftrightarrow 0 \leq \min_{p\in\mathbb{R}^\kappa}\{q^Tp : \exists W \succeq 0 : Qp = \mathcal{A}W\} \\ \Leftrightarrow 0 \leq \min_{p,W}\{q^Tp : Qp - \mathcal{A}W = 0, W \succeq 0\} \\ \Leftrightarrow \{q = Q^T\lambda : \lambda \in \mathbb{R}^{2\kappa+1}, \mathcal{A}^*\lambda \succeq 0\}, \end{array}$$

where the concluding $\Leftrightarrow$ is due to semidefinite duality. Computing $\mathcal{A}^*\lambda$, we arrive at (5.3.9). □

Remark 5.1 *Note that* **C.2** *admits a straightforward modification where the spaces $\mathcal{F}_\ell$ are comprised of trigonometric polynomials $\sum\limits_{i=0}^{\kappa}[p_i\cos(i\omega_\ell s) + q_i\sin(i\omega_\ell s)]$ rather than of algebraic polynomials $\sum_{i=0}^{\kappa} p_i s^i$. Here all we need is a tractable description of the convex hull of the curve*

$$\{[s; \cos(\omega_\ell s); \sin(\omega_\ell s); ...; \cos(\kappa\omega_\ell s); \sin(\kappa\omega_\ell s)] : -1 \leq s \leq 1\}$$

which can be easily extracted from the semidefinite representation of the cone P_+.

Discussion. There are items to note on the results stated in **C**. The bad news is that *understood literally, these results have no direct consequences in our context — when $\mathcal{Z}$ is a box, decision rules* (5.3.7) *never outperform "genuine" affine decision rules with the same information base* (that is, the decision rules (5.3.7) with the spaces of affine functions on the axis in the role of $\mathcal{F}_\ell$).

The explanation is as follows. Consider, instead of (5.3.6), a more general problem, specifically, the uncertain problem

$$\mathcal{C} = \left\{\min_x \left\{c_\zeta^T x + d_\zeta : A_\zeta x - b_\zeta \in \mathbf{K}\right\} : \zeta \in \mathcal{Z}\right\} \atop [c_\zeta, d_\zeta, A_\zeta, b_\zeta : \text{ affine in } \zeta] \tag{5.3.11}$$

where $\mathbf{K}$ is a convex set. Assume that $\mathcal{Z}$ is a direct product of simplexes: $\mathcal{Z} = \Delta_1 \times ... \times \Delta_L$, where Δ_ℓ is a k_ℓ-dimensional simplex (the convex hull of $k_\ell + 1$ affinely independent points in $\mathbb{R}^{k_\ell}$). Assume we want to process the ARC of this problem restricted to the decision rules of the form

$$x_j = \xi_j + \sum_{\ell\in I_j} f_\ell^j(\zeta_\ell), \tag{5.3.12}$$

where ζ_ℓ is the projection of $\zeta \in \mathcal{Z}$ on Δ_ℓ, and the only restriction on the functions f_ℓ^j is that they belong to given families $\mathcal{F}_\ell$ of functions on $\mathbb{R}^{k_\ell}$. We still assume fixed recourse: the columns of A_ζ and the entries in c_ζ associated with adjustable, (i.e., with $I_j \neq \emptyset$) decision variables x_j are independent of ζ.

The above claim that "genuinely affine" decision rules are not inferior as compared to the rules (5.3.7) is nothing but the following simple observation:

Lemma 5.3 *Whenever certain $t \in \mathbb{R}$ is an achievable value of the objective in the ARC of* (5.3.11) *restricted to the decision rules* (5.3.12)*, that is, there exist decision rules of the latter form such that*

$$\left.\begin{array}{l}\sum\limits_{j=1}^n \left[\xi_j + \sum\limits_{\ell\in I_j} f_\ell^j(\zeta_\ell)\right](c_\zeta)_j + d_\zeta \leq t \\ \sum\limits_{j=1}^n \left[\xi_j + \sum\limits_{\ell\in I_j} f_\ell^j(\zeta_\ell)\right]A_\zeta^j - b_\zeta \in \mathbf{K}\end{array}\right\} \quad \begin{array}{rl}\forall\zeta \in & [\zeta_1;...;\zeta_L] \in \mathcal{Z} \\ = & \Delta_1\times...\times\Delta_L,\end{array} \tag{5.3.13}$$

t is also an achievable value of the objective in the ARC of the uncertain problem restricted to affine decision rules with the same information base: there exist affine in ζ_ℓ functions $\phi_\ell^j(\zeta_\ell)$ such that (5.3.13) *remains valid with ϕ_ℓ^j in the role of f_ℓ^j.*

Proof is immediate: since every collection of $k_\ell + 1$ reals can be obtained as the collection of values of an affine function at the vertices of k_ℓ-dimensional simplex, we can find affine functions $\phi_\ell^j(\zeta_\ell)$ such that $\phi_\ell^j(\zeta_\ell) = f_\ell^j(\zeta_\ell)$ whenever ζ_ℓ is a vertex of the simplex Δ_ℓ. When plugging into the left hand sides of the constraints in (5.3.13) the functions $\phi_\ell^j(\zeta_\ell)$ instead of $f_\ell^j(\zeta_\ell)$, these left hand sides become affine functions of ζ (recall that we are in the case of fixed recourse). Due to this affinity and to the fact that $\mathcal{Z}$ is a convex compact set, in order for the resulting constraints to be valid for all $\zeta \in \mathcal{Z}$, it suffices for them to be valid at every one of the extreme points of $\mathcal{Z}$. The components $\zeta_1, ..., \zeta_L$ of such an extreme point ζ are vertices of $\Delta_1, ..., \Delta_L$, and therefore the validity of "ϕ constraints" at ζ is readily given by the validity of the "f constraints" at this point — by construction, at such a point the left hand sides of the "ϕ" and the "f" constraints coincide with each other. □

Does the bad news mean that our effort in **C.1–2** was just wasted? The good news is that this effort still can be utilized. Consider again the case where ζ_ℓ are scalars, assume that $\mathcal{Z}$ is not a box, in which case Lemma 5.3 is not applicable. Thus, we have hope that the ARC of (5.3.6) restricted to the decision rules (5.3.7) is indeed less conservative (has a strictly less optimal value) than the ARC restricted to the affine decision rules. What we need in order to process the former, "more promising," ARC, is a tractable description of the convex hull $\widehat{\mathcal{Z}}$ of the image $\mathcal{U}$ of $\mathcal{Z}$ under the mapping

$$\zeta \mapsto \widehat{\zeta}[\zeta] = \{\zeta_{\ell i}[\zeta_\ell]\}_{\substack{0\leq i\leq m,\\ 1\leq\ell\leq L}}$$

where $\zeta_{\ell 0} = \zeta_\ell$, $\zeta_{\ell i}[\zeta_\ell] = f_{i\ell}(\zeta_\ell)$, $1 \leq i \leq m$, and the functions $f_{i\ell} \in \mathcal{F}_\ell$, $i = 1, ..., m$, span $\mathcal{F}_\ell$. The difficulty is that *with $\mathcal{F}_\ell$ as those considered in* **C.1–2** (these families are "rich enough" for most of applications), *we, as a matter of fact, do not know how to get a tractable representation of $\widehat{\mathcal{Z}}$, unless $\mathcal{Z}$ is a box.* Thus, $\mathcal{Z}$ more complicated than a box seems to be too complex, and when $\mathcal{Z}$ is a box, we gain nothing from allowing for "complex" $\mathcal{F}_\ell$. Nevertheless, we can proceed as follows. Let us include $\mathcal{Z}$, (which is not a box), into a box $\mathcal{Z}^+$, and let us apply the outlined approach to $\mathcal{Z}^+$ in the role of $\mathcal{Z}$, that is, let us try to build a tractable description of the convex hull $\widehat{\mathcal{Z}}^+$ of the image $\mathcal{U}^+$ of $\mathcal{Z}^+$ under the mapping $\zeta \mapsto \widehat{\zeta}[\zeta]$. With luck, (e.g., in situations **C.1–2**), we will succeed, thus getting a tractable representation of $\widehat{\mathcal{Z}}^+$; the latter set is, of course, larger than the "true" set $\widehat{\mathcal{Z}}$ we want to describe. There is another "easy to describe" set that contains $\widehat{\mathcal{Z}}$, namely, the inverse image $\widehat{\mathcal{Z}}^0$ of $\mathcal{Z}$ under the natural projection $\Pi : \{\zeta_{\ell i}\}_{\substack{0\le i\le m,\\ 1\le \ell\le L}} \mapsto \{\zeta_{\ell 0}\}_{1\le\ell\le L}$ that recovers ζ from $\widehat{\zeta}[\zeta]$. And perhaps we are smart enough to find other easy to describe convex sets $\widehat{\mathcal{Z}}^1$,...,$\widehat{\mathcal{Z}}^k$ that contain $\widehat{\mathcal{Z}}$.

Assume, e.g., that $\mathcal{Z}$ is the Euclidean ball $\{\|\zeta\|_2 \le r\}$, and let us take as $\mathcal{Z}^+$ the embedding box $\{\|\zeta\|_\infty \le r\}$.

In the case of **C.1** we have for $i \ge 1$: $\zeta_{\ell i}[\zeta_\ell] = \max[\zeta_\ell, a_{\ell i}]$, whence $|\zeta_{\ell i}[\zeta_\ell]| \le \max[|\zeta_\ell|, |a_{\ell i}|]$. It follows that when $\zeta \in \mathcal{Z}$, we have $\sum_\ell \zeta_{\ell i}^2[\zeta_\ell] \le \sum_\ell \max[\zeta_\ell^2, a_{\ell i}^2] \le \sum_\ell [\zeta_\ell^2 + a_{\ell i}^2] \le r^2 + \sum_\ell a_{\ell i}^2$, and we can take as $\widehat{\mathcal{Z}}^p$, $p = 1, ..., m$, the elliptic cylinders $\{\{\zeta_{\ell i}\}_{\ell,i} : \sum_\ell \zeta_{\ell p}^2 \le r^2 + \sum_\ell a_{\ell p}^2\}$. In the case of **C.2**, we have $\zeta_{\ell i}[\zeta_\ell] = \zeta_\ell^{i+1}$, $1 \le i \le \kappa - 1$, so that $\sum_\ell |\zeta_{\ell i}[\zeta_\ell]| \le \max_{z\in\mathbb{R}^L}\{\sum_\ell |z_\ell|^{i+1} : \|z\|_2 \le r\} = r^{i+1}$. Thus, we can take $\widehat{\mathcal{Z}}^p = \{\{\zeta_{\ell i}\}_{\ell,i} : \sum_\ell |\zeta_{\ell p}| \le r^{p+1}\}$, $1 \le p \le \kappa - 1$.

Since all the easy to describe convex sets $\widehat{\mathcal{Z}}^+$, $\widehat{\mathcal{Z}}^0$,...,$\widehat{\mathcal{Z}}^k$ contain $\widehat{\mathcal{Z}}$, the same is true for the easy to describe convex set

$$\widetilde{\mathcal{Z}} = \widehat{\mathcal{Z}}^+ \cap \widehat{\mathcal{Z}}^0 \cap \widehat{\mathcal{Z}}^1 \cap ... \cap \widehat{\mathcal{Z}}^k,$$

so that the (tractable along with $\widetilde{\mathcal{Z}}$) semi-infinite LO problem

$$\min_{\substack{t,\\ \{X_j(\cdot)\in\mathcal{X}_j\}_{j=1}^n}} \left\{ t : \left. \begin{array}{l} d_{\Pi(\widehat{\zeta})} + \sum\limits_{j=1}^n X_j(\widehat{\zeta})(c_{\Pi(\widehat{\zeta})})_j \le t \\ \sum\limits_{j=1}^n X_j(\widehat{\zeta})A^j_{\Pi(\widehat{\zeta})} - b_{\Pi(\widehat{\zeta})} \ge 0 \end{array} \right\} \forall \widehat{\zeta} = \{\zeta_{\ell i}\} \in \widetilde{\mathcal{Z}} \right\} \tag{S}$$
$$\left[\Pi\left(\{\zeta_{\ell i}\}_{\substack{0\le i\le m,\\ 1\le\ell\le L}}\right) = \{\zeta_{\ell 0}\}_{1\le\ell\le L}, \mathcal{X}_j = \{X_j(\widehat{\zeta}) = \xi_j + \sum_{\substack{\ell\in I_j,\\ 0\le i\le m}} \eta_{\ell i}\zeta_{\ell i}\} \right]$$

is a safe tractable approximation of the ARC of (5.3.6) restricted to decision rules (5.3.7). Note that this approximation is *at least* as flexible as the ARC of (5.3.6) restricted to genuine affine decision rules. Indeed, a rule $X(\cdot) = \{X_j(\cdot)\}_{j=1}^n$ of the latter type is "cut off" the family of all decision rules participating in (S) by the requirement "X_j depend solely on $\zeta_{\ell 0}$, $\ell \in I_j$," or, which is the same, by the requirement $\eta_{\ell i} = 0$ whenever $i > 0$. Since by construction the projection of $\widetilde{\mathcal{Z}}$ on the space of variables $\zeta_{\ell 0}$, $1 \le \ell \le L$, is exactly $\mathcal{Z}$, a pair $(t, X(\cdot))$ is feasible for (S) if and only if it is feasible for the AARC of (5.3.6), the information base being given by $I_1, ..., I_n$. The bottom line is, that when $\mathcal{Z}$ is not a box, the tractable problem (S), while still producing robust feasible decisions, is at least as flexible as the AARC. Whether this "at least as flexible" is or is not "more flexible," depends on the application in question, and since both (S) and AARC are tractable, it is easy to figure out what the true answer is.

Here is a toy example. Let $L = 2$, $n = 2$, and let (5.3.6) be the uncertain problem

$$\left\{ \min_x \left\{ x_2 : \begin{array}{l} x_1 \ge \zeta_1 \\ x_1 \ge -\zeta_1 \\ x_2 \ge x_1 + 3\zeta_1/5 + 4\zeta_2/5 \\ x_2 \ge x_1 - 3\zeta_1/5 - 4\zeta_2/5 \end{array} \right\}, \|\zeta\|_2 \le 1 \right\},$$

with fully adjustable variable x_1 and non-adjustable variable x_2. Due to the extreme simplicity of our problem, we can immediately point out an optimal solution to the *unrestricted* ARC, namely,

$$X_1(\zeta) = |\zeta_1|,\ x_2 \equiv \mathrm{Opt}(\mathrm{ARC}) = \max_{\|\zeta\|_2\leq 1}[|\zeta_1| + |3\zeta_1 + 4\zeta_2|/5] = \frac{4\sqrt{5}}{5} \approx 1.7889.$$

Now let us compare Opt(ARC) with the optimal value Opt(AARC) of the AARC and with the optimal value Opt(RARC) of the restricted ARC where the decision rules are allowed to be affine in $[\zeta_\ell]_\pm$, $\ell = 1, 2$ (as always, $[a]_+ = \max[a, 0]$ and $[a]_- = \max[-a, 0]$). The situation fits **B**, so that we can process the RARC as it is. Noting that $a = [a]_+ - [a]_-$, the decision rules that are affine in $[\zeta_\ell]_\pm$, $\ell = 1, 2$, are exactly the same as the decision rules (5.3.7), where $\mathcal{F}_\ell$, $\ell = 1, 2$, are the spaces of piecewise linear functions on the axis with the only breakpoint 0. We see that *up to the fact that $\mathcal{Z}$ is a circle rather than a square*, the situation fits **C.1** as well, and we can process RARC via its safe tractable approximation (S). Let us look what are the optimal values yielded by these 3 schemes.

• The AARC of our toy problem is

$$\mathrm{Opt}(\mathrm{AARC}) = \min_{x_2,\xi,\eta}\left\{x_2 : \begin{array}{ll} \overbrace{\xi + \eta^T\zeta}^{X_1(\zeta)} \geq |\zeta_1| & (a) \\ x_2 \geq X_1(\zeta) + |3\zeta_1 + 4\zeta_2|/5 & (b) \\ & \\ \forall(\zeta : \|\zeta\|_2 \leq 1) & \end{array}\right\}$$

This problem can be immediately solved. Indeed, (a) should be valid for $\zeta = \zeta^1 \equiv [1; 0]$ and for $\zeta = \zeta^2 \equiv -\zeta^1$, meaning that $X_1(\pm\zeta^1) \geq 1$, whence $\xi \geq 1$. Further, (b) should be valid for $\zeta = \zeta^3 \equiv [3; 4]/5$ and for $\zeta = \zeta^4 \equiv -\zeta^3$, meaning that $x_2 \geq X_1(\pm\zeta^3) + 1$, whence $x_2 \geq \xi + 1 \geq 2$. We see that the optimal value is ≥ 2, and this bound is achievable (we can take $X_1(\cdot) \equiv 1$ and $x_2 = 2$). As a byproduct, in our toy problem the AARC is as conservative as the RC.

• The RARC of our problem as given by **B** is

$$\mathrm{Opt}(\mathrm{RARC}) = \min_{x_2,\xi,\eta,\eta_\pm}\left\{x_2 : \begin{array}{l} \overbrace{\xi + \eta^T\zeta + \eta_+^T\zeta^+ + \eta_-^T\zeta^-}^{X_1(\widehat{\zeta})} \geq |\zeta_1| \\ x_2 \geq X_1(\widehat{\zeta}) + |3\zeta_1 + 4\zeta_2|/5 \\ \\ \forall(\widehat{\zeta} = [\underbrace{\zeta_1; \zeta_2}_{\zeta}; \underbrace{\zeta_1^+; \zeta_2^+}_{\zeta^+}; \underbrace{\zeta_1^-; \zeta_2^-}_{\zeta^-}] \in \widehat{\mathcal{Z}}) \end{array}\right\},$$

$$\widehat{\mathcal{Z}} = \left\{\widehat{\zeta} : \zeta = \zeta^+ - \zeta^-, \zeta^\pm \geq 0, \|\zeta^+ + \zeta^-\|_2 \leq 1\right\}.$$

We can say in advance what are the optimal value and the optimal solution to the RARC — they should be the same as those of the ARC, since the latter, as a matter of fact, admits optimal decision rules that are affine in $|\zeta_1|$, and thus in $[\zeta_\ell]_\pm$. Nevertheless, we have carried out numerical optimization which yielded another optimal solution to the RARC (and thus - to ARC):

$$\begin{array}{l} \mathrm{Opt}(\mathrm{RARC}) = x_2 = 1.7889, \\ \xi = 1.0625, \eta = [0; 0], \eta_+ = \eta_- = [0.0498; -0.4754], \end{array}$$

which corresponds to $X_1(\zeta) = 1.0625 + 0.0498|\zeta_1| - 0.4754|\zeta_2|$.

• The safe tractable approximation of the RARC looks as follows. The mapping $\zeta \mapsto \widehat{\zeta}[\zeta]$ in our case is

$$[\zeta_1; \zeta_2] \mapsto [\zeta_{1,0} = \zeta_1; \zeta_{1,1} = \max[\zeta_1, 0]; \zeta_{2,0} = \zeta_2; \zeta_{2,1} = \max[\zeta_2, 0]],$$

the tractable description of $\widehat{\mathcal{Z}}^+$ as given by **C.1** is

$$\widehat{\mathcal{Z}}^+ = \left\{ \{\zeta_{\ell i}\}_{\substack{i=0,1\\ \ell=1,2}} : \begin{array}{c} -1 \leq \zeta_{\ell 0} \leq 1 \\ 0 \leq \frac{\zeta_{\ell 1}-\zeta_{\ell 0}}{1} \leq \frac{1-\zeta_{\ell 1}}{1} \leq 1 \end{array} \right\}, \ell = 1,2 \Big\}$$

and the sets $\widehat{\mathcal{Z}}^0$, $\widehat{\mathcal{Z}}^1$ are given by

$$\widehat{\mathcal{Z}}^i = \left\{ \{\zeta_{\ell i}\}_{\substack{i=0,1\\ \ell=1,2}} : \zeta_{1i}^2 + \zeta_{2i}^2 \leq 1 \right\}, \; i = 0,1.$$

Consequently, (S) becomes the semi-infinite LO problem

$$\begin{array}{rcl} \mathrm{Opt}(S) & = & \min\limits_{x_2,\xi,\{\eta_{\ell i}\}} \left\{ x_2 : \begin{array}{l} X_1(\widehat{\zeta}) \equiv \xi + \sum_{\substack{\ell=1,2\\ i=0,1}} \eta_{\ell i}\zeta_{\ell i} \geq \zeta_{1,0} \\ X_1(\widehat{\zeta}) \equiv \xi + \sum_{\substack{\ell=1,2\\ i=0,1}} \eta_{\ell i}\zeta_{\ell i} \geq -\zeta_{1,0} \\ x_2 \geq \xi + \sum_{\substack{\ell=1,2\\ i=0,1}} \eta_{\ell i}\zeta_{\ell i} + [3\zeta_{1,0} + 4\zeta_{2,0}]/5 \\ x_2 \geq \xi + \sum_{\substack{\ell=1,2\\ i=0,1}} \eta_{\ell i}\zeta_{\ell i} - [3\zeta_{1,0} + 4\zeta_{2,0}]/5 \end{array} \right. \\ & & \forall \widehat{\zeta} = \{\zeta_{\ell i}\} : \begin{array}{l} -1 \leq \zeta_{\ell 0} \leq 1, \ell = 1,2 \\ 0 \leq \zeta_{\ell 1} - \zeta_{\ell 0} \leq 1 - \zeta_{\ell 1} \leq 1, \ell = 1,2 \\ \zeta_{1i}^2 + \zeta_{2i}^2 \leq 1, \; i = 0,1 \end{array} \Big\}. \end{array}$$

Computation results in

$$\begin{array}{l} \mathrm{Opt}(S) = x_2 = \frac{25+\sqrt{8209}}{60} \approx 1.9267, \\ X_1(\zeta) = \frac{5}{12} - \frac{3}{5}\zeta_{1.0}[\zeta_1] + \frac{6}{5}\zeta_{1,1}[\zeta_1] + \frac{7}{60}\zeta_{2,0}[\zeta_2] = \frac{5}{12} + \frac{3}{5}|\zeta_1| + \frac{7}{60}\zeta_2. \end{array}$$

As it could be expected, we get $2 = \mathrm{Opt}(\mathrm{AARC}) > 1.9267 = \mathrm{Opt}(S) > 1.7889 = \mathrm{Opt}(\mathrm{RARC}) = \mathrm{Opt}(\mathrm{ARC})$. Note that in order to get $\mathrm{Opt}(S) < \mathrm{Opt}(\mathrm{AARC})$, taking into account $\widehat{\mathcal{Z}}^1$ is a must: in the case of **C.1**, whatever be $\mathcal{Z}$ and a box $\mathcal{Z}^+ \supset \mathcal{Z}$, with $\widetilde{\mathcal{Z}} = \widehat{\mathcal{Z}}^+ \cap \widehat{\mathcal{Z}}^0$ we gain nothing as compared to the genuine affine decision rules.

D. *Quadratic decision rules, ellipsoidal uncertainty set.* In this case,

$$\widehat{\zeta}[\zeta] = \left[\begin{array}{c|c} & \zeta^T \\ \hline \zeta & \zeta\zeta^T \end{array}\right]$$

is comprised of the entries of ζ and their pairwise products (so that the associated decision rules (5.3.5) are quadratic in ζ), and $\mathcal{Z}$ is the ellipsoid $\{\zeta \in \mathbb{R}^L : \|Q\zeta\|_2 \leq 1\}$, where Q has a trivial kernel. The convex hull of the image of $\mathcal{Z}$ under the quadratic mapping $\zeta \to \widehat{\zeta}[\zeta]$ is easy to describe:

Lemma 5.4 *In the above notation, the set* $\widehat{\mathcal{Z}} = \mathrm{Conv}\{\widehat{\zeta}[\zeta] : \|Q\zeta\|_2 \leq 1\}$ *is a convex compact set given by the semidefinite representation as follows:*

$$\widehat{\mathcal{Z}} = \left\{ \widehat{\zeta} = \left[\begin{array}{c|c} & v^T \\ \hline v & W \end{array}\right] \in \mathbf{S}^{L+1} : \widehat{\zeta} + \left[\begin{array}{c|c} 1 & \\ \hline & \end{array}\right] \succeq 0, \mathrm{Tr}(QWQ^T) \leq 1 \right\}.$$

Proof. It is immediately seen that it suffices to prove the statement when $Q = I$, which we assume from now on. Besides this, when we add to the mapping $\widehat{\zeta}[\zeta]$ the constant matrix $\left[\begin{array}{c|c} 1 & \\ \hline & \end{array}\right]$, the convex hull of the image of $\mathcal{Z}$ is translated by the same matrix. It follows that all we need is to prove that the convex hull $\mathcal{Q}$ of the image of the unit Euclidean ball under the mapping $\zeta \mapsto \widetilde{\zeta}[\zeta] = \left[\begin{array}{c|c} 1 & \zeta^T \\ \hline \zeta & \zeta\zeta^T \end{array}\right]$ can be represented as

$$\mathcal{Q} = \left\{ \widehat{\zeta} = \left[\begin{array}{c|c} 1 & v^T \\ \hline v & W \end{array}\right] \in \mathbf{S}^{L+1} : \widehat{\zeta} \succeq 0, \mathrm{Tr}(W) \leq 1 \right\}. \tag{5.3.14}$$

Denoting the right hand side in (5.3.14) by $\widehat{\mathcal{Q}}$, both $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ are nonempty convex compact sets. Therefore they coincide if and only if their support functions are identical.[1] We are in the situation where $\mathcal{Q}$ is the convex hull of the set $\left\{\left[\begin{array}{c|c}1 & \zeta^T\\ \hline \zeta & \zeta\zeta^T\end{array}\right] : \zeta^T\zeta \leq 1\right\}$, so that the support function of $\mathcal{Q}$ is

$$\phi(P) = \max_Z \left\{\mathrm{Tr}(PZ) : Z = \left[\begin{array}{c|c}1 & \zeta^T\\ \hline \zeta & \zeta\zeta^T\end{array}\right] : \zeta^T\zeta \leq 1\right\} \qquad \left[P = \left[\begin{array}{c|c}p & q^T\\ \hline q & R\end{array}\right] \in \mathbf{S}^{L+1}\right].$$

We have

$$\begin{array}{l}
\phi(P) = \max\limits_Z \left\{\mathrm{Tr}(PZ) : Z = \left[\begin{array}{c|c}1 & \zeta^T\\ \hline \zeta & \zeta\zeta^T\end{array}\right] \text{ with } \zeta^T\zeta \leq 1\right\}\\
= \max\limits_\zeta \left\{\zeta^T R\zeta + 2q^T\zeta + p : \zeta^T\zeta \leq 1\right\}\\
= \min\limits_\tau \left\{\tau : \tau \geq \zeta^T R\zeta + 2q^T\zeta + p\,\forall(\zeta : \zeta^T\zeta \leq 1)\right\}\\
= \min\limits_\tau \left\{\tau : (\tau - p)t^2 - \zeta^T R\zeta - 2tq^T\zeta \geq 0\,\forall((\zeta,t) : \zeta^T\zeta \leq t^2)\right\}\\
= \min\limits_\tau \left\{\tau : \exists\lambda \geq 0 : (\tau - p)t^2 - \zeta^T R\zeta - 2tq^T\zeta - \lambda(t^2 - \zeta^T\zeta) \geq 0\,\forall(\zeta,t)\right\} \text{ [}\mathcal{S}\text{-Lemma]}\\
= \min\limits_{\tau,\lambda} \left\{\tau : \left[\begin{array}{c|c}\tau - p - \lambda & -q^T\\ \hline -q & \lambda I - R\end{array}\right] \succeq 0, \lambda \geq 0\right\}\\
= \max\limits_{u,v,W,r} \left\{up + 2v^T q + \mathrm{Tr}(RW) : \mathrm{Tr}\left(\left[\begin{array}{c|c}\tau - \lambda & \\ \hline & \lambda I\end{array}\right]\left[\begin{array}{c|c}u & v^T\\ \hline v & W\end{array}\right]\right) + r\lambda\right.\\
\qquad\qquad \left.\equiv \tau\forall(\tau,\lambda), \left[\begin{array}{c|c}u & v^T\\ \hline v & W\end{array}\right] \succeq 0, r \geq 0\right\} \text{ [semidefinite duality]}\\
= \max\limits_{v,W} \left\{p + 2v^T q + \mathrm{Tr}(RW) : \left[\begin{array}{c|c}1 & v^T\\ \hline v & W\end{array}\right] \succeq 0, \mathrm{Tr}(W) \leq 1\right\}\\
= \max\limits_{v,W} \left\{\mathrm{Tr}\left(P\left[\begin{array}{c|c}1 & v^T\\ \hline v & W\end{array}\right]\right) : \left[\begin{array}{c|c}1 & v^T\\ \hline v & W\end{array}\right] \in \widehat{\mathcal{Q}}\right\}.
\end{array}$$

Thus, the support function of $\mathcal{Q}$ indeed is identical to the one of $\widehat{\mathcal{Q}}$. □

Corollary 5.1 *Consider a fixed recourse uncertain LO problem* (5.3.6) *with an ellipsoid as an uncertainty set, where the adjustable decision variables are allowed to be quadratic functions of prescribed portions $P_j\zeta$ of the data. The associated ARC of the problem is computationally tractable and is given by an explicit semidefinite program of the sizes polynomial in those of instances and in the dimension L of the data vector.*

E. <u>*Quadratic decision rules and the intersection of concentric ellipsoids as the uncertainty set.*</u> Here the uncertainty set $\mathcal{Z}$ is $\cap$-ellipsoidal:

$$\begin{array}{c}\mathcal{Z} = \mathcal{Z}_\rho \equiv \{\zeta \in \mathbb{R}^L : \zeta^T Q_j\zeta \leq \rho^2, 1 \leq j \leq J\}\\ \left[Q_j \succeq 0, \sum_j Q_j \succ 0\right]\end{array} \tag{5.3.15}$$

(cf. section 3.3.2), where $\rho > 0$ is an uncertainty level, and, as above, $\widehat{\zeta}[\zeta] = \left[\begin{array}{c|c} & \zeta^T\\ \hline \zeta & \zeta\zeta^T\end{array}\right]$, so that our intention is to process the ARC of an uncertain problem corresponding to quadratic decision rules. As above, all we need is to get a tractable representation of the convex hull of

[1] The support function of a nonempty convex set $X \subset \mathbb{R}^n$ is the function $f(\xi) = \sup_{x\in X} \xi^T x : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$. The fact that two closed nonempty convex sets in $\mathbb{R}^n$ are identical, if and only if their support functions are so, is readily given by the Separation Theorem.

the image of $\mathcal{Z}_\rho$ under the nonlinear mapping $\zeta \mapsto \widehat{\zeta}[\zeta]$. This is essentially the same as to find a similar representation of the convex hull $\widehat{\mathcal{Z}}_\rho$ of the image of $\mathcal{Z}_\rho$ under the nonlinear mapping

$$\zeta \mapsto \widehat{\zeta}_\rho[\zeta] = \left[\begin{array}{c|c} & \zeta^T \\ \hline \zeta & \frac{1}{\rho}\zeta\zeta^T \end{array}\right];$$

indeed, both convex hulls in question can be obtained from each other by simple linear transformations. The advantage of our normalization is that now $\mathcal{Z}_\rho = \rho\mathcal{Z}_1$ and $\widehat{\mathcal{Z}}_\rho = \rho\widehat{\mathcal{Z}}_1$, as it should be for respectable perturbation sets.

While the set $\widehat{\mathcal{Z}}_\rho$ is, in general, computationally intractable, we are about to demonstrate that this set admits a tight tractable approximation, and that the latter induces a tight tractable approximation of the "quadratically adjustable" RC of the Linear Optimization problem in question. The main ingredient we need is as follows:

Lemma 5.5 *Consider the semidefinite representable set*

$$\mathcal{W}_\rho = \rho\mathcal{W}_1,\ \mathcal{W}_1 = \left\{\widehat{\zeta} = \left[\begin{array}{c|c} & v^T \\ \hline v & W \end{array}\right] : \left[\begin{array}{c|c} 1 & v^T \\ \hline v & W \end{array}\right] \succeq 0, \mathrm{Tr}(WQ_j) \leq 1,\ 1 \leq j \leq J\right\}. \tag{5.3.16}$$

Then

$$\forall \rho > 0 : \widehat{\mathcal{Z}}_\rho \subset \mathcal{W}_\rho \subset \widehat{\mathcal{Z}}_{\vartheta\rho}, \tag{5.3.17}$$

where $\vartheta = O(1)\ln(J+1)$ *and* J *is the number of ellipsoids in the description of* $\mathcal{Z}_\rho$.

Proof. Since both $\widehat{\mathcal{Z}}_\rho$ and $\widehat{\mathcal{W}}_\rho$ are nonempty convex compact sets containing the origin and belonging to the subspace $\mathbf{S}_0^{L+1}$ of $\mathbf{S}^{L+1}$ comprised of matrices with the first diagonal entry being zero, to prove (5.3.17) is the same as to verify that the corresponding support functions

$$\phi_{\mathcal{W}_\rho}(P) = \max_{\widehat{\zeta}\in\mathcal{W}_\rho} \mathrm{Tr}(P\widehat{\zeta}),\ \phi_{\widehat{\mathcal{Z}}_\rho}(P) = \max_{\widehat{\zeta}\in\widehat{\mathcal{Z}}_\rho} \mathrm{Tr}(P\widehat{\zeta}),$$

considered as functions of $P \in \mathbf{S}_0^{L+1}$, satisfy the relation

$$\phi_{\widehat{\mathcal{Z}}_\rho}(\cdot) \leq \phi_{\mathcal{W}_\rho}(\cdot) \leq \phi_{\widehat{\mathcal{Z}}_{\vartheta\rho}}(\cdot).$$

Taking into account that $\widehat{\mathcal{Z}}_s = s\widehat{\mathcal{Z}}_1$, $s > 0$, this task reduces to verifying that

$$\phi_{\widehat{\mathcal{Z}}_\rho}(\cdot) \leq \phi_{\mathcal{W}_\rho}(\cdot) \leq \vartheta\phi_{\widehat{\mathcal{Z}}_\rho}(\cdot).$$

Thus, all we should prove is that whenever $P = \left[\begin{array}{c|c} & p^T \\ \hline p & R \end{array}\right] \in \mathbf{S}_0^{L+1}$, one has

$$\max_{\widehat{\zeta}\in\widehat{\mathcal{Z}}_\rho} \mathrm{Tr}(P\widehat{\zeta}) \leq \max_{\widehat{\zeta}\in\mathcal{W}_\rho} \mathrm{Tr}(P\widehat{\zeta}) \leq \vartheta \max_{\widehat{\zeta}\in\widehat{\mathcal{Z}}_\rho} \mathrm{Tr}(P\widehat{\zeta}).$$

Recalling the origin of $\widehat{\mathcal{Z}}_\rho$, the latter relation reads

$$\begin{array}{l} \forall P = \left[\begin{array}{c|c} & p^T \\ \hline p & R \end{array}\right] : \mathrm{Opt}_P(\rho) \equiv \max\limits_{\zeta}\left\{2p^T\zeta + \frac{1}{\rho}\zeta^T R\zeta : \zeta^T Q_j\zeta \leq \rho^2, 1 \leq j \leq J\right\} \\ \qquad \leq \mathrm{SDP}_P(\rho) \equiv \max\limits_{\widehat{\zeta}\in\mathcal{W}_\rho} \mathrm{Tr}(P\widehat{\zeta}) \leq \vartheta\mathrm{Opt}_P(\rho) \equiv \mathrm{Opt}_P(\vartheta\rho). \end{array} \tag{5.3.18}$$

Observe that the three quantities in the latter relation are of the same homogeneity degree w.r.t. $\rho > 0$, so that it suffices to verify this relation when $\rho = 1$, which we assume from now on.

We are about to derive (5.3.18) from the Approximate $\mathcal{S}$-Lemma (Theorem A.8). To this end, let us specify the entities participating in the latter statement as follows:

- $x = [t; \zeta] \in \mathbb{R}^1_t \times \mathbb{R}^L_\zeta$;
- $A = P$, that is, $x^T A x = 2tp^T\zeta + \zeta^T R\zeta$;
- $B = \left[\begin{array}{c|c} 1 & \\ \hline & \end{array}\right]$, that is, $x^T B x = t^2$;
- $B_j = \left[\begin{array}{c|c} & \\ \hline & Q_j \end{array}\right]$, $1 \le j \le J$, that is, $x^T B_j x = \zeta^T Q_j \zeta$;
- $\rho = 1$.

With this setup, the quantity $\mathrm{Opt}(\rho)$ from (A.4.12) becomes nothing but $\mathrm{Opt}_P(1)$, while the quantity $\mathrm{SDP}(\rho)$ from (A.4.13) is

$$\begin{array}{l}
\mathrm{SDP}(1) = \max\limits_X \{\mathrm{Tr}(AX) : \mathrm{Tr}(BX) \le 1, \mathrm{Tr}(B_jX) \le 1,\ 1 \le j \le J, X \succeq 0\} \\
= \max\limits_X \left\{ 2p^Tv + \mathrm{Tr}(RW) : \begin{array}{l} u \le 1 \\ \mathrm{Tr}(WQ_j) \le 1,\ 1 \le j \le J \\ X = \left[\begin{array}{c|c} u & v^T \\ \hline v & W \end{array}\right] \succeq 0 \end{array} \right\} \\
= \max\limits_{v,W} \left\{ 2p^Tv + \mathrm{Tr}(RW) : \begin{array}{l} \mathrm{Tr}(WQ_j) \le 1,\ 1 \le j \le J \\ \left[\begin{array}{c|c} 1 & v^T \\ \hline v & W \end{array}\right] \succeq 0 \end{array} \right\} \\
= \max\limits_{\widehat{\zeta}} \left\{ \mathrm{Tr}(P\widehat{\zeta}) : \widehat{\zeta} = \left[\begin{array}{c|c} & v^T \\ \hline v & W \end{array}\right] : \begin{array}{l} \left[\begin{array}{c|c} 1 & v^T \\ \hline v & W \end{array}\right] \succeq 0 \\ \mathrm{Tr}(WQ_j) \le 1,\ 1 \le j \le J \end{array} \right\} \\
= \mathrm{SDP}_P(1).
\end{array}$$

With these observations, the conclusion (A.4.15) of the Approximate $\mathcal{S}$-Lemma reads

$$\mathrm{Opt}_P(1) \le \mathrm{SDP}_P(1) \le \mathrm{Opt}(\Omega(J)),\ \ \Omega(J) = 9.19\sqrt{\ln(J+1)} \tag{5.3.19}$$

where for $\Omega \ge 1$

$$\begin{array}{l}
\mathrm{Opt}(\Omega) = \max\limits_x \left\{ x^TAx : x^TBx \le 1, x^TB_jx \le \Omega^2 \right\} \\
= \max\limits_{t,\zeta} \left\{ 2tp^T\zeta + \zeta^TR\zeta : t^2 \le 1, \zeta^TQ_j\zeta \le \Omega^2,\ 1 \le j \le J \right\} \\
= \max\limits_{\zeta} \left\{ 2p^T\zeta + \zeta^TR\zeta : \zeta^TQ_j\zeta \le \Omega^2,\ 1 \le j \le J \right\} \\
= \max\limits_{\eta = \Omega^{-1}\zeta} \left\{ \Omega(2p^T\eta) + \Omega^2\eta^TR\eta : \eta^TQ_j\eta \le 1,\ 1 \le j \le J \right\} \\
\le \Omega^2 \max\limits_{\eta} \left\{ 2p^T\eta + \eta^TR\eta : \eta^TQ_j\eta \le 1,\ 1 \le j \le J \right\} \\
= \Omega^2 \mathrm{Opt}_P(1).
\end{array}$$

Setting $\vartheta = \Omega^2(J)$, we see that (5.3.19) implies (5.3.18). □

Corollary 5.2 *Consider a fixed recourse uncertain LO problem* (5.3.6) *with $\cap$-ellipsoidal uncertainty set $\mathcal{Z}_\rho$ (see* (5.3.15)*) where one seeks robust optimal quadratic decision rules:*

$$\begin{array}{c}
x_j = p_j + q_j^T \widehat{P}_j\left(\widehat{\zeta}_\rho[\zeta]\right) \\
\left[\begin{array}{l}
\bullet\ \ \widehat{\zeta}_\rho[\zeta] = \left[\begin{array}{c|c} & \zeta^T \\ \hline \zeta & \frac{1}{\rho}\zeta\zeta^T \end{array}\right] \\
\bullet\ \ \widehat{P}_j : \ \textit{linear mappings from } \mathbf{S}^{L+1} \textit{ to } \mathbb{R}^{m_j} \\
\bullet\ \ p_j \in \mathbb{R}, q_j \in \mathbb{R}^{m_j} : \ \textit{parameters to be specified}
\end{array}\right].
\end{array} \tag{5.3.20}$$

The associated Adjustable Robust Counterpart of the problem admits a safe tractable approximation that is tight within the factor ϑ given by Lemma 5.5.

Here is how the safe approximation of the Robust Counterpart mentioned in Corollary 5.2 can be built:

1. We write down the optimization problem

$$\min_{t,x}\left\{t:\begin{array}{l} a_{0\zeta}^T[t;x]+b_{0\zeta}\equiv t-c_\zeta^Tx-d_\zeta\geq 0\\ a_{i\zeta}^T[t;x]+b_{i,\zeta}\equiv A_{i\zeta}^Tx-b_{i\zeta}\geq 0,\ i=1,...,m\end{array}\right\}\qquad (P)$$

   where $A_{i\zeta}^T$ is i-th row in A_ζ and $b_{i\zeta}$ is i-th entry in b_ζ;

2. We plug into the $m+1$ constraints of (P), instead of the original decision variables x_j, the expressions $p_j+q_j^T\widehat{P}_j\left(\widehat{\zeta}_\rho[\zeta]\right)$, thus arriving at the optimization problem of the form

$$\min_{[t;y]}\left\{t:\alpha_{i\widehat{\zeta}}^T[t;y]+\beta_{i\widehat{\zeta}}\geq 0,\ 0\leq i\leq m\right\},\qquad (P')$$

   where y is the collection of coefficients p_j,q_j of the quadratic decision rules, $\widehat{\zeta}$ is our new uncertain data — a matrix from $\mathbf{S}_0^{L+1}$ (see p. 233), and $\alpha_{i\widehat{\zeta}}$, $\beta_{i\widehat{\zeta}}$ are affine in $\widehat{\zeta}$, the affinity being ensured by the assumption of fixed recourse. The "true" quadratically adjustable RC of the problem of interest is the semi-infinite problem

$$\min_{[t;y]}\left\{t:\forall\widehat{\zeta}\in\widehat{\mathcal{Z}}_\rho:\alpha_{i\widehat{\zeta}}^T[t;y]+\beta_{i\widehat{\zeta}}\geq 0,\ 0\leq i\leq m\right\}\qquad (R)$$

   obtained from (P') by requiring the constraints to remain valid for all $\widehat{\zeta}\in\widehat{\mathcal{Z}}_\rho$, the latter set being the convex hull of the image of $\mathcal{Z}_\rho$ under the mapping $\zeta\mapsto\widehat{\zeta}_\rho[\zeta]$. The semi-infinite problem (R) in general is intractable, and we replace it with its safe tractable approximation

$$\min_{[t;y]}\left\{t:\forall\widehat{\zeta}\in\mathcal{W}_\rho:\alpha_{i\widehat{\zeta}}^T[t;y]+\beta_{i\widehat{\zeta}}\geq 0,\ 0\leq i\leq m\right\},\qquad (R')$$

   where $\mathcal{W}_\rho$ is the semidefinite representable convex compact set defined in Lemma 5.5. By Theorem 1.1, (R') is tractable and can be straightforwardly converted into a semidefinite program of sizes polynomial in $n=\dim x$, m and $L=\dim\zeta$. Here is the conversion: recalling the structure of $\widehat{\zeta}$ and setting $z=[t;x]$, we can rewrite the body of i-th constraint in (R') as

$$\alpha_{i\widehat{\zeta}}^Tz+\beta_{i\widehat{\zeta}}\equiv a_i[z]+\mathrm{Tr}\Big(\underbrace{\left[\begin{array}{c|c} & v^T\\ \hline v & W\end{array}\right]}_{\widehat{\zeta}}\left[\begin{array}{c|c} & p_i^T[z]\\ \hline p_i[z] & P_i[z]\end{array}\right]\Big),$$

   where $a_i[z]$, $p_i[z]$ and $P_i[z]=P_i^T[z]$ are affine in z. Therefore, invoking the definition of $\mathcal{W}_\rho=\rho\mathcal{W}_1$ (see Lemma 5.5), the RC of the i-th semi-infinite constraint in (R') is the first predicate in the following chain of equivalences:

$$\min_{v,W}\left\{a_i[z]+2\rho v^Tp_i[z]+\rho\mathrm{Tr}(WP_i[z]):\right.\\ \left.\left[\begin{array}{c|c}1 & v^T\\ \hline v & W\end{array}\right]\succeq 0,\mathrm{Tr}(WQ_j)\leq 1,\ 1\leq j\leq J\right\}\geq 0\qquad (a_i)$$

$$\Updownarrow$$

$$\exists\lambda^i=[\lambda_1^i;...;\lambda_J^i]:\left\{\begin{array}{l}\lambda^i\geq 0\\ \left[\begin{array}{c|c}a_i[z]-\sum_j\lambda_j^i & \rho p_i^T[z]\\ \hline \rho p_i[z] & \rho P_i[z]+\sum_j\lambda_j^iQ_j\end{array}\right]\succeq 0\end{array}\right.\qquad (b_i)$$

where $\Updownarrow$ is given by Semidefinite Duality. Consequently, we can reformulate (R') equivalently as the semidefinite program

$$\min_{\substack{z=[t;y]\\ \{\lambda_j^i\}}} \left\{ t: \begin{array}{l} \left[\begin{array}{c|c} a_i[z] - \sum_j \lambda_j^i & \rho p_i^T[z] \\ \hline \rho p_i[z] & \rho P_i[z] + \sum_j \lambda_j^i Q_j \end{array}\right] \succeq 0 \\ \lambda_j^i \geq 0,\, 0 \leq i \leq m, 1 \leq j \leq J \end{array} \right\}.$$

The latter SDP is a ϑ-tight safe tractable approximation of the quadratically adjustable RC with ϑ given by Lemma 5.5.

5.3.3 The AARC of Uncertain Linear Optimization Problem Without Fixed Recourse

We have seen that the AARC of an uncertain LO problem

$$\begin{array}{c} \mathcal{C} = \left\{ \min_x \left\{ c_\zeta^T x + d_\zeta : A_\zeta x \geq b_\zeta \right\} : \zeta \in \mathcal{Z} \right\} \\ [c_\zeta, d_\zeta, A_\zeta, b_\zeta : \text{ affine in } \zeta] \end{array} \tag{5.3.21}$$

with computationally tractable convex compact uncertainty set $\mathcal{Z}$ *and with fixed recourse* is computationally tractable. What happens when the assumption of fixed recourse is removed? The answer is that in general the AARC can become intractable (see [14]). However, we are about to demonstrate that for an ellipsoidal uncertainty set $\mathcal{Z} = \mathcal{Z}_\rho = \{\zeta : \|Q\zeta\|_2 \leq \rho\}$, $\mathrm{Ker}Q = \{0\}$, the AARC is computationally tractable, and for the $\cap$-ellipsoidal uncertainty set $\mathcal{Z} = \mathcal{Z}_\rho$ given by (5.3.15), the AARC admits a tight safe tractable approximation. Indeed, for affine decision rules

$$x_j = X_j(P_j\zeta) \equiv p_j + q_j^T P_j \zeta$$

the AARC of (5.3.21) is the semi-infinite problem of the form

$$\min_{z=[t;y]} \left\{ t : \forall \zeta \in \mathcal{Z}_\rho : a_i[z] + 2b_i^T[z]\zeta + \zeta^T C_i[z]\zeta \leq 0,\, 0 \leq i \leq m \right\}, \tag{5.3.22}$$

where $y = \{p_j, q_j\}_{j=1}^n$ and $a_i[z], b_i[z], C_i[z]$ are real/vector/symmetric matrix affinely depending on $z = [t; y]$. Consider the case of $\cap$-ellipsoidal uncertainty:

$$\mathcal{Z}_\rho = \{\zeta : \zeta^T Q_j \zeta \leq \rho^2,\, 1 \leq j \leq J\} \qquad [Q_j \succeq 0, \textstyle\sum_j Q_j \succ 0].$$

For a fixed i, $0 \leq i \leq m$, let us set $A_{i,z} = \left[\begin{array}{c|c} & b_i^T[z] \\ \hline b_i[z] & C_i[z] \end{array}\right]$, $B = \left[\begin{array}{c|c} 1 & \\ \hline & \end{array}\right]$, $B_j = \left[\begin{array}{c|c} & \\ \hline & Q_j \end{array}\right]$, $1 \leq j \leq J$, and observe that clearly

$$\begin{array}{l} \mathrm{Opt}_{i,z}(\rho) := \max\limits_{\eta=[\tau;\zeta]} \left\{ \eta^T A_{i,z}\eta \equiv 2\tau b_i^T[z]\zeta : \eta^T B\eta \equiv \tau^2 \leq 1, \eta^T B_j \eta \equiv \zeta^T Q_j \zeta \leq \rho^2, 1 \leq j \leq J \right\} \\ \leq \mathrm{SDP}_{i,z} := \min_{\lambda \geq 0} \left\{ \lambda_0 + \rho^2 \sum_{j=1}^J \lambda_j : \lambda_0 B + \sum_j \lambda_j B_j \succeq A_{i,z} \right\}, \end{array}$$

so that the explicit system of LMIs

$$\lambda_0 B + \sum_{j=1}^J \lambda_j B_j \succeq A_{i,z}, \lambda_0 + \rho^2 \sum_j \lambda_j \leq -a_i[z], \lambda \geq 0 \tag{5.3.23}$$

in variables z, λ is a safe tractable approximation of the i-th semi-infinite constraint

$$a_i[z] + 2b_i[z]\zeta + \zeta^T C[z]\zeta \leq 0 \;\forall \zeta \in \mathcal{Z}_\rho \tag{5.3.24}$$

appearing in (5.3.22). Let us prove that this approximation is tight within the factor ϑ which is 1 when $J = 1$ and is $9.19\sqrt{\ln(J)}$ otherwise. All we need to prove is that if z cannot be extended to a feasible solution of (5.3.23), z is infeasible for the semi-infinite constraint in question at the uncertainty level $\vartheta\rho$, or, which is clearly the same, that $\mathrm{Opt}_{i,z}(\vartheta\rho) > -a_i[z]$. When z cannot be extended to a feasible solution to (5.3.23), we have $\mathrm{SDP}_{i,z} > -a_i[z]$. Invoking Approximate $\mathcal{S}$-Lemma (Theorem A.8) for the data $A = A_{i,z}$, B, $\{B_j\}$, there exists $\bar{\eta} = [\bar{\tau}; \bar{\zeta}]$ such that $\bar{\tau}^2 = \bar{\eta}^T B\bar{\eta} \leq 1$, $\bar{\zeta}^T Q_j \bar{\zeta} = \bar{\eta}^T B_j \bar{\eta} \leq \vartheta^2\rho^2$, $1 \leq j \leq J$, and $\bar{\eta}^T A_{i,z}\bar{\eta} \geq \mathrm{SDP}_{i,z}$. Since $|\bar{\tau}| \leq 1$ and $\bar{\zeta}^T Q_j \bar{\zeta} \leq \vartheta^2\rho^2$, we have

$$\mathrm{Opt}_{i,z}(\vartheta\rho) \geq \bar{\eta}^T A_{i,z}\bar{\eta} \geq \mathrm{SDP}_{i,z} > -a_i[z],$$

as claimed.

We have arrived at the following result:

> *The AARC of an arbitrary uncertain LO problem, the uncertainty set being the intersection of J ellipsoids centered at the origin, is computationally tractable, provided $J = 1$, and admits safe tractable approximation, tight within the factor* $9.19\sqrt{\ln(J)}$ *when $J > 1$.*

In fact the above approach can be extended even slightly beyond just affine decision rules. Specifically, in the case of an uncertain LO we could allow for the adjustable "fixed recourse" variables x_j — those for which all the coefficients in the objective and the constraints of instances are certain — to be *quadratic* in $P_j\zeta$, and for the remaining "non-fixed recourse" adjustable variables to be affine in $P_j\zeta$. Indeed, this modification does not alter the structure of (5.3.22).

5.3.4 Illustration: the AARC of Multi-Period Inventory Affected by Uncertain Demand

We are about to illustrate the AARC methodology by its application to the simple multi-product multi-period inventory model presented in Example 5.1 (see also p. 211).

Building the AARC of (5.2.3). We first decide on the information base of the "actual decisions" — vectors w_t of replenishment orders of instants $t = 1, ..., N$. Assuming that the part of the uncertain data, (i.e., of the demand trajectory $\zeta = \zeta^N = [\zeta_1; ...; \zeta_N]$) that becomes known when the decision on w_t should be made is the vector $\zeta^{t-1} = [\zeta_1; ...; \zeta_{t-1}]$ of the demands in periods preceding time t, we introduce affine decision rules

$$w_t = \omega_t + \Omega_t \zeta^{t-1} \tag{5.3.25}$$

for the orders; here ω_t, Ω_t form the coefficients of the decision rules we are seeking.

The remaining variables in (5.2.3), with a single exception, are analysis variables, and we allow them to be arbitrary affine functions of the entire demand trajectory ζ^N:

$$\begin{array}{lll} x_t = \xi_t + \Xi_t\zeta^N,\ t = 2, ..., N+1 & \text{[states]} \\ y_t = \eta_t + H_t\zeta^N,\ t = 1, ..., N & \text{[upper bounds on } [x_t]_+] \\ z_t = \pi_t + \Pi_t\zeta^N,\ t = 1, ..., N & \text{[upper bounds on } [x_t]_-]. \end{array} \tag{5.3.26}$$

The only remaining variable C — the upper bound on the inventory management cost we intend to minimize — is considered as non-adjustable.

We now plug the affine decision rules in the objective and the constraints of (5.2.3), and require the resulting relations to be satisfied for all realizations of the uncertain data ζ^N from a given uncertainty set $\mathcal{Z}$, thus arriving at the AARC of our inventory model:

$$\begin{array}{l} \text{minimize} \qquad C \\ \text{s.t. } \forall \zeta^N \in \mathcal{Z}: \\ C \geq \sum_{t=1}^N \left[c_{\mathrm{h},t}^T[\eta_t + H_t\zeta^N] + c_{\mathrm{b},t}^T[\pi_t + \Pi_t\zeta^N] + c_{\mathrm{o},t}^T[\omega_t + \Omega_t\zeta^{t-1}] \right] \\ \xi_t + \Xi_t\zeta^N = \left\{ \begin{array}{l} \xi_{t-1} + \Xi_{t-1}\zeta^N + [\omega_t + \Omega_t\zeta^{t-1}] - \zeta_t,\ 2 \leq t \leq N \\ x_0 + \omega_1 - \zeta_1,\ t = 1 \end{array} \right. \\ \eta_t + H_t\zeta^N \geq 0,\ \eta_t + H_t\zeta^N \geq \xi_t + \Xi_t\zeta^N,\ 1 \leq t \leq N \\ \pi_t + \Pi_t\zeta^N \geq 0,\ \pi_t + \Pi_t\zeta^N \geq -\xi_t - \Xi_t\zeta^N,\ 1 \leq t \leq N \\ \underline{w}_t \leq \omega_t + \Omega_t\zeta^{t-1} \leq \overline{w}_t,\ 1 \leq t \leq N \\ q^T\left[\eta_t + H_t\zeta^N\right] \leq r \end{array} \tag{5.3.27}$$

the variables being C and the coefficients $\omega_t, \Omega_t, ..., \pi_t, \Pi_t$ of the affine decision rules.

We see that the problem in question has fixed recourse (it always is so when the uncertainty affects just the constant terms in conic constraints) and is nothing but an explicit semi-infinite LO program. Assuming the uncertainty set $\mathcal{Z}$ to be computationally tractable, we can invoke Theorem 1.1 and reformulate this semi-infinite problem as a computationally tractable one. For example, with *box uncertainty*:

$$\mathcal{Z} = \{\zeta^N \in \mathbb{R}_+^{N\times d} : \underline{\zeta}_t \leq \zeta_t \leq \overline{\zeta}_t,\ 1 \leq t \leq N\},$$

the semi-infinite LO program (5.3.27) can be immediately rewritten as an explicit "certain" LO program. Indeed, after replacing the semi-infinite coordinate-wise vector inequalities/equations appearing in (5.3.27) by equivalent systems of scalar semi-infinite inequalities/equations and representing the semi-infinite linear equations by pairs of opposite semi-infinite linear inequalities, we end up with a semi-infinite optimization program with a certain linear objective and finitely many constraints of the form

$$\forall \left(\zeta_t^i \in [\underline{\zeta}_t^i, \overline{\zeta}_t^i], t \leq N, i \leq d\right) : p^\ell[y] + \sum_{i,t} \zeta_t^i p_{ti}^\ell[y] \leq 0$$

(ℓ is the serial number of the constraint, y is the vector comprised of the decision variables in (5.3.27), and $p^\ell[y]$, $p_{ti}^\ell[y]$ are given affine functions of y). The above semi-infinite constraint can be represented by a system of linear inequalities

$$\begin{array}{l} \underline{\zeta}_t^i p_{ti}^\ell[y] \leq u_{ti}^\ell \\ \overline{\zeta}_t^i p_{ti}^\ell[y] \leq u_{ti}^\ell \\ p^\ell[y] + \sum_{t,i} u_{ti}^\ell \leq 0, \end{array}$$

in variables y and additional variables u_{ti}^ℓ. Putting all these systems of inequalities together and augmenting the resulting system of linear constraints with our original objective to be minimized, we end up with an explicit LO program that is equivalent to (5.3.27).

Some remarks are in order:

1. We could act similarly when building the AARC of any uncertain LO problem with fixed recourse and "well-structured" uncertainty set, e.g., one given by an explicit polyhedral/conic quadratic/semidefinite representation. In the latter case, the resulting tractable reformulation of the AARC would be an explicit linear/conic quadratic/semidefinite program of sizes that are polynomial in the sizes of the instances and in the size of conic description of the uncertainty set. Moreover, the "tractable reformulation" of the AARC can be built automatically, by a kind of compilation.

2. Note how flexible the AARC approach is: we could easily incorporate additional constraints, (e.g., those forbidding backlogged demand, expressing lags in acquiring information on past demands and/or lags in executing the replenishment orders, etc.). Essentially, the only thing that matters is that we are dealing with an uncertain LO problem with fixed recourse. This is in sharp contrast with the ARC. As we have already mentioned, there is, essentially, only one optimization technique — Dynamic Programming — that with luck can be used to process the (general-type) ARC numerically. To do so, one needs indeed a lot of luck — to be "computationally tractable," Dynamic Programming imposes many highly "fragile" limitations on the structure and the sizes of instances. For example, the effort to solve the "true" ARC of our toy Inventory problem by Dynamic Programming blows up exponentially with the number of products d (we can say that $d = 4$ is already "too big"); in contrast to this, the AARC does not suffer of "curse of dimensionality" and scales reasonably well with problem's sizes.

3. Note that we have no difficulties processing uncertainty-affected *equality constraints* (such as state equations above) — this is something that we cannot afford with the usual — non-adjustable — RC (how could an equation remain valid when the variables are kept constant, and the coefficients are perturbed?).

4. Above, we "immunized" affine decision rules against uncertainty in the worst-case-oriented fashion — by requiring the constraints to be satisfied for *all* realizations of uncertain data from $\mathcal{Z}$. Assuming ζ to be random, we could replace the worst-case interpretation of the uncertain constraints with their chance constrained interpretation. To process the "chance constrained" AARC, we could use all the "chance constraint machinery" we have developed so far for the RC, exploiting the fact that for fixed recourse there is no essential difference between the structure of the RC and that of the AARC.

Of course, all the nice properties of the AARC we have just mentioned have their price — in general, as in our toy inventory example, we have no idea of how much we lose in terms of optimality when passing from general decision rules to affine rules. At present, we are not aware of any theoretical tools for evaluating such a loss. Moreover, it is easy to build examples showing that sticking to affine decision rules can indeed be costly; it even may happen that the AARC is infeasible, while the ARC is not. Much more surprising is the fact that there are meaningful situations where the AARC is unexpectedly good. Here we present a single simple example.

Consider our inventory problem in the single-product case with added constraints that no backlogged demand is allowed and that the amount of product in the inventory should remain between two given positive bounds. Assuming box uncertainty in the demand, the "true" ARC of the uncertain problem is well within the grasp of Dynamic Programming, and thus we can measure the "non-optimality" of affine decision rules experimentally — by comparing the optimal values of the true ARC with those of the AARC as well as of the non-adjustable RC. To this end, we generated at random several hundreds of data sets for the problem with time horizon

$N = 10$ and filtered out all data sets that led to infeasible ARC (it indeed can be infeasible due to the presence of upper and lower bounds on the inventory level and the fact that we forbid backlogged demand). We did our best to get as rich a family of examples as possible — those with time-independent and with time-dependent costs, various levels of demand uncertainty (from 10% to 50%), etc. We then solved ARCs, AARCs and RCs of the remaining "well-posed" problems — the ARCs by Dynamic Programming, the AARCs and RCs — by reduction to explicit LO programs. The number of "well-posed" problems we processed was 768, and the results were as follows:

1. To our great surprise, *in every one of the 768 cases we have analyzed, the computed optimal values of the "true" ARC and the AARC were identical.* Thus, there is an "experimental evidence" that in the case of our single-product inventory problem, the affine decision rules allow one to reach "true optimality."

   Quite recently, D. Bertsimas, D. Iancu and P. Parrilo have demonstrated [30] that the above "experimental evidence" has solid theoretical reasons, specifically, they have established the following remarkable and unexpected result:

   Consider the multi-stage uncertainty-affected decision making problem

$$\left\{ \min_{C,x,w} \left\{ C : \begin{array}{l} C \geq \sum_{t=1}^{N} [c_t w_t + h_t(x_t)] \\ x_t = \alpha_t x_{t-1} + \beta_t w_t + \gamma_t \zeta_t,\ 1 \leq t \leq N \\ \underline{w}_t \leq w_t \leq \overline{w}_t,\ 1 \leq t \leq N \end{array} \right\} : \zeta^N = [\zeta_1; ...; \zeta_N] \in \mathcal{Z} \subset \mathbb{R}^N \right\}$$

   with uncertain data $\zeta^N = [\zeta_1; ...; \zeta_N]$, the variables in the problem being C (non-adjustable), w_t (allowed to depend on the "past demands" ζ^{t-1}), $1 \leq t \leq N$, and x_t (fully adjustable – allowed to depend on ζ^N); here the functions $h_t(\cdot)$ are convex functions on the axis. Assume, further, that $\mathcal{Z}$ is a box, and consider the ARC and the AARC of the problem, that is, the infinite-dimensional problems

$$\min_{C,x(\cdot),w(\cdot),z(\cdot)} \left\{ C : \begin{array}{l} C \geq \sum_{t=1}|^N [c_t w_t(\zeta^{t-1}) + z_t(\zeta^N)] \\ x_t(\zeta^N) = \alpha_t x_{t-1}(\zeta^N) + \beta_t w_t(\zeta^{t-1}) + \gamma_t \zeta_t,\ 1 \leq t \leq N \\ z_t(\zeta^N) \geq h_t(x_t(\zeta^N)), 1 \leq t \leq N \\ \underline{w}_t \leq w_t(\zeta^{t-1}) \leq \overline{w}_t, 1 \leq t \leq N \end{array} \right\} \forall \zeta^N \in \mathcal{Z} \right\}$$

   where the optimization is taken over arbitrary functions $x_t(\zeta^N)$, $w_t(\zeta^{t-1})$, $z_t(\zeta^N)$ (ARC) and over affine functions $x_t(\zeta^T$, $z_t(\zeta^t)$, $w_t(\zeta^{t-1})$ (AARC); here slacks $z_t(\cdot)$ are upper bounds on costs $h_t(x_t(\cdot))$. The the optimal solution to the AARC is an optimal solution to the ARC as well.

   Note that the single product version of our Inventory Management problem satisfies the premise of the latter result, provided that the uncertainty set is a box (as is the case in the experiments we have reported), and the corresponding functions $h_t(\cdot)$ are not only convex, but also piecewise linear, the domain of h_t being $x \leq r/q$; in this case what was called AARC of the problem, is nothing but our AARC (5.3.26) – (5.3.27) where we further restrict $z_t(\cdot)$ to be identical to $y_t(\cdot)$. Thus, in the single product case the AARC of the Inventory Management problem in question is *equivalent* to its ARC.

   Toe the best of our knowledge, the outlined result of Bertsimas, Iancu and Parrilo yields the only known for the time being generic example of a meaningful multi-stage uncertainty affected decision making problem where the affine decision rules are provably optimal. This remarkable result is very "fragile," e.g., it cannot be extended on multi-product inventory, or on the case when aside of bounds on replenishment orders in every period there are

Range of $\frac{\text{Opt(RC)}}{\text{Opt(AARC)}}$	1	$(1,2]$	$(2,10]$	$(10,1000]$	∞
Frequency in the sample	38%	23%	14%	11%	15%

Table 5.1: Experiments with ARCs, AARCs and RCs of randomly generated single-product inventory problems affected by uncertain demand.

bounds on cumulative replenishment orders, etc. It should be added that the phenomenon in question seems to be closely related to our intention to optimize the *guaranteed*, (i.e., the worst-case, w.r.t. demand trajectories from the uncertainty set), inventory management cost. When optimizing the "average" cost, the ARC frequently becomes significantly less expensive than the AARC.[2]

2. The (equal to each other) optimal values of the ARC and the AARC in many cases were much better than the optimal value of the RC, as it is seen from table 5.1. In particular, in 40% of the cases the RC was at least twice as bad in terms of the (worst-case) inventory management cost as the ARC/AARC, and in 15% of the cases the RC was in fact infeasible.

The bottom line is twofold. First, we see that in multi-stage decision making there exist meaningful situations where the AARC, while "not less computationally tractable" than the RC, is much more flexible and much less conservative. Second, the AARC is not necessarily "significantly inferior" as compared to the ARC.

5.4 Adjustable Robust Optimization and Synthesis of Linear Controllers

While the usefulness of affine decision rules seems to be heavily underestimated in the "OR-style multi-stage decision making," they play one of the central roles in Control. Our next goal is to demonstrate that the use of AARC can render important Control implications.

5.4.1 Robust Affine Control over Finite Time Horizon

Consider a discrete time linear dynamical system

$$\begin{array}{rcl} x_0 &=& z \\ x_{t+1} &=& A_t x_t + B_t u_t + R_t d_t \ \ ,\ t=0,1,... \\ y_t &=& C_t x_t + D_t d_t \end{array} \tag{5.4.1}$$

where $x_t \in \mathbb{R}^{n_x}$, $u_t \in \mathbb{R}^{n_u}$, $y_t \in \mathbb{R}^{n_y}$ and $d_t \in \mathbb{R}^{n_d}$ are the state, the control, the output and the exogenous input (disturbance) at time t, and A_t, B_t, C_t, D_t, R_t are known matrices of appropriate dimension.

[2] On this occasion, it is worthy of mention that affine decision rules were proposed many years ago, in the context of Multi-Stage Stochastic Programming, by A. Charnes. In Stochastic Programming, people are indeed interested in optimizing the expected value of the objective, and soon it became clear that in this respect, the affine decision rules can be pretty far from being optimal. As a result, the simple — and extremely useful from the computational perspective — concept of affine decision rules remained completely forgotten for many years.

Notational convention. Below, given a sequence of vectors $e_0, e_1, ...$ and an integer $t \geq 0$, we denote by e^t the initial fragment of the sequence: $e^t = [e_0; ...; e_t]$. When t is negative, e^t, by definition, is the zero vector.

Affine control laws. A typical problem of (finite-horizon) Linear Control associated with the "open loop" system (5.4.1) is to "close" the system by a non-anticipative affine output-based control law

$$u_t = g_t + \sum_{\tau=0}^{t} G_{t\tau} y_\tau \tag{5.4.2}$$

(here the vectors g_t and matrices $G_{t\tau}$ are the parameters of the control law). The closed loop system (5.4.1), (5.4.2) is required to meet prescribed design specifications. We assume that these specifications are represented by a system of linear inequalities

$$Aw^N \leq b \tag{5.4.3}$$

on the *state-control trajectory* $w^N = [x_0; ...; x_{N+1}; u_0; ...; u_N]$ over a given finite time horizon $t = 0, 1, ..., N$.

An immediate observation is that for a given control law (5.4.2) the dynamics (5.4.1) specifies the trajectory as an affine function of the initial state z and the sequence of disturbances $d^N = (d_0, ..., d_N)$:

$$w^N = w_0^N[\gamma] + W^N[\gamma]\zeta, \; \zeta = (z, d^N),$$

where $\gamma = \{g_t, G_{t\tau}, 0 \leq \tau \leq t \leq N\}$, is the "parameter" of the underlying control law (5.4.2). Substituting this expression for w^N into (5.4.3), we get the following system of constraints on the decision vector γ:

$$A\left[w_0^N[\gamma] + W^N[\gamma]\zeta\right] \leq b. \tag{5.4.4}$$

If the disturbances d^N and the initial state z are certain, (5.4.4) is "easy" — it is a system of constraints on γ with certain data. Moreover, in the case in question we lose nothing by restricting ourselves with "off-line" control laws (5.4.2) — those with $G_{t\tau} \equiv 0$; when restricted onto this subspace, let it be called Γ, in the γ space, the function $w_0^N[\gamma] + W^N[\gamma]\zeta$ turns out to be bi-affine in γ and in ζ, so that (5.4.4) reduces to a system of explicit linear inequalities on $\gamma \in \Gamma$. Now, when the disturbances and/or the initial state are *not* known in advance, (which is the only case of interest in Robust Control), (5.4.4) becomes an uncertainty-affected system of constraints, and we could try to solve the system in a robust fashion, e.g., to seek a solution γ that makes the constraints feasible for all realizations of $\zeta = (z, d^N)$ from a given uncertainty set $\mathcal{ZD}^N$, thus arriving at the system of semi-infinite scalar constraints

$$A\left[w_0^N[\gamma] + W^N[\gamma]\zeta\right] \leq b \; \forall \zeta \in \mathcal{ZD}^N. \tag{5.4.5}$$

Unfortunately, the semi-infinite constraints in this system are *not* bi-affine, since the dependence of w_0^N, W^N on γ is highly nonlinear, unless γ is restricted to vary in Γ. Thus, when seeking "on-line" control laws (those where $G_{t\tau}$ can be nonzero), (5.4.5) becomes a system of highly nonlinear semi-infinite constraints and as such seems to be severely computationally intractable (the feasible set corresponding to (5.4.4) can be in fact nonconvex). One possibility to circumvent this difficulty would be to switch from control laws that are affine in the outputs y_t to those affine in disturbances and the initial state (cf. approach of [51]). This, however, could be problematic in the situations when we do not observe z and d_t directly. The good news is that we can overcome this difficulty without requiring d_t and z to be observable, the remedy being a suitable re-parameterization of affine control laws.

5.4.2 Purified-Output-Based Representation of Affine Control Laws and Efficient Design of Finite-Horizon Linear Controllers

Imagine that in parallel with controlling (5.4.1) with the aid of a non-anticipating output-based control law $u_t = U_t(y_0, ..., y_t)$, we run the *model* of (5.4.1) as follows:

$$\begin{array}{rcl} \widehat{x}_0 &=& 0 \\ \widehat{x}_{t+1} &=& A_t \widehat{x}_t + B_t u_t \\ \widehat{y}_t &=& C_t \widehat{x}_t \\ v_t &=& y_t - \widehat{y}_t. \end{array} \tag{5.4.6}$$

Since we know past controls, we can run this system in an "on-line" fashion, so that the *purified output* v_t becomes known when the decision on u_t should be made. An immediate observation is that *the purified outputs are completely independent of the control law in question — they are affine functions of the initial state and the disturbances* $d_0, ..., d_t$, *and these functions are readily given by the dynamics of* (5.4.1).

Indeed, from the descriptions of the open-loop system and the model, it follows that the differences $\delta_t = x_t - \widehat{x}_t$ evolve with time according to the equations

$$\begin{array}{rcl} \delta_0 &=& z \\ \delta_{t+1} &=& A_t + R_t d_t,\ t = 0, 1, ... \end{array}$$

while

$$v_t = C_t \delta_t + D_t d_t.$$

From these relations it follows that

$$v_t = \mathcal{V}_t^d d^t + \mathcal{V}_t^z z \tag{5.4.7}$$

with matrices $\mathcal{V}_t^d$, $\mathcal{V}_t^z$ depending solely on the matrices $A_\tau, B_\tau, ...,\ 0 \leq \tau \leq t$, and readily given by these matrices.

Now, it was mentioned that $v_0, ..., v_t$ are known when the decision on u_t should be made, so that we can consider *purified-output-based* (POB) affine control laws

$$u_t = h_t + \sum\nolimits_{\tau=0}^t H_{t\tau} v_\tau.$$

The complete description of the dynamical system "closed" by this control is

$$\begin{array}{|l|} \hline \underline{\text{plant:}} \\ (a): \left\{ \begin{array}{rcl} x_0 &=& z \\ x_{t+1} &=& A_t x_t + B_t u_t + R_t d_t \\ y_t &=& C_t x_t + D_t d_t \end{array} \right. \\ \hline \underline{\text{model:}} \\ (b): \left\{ \begin{array}{rcl} \widehat{x}_0 &=& 0 \\ \widehat{x}_{t+1} &=& A_t \widehat{x}_t + B_t u_t \\ \widehat{y}_t &=& C_t \widehat{x}_t \end{array} \right. \\ \hline \underline{\text{purified outputs:}} \\ (c): \quad v_t \;=\; y_t - \widehat{y}_t \\ \hline \underline{\text{control law:}} \\ (d): \quad u_t \;=\; h_t + \sum\limits_{\tau=0}^{t} H_{t\tau} v_\tau \\ \hline \end{array} \tag{5.4.8}$$

The main result. We are about to prove the following simple and fundamental fact:

Theorem 5.3

(i) *For every affine control law in the form of* (5.4.2)*, there exists a control law in the form of* (5.4.8.d) *that, whatever be the initial state and a sequence of inputs, results in exactly the same state-control trajectories of the closed loop system;*

(ii) *Vice versa, for every affine control law in the form of* (5.4.8.d)*, there exists a control law in the form of* (5.4.2) *that, whatever be the initial state and a sequence of inputs, results in exactly the same state-control trajectories of the closed loop system;*

(iii) [bi-affinity] *The state-control trajectory w^N of closed loop system* (5.4.8) *is affine in z, d^N when the parameters $\eta = \{h_t, H_{t\tau}\}_{0\leq\tau\leq t\leq N}$ of the underlying control law are fixed, and is affine in η when z, d^N are fixed:*

$$w^N = \omega[\eta] + \Omega_z[\eta]z + \Omega_d[\eta]d^N \tag{5.4.9}$$

for some vectors $\omega[\eta]$ and matrices $\Omega_z[\eta]$, $\Omega_d[\eta]$ depending affinely on η.

Proof. (i): Let us fix an affine control law in the form of (5.4.2), and let $x_t = X_t(z, d^{t-1})$, $u_t = U_t(z, d^t)$, $y_t = Y_t(z, d^t)$, $v_t = V_t(z, d^t)$ be the corresponding states, controls, outputs, and purified outputs. To prove (i) it suffices to show that for every $t \geq 0$ with properly chosen vectors q_t and matrices $Q_{t\tau}$ one has

$$\forall (z, d^t) : Y_t(z, d^t) = q_t + \sum_{\tau=0}^{t} Q_{t\tau} V_\tau(z, d^\tau). \tag{I$_t$}$$

Indeed, given the validity of these relations and taking into account (5.4.2), we would have

$$U_t(z, d^t) \equiv g_t + \sum_{\tau=0}^{t} G_{t\tau} Y_\tau(z, d^\tau) \equiv h_t + \sum_{\tau=0}^{t} H_{t\tau} V(z, d^\tau) \tag{II$_t$}$$

with properly chosen h_t, $H_{t\tau}$, so that the control law in question can indeed be represented as a linear control law via purified outputs.

We shall prove (I$_t$) by induction in t. The base $t = 0$ is evident, since by (5.4.8.a–c) we merely have $Y_0(z, d^0) \equiv V_0(z, d^0)$. Now let $s \geq 1$ and assume that relations (I$_t$) are valid for $0 \leq t < s$. Let us prove the validity of (I$_s$). From the validity of (I$_t$), $t < s$, it follows that the relations (II$_t$), $t < s$, take place, whence, by the description of the model system, $\widehat{x}_s = \widehat{X}_s(z, d^{s-1})$ is affine in the purified outputs, and consequently the same is true for the model outputs $\widehat{y}_s = \widehat{Y}_s(z, d^{s-1})$:

$$\widehat{Y}_s(z, d^{s-1}) = p_s + \sum_{\tau=0}^{s-1} P_{s\tau} V_\tau(z, d^\tau).$$

We conclude that with properly chosen p_s, $P_{s\tau}$ we have

$$Y_s(z, d^s) \equiv \widehat{Y}_s(z, d^{s-1}) + V_s(z, d^s) = p_s + \sum_{\tau=0}^{s-1} P_{s\tau} V_\tau(z, d^\tau) + V_s(z, d^s),$$

as required in (I$_s$). Induction is completed, and (i) is proved.

(ii): Let us fix a linear control law in the form of (5.4.8.d), and let $x_t = X_t(z, d^{t-1})$, $\widehat{x}_t = \widehat{X}_t(z, d^{t-1})$, $u_t = U_t(z, d^t)$, $y_t = Y_t(z, d^t)$, $v_t = V_t(z, d^t)$ be the corresponding actual and model states, controls, and actual and purified outputs. We should verify that the state-control dynamics in question can be obtained from an appropriate control law in the form of (5.4.2). To this end, similarly to the proof of (i), it suffices to show that for every $t \geq 0$ one has

$$V_t(z, d^t) \equiv q_t + \sum_{\tau=0}^{t} Q_{t\tau} Y_\tau(z, d^\tau) \qquad (\mathrm{III}_t)$$

with properly chosen q_t, $Q_{t\tau}$. We again apply induction in t. The base $t = 0$ is again trivially true due to $V_0(z, d^0) \equiv Y_0(z, d^0)$. Now let $s \geq 1$, and assume that relations (III_t) are valid for $0 \leq t < s$, and let us prove that (III_s) is valid as well. From the validity of (III_t), $t < s$, and from (5.4.8.d) it follows that

$$t < s \Rightarrow U_t(z, d^t) = c_t + \sum_{\tau=0}^{t} C_{t\tau} Y_\tau(z, d^\tau)$$

with properly chosen c_t and $C_{t\tau}$. From these relations and the description of the model system it follows that its state $\widehat{X}_s(z, d^{s-1})$ at time s, and therefore the model output $\widehat{Y}_s(z, d^{s-1})$, are affine functions of $Y_0(z, d^0)$,..., $Y_{s-1}(z, d^{s-1})$:

$$\widehat{Y}_s(z, d^{s-1}) = p_s + \sum_{\tau=0}^{s-1} P_{s\tau} Y_\tau(z, d^\tau)$$

with properly chosen p_s, $P_{s\tau}$. It follows that

$$V_s(z, d^s) \equiv Y_s(z, d^s) - \widehat{Y}_s(z, d^{s-1}) = Y_s(z, d^s) - p_s - \sum_{\tau=0}^{s-1} P_{s\tau} Y_\tau(z, d^\tau),$$

as required in (III_s). Induction is completed, and (ii) is proved.

(iii): For $0 \leq s \leq t$ let

$$A_s^t = \begin{cases} \prod_{r=s}^{t-1} A_r, & s < t \\ I, & s = t \end{cases}$$

Setting $\delta_t = x_t - \widehat{x}_t$, we have by (5.4.8.$a$–$b$)

$$\delta_{t+1} = A_t \delta_t + R_t d_t,\ \delta_0 = z \Rightarrow \delta_t = A_0^t z + \sum_{s=0}^{t-1} A_{s+1}^t R_s d_s$$

(from now on, sums over empty index sets are zero), whence

$$v_\tau = C_\tau \delta_\tau + D_\tau d_\tau = C_\tau A_0^\tau z + \sum_{s=0}^{\tau-1} C_\tau A_{s+1}^\tau R_s d_s + D_\tau d_\tau. \qquad (5.4.10)$$

Therefore control law (5.4.8.d) implies that

$$\begin{array}{rcl} u_t &=& h_t + \sum\limits_{\tau=0}^{t} H_{t\tau} v_\tau = \underbrace{h_t}_{\nu_t[\eta]} + \underbrace{\left[\sum\limits_{\tau=0}^{t} H_{t\tau} C_\tau A_0^\tau\right]}_{N_t[\eta]} z \\ && + \sum\limits_{s=0}^{t-1} \underbrace{\left[H_{ts} D_s + \sum\limits_{\tau=s+1}^{t} H_{t\tau} C_\tau A_{s+1}^\tau R_s\right]}_{N_{ts}[\eta]} d_s + \underbrace{H_{tt} D_t}_{N_{tt}[\eta]} d_t \\ &=& \nu_t[\eta] + N_t[\eta] z + \sum\limits_{s=0}^{t} N_{ts}[\eta] d_s, \end{array} \tag{5.4.11}$$

whence, invoking (5.4.8.a),

$$\begin{array}{rcl} x_t &=& A_0^t z + \sum\limits_{\tau=0}^{t-1} A_{\tau+1}^t [B_\tau u_\tau + R_\tau d_\tau] = \underbrace{\left[\sum\limits_{\tau=0}^{t-1} A_{\tau+1}^t B_\tau h_t\right]}_{\mu_t[\eta]} \\ && + \underbrace{\left[A_0^t + \sum\limits_{\tau=0}^{t-1} A_{\tau+1}^t B_\tau N_\tau[\eta]\right]}_{M_t[\eta]} z \\ && + \sum\limits_{s=0}^{t-1} \underbrace{\left[\sum\limits_{\tau=s}^{t-1} A_{\tau+1}^t B_\tau N_{\tau s}[\eta] + A_{s+1}^t B_s R_s\right]}_{M_{ts}[\eta]} d_s \\ &=& \mu_t[\eta] + M_t[\eta] z + \sum\limits_{s=0}^{t-1} M_{ts}[\eta] d_s. \end{array} \tag{5.4.12}$$

We see that the states x_t, $0 \leq t \leq N+1$, and the controls u_t, $0 \leq t \leq N$, of the closed loop system (5.4.8) are affine functions of z, d^N, and the corresponding "coefficients" $\mu_t[\eta]$,...,$N_{ts}[\eta]$ are affine vector- and matrix-valued functions of the parameters $\eta = \{h_t, H_{t\tau}\}_{0 \leq \tau \leq t \leq N}$ of the underlying control law (5.4.8.d). □

The consequences. The representation (5.4.8.d) of affine control laws is incomparably better suited for design purposes than the representation (5.4.2), since, as we know from Theorem 5.3.(iii), *with controller* (5.4.8.d), *the state-control trajectory* w^N *becomes bi-affine in* $\zeta = (z, d^N)$ *and in the parameters* $\eta = \{h_t, H_{t\tau}, 0 \leq \tau \leq t \leq N\}$ *of the controller:*

$$w^N = \omega^N[\eta] + \Omega^N[\eta]\zeta \tag{5.4.13}$$

with vector- and matrix-valued functions $\omega^N[\eta]$, $\Omega^N[\eta]$ *affinely depending on* η *and readily given by the dynamics* (5.4.1). Substituting (5.4.13) into (5.4.3), we arrive at the system of semi-infinite *bi-affine* scalar inequalities

$$A\left[\omega^N[\eta] + \Omega^N[\eta]\zeta\right] \leq b \tag{5.4.14}$$

in variables η, and can use the tractability results from lectures 1, 4 in order to solve efficiently the RC/GRC of this uncertain system of scalar linear constraints. For example, we can process efficiently the GRC setting of the semi-infinite constraints (5.4.13)

$$\begin{array}{r} a_i^T\left[\omega^N[\eta] + \Omega^N[\eta][z; d^N]\right] - b_i \leq \alpha_i^z \mathrm{dist}(z, \mathcal{Z}) + \alpha_d^i \mathrm{dist}(d^N, \mathcal{D}^N) \\ \forall [z; d^N]\ \forall i = 1, ..., I \end{array} \tag{5.4.15}$$

where $\mathcal{Z}$, $\mathcal{D}^N$ are "good," (e.g., given by strictly feasible semidefinite representations), closed convex normal ranges of z, d^N, respectively, and the distances are defined via the $\|\cdot\|_\infty$ norms (this setting corresponds to the "structured" GRC, see Definition 4.3). By the results of section 4.3, system (5.4.15) is equivalent to the system of constraints

$$\begin{array}{ll} \forall(i, 1 \le i \le I): & \\ (a) \quad a_i^T\left[\omega^N[\eta] + \Omega^N[\eta][z;d^N]\right] - b_i \le 0 \;\forall [z;d^N] \in \mathcal{Z}\times\mathcal{D}^N & \\ (b) \quad \|a_i^T\Omega_z^N[\eta]\|_1 \le \alpha_z^i \qquad (c) \quad \|a_i^T\Omega_d^N[\eta]\|_1 \le \alpha_d^i, & \end{array} \tag{5.4.16}$$

where $\Omega^N[\eta] = \left[\Omega_z^N[\eta], \Omega_d^N[\eta]\right]$ is the partition of the matrix $\Omega^N[\eta]$ corresponding to the partition $\zeta = [z;d^N]$. Note that in (5.4.16), the semi-infinite constraints (a) admit explicit semidefinite representations (Theorem 1.1), while constraints $(b-c)$ are, essentially, just linear constraints on η and on α_z^i, α_d^i. As a result, (5.4.16) can be thought of as a computationally tractable system of convex constraints on η *and on the sensitivities* α_z^i, α_d^i, and we can minimize under these constraints a "nice," (e.g., convex), function of η and the sensitivities. Thus, after passing to the POB representation of affine control laws, we can process efficiently specifications expressed by systems of linear inequalities, to be satisfied in a robust fashion, on the (finite-horizon) state-control trajectory.

The just summarized nice consequences of passing to the POB control laws are closely related to the tractability of AARCs of uncertain LO problems with fixed recourse, specifically, as follows. Let us treat the state equations (5.4.1) coupled with the design specifications (5.4.3) as a system of uncertainty-affected linear constraints on the state-control trajectory w, the uncertain data being $\zeta = [z;d^N]$. Relations (5.4.10) say that the purified outputs v_t are known in advance, completely independent of what the control law in use is, *linear* functions of ζ. With this interpretation, a POB control law becomes a collection of affine decision rules that specify the decision variables u_t as affine functions of $P_t\zeta \equiv [v_0;v_1;...;v_t]$ and simultaneously, via the state equations, specify the states x_t as affine functions of $P_{t-1}\zeta$. Thus, when looking for a POB control law that meets our design specifications in a robust fashion, we are doing nothing but solving the RC (or the GRC) of an uncertain LO problem in affine decision rules possessing a prescribed "information base." On closest inspection, this uncertain LO problem is with fixed recourse, and therefore its robust counterparts are computationally tractable.

Remark 5.2 *It should be stressed that the re-parameterization of affine control laws underlying Theorem 5.3 (and via this Theorem — the nice tractability results we have just mentioned) is nonlinear. As a result, it can be of not much use when we are optimizing over affine control laws satisfying additional restrictions rather than over all affine control laws.*

Assume, e.g., that we are seeking control in the form of a simple output-based linear feedback:

$$u_t = G_t y_t.$$

This requirement is just a system of simple linear constraints on the parameters of the control law in the form of (5.4.2), which, however, does not help much, since, as we have already explained, optimization over control laws in this form is by itself difficult. And when passing to affine control laws in the form of (5.4.8.d), the requirement that our would-be control should be a linear output-based feedback becomes a system of highly nonlinear constraints on our new design parameters η, and the synthesis again turns out to be difficult.

Example: Controlling finite-horizon gains. Natural design specification pertaining to finite-horizon Robust Linear Control are in the form of bounds on finite-horizon *gains* $\mathrm{z2x}^N$, $\mathrm{z2u}^N$, $\mathrm{d2x}^N$, $\mathrm{d2u}^N$ defined as follows: with a linear, (i.e., with $h_t \equiv 0$) control law (5.4.8.d), the states x_t and the controls u_t are linear functions of z and d^N:

$$x_t = X_t^z[\eta]z + X_t^d[\eta]d^N,\ u_t = U_t^z[\eta]z + U_t^d[\eta]d^N$$

with matrices $X_t^z[\eta]$,...,$U_t^d[\eta]$ affinely depending on the parameters η of the control law. Given t, we can define the z to x_t *gains* and the *finite-horizon* z to x gain as $\mathrm{z2x}_t(\eta) = \max\limits_z\{\|X_t^z[\eta]z\|_\infty : \|z\|_\infty \leq 1\}$ and $\mathrm{z2x}^N(\eta) = \max\limits_{0\leq t\leq N} \mathrm{z2x}_t(\eta)$. The definitions of the z to u gains $\mathrm{z2u}_t(\eta)$, $\mathrm{z2u}^N(\eta)$ and the "disturbance to x/u" gains $\mathrm{d2x}_t(\eta)$, $\mathrm{d2x}^N(\eta)$, $\mathrm{d2u}_t(\eta)$, $\mathrm{d2u}^N(\eta)$ are completely similar, e.g., $\mathrm{d2u}_t(\eta) = \max\limits_{d^N}\{\|U_t^d[\eta]d^N\|_\infty : \|d^N\|_\infty \leq 1\}$ and $\mathrm{d2u}^N(\eta) = \max\limits_{0\leq t\leq N} \mathrm{d2u}_t(\eta)$. The finite-horizon gains clearly are nonincreasing functions of the time horizon N and have a transparent Control interpretation; e.g., $\mathrm{d2x}^N(\eta)$ ("peak to peak d to x gain") is the largest possible perturbation in the states x_t, $t = 0, 1, ..., N$ caused by a unit perturbation of the sequence of disturbances d^N, both perturbations being measured in the $\|\cdot\|_\infty$ norms on the respective spaces. Upper bounds on ^{N}-gains (and on *global* gains like $\mathrm{d2x}^\infty(\eta) = \sup_{N\geq 0} \mathrm{d2x}^N(\eta)$) are natural Control specifications. With our purified-output-based representation of linear control laws, the finite-horizon specifications of this type result in explicit systems of linear constraints on η and thus can be processed routinely via LO. For example, an upper bound $\mathrm{d2x}^N(\eta) \leq \lambda$ on $\mathrm{d2x}^N$ gain is equivalent to the requirement $\sum_j |(X_t^d[\eta])_{ij}| \leq \lambda$ for all i and all $t \leq N$; since X_t^d is affine in η, this is just a system of linear constraints on η and on appropriate slack variables. Note that imposing bounds on the gains can be interpreted as passing to the GRC (5.4.15) in the case where the "desired behavior" merely requires $w^N = 0$, and the normal ranges of the initial state and the disturbances are the origins in the corresponding spaces: $\mathcal{Z} = \{0\}$, $\mathcal{D}^N = \{0\}$.

Non-affine control laws

So far, we focused on synthesis of finite-horizon *affine* POB controllers. Acting in the spirit of section 5.3.2, we can handle also synthesis of *quadratic* POB control laws — those where every entry of u_t, instead of being affine in the purified outputs $v^t = [v_0; ...; v_t]$, is allowed to be a quadratic function of v^t. Specifically, assume that we want to "close" the open loop system (5.4.1) by a non-anticipating control law in order to ensure that the state-control trajectory w^N of the closed loop system satisfies a given system S of linear constraints in a robust fashion, that is, for all realizations of the "uncertain data" $\zeta = [z; d^N]$ from a given uncertainty set $\mathcal{Z}_\rho^N = \rho\mathcal{Z}^N$ ($\rho > 0$ is, as always, the uncertainty level, and $\mathcal{Z} \ni 0$ is a closed convex set of "uncertain data of magnitude ≤ 1"). Let us use a *quadratic* POB control law in the form of

$$u_t^i = h_{it}^0 + h_{i,t}^T v^t + \frac{1}{\rho}[v^t]^T H_{i,t} v^t, \tag{5.4.17}$$

where u_t^i is i-th coordinate of the vector of controls at instant t, and h_{it}^0, h_{it} and H_{it} are, respectively, real, vector, and matrix parameters of the control law.[3] On a finite time horizon $0 \leq t \leq N$, such a quadratic control law is specified by ρ and the finite-dimensional vector $\eta = \{h_{it}^0, h_{it}, H_{it}\}_{\substack{1\leq i\leq \dim u\\ 0\leq t\leq N}}$. Now note that the purified outputs are well defined for any non-anticipating control law, not necessary affine, and they are *independent of the control law linear*

[3]The specific way in which the uncertainty level ρ affects the controls is convenient technically and is of no practical importance, since "in reality" the uncertainty level is a known constant.

functions of $\zeta^t \equiv [z; d^t]$*. The coefficients of these linear functions are readily given by the data* $A_\tau, ..., D_\tau$, $0 \le \tau \le t$ (see (5.4.7)). With this in mind, we see that *the controls, as given by* (5.4.17)*, are quadratic functions of the initial state and the disturbances, the coefficients of these quadratic functions being affine in the vector* η *of parameters of our quadratic control law:*

$$u_t^i = \mathcal{U}_{it}^{(0)}[\eta] + [z; d^t]^T \mathcal{U}_{it}^{(1)}[\eta] + \frac{1}{\rho}[z; d^t]^T \mathcal{U}_{it}^{(2)}[\eta][z; d^t] \tag{5.4.18}$$

with *affine in* η reals/vectors/matrices $\mathcal{U}_{it}^{(\kappa)}[\eta]$, $\kappa = 0, 1, 2$. Plugging these representations of the controls into the state equations of the open loop system (5.4.1), we conclude that the states x_t^j of the closed loop system obtained by "closing" (5.4.1) by the quadratic control law (5.4.17), have the same "affine in η, quadratic in $[z; d^t]$" structure as the controls:

$$x_t^i = \mathcal{X}_{jt}^{(0)}[\eta] + [z; d^{t-1}]^T \mathcal{X}_{jt}^{(1)}[\eta] + \frac{1}{\rho}[z; d^{t-1}]^T \mathcal{X}_{jt}^{(2)}[\eta][z; d^{t-1}] \tag{5.4.19}$$

with affine in η reals/vectors/matrices $\mathcal{X}_{jt}^{(\kappa)}$, $\kappa = 0, 1, 2$.

Plugging representations (5.4.18), (5.4.19) into the system S of our target constraints, we end up with a system of semi-infinite constraints on the parameters η of the control law, specifically, the system

$$a_k[\eta] + 2\zeta^T p_k[\eta] + \frac{1}{\rho}\zeta^T R_k[\eta]\zeta \le 0 \,\forall \zeta = [z; d^N] \in \mathcal{Z}_\rho^N = \rho\mathcal{Z}^N, \; k = 1, ..., K, \tag{5.4.20}$$

where $a_k[\eta]$, $p_k[\eta]$ and $R_k[\zeta]$ are affine in η. Setting $P_k[\eta] = \left[\begin{array}{c|c} & p_k^T[\eta] \\ \hline p_k[\eta] & R_k[\eta] \end{array}\right]$, $\widehat{\zeta}_\rho[\zeta] = \left[\begin{array}{c|c} & \zeta^T \\ \hline \zeta & \zeta\zeta^T \end{array}\right]$ and denoting by $\widehat{\mathcal{Z}}_\rho^N$ the convex hull of the image of the set $\mathcal{Z}_\rho^N$ under the mapping $\zeta \mapsto \widehat{\zeta}_\rho[\zeta]$, system (5.4.20) can be rewritten equivalently as

$$a_k[\eta] + \mathrm{Tr}(P_k[\eta]\widehat{\zeta}) \le 0 \;\forall (\widehat{\zeta} \in \widehat{\mathcal{Z}}_\rho^N \equiv \rho\widehat{\mathcal{Z}}_1^N, k = 1, ..., K) \tag{5.4.21}$$

and we end up with a system of semi-infinite bi-affine scalar inequalities. From the results of section 5.3.2 it follows that this semi-infinite system:

- is computationally tractable, provided that $\mathcal{Z}^N$ is an ellipsoid $\{\zeta : \zeta^T Q\zeta \le 1\}$, $Q \succ 0$. Indeed, here $\widehat{\mathcal{Z}}_1^N$ is the semidefinite representable set
$$\{\left[\begin{array}{c|c} & \omega^T \\ \hline \omega & \Omega \end{array}\right] : \left[\begin{array}{c|c} 1 & \omega^T \\ \hline \omega & \Omega \end{array}\right] \succeq 0, \mathrm{Tr}(\Omega Q) \le 1\};$$

- admits a safe tractable approximation tight within the factor $\vartheta = O(1)\ln(J+1)$, provided that $\mathcal{Z}^N$ is the $\cap$-ellipsoidal uncertainty set $\{\zeta : \zeta^T Q_j \zeta \le 1, 1 \le j \le J\}$, where $Q_j \succeq 0$ and $\sum_j Q_j \succ 0$. This approximation is obtained when replacing the "true" uncertainty set $\widehat{\mathcal{Z}}_\rho^N$ with the semidefinite representable set
$$\mathcal{W}_\rho = \rho\{\left[\begin{array}{c|c} & \omega^T \\ \hline \omega & \Omega \end{array}\right] : \left[\begin{array}{c|c} 1 & \omega^T \\ \hline \omega & \Omega \end{array}\right] \succeq 0, \mathrm{Tr}(\Omega Q_j) \le 1, 1 \le j \le J\}$$
(recall that $\widehat{\mathcal{Z}}_\rho^N \subset \mathcal{W}_\rho \subset \widehat{\mathcal{Z}}_{\vartheta\rho}^N$).

5.4.3 Handling Infinite-Horizon Design Specifications

One might think that the outlined reduction of (discrete time) Robust Linear Control problems to Convex Programming, based on passing to the POB representation of affine control laws and deriving tractable reformulations of the resulting semi-infinite bi-affine scalar inequalities is intrinsically restricted to the case of finite-horizon control specifications. In fact our approach is well suited for handling infinite-horizon specifications — those imposing restrictions on the asymptotic behavior of the closed loop system. Specifications of the latter type usually have to do with the *time-invariant* open loop system (5.4.1):

$$\begin{array}{rcl} x_0 &=& z \\ x_{t+1} &=& Ax_t + Bu_t + Rd_t \ , \ t = 0, 1, ... \\ y_t &=& Cx_t + Dd_t \end{array} \tag{5.4.22}$$

From now on we assume that *the open loop system* (5.4.22) *is stable*, that is, the spectral radius of A is < 1 (in fact this restriction can be somehow circumvented, see below). Imagine that we "close" (5.4.22) by a *nearly time-invariant* POB control law *of order* k, that is, a law of the form

$$u_t = h_t + \sum\nolimits_{s=0}^{k-1} H_s^t v_{t-s}, \tag{5.4.23}$$

where $h_t = 0$ for $t \geq N_*$ and $H_\tau^t = H_\tau$ for $t \geq N_*$ for a certain *stabilization time* N_*. From now on, all entities with negative indices are set to 0. While the "time-varying" part $\{h_t, H_\tau^t, 0 \leq t < N_*\}$ of the control law can be used to adjust the finite-horizon behavior of the closed loop system, its asymptotic behavior is as if the law were time-invariant: $h_t \equiv 0$ and $H_\tau^t \equiv H_\tau$ for all $t \geq 0$. Setting $\delta_t = x_t - \widehat{x}_t$, $H^t = [H_0^t, ..., H_{k-1}^t]$, $H = [H_0, ..., H_{k-1}]$, the dynamics (5.4.22), (5.4.6), (5.4.23) is given by

$$\begin{array}{rcl} \overbrace{\begin{bmatrix} x_{t+1} \\ \delta_{t+1} \\ \delta_t \\ \vdots \\ \delta_{t-k+2} \end{bmatrix}}^{\omega_{t+1}} &=& \overbrace{\left[\begin{array}{c|cccc} A & BH_0^tC & BH_1^tC & \dots & BH_{k-1}^tC \\ \hline & A & & & \\ & & A & & \\ & & & \ddots & \\ & & & & A \end{array}\right]}^{A_+[H^t]} \omega_t \\ && + \overbrace{\left[\begin{array}{c|cccc} R & BH_0^tD & BH_1^tD & \dots & BH_{k-1}^tD \\ \hline & R & & & \\ & & R & & \\ & & & \ddots & \\ & & & & R \end{array}\right]}^{R_+[H^t]} \begin{bmatrix} d_t \\ d_t \\ d_{t-1} \\ \vdots \\ d_{t-k+1} \end{bmatrix} \\ && + \begin{bmatrix} Bh_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ t = 0, 1, 2, ..., \\ u_t &=& h_t + \sum_{\nu=0}^{k-1} H_\nu^t [C\delta_{t-\nu} + Dd_{t-\nu}]. \end{array} \tag{5.4.24}$$

We see that starting with time N_*, dynamics (5.4.24) is exactly as if the underlying control law were the time invariant POB law with the parameters $h_t \equiv 0$, $H^t \equiv H$. Moreover, since A is stable, we see that *system* (5.4.24) *is stable independently of the parameter* H *of the control* *law, and the resolvent* $\mathcal{R}_H(s) := (sI - A_+[H])^{-1}$ *of* $A_+[H]$ *is the affine in* H *matrix*

$$\left[\begin{array}{c|c|c|c|c} \mathcal{R}_A(s) & \mathcal{R}_A(s)BH_0C\mathcal{R}_A(s) & \mathcal{R}_A(s)BH_1C\mathcal{R}_A(s) & \ldots & \mathcal{R}_A(s)BH_{k-1}C\mathcal{R}_A(s) \\ \hline & \mathcal{R}_A(s) & & & \\ \hline & & \mathcal{R}_A(s) & & \\ \hline & & & \ddots & \\ \hline & & & & \mathcal{R}_A(s) \end{array}\right], \tag{5.4.25}$$

where $\mathcal{R}_A(s) = (sI - A)^{-1}$ *is the resolvent of* A.

Now imagine that the sequence of disturbances d_t is of the form $d_t = s^t d$, where $s \in \mathbb{C}$ differs from 0 and from the eigenvalues of A. From the stability of (5.4.24) it follows that as $t \to \infty$, the solution ω_t of the system, independently of the initial state, approaches the "steady-state" solution $\widehat{\omega}_t = s^t\mathcal{H}(s)d$, where $\mathcal{H}(s)$ is certain matrix. In particular, the state-control vector $w_t = \begin{bmatrix} x_t \\ u_t \end{bmatrix}$ approaches, as $t \to \infty$, the trajectory $\widehat{w}_t = s^t\mathcal{H}_{xu}(s)d$. The associated *disturbance-to-state/control transfer matrix* $\mathcal{H}_{xu}(s)$ is easily computable:

$$\mathcal{H}_{xu}(s) = \left[\begin{array}{c} \overbrace{\mathcal{R}_A(s)\left[R + \sum_{\nu=0}^{k-1} s^{-\nu}BH_\nu\left[D + C\mathcal{R}_A(s)R\right]\right]}^{\mathcal{H}_x(s)} \\ \hline \underbrace{\left[\sum_{\nu=0}^{k-1} s^{-\nu}H_\nu\right]\left[D + C\mathcal{R}_A(s)R\right]}_{\mathcal{H}_u(s)} \end{array}\right]. \tag{5.4.26}$$

The crucial fact is *that the transfer matrix* $\mathcal{H}_{xu}(s)$ *is affine in the parameters* $H = [H_0, ..., H_{k-1}]$ *of the nearly time invariant control law* (5.4.23). As a result, *design specifications representable as explicit convex constraints on the transfer matrix* $\mathcal{H}_{xu}(s)$ (these are typical specifications in infinite-horizon design of linear controllers) *are equivalent to explicit convex constraints on the parameters* H *of the underlying POB control law and therefore can be processed efficiently via Convex Optimization.*

Example: Discrete time H_∞ control. Discrete time H_∞ design specifications impose constraints on the behavior of the transfer matrix along the unit circumference $s = \exp\{\imath\omega\}$, $0 \le \omega \le 2\pi$, that is, on the steady state response of the closed loop system to a disturbance in the form of a harmonic oscillation.[4]. A rather general form of these specifications is a system of constraints

$$\|Q_i(s) - M_i(s)\mathcal{H}_{xu}(s)N_i(s)\| \le \tau_i \quad \forall (s = \exp\{\imath\omega\} : \omega \in \Delta_i), \tag{5.4.27}$$

where $Q_i(s)$, $M_i(s)$, $N_i(s)$ are given rational matrix-valued functions with no singularities on the unit circumference $\{s : |s| = 1\}$, $\Delta_i \subset [0, 2\pi]$ are given segments, and $\|\cdot\|$ is the standard matrix norm (the largest singular value).

We are about to demonstrate that constraints (5.4.27) can be represented by an explicit finite system of LMIs; as a result, specifications (5.4.27) can be efficiently processed numerically. Here is the derivation. Both "transfer functions" $\mathcal{H}_x(s)$, $\mathcal{H}_u(s)$ are of the form $q^{-1}(s)Q(s, H)$,

[4]The entries of $\mathcal{H}_x(s)$ and $\mathcal{H}_u(s)$, restricted onto the unit circumference $s = \exp\{\imath\omega\}$, have very transparent interpretation. Assume that the only nonzero entry in the disturbances is the j-th one, and it varies in time as a harmonic oscillation of unit amplitude and frequency ω. The steady-state behavior of i-th state then will be a harmonic oscillation of the same frequency, but with another amplitude, namely, $|(\mathcal{H}_x(\exp\{\imath\omega\}))_{ij}|$ and phase shifted by $\arg(\mathcal{H}_x(\exp\{\imath\omega\})_{ij})$. Thus, the *state-to-input frequency responses* $(\mathcal{H}_x(\exp\{\imath\omega\}))_{ij}$ explain the steady-state behavior of states when the input is comprised of harmonic oscillations. The interpretation of the *control-to-input frequency responses* $(\mathcal{H}_u(\exp\{\imath\omega\}))_{ij}$ is completely similar.

where $q(s)$ is a scalar polynomial independent of H, and $Q(s,H)$ is a matrix-valued polynomial of s with coefficients *affinely depending on* H. With this in mind, we see that the constraints are of the generic form

$$\|p^{-1}(s)P(s,H)\| \leq \tau\,\forall(s=\exp\{\imath\omega\}:\omega\in\Delta), \tag{5.4.28}$$

where $p(\cdot)$ is a scalar polynomial independent of H and $P(s,H)$ is a polynomial in s with $m\times n$ matrix coefficients affinely depending on H. Constraint (5.4.28) can be expressed equivalently by the semi-infinite matrix inequality

$$\left[\begin{array}{cc} \tau I_m & P(z,H)/p(z) \\ (P(z,H))^*/(p(z))^* & \tau I_n \end{array}\right] \succeq 0\,\forall(z=\exp\{\imath\omega\}:\omega\in\Delta)$$

(* stands for the Hermitian conjugate, $\Delta\subset[0,2\pi]$ is a segment) or, which is the same,

$$\begin{array}{rcl} S_{H,\tau}(\omega) & \equiv & \left[\begin{array}{cc} \tau p(\exp\{\imath\omega\})(p(\exp\{\imath\omega\}))^* I_m & (p(\exp\{\imath\omega\}))^* P(\exp\{\imath\omega\},H) \\ p(\exp\{\imath\omega\})(P(\exp\{\imath\omega\},H))^* & \tau p(\exp\{\imath\omega\})(p(\exp\{\imath\omega\}))^* I_n \end{array}\right] \\ & \succeq & 0\,\forall\omega\in\Delta. \end{array}$$

Observe that $S_{H,\tau}(\omega)$ is a trigonometric polynomial taking values in the space of Hermitian matrices of appropriate size, the coefficients of the polynomial being affine in H,τ. It is known [49] that the cone $\mathcal{P}_m$ of (coefficients of) all Hermitian matrix-valued trigonometric polynomials $S(\omega)$ of degree $\leq m$, which are $\succeq 0$ for all $\omega\in\Delta$, is semidefinite representable, i.e., there exists an explicit LMI

$$\mathcal{A}(S,u)\succeq 0$$

in variables S (the coefficients of a polynomial $S(\cdot)$) and additional variables u such that $S(\cdot)\in\mathcal{P}_m$ if and only if S can be extended by appropriate u to a solution of the LMI. Consequently, the relation

$$\mathcal{A}(S_{H,\tau},u)\succeq 0, \tag{$*$}$$

which is an LMI in H,τ,u, is a semidefinite representation of (5.4.28): H,τ solve (5.4.28) if and only if there exists u such that H,τ,u solve ($*$).

5.4.4 Putting Things Together: Infinite- and Finite-Horizon Design Specifications

For the time being, we have considered optimization over purified-output-based affine control laws in two different settings, finite- and infinite-horizon design specifications. In fact we can to some extent combine both settings, thus seeking affine purified-output-based controls ensuring both a good steady-state behavior of the closed loop system and a "good transition" to this steady-state behavior. The proposed methodology will become clear from the example that follows.

Consider the open-loop time-invariant system representing the discretized double-pendulum depicted on figure 5.2. The dynamics of the continuous time prototype plant is given by

$$\begin{array}{rcl} \dot{x} & = & A_c x + B_c u + R_c d \\ y & = & Cx, \end{array}$$

where

$$A_c=\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{array}\right],\ B_c=\left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array}\right],\ R_c=\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -1 \end{array}\right],\ C=\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

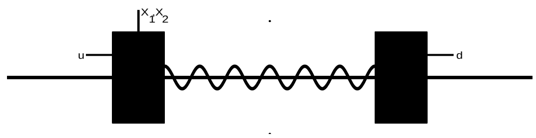


Figure 5.2: Double pendulum: two masses linked by a spring sliding without friction along a rod. Position and velocity of the first mass are observed.

(x_1, x_2 are the position and the velocity of the first mass, and x_3, x_4 those of the second mass). The discrete time plant we will actually work with is

$$\begin{array}{rcl} x_{t+1} & = & A_0 x_t + Bu_t + Rd_t \\ y_t & = & Cx_t \end{array} \tag{5.4.29}$$

where $A_0 = \exp\{\Delta \cdot A_c\}$, $B = \int\limits_0^{\Delta} \exp\{sA_c\}B_c ds$, $R = \int\limits_0^{\Delta} \exp\{sA_c\}R_c ds$. System (5.4.29) is not stable (absolute values of all eigenvalues of A_0 are equal to 1), which seemingly prevents us from addressing infinite-horizon design specifications via the techniques developed in section 5.4.3. The simplest way to circumvent the difficulty is to augment the original plant by a stabilizing time-invariant linear feedback; upon success, we then apply the purified-output-based synthesis to the augmented, already stable, plant. Specifically, let us look for a controller of the form

$$u_t = Ky_t + w_t. \tag{5.4.30}$$

With such a controller, (5.4.29) becomes

$$\begin{array}{rcl} x_{t+1} & = & Ax_t + Bw_t + Rd_t, \quad A = A_0 + BKC \\ y_t & = & Cx_t. \end{array} \tag{5.4.31}$$

If K is chosen in such a way that the matrix $A = A_0 + BKC$ is stable, we can apply all our purified-output-based machinery to the plant (5.4.31), with w_t in the role of u_t, however keeping in mind that the "true" controls u_t will be $Ky_t + w_t$.

For our toy plant, a stabilizing feedback K can be found by "brute force" — by generating a random sample of matrices of the required size and selecting from this sample a matrix, if any, which indeed makes (5.4.31) stable. Our search yielded feedback matrix $K = [-0.6950, -1.7831]$, with the spectral radius of the matrix $A = A_0 + BKC$ equal to 0.87. From now on, we focus on the resulting plant (5.4.31), which we intend to "close" by a control law from $\mathcal{C}_{8,0}$, where $\mathcal{C}_{k,0}$ is the family of all time invariant control laws of the form

$$w_t = \sum_{\tau=0}^{t} H_{t-\tau} v_\tau \quad \left[\begin{array}{rcl} v_t & = & y_t - C\widehat{x}_t, \\ \widehat{x}_{t+1} & = & A\widehat{x}_t + Bw_t,\ \widehat{x}_0 = 0 \end{array} \right] \tag{5.4.32}$$

where $H_s = 0$ when $s \geq k$. Our goal is to pick in $\mathcal{C}_{8,0}$ a control law with desired properties (to be precisely specified below) expressed in terms of the following 6 criteria:

- the four peak to peak gains z2x, z2u, d2x, d2u defined on p. 248;
- the two H_∞ gains

$$H_{\infty,x} = \max_{|s|=1,i,j} |(\mathcal{H}_x(s))|_{ij}, \quad H_{\infty,u} = \max_{|s|=1,i,j} |(\mathcal{H}_u(s))|_{ij},$$

Optimized criterion	Resulting values of the criteria					
	$z2x^{40}$	$z2u^{40}$	$d2x^{40}$	$d2u^{40}$	$H_{\infty,x}$	$H_{\infty,u}$
$z2x^{40}$	<u>25.8</u>	205.8	1.90	3.75	10.52	5.87
$z2u^{40}$	58.90	<u>161.3</u>	1.90	3.74	39.87	20.50
$d2x^{40}$	5773.1	13718.2	<u>1.77</u>	6.83	1.72	4.60
$d2u^{40}$	1211.1	4903.7	1.90	<u>2.46</u>	66.86	33.67
$H_{\infty,x}$	121.1	501.6	1.90	5.21	<u>1.64</u>	5.14
$H_{\infty,u}$	112.8	460.4	1.90	4.14	8.13	<u>1.48</u>
	z2x	z2u	d2x	d2u	$H_{\infty,x}$	$H_{\infty,u}$
(5.4.34)	31.59	197.75	1.91	4.09	1.82	2.04
(5.4.35)	2.58	0.90	1.91	4.17	1.77	1.63

Table 5.2: Gains for time invariant control laws of order 8 yielded by optimizing, one at a time, the criteria $z2x^{40}$,...,$H_{\infty,u}$ over control laws from $\mathcal{F} = \{\eta \in \mathcal{C}_{8,0} : d2x^{40}[\eta] \leq 1.90\}$ (first six lines), and by solving programs (5.4.34), (5.4.35) (last two lines).

where $\mathcal{H}_x$ and $\mathcal{H}_u$ are the transfer functions from the disturbances to the states and the controls, respectively.

Note that while the purified-output-based control w_t we are seeking is defined in terms of the stabilized plant (5.4.31), the criteria z2u,d2u, $H_{\infty,u}$ are defined in terms of the original controls $u_t = Ky_t + w_t = KCx_t + w_t$ affecting the actual plant (5.4.29).

In the synthesis we are about to describe our primary goal is to minimize the global disturbance to state gain d2x, while the secondary goal is to avoid too large values of the remaining criteria. We achieve this goal as follows.

Step 1: Optimizing d2x. As it was explained on p. 248, the optimization problem

$$\mathrm{Opt}_{\mathrm{d2x}}(k,0;N_+) = \min_{\eta\in\mathcal{C}_{k,0}} \max_{0\leq t\leq N_+} \mathrm{d2x}_t[\eta] \tag{5.4.33}$$

is an explicit convex program (in fact, just an LO), and its optimal value is a lower bound on the best possible global gain d2x achievable with control laws from $\mathcal{C}_{k,0}$. In our experiment, we solve (5.4.33) for $k = 8$ and $N_+ = 40$, arriving at $\mathrm{Opt}_{\mathrm{d2x}}(8,0;40) = 1.773$. The global d2x gain of the resulting time-invariant control law is 1.836 — just 3.5% larger than the outlined lower bound. We conclude that the control yielded by the solution to (5.4.33) is nearly the best one, in terms of the global d2x gain, among time-invariant controls of order 8. At the same time, part of the other gains associated with this control are far from being good, see line "$d2x^{40}$" in table 5.2.

Step 2: Improving the remaining gains. To improve the "bad" gains yielded by the nearly d2x-optimal control law we have built, we act as follows: we look at the family $\mathcal{F}$ of all time invariant control laws of order 8 with the finite-horizon d2x gain $\mathrm{d2x}^{40}[\eta] = \max\limits_{0\leq t\leq 40} \mathrm{d2x}_t[\eta]$ not exceeding 1.90 (that is, look at the controls from $\mathcal{C}_{8,0}$ that are within 7.1% of the optimum in terms of their $d2x^{40}$ gain) and act as follows:

A. We optimize over $\mathcal{F}$, one at a time, every one of the remaining criteria $\mathrm{z2x}^{40}[\eta] = \max\limits_{0\leq t\leq 40} \mathrm{z2x}_t[\eta]$, $\mathrm{z2u}^{40}[\eta] = \max\limits_{0\leq t\leq 40} \mathrm{z2u}_t[\eta]$, $\mathrm{d2u}^{40}[\eta] = \max\limits_{0\leq t\leq 40} \mathrm{d2u}_t[\eta]$, $H_{\infty,x}[\eta]$, $H_{\infty,u}[\eta]$, thus obtaining "reference values" of these criteria; these are lower bounds on the optimal values of the corresponding global gains, optimization being carried out over the set $\mathcal{F}$. These lower bounds are the underlined data in table 5.2.

B. We then minimize over $\mathcal{F}$ the "aggregated gain"

$$\frac{\mathrm{z2x}^{40}[\eta]}{25.8} + \frac{\mathrm{z2u}^{40}[\eta]}{161.3} + \frac{\mathrm{d2u}^{40}[\eta]}{2.46} + \frac{H_{\infty,x}[\eta]}{1.64} + \frac{H_{\infty,u}[\eta]}{1.48} \tag{5.4.34}$$

(the denominators are exactly the aforementioned reference values of the corresponding gains). The global gains of the resulting time-invariant control law of order 8 are presented in the "(5.4.34)" line of table 5.2.

Step 3: Finite-horizon adjustments. Our last step is to improve the z2x and z2u gains by passing from a time invariant affine control law of order 8 to a nearly time invariant law of order 8 with stabilization time $N_* = 20$. To this end, we solve the convex optimization problem

$$\min_{\eta\in\mathcal{C}_{8,20}} \left\{ \mathrm{z2x}^{50}[\eta] + \mathrm{z2u}^{50}[\eta] : \begin{array}{rcl} \mathrm{d2x}^{50}[\eta] & \leq & 1.90 \\ \mathrm{d2u}^{50}[\eta] & \leq & 4.20 \\ H_{\infty,x}[\eta] & \leq & 1.87 \\ H_{\infty,u}[\eta] & \leq & 2.09 \end{array} \right\} \tag{5.4.35}$$

(the right hand sides in the constraints for $\mathrm{d2u}^{50}[\cdot]$, $H_{\infty,x}[\cdot]$, $H_{\infty,u}[\cdot]$ are the slightly increased (by 2.5%) gains of the time invariant control law obtained in Step 2). The global gains of the resulting control law are presented in the last line of table 5.2, see also figure 5.3. We see that finite-horizon adjustments allow us to reduce by orders of magnitude the global z2x and z2u gains and, as an additional bonus, result in a substantial reduction of H_∞-gains.

Simple as this control problem may be, it serves well to demonstrate the importance of purified-output-based representation of affine control laws and the associated possibility to express various control specifications as explicit convex constraints on the parameters of such laws.

5.5 Exercises

Exercise 5.1 Consider a discrete time linear dynamical system

$$\begin{array}{rcl} x_0 & = & z \\ x_{t+1} & = & A_t x_t + B_t u_t + R_t d_t,\ t = 0, 1, ... \end{array} \tag{5.5.1}$$

where $x_t \in \mathbb{R}^n$ are the states, $u_t \in \mathbb{R}^m$ are the controls, and $d_t \in \mathbb{R}^k$ are the exogenous disturbances. We are interested in the behavior of the system on the finite time horizon $t = 0, 1, ..., N$. A "desired behavior" is given by the requirement

$$\|Pw^N - q\|_\infty \leq R \tag{5.5.2}$$

on the state-control trajectory $w^N = [x_0; ...; x_{N+1}; u_0; ...; u_N]$.

Let us treat $\zeta = [z; d_0; ...; d_N]$ as an uncertain perturbation with perturbation structure $(\mathcal{Z}, \mathcal{L}, \|\cdot\|_r)$, where

$$\mathcal{Z} = \{\zeta : \|\zeta - \bar{\zeta}\|_s \leq R\},\ \mathcal{L} = \mathbb{R}^L \qquad [L = \dim\zeta]$$

and $r, s \in [1, \infty]$, so that (5.5.1), (5.5.2) become a system of uncertainty-affected linear constraints on w^N. We want to process the Affinely Adjustable GRC of the system, where u_t are allowed to be affine functions of the initial state z and the vector of disturbances $d^t = [d_0; ...; d_t]$ up to time t, and the states x_t are allowed to be affine functions of z and d^{t-1}. We wish to minimize the corresponding global sensitivity.

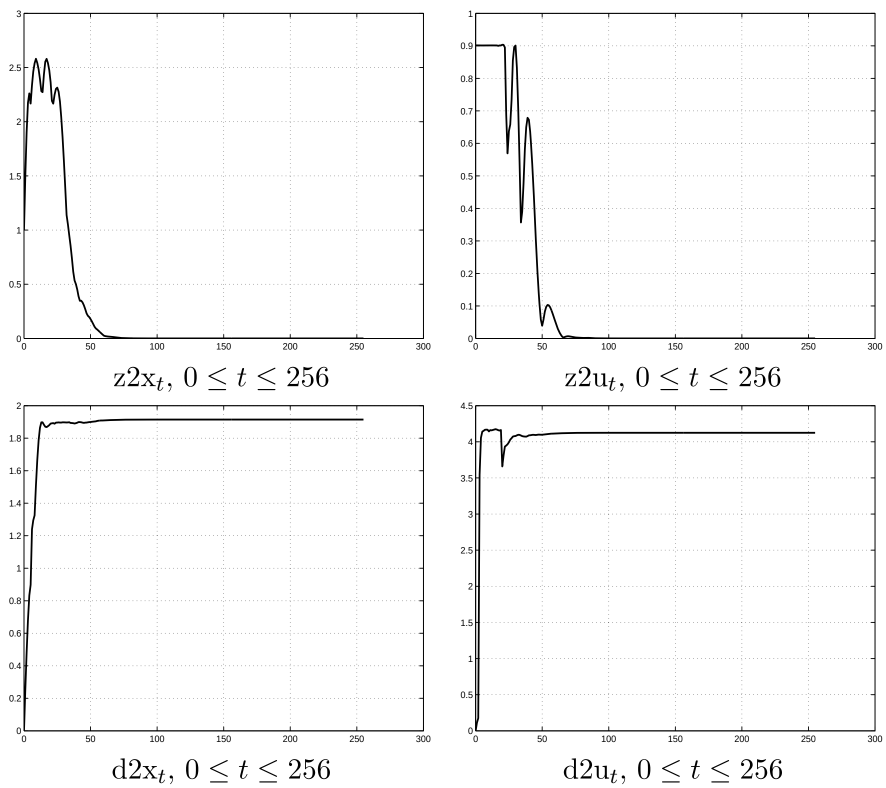


Figure 5.3: Frequency responses and gains of control law given by solution to (5.4.35).

In control terms: we want to "close" the open-loop system (5.5.1) with a non-anticipative affine control law

$$u_t = U_t^z z + U_t^d d^t + u_t^0 \tag{5.5.3}$$

based on observations of initial states and disturbances up to time t in such a way that the "closed loop system" (5.5.1), (5.5.3) exhibits the desired behavior in a robust w.r.t. the initial state and the disturbances fashion.

Write down the AAGRC of our uncertain problem as an explicit convex program with efficiently computable constraints.

Exercise 5.2 Consider the modification of Exercise 5.1 where the cone $\mathcal{L} = \mathbb{R}^L$ is replaced with

$$\mathcal{L} = \{[0; d_0; ...; d_N] : d_t \geq 0,\ 0 \leq t \leq N\},$$

and solve the corresponding version of the Exercise.

Exercise 5.3 Consider the simplest version of Exercise 5.1, where (5.5.1) reads

$$\begin{array}{rcl} x_0 &=& z \in \mathbb{R} \\ x_{t+1} &=& x_t + u_t - d_t,\ t = 0, 1, ..., 15, \end{array}$$

(5.5.2) reads

$$|\theta x_t| = 0,\ t = 1, 2, ..., 16,\ \ |u_t| = 0,\ t = 0, 1, ..., 15$$

and the perturbation structure is

$$\mathcal{Z} = \{[z; d_0; ...; d_{15}] = 0\} \subset \mathbb{R}^{17},\ \mathcal{L} = \{[0; d_0; d_1; ...; d_{15}]\}, \|\zeta\| \equiv \|\zeta\|_2.$$

Assuming the same "adjustability status" of u_t and x_t as in Exercise 5.1,

1. Represent the AAGRC of (the outlined specializations of) (5.5.1), (5.5.2), where the goal is to minimize the global sensitivity, as an explicit convex program;

2. Interpret the AAGRC in Control terms;

3. Solve the AAGRC for the values of θ equal to 1.e6, 10, 2, 1.

Exercise 5.4 Consider a communication network — an oriented graph G with the set of nodes $V = \{1, ..., n\}$ and the set of arcs Γ. Several ordered pairs of nodes (i, j) are marked as "source-sink" nodes and are assigned traffic d_{ij} — the amount of information to be transmitted from node i to node j per unit time; the set of all source-sink pairs is denoted by $\mathcal{J}$. Arcs $\gamma \in \Gamma$ of a communication network are assigned with capacities — upper bounds on the total amount of information that can be sent through the arc per unit time. We assume that the arcs already possess certain capacities p_γ, which can be further increased; the cost of a unit increase of the capacity of arc γ is a given constant c_γ.

1) Assuming the demands d_{ij} certain, formulate the problem of finding the cheapest extension of the existing network capable to ensure the required source-sink traffic as an LO program.

2) Now assume that the vector of traffic $d = \{d_{ij} : (i, j) \in \mathcal{J}\}$ is uncertain and is known to run through a given semidefinite representable compact uncertainty set $\mathcal{Z}$. Allowing the amounts x_γ^{ij} of information with origin i and destination j traveling through the arc γ to depend affinely on traffic, build the AARC of the (uncertain version of the) problem from 1). Consider

two cases: (a) for every $(i,j) \in \mathcal{J}$, x_γ^{ij} can depend affinely solely on d_{ij}, and (b) x_γ^{ij} can depend affinely on the entire vector d. Are the resulting problems computationally tractable?

3) Assume that the vector d is random, and its components are independent random variables uniformly distributed in given segments Δ_{ij} of positive lengths. Build the chance constrained versions of the problems from 2).

Appendix A

Notation and Prerequisites

A.1 Notation

• $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ stand for the sets of all integers, reals, and complex numbers, respectively.
• $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$ stand for the spaces of complex, respectively, real $m \times n$ matrices. We write $\mathbb{C}^n$ and $\mathbb{R}^n$ as shorthands for $\mathbb{C}^{n\times 1}$, $\mathbb{R}^{n\times 1}$, respectively.
For $A \in \mathbb{C}^{m\times n}$, A^T stands for the transpose, and A^H for the conjugate transpose of A:

$$(A^H)_{rs} = \overline{A_{sr}},$$

where $\overline{z}$ is the conjugate of $z \in \mathbb{C}$.
• Both $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$ are equipped with the inner product

$$\langle A, B\rangle = \mathrm{Tr}(AB^H) = \sum_{r,s} A_{rs}\overline{B_{rs}}.$$

The norm associated with this inner product is denoted by $\|\cdot\|_2$.
• For $p \in [1, \infty]$, we define the p-norms $\|\cdot\|_p$ on $\mathbb{C}^n$ and $\mathbb{R}^n$ by the relation

$$\|x\|_p = \left\{\begin{array}{ll} \left(\sum_i |x_i|^p\right)^{1/p}, & 1 \le p < \infty \\ \lim_{p\to\infty} \|x\|_p = \max_i |x_i|, & p = \infty \end{array}\right., \ 1 \le p \le \infty.$$

Note that when $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are conjugates of each other:

$$\|x\|_p = \max_{y:\|y\|_q\le 1} |\langle x, y\rangle|.$$

In particular, $|\langle x, y\rangle| \le \|x\|_p\|y\|_q$ (Hölder inequality).
• We use the notation I_m, $0_{m\times n}$ for the unit $m \times m$, respectively, the zero $m \times n$ matrices.
• $\mathbf{H}^m$, $\mathbf{S}^m$ are real vector spaces of $m \times m$ Hermitian, respectively, real symmetric matrices. Both are Euclidean spaces w.r.t. the inner product $\langle\cdot,\cdot\rangle$.
• We use "MATLAB notation": when $A_1, ..., A_k$ are matrices with the same number of rows, $[A_1, ..., A_k]$ denotes the matrix with the same number of rows obtained by writing, from left to right, first the columns of A_1, then the columns of A_2, and so on. When $A_1, ..., A_k$ are matrices with the same number of columns, $[A_1; A_2; ...; A_k]$ stands for the matrix with the same number of columns obtained by writing, from top to bottom, first the rows of A_1, then the rows of A_2, and so on.

• For a Hermitian/real symmetric $m \times m$ matrix A, $\lambda(A)$ is the vector of eigenvalues $\lambda_r(A)$ of A taken with their multiplicities in the non-ascending order:

$$\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_m(A).$$

• For an $m \times n$ matrix A, $\sigma(A) = (\sigma_1(A), ..., \sigma_n(A))^T$ is the vector of singular values of A:

$$\sigma_r(A) = \lambda_r^{1/2}(A^H A),$$

and

$$\|A\|_{2,2} = \|A\| = \sigma_1(A) = \max\{\|Ax\|_2 : x \in \mathbb{C}^n, \|x\|_2 \leq 1\}$$

(by evident reasons, when A is real, one can replace $\mathbb{C}^n$ in the right hand side with $\mathbb{R}^n$).
• For Hermitian/real symmetric matrices A, B, we write $A \succeq B$ ($A \succ B$) to express that $A - B$ is positive semidefinite (resp., positive definite).

A.2 Conic Programming

A.2.1 Euclidean Spaces, Cones, Duality

Euclidean spaces

A *Euclidean space* is a finite dimensional linear space over reals equipped with an *inner product* $\langle x, y\rangle_E$ — a bilinear and symmetric real-valued function of $x, y \in E$ such that $\langle x, x\rangle_E > 0$ whenever $x \neq 0$.

Example: The standard Euclidean space $\mathbb{R}^n$. This space is comprised of n-dimensional real column vectors with the standard coordinate-wise linear operations and the inner product $\langle x, y\rangle_{\mathbb{R}^n} = x^T y$. $\mathbb{R}^n$ is a universal example of an Euclidean space: for every Euclidean n-dimensional space $(E, \langle\cdot,\cdot\rangle_E)$ there exists a one-to-one linear mapping $x \mapsto Ax : \mathbb{R}^n \to E$ such that $x^T y \equiv \langle Ax, Ay\rangle_E$. All we need in order to build such a mapping, is to find an *orthonormal basis* $e_1, ..., e_n$, $n = \dim E$, in E, that is, a basis such that $\langle e_i, e_j\rangle_E = \delta_{ij} \equiv \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$; such a basis always exists. Given an orthonormal basis $\{e_i\}_{i=1}^n$, a one-to-one mapping $A : \mathbb{R}^n \to E$ preserving the inner product is given by $Ax = \sum_{i=1}^n x_i e_i$.

Example: The space $\mathbb{R}^{m\times n}$ of $m \times n$ real matrices with the Frobenius inner product. The elements of this space are $m \times n$ real matrices with the standard linear operations and the inner product $\langle A, B\rangle_F = \mathrm{Tr}(AB^T) = \sum_{i,j} A_{ij} B_{ij}$.

Example: The space $\mathbf{S}^n$ of $n \times n$ real symmetric matrices with the Frobenius inner product. This is the subspace of $\mathbb{R}^{n\times n}$ comprised of all symmetric $n \times n$ matrices; the inner product is inherited from the embedding space. Of course, for symmetric matrices, this product can be written down without transposition:

$$A, B \in \mathbf{S}^n \Rightarrow \langle A, B\rangle_F = \mathrm{Tr}(AB) = \sum_{i,j} A_{ij} B_{ij}.$$

Example: The space $\mathbf{H}^n$ of $n \times n$ Hermitian matrices with the Frobenius inner product. This is the real linear space comprised of $n \times n$ Hermitian matrices; the inner product is

$$\langle A, B\rangle = \mathrm{Tr}(AB^H) = \mathrm{Tr}(AB) = \sum_{i,j=1}^n A_{ij} \overline{B_{ij}}.$$

Linear forms on Euclidean spaces

Every homogeneous linear form $f(x)$ on a Euclidean space $(E, \langle\cdot,\cdot\rangle_E)$ can be represented in the form $f(x) = \langle e_f, x\rangle_E$ for certain vector $e_f \in E$ uniquely defined by $f(\cdot)$. The mapping $f \mapsto e_f$ is a one-to-one linear mapping of the space of linear forms on E onto E.

Conjugate mapping

Let $(E, \langle\cdot,\cdot\rangle_E)$ and $(F, \langle\cdot\cdot\rangle_F)$ be Euclidean spaces. For a linear mapping $A : E \to F$ and every $f \in F$, the function $\langle Ae, f\rangle_F$ is a linear function of $e \in E$ and as such it is representable as $\langle e, A^* f\rangle_E$ for certain uniquely defined vector $A^* f \in E$. It is immediately seen that the mapping $f \mapsto A^* f$ is a linear mapping of F into E; the characteristic identity specifying this mapping is

$$\langle Ae, f\rangle_F = \langle e, A^* f\rangle \ \forall (e \in E, f \in F).$$

The mapping A^* is called *conjugate* to A. It is immediately seen that the conjugation is a linear operation with the properties $(A^*)^* = A$, $(AB)^* = B^* A^*$. If $\{e_j\}_{j=1}^m$ and $\{f_i\}_{i=1}^n$ are orthonormal bases in E, F, then every linear mapping $A : E \to F$ can be associated with the matrix $[a_{ij}]$ ("matrix of the mapping in the pair of bases in question") according to the identity

$$A \sum_{j=1}^m x_j e_j = \sum_i \left[\sum_j a_{ij} x_j \right] f_i$$

(in other words, a_{ij} is the i-th coordinate of the vector Ae_j in the basis $f_1, ..., f_n$). With this representation of linear mappings by matrices, the matrix representing A^* in the pair of bases $\{f_i\}$ in the argument and $\{e_j\}$ in the image spaces of A^* is the transpose of the matrix representing A in the pair of bases $\{e_j\}$, $\{f_i\}$.

Cones in Euclidean space

A nonempty subset $\mathbf{K}$ of a Euclidean space $(E, \langle\cdot,\cdot\rangle_E)$ is called a cone, if it is a convex set comprised of rays emanating from the origin, or, equivalently, whenever $t_1, t_2 \geq 0$ and $x_1, x_2 \in \mathbf{K}$, we have $t_1 x_1 + t_2 x_2 \in \mathbf{K}$.

A cone $\mathbf{K}$ is called *regular*, if it is closed, possesses a nonempty interior and is *pointed* — does not contain lines, or, which is the same, is such that $a \in \mathbf{K}$, $-a \in \mathbf{K}$ implies that $a = 0$.

Dual cone. If $\mathbf{K}$ is a cone in a Euclidean space $(E, \langle\cdot,\cdot\rangle_E)$, then the set

$$\mathbf{K}^* = \{e \in E : \langle e, h\rangle_E \geq 0 \, \forall h \in \mathbf{K}\}$$

also is a cone called the cone *dual* to $\mathbf{K}$. The dual cone always is closed. The cone dual to dual is the closure of the original cone: $(\mathbf{K}^*)^* = \mathrm{cl}\,\mathbf{K}$; in particular, $(\mathbf{K}^*)^* = \mathbf{K}$ for every closed cone $\mathbf{K}$. The cone $\mathbf{K}^*$ possesses a nonempty interior if and only if $\mathbf{K}$ is pointed, and $\mathbf{K}^*$ is pointed if and only if $\mathbf{K}$ possesses a nonempty interior; in particular, $\mathbf{K}$ is regular if and only if $\mathbf{K}^*$ is so.

Example: Nonnegative ray and nonnegative orthants. The simplest one-dimensional cone is the nonnegative ray $\mathbb{R}_+ = \{t \geq 0\}$ on the real line $\mathbb{R}^1$. The simplest cone in $\mathbb{R}^n$ is the *nonnegative orthant* $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$. This cone is regular and self-dual: $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$.

Example: Lorentz cone $\mathbf{L}^n$. The cone $\mathbf{L}^n$ "lives" in $\mathbb{R}^n$ and is comprised of all vectors $x = [x_1; ...; x_n] \in \mathbb{R}^n$ such that $x_n \geq \sqrt{\sum_{j=1}^{n-1} x_j^2}$; same as $\mathbb{R}^n_+$, the Lorentz cone is regular and self-dual.

By definition, $\mathbf{L}^1 = \mathbb{R}_+$ is the nonnegative orthant; this is in full accordance with the "general" definition of a Lorentz cone combined with the standard convention "a sum over an empty set of indices is 0."

Example: Semidefinite cone $\mathbf{S}^n_+$. The cone $\mathbf{S}^n_+$ "lives" in the Euclidean space $\mathbf{S}^n$ of $n\times n$ symmetric matrices equipped with the Frobenius inner product. The cone is comprised of all $n \times n$ symmetric *positive semidefinite* matrices A, i.e., matrices $A \in \mathbf{S}^n$ such that $x^TAx \geq 0$ for all $x \in \mathbb{R}^n$, or, equivalently, such that all eigenvalues of A are nonnegative. Same as $\mathbb{R}^n_+$ and $\mathbf{L}^n$, the cone $\mathbf{S}^n_+$ is regular and self-dual.

Example: Hermitian semidefinite cone $\mathbf{H}^n_+$. This cone "lives" in the space $\mathbf{H}^n$ of $n\times n$ Hermitian matrices and is comprised of all positive semidefinite Hermitian $n \times n$ matrices; it is regular and self-dual.

A.2.2 Conic Problems and Conic Duality

Conic problem

A *conic problem* is an optimization problem of the form

$$\mathrm{Opt}(P) = \min_x \left\{ \langle c, x\rangle_E : \begin{array}{l} A_ix - b_i \in \mathbf{K}_i,\ i = 1, ..., m, \\ Ax = b \end{array} \right\} \tag{P}$$

where

- $(E, \langle\cdot,\cdot\rangle_E)$ is a Euclidean space of *decision vectors* x and $c \in E$ is the *objective*;
- A_i, $1 \leq i \leq m$, are linear maps from E into Euclidean spaces $(F_i, \langle\cdot,\cdot\rangle_{F_i})$, $b_i \in F_i$ and $\mathbf{K}_i \subset F_i$ are regular cones;
- A is a linear mapping from E into a Euclidean space $(F, \langle\cdot,\cdot\rangle_F)$ and $b \in F$.

Examples: Linear, Conic Quadratic and Semidefinite Optimization. We will be especially interested in the three generic conic problems as follows:

- *Linear Optimization*, or *Linear Programming:* this is the family of all conic problems associated with nonnegative orthants $\mathbb{R}^m_+$, that is, the family of all usual LPs $\min_x\{c^Tx : Ax - b \geq 0\}$;
- *Conic Quadratic Optimization*, or *Conic Quadratic Programming*, or *Second Order Cone Programming:* this is the family of all conic problems associated with the cones that are *finite direct products* of Lorentz cones, that is, the conic programs of the form

  $$\min_x \left\{ c^Tx : [A_1; ...; A_m]x - [b_1; ...; b_m] \in \mathbf{L}^{k_1} \times ... \times \mathbf{L}^{k_m} \right\}$$

  where A_i are $k_i \times \dim x$ matrices and $b_i \in \mathbb{R}^{k_i}$. The "Mathematical Programming" form of such a program is

  $$\min_x \left\{ c^Tx : \|\bar{A}_ix - \bar{b}_i\|_2 \leq \alpha_i^Tx - \beta_i,\ 1 \leq i \leq m \right\},$$

  where $A_i = [\bar{A}_i; \alpha_i^T]$ and $b_i = [\bar{b}_i; \beta_i]$, so that α_i is the last row of A_i, and β_i is the last entry of b_i;

- *Semidefinite Optimization*, or *Semidefinite Programming:* this is the family of all conic problems associated with the cones that are *finite direct products* of Semidefinite cones, that is, the conic programs of the form

$$\min_x \left\{ c^T x : A_i^0 + \sum_{j=1}^{\dim x} x_j A_i^j \succeq 0,\ 1 \le i \le m \right\},$$

  where A_i^j are symmetric matrices of appropriate sizes.

A.2.3 Conic Duality

Conic duality — derivation

The origin of conic duality is the desire to find a systematic way to bound from below the optimal value in a conic problem (P). This way is based on *linear aggregation* of the constraints of (P), namely, as follows. Let $y_i \in \mathbf{K}_i^*$ and $z \in F$. By the definition of the dual cone, for every x feasible for (P) we have

$$\langle A_i^* y_i, x\rangle_E - \langle y_i, b_i\rangle_{F_i} \equiv \langle y_i, Ax_i - b_i\rangle_{F_i} \ge 0,\ 1 \le i \le m,$$

and of course

$$\langle A^* z, x\rangle_E - \langle z, b\rangle_F = \langle z, Ax - b\rangle_F = 0.$$

Summing up the resulting inequalities, we get

$$\langle A^* z + \sum_i A_i^* y_i, x\rangle_E \ge \langle z, b\rangle_F + \sum_i \langle y_i, b_i\rangle_{F_i}. \tag{C}$$

By its origin, this scalar linear inequality on x is a consequence of the constraints of (P), that is, it is valid for all feasible solutions x to (P). It may happen that the left hand side in this inequality is, identically in $x \in E$, equal to the objective $\langle c, x\rangle_E$; this happens if and only if

$$A^* z + \sum_i A_i^* y_i = c.$$

Whenever it is the case, the right hand side of (C) is a valid lower bound on the optimal value in (P). The dual problem is nothing but the problem

$$\mathrm{Opt}(D) = \max_{z,\{y_i\}} \left\{ \langle z, b\rangle_F + \sum_i \langle y_i, b_i\rangle_{F_i} : \begin{array}{c} y_i \in \mathbf{K}_i^*,\ 1 \le i \le m, \\ A^* z + \sum_i A_i^* y_i = c \end{array} \right\} \tag{D}$$

of maximizing this lower bound.

By the origin of the dual problem, we have

Weak Duality: *One has* $\mathrm{Opt}(D) \le \mathrm{Opt}(P)$.

We see that (D) is a conic problem. A nice and important fact is that *conic duality is symmetric.*

Symmetry of Duality: *The conic dual to* (D) *is (equivalent to)* (P).

Proof. In order to apply to (D) the outlined recipe for building the conic dual, we should rewrite (D) as a *minimization* problem

$$-\mathrm{Opt}(D) = \min_{z,\{y_i\}} \left\{ \langle z, -b\rangle_F + \sum_i \langle y_i, -b_i\rangle_{F_i} : \begin{array}{l} y_i \in \mathbf{K}_i^*, 1 \le i \le m \\ A^*z + \sum_i A_i^* y_i = c \end{array} \right\}; \qquad (D')$$

the corresponding space of decision vectors is the direct product $F \times F_1 \times ... \times F_m$ of Euclidean spaces equipped with the inner product

$$\langle [z; y_1, ..., y_m], [z'; y_1', ..., y_m'] \rangle = \langle z, z'\rangle_F + \sum_i \langle y_i, y_i'\rangle_{F_i}.$$

The above "duality recipe" as applied to (D') reads as follows: pick weights $\eta_i \in (\mathbf{K}_i^*)^* = \mathbf{K}_i$ and $\zeta \in E$, so that the scalar inequality

$$\underbrace{\langle \zeta, A^*z + \sum_i A_i^* y_i\rangle_E + \sum_i \langle \eta_i, y_i\rangle_{F_i}}_{=\langle A\zeta, z\rangle_F + \sum_i \langle A_i\zeta + \eta_i, y_i\rangle_{F_i}} \ge \langle \zeta, c\rangle_E \qquad (C')$$

in variables z, $\{y_i\}$ is a consequence of the constraints of (D'), and impose on the "aggregation weights" $\zeta, \{\eta_i \in \mathbf{K}_i\}$ an additional restriction that the left hand side in this inequality is, identically in $z, \{y_i\}$, equal to the objective of (D'), that is, the restriction that

$$A\zeta = -b,\ A_i\zeta + \eta_i = -b_i,\ 1 \le i \le m,$$

and maximize under this restriction the right hand side in (C'), thus arriving at the problem

$$\max_{\zeta,\{\eta_i\}} \left\{ \langle c, \zeta\rangle_E : \begin{array}{l} \mathbf{K}_i \ni \eta_i = A_i[-\zeta] - b_i, 1 \le i \le m \\ A[-\zeta] = b \end{array} \right\}.$$

Substituting $x = -\zeta$, the resulting problem, after eliminating η_i variables, is nothing but

$$\max_x \left\{ -\langle c, x\rangle_E : \begin{array}{l} A_i x - b_i \in \mathbf{K}_i,\ 1 \le i \le m \\ Ax = b \end{array} \right\},$$

which is equivalent to (P). □

Conic Duality Theorem

A conic program (P) is called *strictly feasible*, if it admits a feasible solution $\bar{x}$ such that $A_i\bar{x} = -b_i \in \mathrm{int}K_i$, $i = 1, ..., m$.

Conic Duality Theorem is the following statement resembling very much the standard Linear Programming Duality Theorem:

Theorem A.1 [Conic Duality Theorem] *Consider a primal-dual pair of conic problems* (P), (D). *Then*

(i) [Weak Duality] *One has* $\mathrm{Opt}(D) \le \mathrm{Opt}(P)$.

(ii) [Symmetry] *The duality is symmetric:* (D) *is a conic problem, and the problem dual to* (D) *is (equivalent to)* (P).

(iii) [Strong Duality] *If one of the problems* (P), (D) *is strictly feasible and bounded, then the other problem is solvable, and* $\mathrm{Opt}(P) = \mathrm{Opt}(D)$.

If both the problems are strictly feasible, then both are solvable with equal optimal values.

Proof. We have already verified Weak Duality and Symmetry. Let us prove the first claim in Strong Duality. By Symmetry, we can restrict ourselves to the case when the strictly feasible and bounded problem is (P).

Consider the following two sets in the Euclidean space $G = \mathbb{R} \times F \times F_1 \times ... \times F_m$:

$$\begin{array}{rcl} T &=& \{[t; z; y_1; ...; y_m] : \exists x : t = \langle c, x\rangle_E; y_i = A_i x - b_i, 1 \leq i \leq m; \\ && z = Ax - b\}, \\ S &=& \{[t; z; y_1; ...; y_m] : t < \mathrm{Opt}(P), y_1 \in \mathbf{K}_1, ..., y_m \in \mathbf{K}_m, z = 0\}. \end{array}$$

The sets T and S clearly are convex and nonempty; observe that they do not intersect. Indeed, assuming that $[t; z; y_1; ...; y_m] \in S \cap T$, we should have $t < \mathrm{Opt}(P)$, and $y_i \in \mathbf{K}_i$, $z = 0$ (since the point is in S), and at the same time for certain $x \in E$ we should have $t = \langle c, x\rangle_E$ and $A_i x - b_i = y_i \in \mathbf{K}_i$, $Ax - b = z = 0$, meaning that there exists a feasible solution to (P) with the value of the objective $< \mathrm{Opt}(P)$, which is impossible. Since the convex and nonempty sets S and T do not intersect, they can be separated by a linear form: there exists $[\tau; \zeta; \eta_1; ...; \eta_m] \in G = \mathbb{R} \times F \times F_1 \times ... \times F_m$ such that

$$\begin{array}{ll} (a) & \sup\limits_{[t;z;y_1;...;y_m]\in S} \langle [\tau; \zeta; \eta_1; ...; \eta_m], [t; z; y_1; ...; y_m]\rangle_G \\ & \leq \inf\limits_{[t;z;y_1;...;y_m]\in T} \langle [\tau; \zeta; \eta_1; ...; \eta_m], [t; z; y_1; ...; y_m]\rangle_G, \\ (b) & \inf\limits_{[t;z;y_1;...;y_m]\in S} \langle [\tau; \zeta; \eta_1; ...; \eta_m], [t; z; y_1; ...; y_m]\rangle_G \\ & < \sup\limits_{[t;z;y_1;...;y_m]\in T} \langle [\tau; \zeta; \eta_1; ...; \eta_m], [t; z; y_1; ...; y_m]\rangle_G, \end{array}$$

or, which is the same,

$$\begin{array}{ll} (a) & \sup\limits_{t<\mathrm{Opt}(P), y_i\in\mathbf{K}_i} [\tau t + \sum_i \langle \eta_i, y_i\rangle_{F_i}] \\ & \leq \inf\limits_{x\in E} [\tau\langle c, x\rangle_E + \langle \zeta, Ax - b\rangle_F + \sum_i \langle \eta_i, A_i x - b_i\rangle_{F_i}], \\ (b) & \inf\limits_{t<\mathrm{Opt}(P), y_i\in\mathbf{K}_i} [\tau t + \sum_i \langle \eta_i, y_i\rangle_{F_i}] \\ & < \sup\limits_{x\in E} [\tau\langle c, x\rangle + \langle \zeta, Ax - b\rangle_F + \sum_i \langle \eta_i, Ax - b_i\rangle_{F_i}]. \end{array} \tag{A.2.1}$$

Since the left hand side in (A.2.1.a) is finite, we have

$$\tau \geq 0,\ -\eta_i \in \mathbf{K}_i^*, 1 \leq i \leq m, \tag{A.2.2}$$

whence the left hand side in (A.2.1.a) is equal to $\tau\mathrm{Opt}(P)$. Since the right hand side in (A.2.1.a) is finite and $\tau \geq 0$, we have

$$A^*\zeta + \sum_i A_i^*\eta_i + \tau c = 0 \tag{A.2.3}$$

and the right hand side in (a) is $\langle -\zeta, b\rangle_F - \sum_i \langle \eta_i, b_i\rangle_{F_i}$, so that (A.2.1.$a$) reads

$$\tau\mathrm{Opt}(P) \leq \langle -\zeta, b\rangle_F - \sum_i \langle \eta_i, b_i\rangle_{F_i}. \tag{A.2.4}$$

We claim that $\tau > 0$. Believing in our claim, let us extract from it Strong Duality. Indeed, setting $y_i = -\eta_i/\tau$, $z = -\zeta/\tau$, (A.2.2), (A.2.3) say that $z, \{y_i\}$ is a feasible solution for (D), and by (A.2.4) the value of the dual objective at this dual feasible solution is $\geq \mathrm{Opt}(P)$. By Weak

Duality, this value cannot be larger than $\mathrm{Opt}(P)$, and we conclude that our solution to the dual is in fact an optimal one, and that $\mathrm{Opt}(P) = \mathrm{Opt}(D)$, as claimed.

It remains to prove that $\tau > 0$. Assume this is not the case; then $\tau = 0$ by (A.2.2). Now let $\bar{x}$ be a strictly feasible solution to (P). Taking inner product of both sides in (A.2.3) with $\bar{x}$, we have

$$\langle \zeta, A\bar{x}\rangle_F + \sum_i \langle \eta_i, A_i\bar{x}\rangle_{F_i} = 0,$$

while (A.2.4) reads

$$-\langle \zeta, b\rangle_F - \sum_i \langle \eta_i, b_i\rangle_{F_i} \geq 0.$$

Summing up the resulting inequalities and taking into account that $\bar{x}$ is feasible for (P), we get

$$\sum_i \langle \eta_i, A_i\bar{x} - b_i\rangle \geq 0.$$

Since $A_i\bar{x} - b_i \in \mathrm{int}\mathbf{K}_i$ and $\eta_i \in -\mathbf{K}_i^*$, the inner products in the left hand side of the latter inequality are nonpositive, and i-th of them is zero if and only if $\eta_i = 0$; thus, the inequality says that $\eta_i = 0$ for all i. Adding this observation to $\tau = 0$ and looking at (A.2.3), we see that $A^*\zeta = 0$, whence $\langle \zeta, Ax\rangle_F = 0$ for all x and, in particular, $\langle \zeta, b\rangle_F = 0$ due to $b = A\bar{x}$. The bottom line is that $\langle \zeta, Ax - b\rangle_F = 0$ for all x. Now let us look at (A.2.1.b). Since $\tau = 0$, $\eta_i = 0$ for all i and $\langle \zeta, Ax - b\rangle_F = 0$ for all x, both sides in this inequality are equal to 0, which is impossible. We arrive at a desired contradiction.

We have proved the first claim in Strong Duality. The second claim there is immediate: if both (P), (D) are strictly feasible, then both problems are bounded as well by Weak Duality, and thus are solvable with equal optimal values by the already proved part of Strong Duality. $\square$

Optimality conditions in Conic Programming

Optimality conditions in Conic Programming are given by the following statement:

Theorem A.2 *Consider a primal-dual pair (P), (D) of conic problems, and let both problems be strictly feasible. A pair $(x, \xi \equiv [z; y_1; ...; y_m])$ of feasible solutions to (P) and (D) is comprised of optimal solutions to the respective problems if and only if*

(i) [Zero duality gap] *One has*

$$\begin{array}{rcl}\mathrm{DualityGap}(x;\xi) & := & \langle c, x\rangle_E - [\langle z, b\rangle_F + \sum_i \langle b_i, y_i\rangle_{F_i}] \\ & = & 0,\end{array}$$

same as if and only if

(ii) [Complementary slackness]

$$\forall i : \langle y_i, A_i x_i - b_i\rangle_{F_i} = 0.$$

Proof. By Conic Duality Theorem, we are in the situation when $\mathrm{Opt}(P) = \mathrm{Opt}(D)$. Therefore

$$\mathrm{DualityGap}(x;\xi) = \underbrace{[\langle c, x\rangle_E - \mathrm{Opt}(P)]}_{a} + \underbrace{\left[\mathrm{Opt}(D) - \left[\langle z, b\rangle_F + \sum_i \langle b_i, y_i\rangle_{F_i}\right]\right]}_{b}$$

Since x and ξ are feasible for the respective problems, the duality gap is nonnegative and it can vanish if and only if $a = b = 0$, that is, if and only if x and ξ are optimal solutions to the respective problems, as claimed in (i). To prove (ii), note that since x is feasible, we have

$$Ax = b,\ A_i x - b_i \in \mathbf{K}_i,\ c = A^* z + \sum_i A_i^* y_i, y_i \in \mathbf{K}_i^*,$$

whence

$$\begin{array}{l}\mathrm{DualityGap}(x;\xi) = \langle c, x\rangle_E - [\langle z, b\rangle_F + \sum_i \langle b_i, y_i\rangle_{F_i}] \\ = \langle A^* z + \sum_i A_i^* y_i, x\rangle_E - [\langle z, b\rangle_F + \sum_i \langle b_i, y_i\rangle_{F_i}] \\ = \underbrace{\langle z, Ax - b\rangle_F}_{=0} + \sum_i \underbrace{\langle y_i, A_i x - b_i\rangle_{F_i}}_{\geq 0},\end{array}$$

where the nonnegativity of the terms in the last $\sum_i$ follows from $y_i \in \mathbf{K}_i^*, A_i x_i - b_i \in \mathbf{K}_i$. We see that the duality gap, as evaluated at a pair of primal-dual feasible solutions, vanishes if and only if the complementary slackness holds true, and thus (ii) is readily given by (i). □

A.2.4 Conic Representations of Sets and Functions

Conic representations of sets

When asked whether the optimization programs

$$\min_y \sum_{i=1}^m |a_i^T y - b_i| \tag{A.2.5}$$

and

$$\min_y \max_{1\leq i\leq m} |a_i^T y - b_i| \tag{A.2.6}$$

are Linear Optimization programs, the answer definitely will be ”yes”, in spite of the fact that an LO program is defined as

$$\min_x \left\{c^T x : Ax \geq b, Px = p\right\} \tag{A.2.7}$$

and neither (A.2.5), nor (A.2.6) are in this form. What the “yes” answer actually means, is that both (A.2.5) and (A.2.6) can be straightforwardly *reduced to*, or, which is the same, *represented by* LO programs, e.g., the LO program

$$\min_{y,u} \left\{\sum_{i=1}^m u_i : -u_i \leq a_i^T y - b_i \leq u_i,\ 1 \leq i \leq m\right\} \tag{A.2.8}$$

in the case of (A.2.5), and the LO program

$$\min_{y,t} \left\{t : -t \leq a_i^T y - b_i \leq t,\ 1 \leq i \leq m\right\} \tag{A.2.9}$$

in the case of (A.2.6).

An “in-depth” explanation of what actually takes place in these and similar examples is as follows.

1. The "initial form" of a typical Mathematical Programming problem is $\min_{v\in V} f(v)$, where $f(v) : V \to \mathbb{R}$ is the objective, and $V \subset \mathbb{R}^n$ is the feasible set of the problem. It is technically convenient to assume that the objective is "as simple as possible" — just linear: $f(v) = e^T v$; this assumption does not restrict generality, since we can always pass from the original problem, given in the form $\min_{v\in V} \phi(v)$, to the equivalent problem

$$\min_{y=[v;s]} \left\{ c^T y \equiv s : y \in Y = \{[v; s] : v \in V, s \geq \phi(v)\} \right\}.$$

Thus, from now on we assume w.l.o.g. that the original problem is

$$\min_y \left\{ d^T y : y \in Y \right\}. \tag{A.2.10}$$

2. All we need in order to reduce (A.2.10) to an LO program is what is called a *polyhedral representation* of Y, that is, a representation of the form

$$U = \{y \in \mathbb{R}^n : \exists u : Ay + Bu - b \in \mathbb{R}^N_+\}.$$

Indeed, given such a representation, we can reformulate (A.2.10) as the LO program

$$\min_{x=[y;u]} \left\{ c^T x := d^T y : \mathcal{A}(x) := Ay + Bu - b \geq 0 \right\}.$$

For example, passing from (A.2.5) to (A.2.8), we first rewrite the original problem as

$$\min_{t,y} \left\{ t : \sum_i |a_i^T y - b_i| \leq t \right\}$$

and then point out a polyhedral representation

$$\begin{array}{l} \{[y;t] : \sum_i |a_i^T y - b_i| \leq t\} \\ = \{[y;t] : \exists u : \underbrace{\left\{ \begin{array}{l} u_i - a_i^T y + b_i \geq 0, \\ u_i + a_i^T y - b_i \geq 0, \\ t - \sum_i u_i \geq 0 \end{array} \right.}_{A[y;t]+Bu-b\geq 0} \} \end{array}$$

of the feasible set of the latter problem, thus ending up with reformulating the problem of interest as an LO program in variables y, t, u. The course of actions for (A.2.6) is completely similar, up to the fact that after "linearizing the objective" we get the optimization problem

$$\min_{y,t} \left\{ t : -t \leq a_i^T y - b_i \leq t, \, 1 \leq i \leq m \right\}$$

where the feasible set is polyhedral "as it is" (i.e., with polyhedral representation not requiring u-variables).

The notion of polyhedral representation naturally extends to conic problems, specifically, as follows. Let $\mathcal{K}$ be a family of regular cones, every one "living" in its own Euclidean space. A set $Y \subset \mathbb{R}^n$ is called *$\mathcal{K}$-representable*, if it can be represented in the form

$$Y = \{y \in \mathbb{R}^n : \exists u \in \mathbb{R}^m : Ay + Bu - b \in \mathbf{K}\}, \tag{A.2.11}$$

where $\mathbf{K} \in \mathcal{K}$ and A, B, b are matrices and vectors of appropriate dimensions. A representation of Y of the form (A.2.11), (i.e., the corresponding collection $A, B, b, \mathbf{K}$), is called a *$\mathcal{K}$-representation* ($\mathcal{K}$-r. for short) of Y.

Geometrically, a $\mathcal{K}$-r. of Y is the representation of Y as the *projection* on the space of y variables of the set $Y_+ = \{[y;u] : Ax + Bu - b \in \mathbf{K}\}$, which, in turn, is given as the inverse image of a cone $\mathbf{K} \in \mathcal{K}$ under the affine mapping $[y;u] \mapsto Ay + Bu - b$.

The role of the notion of a conic representation stems from the fact that given a $\mathcal{K}$-r. of the feasible domain Y of (A.2.10), we can immediately rewrite this optimization program as a conic program involving a cone from the family $\mathcal{K}$, specifically, as the program

$$\min_{x=[y;u]} \left\{ c^T x := d^T y : \mathcal{A}(x) := Ay + Bu - b \in \mathbf{K} \right\}. \tag{A.2.12}$$

In particular,

- When $\mathcal{K} = \mathcal{LO}$ is the family of all nonnegative orthants (or, which is the same, the family of all finite direct products of nonnegative rays), a $\mathcal{K}$-representation of Y allows one to rewrite (A.2.10) as a Linear program;
- When $\mathcal{K} = \mathcal{CQO}$ is the family of all finite direct products of Lorentz cones, a $\mathcal{K}$-representation of Y allows one to rewrite (A.2.10) as a Conic Quadratic program;
- When $\mathcal{K} = \mathcal{SDO}$ is the family of all finite direct products of positive semidefinite cones, a $\mathcal{K}$-representation of Y allows one to rewrite (A.2.10) as a Semidefinite program.

Note that a $\mathcal{K}$-representable set is always convex.

Elementary calculus of $\mathcal{K}$-representations

It turns out that when the family of cones $\mathcal{K}$ is "rich enough," $\mathcal{K}$-representations admit a kind of simple "calculus" that allows to convert $\mathcal{K}$-r.'s of operands participating in a standard convexity-preserving operation, like taking intersection, into a $\mathcal{K}$-r. of the result of this operation. "Richness" here means that $\mathcal{K}$

- contains a nonnegative ray $\mathbb{R}_+$;
- is closed w.r.t. taking finite direct products: whenever $\mathbf{K}_i \in \mathcal{K}$, $1 \le i \le m < \infty$, one has $\mathbf{K}_1 \times ... \times \mathbf{K}_m \in \mathcal{K}$;
- is closed w.r.t. passing from a cone to its dual: whenever $\mathbf{K} \in \mathcal{K}$, one has $\mathbf{K}^* \in \mathcal{K}$.

In particular, every one of the three aforementioned families of cones $\mathcal{LO}$, $\mathcal{CQO}$, $\mathcal{SDO}$ is rich.

We present here the most basic and most frequently used "calculus rules" (for more rules and for instructive examples of $\mathcal{LO}$-, $\mathcal{CQO}$-, and $\mathcal{SDO}$-representable sets, see [9]). Let $\mathcal{K}$ be a rich family of cones. Then

1. [taking finite intersections] If the sets $Y_i \subset \mathbb{R}^n$ are $\mathcal{K}$-representable, $1 \le i \le m$, then so is their intersection $Y = \bigcap\limits_{i=1}^m Y_i$.
   Indeed, if $Y_i = \{y \in \mathbb{R}^n : \exists u_i : A_i x + B_i u - b_i \in \mathbf{K}_i$ with $\mathbf{K}_i \in \mathcal{K}$, then
   $$\begin{array}{rl} Y = & \{y \in \mathbb{R}^n : \exists u = [u_1; ...; u_m] : \\ & \quad [A_1; ...; A_m]y + \mathrm{Diag}\{B_1, ..., B_m\}[u_1; ...; u_m] - [b_1; ...; b_m] \\ & \quad \in \mathbf{K} := \mathbf{K}_1 \times ... \times \mathbf{K}_m\}, \end{array}$$
   and $\mathbf{K} \in \mathcal{K}$, since $\mathcal{K}$ is closed w.r.t. taking finite direct products.

2. [taking finite direct products] If the sets $Y_i \subset \mathbb{R}^{n_i}$ are $\mathcal{K}$-representable, $1 \leq i \leq m$, then so is their direct product $Y = Y_1 \times ... \times Y_m$.
   Indeed, if $Y_i = \{y \in \mathbb{R}^n : \exists u_i : A_i x + B_i u - b_i \in \mathbf{K}_i$ with $\mathbf{K}_i \in \mathcal{K}$, then
   $$\begin{array}{rl} Y = & \{y = [y_1; ...; y_m] \in \mathbb{R}^{n_1+...+n_m} : \exists u = [u_1; ...; u_m] : \\ & \quad \mathrm{Diag}\{A_1, ..., A_m]y + \mathrm{Diag}\{B_1, ..., B_m\}[u_1; ...; u_m] - [b_1; ...; b_m] \\ & \quad \in \mathbf{K} := \mathbf{K}_1 \times ... \times \mathbf{K}_m\}, \end{array}$$
   and, as above, $\mathbf{K} \in \mathcal{K}$.
3. [taking inverse affine images] Let $Y \subset \mathbb{R}^n$ be $\mathcal{K}$-representable, let $z \mapsto Pz + p : \mathbb{R}^N \to \mathbb{R}^n$ be an affine mapping. Then the inverse affine image $Z = \{z : Pz + p \in Y\}$ of Y under this mapping is $\mathcal{K}$-representable.
   Indeed, if $Y = \{y \in \mathbb{R}^n : \exists u : Ay + Bu - b \in \mathbf{K}\}$ with $\mathbf{K} \in \mathcal{K}$, then
   $$Z = \{z \in \mathbb{R}^N : \exists u : \underbrace{A[Pz + p] + Bu - b}_{\equiv \widetilde{A}z + Bu - \widetilde{b}} \in \mathbf{K}\}.$$
4. [taking affine images] If a set $Y \subset \mathbb{R}^n$ is $\mathcal{K}$-representable and $y \mapsto z = Py + p : \mathbb{R}^n \to \mathbb{R}^m$ is an affine mapping, then the image $Z = \{z = Py + p : y \in Y\}$ of Y under the mapping is $\mathcal{K}$-representable.
   Indeed, if $Y = \{y \in \mathbb{R}^n : \exists u : Au + Bu - b \in \mathbf{K}\}$, then
   $$Z = \{z \in \mathbb{R}^m : \exists [y; u] : \underbrace{\left[\begin{array}{c} Py + p - z \\ -Py - p + z \\ Ay + Bu - b \end{array}\right]}_{\equiv \widetilde{A}z + \widetilde{B}[y;u] - \widetilde{b}} \in \mathbf{K}_+ := \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbf{K}\},$$
   and the cone $\mathbf{K}_+$ belongs to $\mathcal{K}$ as the direct product of several nonnegative rays (every one of them belongs to $\mathcal{K}$) and the cone $\mathbf{K} \in \mathcal{K}$.

Note that the above "calculus rules" are "completely algorithmic" — a $\mathcal{K}$-r. of the result of an operation is readily given by $\mathcal{K}$-r.'s of the operands.

Conic representation of functions

By definition, the *epigraph* of a function $f(y) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the set

$$\mathrm{Epi}\{f\} = \{[y; t] \in \mathbb{R}^n \times \mathbb{R} : t \geq f(y)\}.$$

Note that a function is convex if and only if its epigraph is so.

Let $\mathcal{K}$ be a family of regular cones. A function f is called *$\mathcal{K}$-representable*, if its epigraph is so:

$$\mathrm{Epi}\{f\} := \{[y, t] : \exists u : Ay + ta + Bu - b \in \mathbf{K}\} \tag{A.2.13}$$

with $\mathbf{K} \in \mathcal{K}$. A $\mathcal{K}$-representation ($\mathcal{K}$-r. for short) of a function is, by definition, a $\mathcal{K}$-r. of its epigraph. Since $\mathcal{K}$-representable sets always are convex, so are $\mathcal{K}$-representable functions.

Examples of $\mathcal{K}$-r.'s of functions:

- the function $f(y) = |y| : \mathbb{R} \to \mathbb{R}$ is $\mathcal{LO}$-representable:
  $$\{[y; t] : t \geq |y|\} = \{[y; t] : A[y; t] := [t - y; t + y] \in \mathbb{R}^2_+\};$$

- the function $f(y) = \|y\|_2 : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{CQO}$-representable:

$$\{[y;t] \in \mathbb{R}^{n+1} : t \geq \|y\|_2\} = \{[y;t] \in \mathbf{L}^{n+1}\};$$

- the function $f(y) = \lambda_{\max}(y) : \mathbf{S}^n \to \mathbb{R}$ (the maximal eigenvalue of a symmetric matrix y) is $\mathcal{SDO}$-representable:

$$\{[y;t] \in \mathbf{S}^n \times \mathbb{R} : t \geq \lambda_{\max}(y)\} = \{[y;t] : \mathcal{A}[y;t] := tI_n - y \in \mathbf{S}^n_+\}.$$

Observe that a $\mathcal{K}$-r. (A.2.13) of a function f induces $\mathcal{K}$-r.'s of its *level sets* $\{y : f(y) \leq c\}$:

$$\{y : f(y) \leq c\} = \{y : \exists u : Ay + Bu - [b - ca] \in \mathbf{K}\}.$$

This explains the importance of $\mathcal{K}$-representations of functions: usually, the feasible set Y of a convex problem (A.2.10) is given by a system of convex constraints:

$$Y = \{y : f_i(y) \leq 0,\, 1 \leq i \leq m\}.$$

If now all functions f_i are $\mathcal{K}$-representable, then, by the above observation and by the "calculus rule" related to intersections, Y is $\mathcal{K}$-representable as well, and a $\mathcal{K}$-r. of Y is readily given by $\mathcal{K}$-r.'s of f_i.

$\mathcal{K}$-representable functions admit simple calculus, which is similar to the one of $\mathcal{K}$-representable sets, and is equally algorithmic; for details and instructive examples, see [9].

A.3 Efficient Solvability of Convex Programming

The goal of this section is to explain the precise meaning of the informal (and in fact slightly exaggerated) claim,

> *An optimization problem with convex efficiently computable objective and constraints is efficiently solvable.*

that on many different occasions was reiterated in the main body of the book. Our exposition follows the one from [9, chapter 5].

A.3.1 Generic Convex Programs and Efficient Solution Algorithms

In what follows, it is convenient to represent optimization programs as

$$(p): \quad \mathrm{Opt}(p) = \min_x \left\{ p_0(x) : x \in X(p) \subset \mathbb{R}^{n(p)} \right\},$$

where $p_0(\cdot)$ and $X(p)$ are the objective, which we assume to be a real-valued function on $\mathbb{R}^{n(p)}$, and the feasible set of program (p), respectively, and $n(p)$ is the dimension of the decision vector.

A generic optimization problem

A *generic optimization program* $\mathcal{P}$ is a collection of optimization programs (p) ("instances of $\mathcal{P}$") such that every instance of $\mathcal{P}$ is identified by a finite-dimensional *data vector* $\mathrm{data}(p)$; the dimension of this vector is called the *size* $\mathrm{Size}(p)$ of the instance:

$$\mathrm{Size}(p) = \dim \mathrm{data}(p).$$

For example, *Linear Optimization* is a generic optimization problem $\mathcal{LO}$ with instances of the form

$$(p): \quad \min_x \left\{c_p^T x : x \in X(p) := \{x : A_p x - b_p \geq 0\}\right\} \qquad [A_p : m(p) \times n(p)],$$

where $m(p), n(p), c_p, A_p, b_p$ can be arbitrary. The data of an instance can be identified with the vector

$$\mathrm{data}(p) = [m(p); n(p); c_p; b_p; A_p^1; ...; A_p^{n(p)}],$$

where A_p^i is i-th column in A_p.

Similarly, *Conic Quadratic Optimization* is a generic optimization problem $\mathcal{CQO}$ with instances

$$\begin{array}{c} (p): \quad \min_x \left\{c_p^T x : x \in X(p)\right\}, \\ X(p) := \{x : \|A_{pi}x - b_{pi}\|_2 \leq e_{pi}^T x - d_{pi},\ 1 \leq i \leq m(p)\} \end{array} \qquad [A_{pi} : k_i(p) \times n(p)].$$

The data of an instance can be defined as the vector obtained by listing, in a fixed order, the dimensions $m(p)$, $n(p)$, $\{k_i(p)\}_{i=1}^{m(p)}$ and the entries of the reals d_{pi}, vectors c_p, b_{pi}, e_{pi} and the matrices A_{pi}^ℓ.

Finally, *Semidefinite Optimization* is a generic optimization problem $\mathcal{SDO}$ with instances of the form

$$\begin{array}{c} (p): \ \min_x \left\{c_p^T x : x \in X(p) := \{x : A_p^i(x) \succeq 0, 1 \leq i \leq m(p)\}\right\} \\ A_p^i(x) = A_{pi}^0 + x_1 A_{pi}^1 + ... + x_{n(p)} A_{pi}^{n(p)}, \end{array}$$

where A_{pi}^ℓ are symmetric matrices of size $k_i(p)$. The data of an instance can be defined in the same fashion as in the case of $\mathcal{CQO}$.

Approximate solutions

In order to quantify the quality of a candidate solution of an instance (p) of a generic problem $\mathcal{P}$, we assume that $\mathcal{P}$ is equipped with an *infeasibility measure* $\mathrm{Infeas}_{\mathcal{P}}(p,x)$ — a real-valued nonnegative function of an instance $(p) \in \mathcal{P}$ and a candidate solution $x \in \mathbb{R}^{n(p)}$ to the instance such that $x \in X(p)$ if and only if $\mathrm{Infeas}_{\mathcal{P}}(p,x) = 0$.

Given an infeasibility measure and a tolerance $\epsilon > 0$, we define an ϵ *solution* to an instance $(p) \in \mathcal{P}$ as a point $x_\epsilon \in \mathbb{R}^{n(p)}$ such that

$$p_0(x_\epsilon) - \mathrm{Opt}(p) \leq \epsilon \ \& \ \mathrm{Infeas}_{\mathcal{P}}(p, x_\epsilon) \leq \epsilon.$$

For example, a natural infeasibility measure for a generic optimization problem $\mathcal{P}$ with instances of the form

$$(p): \quad \min_x \{p_0(x) : x \in X(p) := \{x : p_i(x) \leq 0,\ 1 \leq i \leq m(p)\}\} \tag{A.3.1}$$

is

$$\mathrm{Infeas}_{\mathcal{P}}(p,x) = \max\left[0, p_1(x), p_2(x), ..., p_{m(p)}(x)\right]; \tag{A.3.2}$$

this recipe, in particular, can be applied to the generic problems $\mathcal{LO}$ and $\mathcal{CQO}$. A natural infeasibility measure for $\mathcal{SDO}$ is

$$\mathrm{Infeas}_{\mathcal{SDO}}(p,x) = \min\left\{t \geq 0 : A_p^i(x) + tI_{k_i(p)} \succeq 0,\ 1 \leq i \leq m(p)\right\}.$$

Convex generic optimization problems

A generic problem $\mathcal{P}$ is called *convex*, if for every instance (p) of the problem, $p_0(x)$ and $\mathrm{Infeas}_{\mathcal{P}}(p,x)$ are convex functions of $x \in \mathbb{R}^{n(p)}$. Note that then $X(p) = \{x \in \mathbb{R}^{n(p)} : \mathrm{Infeas}_{\mathcal{P}}(p,x) \leq 0\}$ is a convex set for every $(p) \in \mathcal{P}$.

For example, $\mathcal{LO}$, $\mathcal{CQO}$ and $\mathcal{SDO}$ with the just defined infeasibility measures are generic convex programs. The same is true for generic problems with instances (A.3.1) and infeasibility measure (A.3.2), provided that all instances are convex programs, i.e., $p_0(x), p_1(x), ..., p_{m(p)}(x)$ are restricted to be real-valued *convex* functions on $\mathbb{R}^{n(p)}$.

A solution algorithm

A solution algorithm $\mathcal{B}$ for a generic problem $\mathcal{P}$ is a code for the Real Arithmetic Computer — an idealized computer capable to store real numbers and to carry out the operations of Real Arithmetics (the four arithmetic operations, comparisons and computing elementary functions like $\sqrt{\cdot}$, $\exp\{\cdot\}$, $\sin(\cdot)$) with real arguments. Given on input the data vectors $\mathrm{data}(p)$ of an instance $(p) \in \mathcal{P}$ and a tolerance $\epsilon > 0$ and executing on this input the code $\mathcal{B}$, the computer should eventually stop and output

— either a vector $x_\epsilon \in \mathbb{R}^{n(p)}$ that must be an ϵ solution to (p),

— or a correct statement "(p) is infeasible"/"(p) is not below bounded."

The *complexity* of the generic problem $\mathcal{P}$ with respect to a solution algorithm $\mathcal{B}$ is quantified by the function $\mathrm{Compl}_{\mathcal{P}}(p,\epsilon)$; the value of this function at a pair $(p) \in \mathcal{P}$, $\epsilon > 0$ is exactly the number of elementary operations of the Real Arithmetic Computer in the course of executing the code $\mathcal{B}$ on the input $(\mathrm{data}(p), \epsilon)$.

Polynomial time solution algorithms

A solution algorithm for a generic problem $\mathcal{P}$ is called *polynomial time* ("efficient"), if the complexity of solving instances of $\mathcal{P}$ within (an arbitrary) accuracy $\epsilon > 0$ is bounded by a polynomial in the size of the instance and the *number of accuracy digits* $\mathrm{Digits}(p,\epsilon)$ in an ϵ solution:

$$\mathrm{Compl}_{\mathcal{P}}(p,\epsilon) \leq \chi\left(\mathrm{Size}(p)\mathrm{Digits}(p,\epsilon)\right)^{\chi},$$
$$\mathrm{Size}(p) = \dim \mathrm{data}(p),\ \mathrm{Digits}(p,\epsilon) = \ln\left(\frac{\mathrm{Size}(p)+\|\mathrm{data}(p)\|_1+\epsilon^2}{\epsilon}\right);$$

from now on, χ stands for various "characteristic constants" (not necessarily identical to each other) of the generic problem in question, i.e., for positive quantities depending on $\mathcal{P}$ and independent of $(p) \in \mathcal{P}$ and $\epsilon > 0$. Note also that while the "strange" numerator in the fraction participating in the definition of Digits arises by technical reasons, the number of accuracy digits for small $\epsilon > 0$ becomes independent of this numerator and close to $\ln(1/\epsilon)$.

A generic problem $\mathcal{P}$ is called *polynomially solvable* ("computationally tractable"), if it admits a polynomial time solution algorithm.

A.3.2 Polynomial Solvability of Generic Convex Programming Problems

The main fact about generic convex problems that underlies the remarkable role played by these problems in Optimization is that *under minor non-restrictive technical assumptions, a generic convex problem*, in contrast to typical generic non-convex problems, *is computationally tractable.*

The just mentioned "minor non-restrictive technical assumptions" are those of *polynomial computability*, *polynomial growth*, and *polynomial boundedness of feasible sets*.

Polynomial computability

A generic convex optimization problem $\mathcal{P}$ is called *polynomially computable*, if it can be equipped with two codes, $\mathcal{O}$ and $\mathcal{C}$, for the Real Arithmetic Computer, such that:

• for every instance $(p) \in \mathcal{P}$ and any candidate solution $x \in \mathbb{R}^{n(p)}$ to the instance, executing $\mathcal{O}$ on the input $(\text{data}(p), x)$ takes a polynomial in $\text{Size}(p)$ number of elementary operations and produces a value and a subgradient of the objective $p_0(\cdot)$ at the point x;

• for every instance $(p) \in \mathcal{P}$, any candidate solution $x \in \mathbb{R}^{n(p)}$ to the instance and any $\epsilon > 0$, executing $\mathcal{C}$ on the input $(\text{data}(p), x, \epsilon)$ takes a polynomial in $\text{Size}(p)$ and $\text{Digits}(p, \epsilon)$ number of elementary operations and results

— either in a correct claim that $\text{Infeas}_{\mathcal{P}}(p, x) \leq \epsilon$,

— or in a correct claim that $\text{Infeas}_{\mathcal{P}}(p, x) > \epsilon$ and in computing a linear form $e \in \mathbb{R}^{n(p)}$ that separates x and the set $\{y : \text{Infeas}_{\mathcal{P}}(p, y) \leq \epsilon\}$, so that

$$\forall (y, \text{Infeas}_{\mathcal{P}}(p, y) \leq \epsilon) : e^T y < e^T x.$$

Consider, for example, a generic convex program $\mathcal{P}$ with instances of the form (A.3.1) and the infeasibility measure (A.3.2) and assume that the functions $p_0(\cdot), p_1(\cdot), ..., p_{m(p)}(\cdot)$ are real-valued and convex for all instances of $\mathcal{P}$. Assume, moreover, that the objective and the constraints of instances are efficiently computable, meaning that there exists a code $\mathcal{CO}$ for the Real Arithmetic Computer, which being executed on an input of the form $(\text{data}(p), x \in \mathbb{R}^{n(p)})$ computes in a polynomial in $\text{Size}(p)$ number of elementary operations the values and subgradients of $p_0(\cdot)$, $p_1(\cdot)$,..., $p_{m(p)}(\cdot)$ at x. In this case, $\mathcal{P}$ is polynomially computable. Indeed, the code $\mathcal{O}$ allowing to compute in polynomial time the value and a subgradient of the objective at a given candidate solution is readily given by $\mathcal{CO}$. In order to build $\mathcal{C}$, let us execute $\mathcal{CO}$ on an input $(\text{data}(p), x)$ and compare the quantities $p_i(x)$, $1 \leq i \leq m(p)$, with ϵ. If $p_i(x) \leq \epsilon$, $1 \leq i \leq m(p)$, we output the correct claim that $\text{Infeas}_{\mathcal{P}}(p, x) \leq \epsilon$, otherwise we output a correct claim that $\text{Infeas}_{\mathcal{P}}(p, x) > \epsilon$ and return, as e, a subgradient, taken at x, of a constraint $p_{i(x)}(\cdot)$, where $i(x) \in \{1, 2, ..., m(p)\}$ is such that $p_{i(x)}(x) > \epsilon$.

By the reasons outlined above, the generic problems $\mathcal{LO}$ and $\mathcal{CQO}$ of Linear and Conic Quadratic Optimization are polynomially computable. The same is true for Semidefinite Optimization, see [9, chapter 5].

Polynomial growth

We say that $\mathcal{P}$ is of *polynomial growth*, if for properly chosen $\chi > 0$ one has

$$\begin{array}{l}\forall ((p) \in \mathcal{P}, x \in \mathbb{R}^{n(p)}) : \\ \max\left[|p_0(x)|, \text{Infeas}_{\mathcal{P}}(p, x)\right] \leq \chi \left(\text{Size}(p) + \|\text{data}(p)\|_1\right)^{\chi \text{Size}^{\chi}(p)}.\end{array}$$

For example, the generic problems of Linear, Conic Quadratic and Semidefinite Optimization clearly are with polynomial growth.

Polynomial boundedness of feasible sets

We say that $\mathcal{P}$ is with polynomially bounded feasible sets, if for properly chosen $\chi > 0$ one has

$$\forall ((p) \in \mathcal{P}) : x \in X(p) \Rightarrow \|x\|_\infty \leq \chi \left(\text{Size}(p) + \|\text{data}(p)\|_1\right)^{\chi \text{Size}^{\chi}(p)}.$$

While the generic convex problems $\mathcal{LO}$, $\mathcal{CQO}$, and $\mathcal{SDO}$ are polynomially computable and with polynomial growth, neither one of these problems (same as neither one of other natural generic convex problems) "as it is" possesses polynomially bounded feasible sets. We, however, can enforce the latter property by passing from a generic problem $\mathcal{P}$ to its "bounded version" $\mathcal{P}_{\rm b}$ as follows: the instances of $\mathcal{P}_{\rm b}$ are the instances (p) of $\mathcal{P}$ augmented by bounds on the variables; thus, an instance $(p_+) = (p, R)$ of $\mathcal{P}_{\rm b}$ is of the form

$$(p, R): \quad \min_x \left\{ p_0(x) : x \in X(p, R) = X(p) \cap \{x \in \mathbb{R}^{n(p)} : \|x\|_\infty \leq R\} \right\}$$

where (p) is an instance of $\mathcal{P}$ and $R > 0$. The data of (p, R) is the data of (p) augmented by R, and

$$\text{Infeas}_{\mathcal{P}_{\rm b}}((p, R), x) = \text{Infeas}_{\mathcal{P}}(p, x) + \max[\|x\|_\infty - R, 0].$$

Note that $\mathcal{P}_{\rm b}$ inherits from $\mathcal{P}$ the properties of polynomial computability and/or polynomial growth, if any, and always is with polynomially bounded feasible sets. Note also that R can be really large, like $R = 10^{100}$, which makes the "expressive abilities" of $\mathcal{P}_{\rm b}$, for all practical purposes, as strong as those of $\mathcal{P}$. Finally, we remark that the "bounded versions" of $\mathcal{LO}$, $\mathcal{CQO}$, and $\mathcal{SDO}$ are sub-problems of the original generic problems.

Main result

The main result on computational tractability of Convex Programming is the following:

Theorem A.3 *Let $\mathcal{P}$ be a polynomially computable generic convex program with a polynomial growth that possesses polynomially bounded feasible sets. Then $\mathcal{P}$ is polynomially solvable.*

As a matter of fact, "in real life" the only restrictive assumption in Theorem A.3 is the one of polynomial computability. This is the assumption that is usually violated when speaking about *semi-infinite* convex programs like the RCs of uncertain conic problems

$$\min_x \left\{ c_p^T x : x \in X(p) = \{x \in \mathbb{R}^{n(p)} : A_{p\zeta} x + a_{p\zeta} \in \mathbf{K} \,\forall (\zeta \in \mathcal{Z})\} \right\}.$$

associated with simple *non-polyhedral* cones $\mathbf{K}$. Indeed, when $\mathbf{K}$ is, say, a Lorentz cone, so that

$$X(p) = \{x : \|B_{p\zeta} x + b_{p\zeta}\|_2 \leq c_{p\zeta}^T x + d_{p\zeta} \,\forall (\zeta \in \mathcal{Z})\},$$

to compute the natural infeasibility measure

$$\min \left\{ t \geq 0 : \|B_{p\zeta} x + b_{p\zeta}\|_2 \leq c_{p\zeta}^T x + d_{p\zeta} + t \,\forall (\zeta \in \mathcal{Z}) \right\}$$

at a given candidate solution x means to *maximize* the function $f_x(\zeta) = \|B_{p\zeta} x + b_{p\zeta}\|_2 - c_{p\zeta}^T x - d_{p\zeta}$ over the uncertainty set $\mathcal{Z}$. When the uncertain data are affinely parameterized by ζ, this requires a *maximization of a nonlinear convex function $f_x(\zeta)$ over $\zeta \in \mathcal{Z}$*, and this problem can be (and generically is) computationally intractable, even when $\mathcal{Z}$ is a simple convex set. It becomes also clear why the outlined difficulty does not occur in uncertain LO with the data affinely parameterized by ζ: here $f_x(\zeta)$ is an affine function of ζ, and as such can be efficiently maximized over $\mathcal{Z}$, provided the latter set is convex and "not too complicated."

A.3.3 "What is Inside": Efficient Black-Box-Oriented Algorithms in Convex Optimization

Theorem A.3 is a direct consequence of a fact that is instructive in its own right and has to do with "black-box-oriented" Convex Optimization, specifically, with solving an optimization problem

$$\min_{x\in X} f(x), \tag{A.3.3}$$

where

- $X \subset \mathbb{R}^n$ is a solid (a convex compact set with a nonempty interior) known to belong to a given Euclidean ball $E_0 = \{x : \|x\|_2 \leq R\}$ and represented by a *Separation oracle* — a routine that, given on input a point $x \in \mathbb{R}^n$, reports whether $x \in X$, and if it is not the case, returns a vector $e \neq 0$ such that

$$e^T x \geq \max_{y\in X} e^T y;$$

- f is a convex real-valued function on $\mathbb{R}^n$ represented by a *First Order oracle* that, given on input a point $x \in \mathbb{R}^n$, returns the value and a subgradient of f at x.

In addition, we assume that we know in advance an $r > 0$ such that X contains a Euclidean ball of the radius r (the center of this ball can be unknown).

Theorem A.3 is a straightforward consequence of the following important fact:

Theorem A.4 [9, Theorem 5.2.1] *There exists a Real Arithmetic algorithm (the Ellipsoid method) that, as applied to* (A.3.3)*, the required accuracy being* $\epsilon > 0$*, finds a feasible* ϵ *solution* x_ϵ *to the problem (i.e.,* $x_\epsilon \in X$ *and* $f(x_\epsilon) - \min_X f \leq \epsilon$*) after at most*

$$\begin{array}{c} N(\epsilon) = \mathrm{Ceil}\left(2n^2\left[\ln\left(\frac{R}{r}\right) + \ln\left(\frac{\epsilon+\mathrm{Var}_R(f)}{\epsilon}\right)\right]\right) + 1 \\ \mathrm{Var}_R(f) = \max_{\|x\|_2\leq R} f(x) - \min_{\|x\|_2\leq R} f(x) \end{array}$$

steps, with a step reducing to a single call to the Separation and to the First Order oracles accompanied by $O(1)n^2$ *additional arithmetic operations to process the answers of the oracles. Here* $O(1)$ *is an absolute constant.*

Recently, the Ellipsoid method was equipped with "on line" accuracy certificates, which yield a slightly strengthened version of the above theorem, namely, as follows:

Theorem A.5 [74] *Consider problem* (A.3.3) *and assume that*

• $X \in \mathbb{R}^n$ *is a solid contained in the centered at the origin Euclidean ball* E_0 *of a known in advance radius* R *and given by a Separation oracle that, given on input a point* $x \in \mathbb{R}^n$*, reports whether* $x \in \mathrm{int}X$*, and if it is not the case, returns a nonzero* e *such that* $e^T x \geq \max_{y\in X} e^T y$*;*

• $f : \mathrm{int}X \to \mathbb{R}$ *is a convex function represented by a First Order oracle that, given on input a point* $x \in \mathrm{int}X$*, reports the value* $f(x)$ *and a subgradient* $f'(x)$ *of* f *at* x*. In addition, assume that* f *is semibounded on* X*, meaning that* $V_X(f) \equiv \sup_{x,y\in\mathrm{int}X}(y-x)^T f'(x) < \infty$*.*

There exists an explicit Real Arithmetic algorithm that, given on input a desired accuracy $\epsilon > 0$*, terminates with a strictly feasible* ϵ*-solution* x_ϵ *to the problem (*$x_\epsilon \in \mathrm{int}X$*,* $f(x_\epsilon) - \inf_{x\in\mathrm{int}X} f(x) \leq \epsilon$*) after at most*

$$N(\epsilon) = O(1)\left(n^2\left[\ln\left(\frac{nR}{r}\right) + \ln\left(\frac{\epsilon + V_X(f)}{\epsilon}\right)\right]\right)$$

steps, with a step reducing to a single call to the Separation and to the First Order oracles accompanied by $O(1)n^2$ additional arithmetic operations to process the answers of the oracles. Here r is the supremum of the radii of Euclidean balls contained in X, and $O(1)$'s are absolute constants.

The progress, as compared to Theorem A.3, is that now we do not need a priori knowledge of $r > 0$ such that X contains a Euclidean ball of radius r, f is allowed to be undefined outside of $\mathrm{int}X$ and the role of $\mathrm{Var}_R(f)$ (the quantity that now can be $+\infty$) is played by $V_X(f) \le \sup_{\mathrm{int}X} f - \inf_{\mathrm{int}X} f$.

A.4 Miscellaneous

A.4.1 Matrix Cube Theorems

Matrix Cube Theorem, Complex Case

The "Complex Matrix Cube" problem is as follows:

CMC: *Let m, $p_1, q_1, ..., p_L, q_L$ be positive integers, and $A \in \mathbf{H}^m_+$, $L_j \in \mathbb{C}^{p_j \times m}$, $R_j \in \mathbb{C}^{q_j \times m}$ be given matrices, $L_j \neq 0$. Let also a partition $\{1, 2, ..., L\} = I^{\mathrm{r}}_{\mathrm{s}} \cup I^{\mathrm{c}}_{\mathrm{s}} \cup I^{\mathrm{c}}_{\mathrm{f}}$ of the index set $\{1, ..., L\}$ into three non-overlapping sets be given, and let $p_j = q_j$ for $j \in I^{\mathrm{r}}_{\mathrm{s}} \cup I^{\mathrm{c}}_{\mathrm{s}}$. With these data, we associate a parametric family of "matrix boxes"*

$$\mathcal{U}[\rho] = \left\{ A + \rho \sum_{j=1}^{L} [L_j^H \Theta^j R_j + R_j^H [\Theta^j]^H L_j] : \begin{array}{c} \Theta^j \in \mathcal{Z}^j, \\ 1 \le j \le L \end{array} \right\} \subset \mathbf{H}^m, \tag{A.4.1}$$

where $\rho \ge 0$ is the parameter and

$$\mathcal{Z}^j = \left\{ \begin{array}{l} \{\Theta^j = \theta I_{p_j} : \theta \in \mathbb{R}, |\theta| \le 1\}, \quad j \in I^{\mathrm{r}}_{\mathrm{s}} \\ \qquad [\text{“real scalar perturbation blocks”}] \\ \{\Theta^j = \theta I_{p_j} : \theta \in \mathbb{C}, |\theta| \le 1\}, \quad j \in I^{\mathrm{c}}_{\mathrm{s}} \\ \quad [\text{“complex scalar perturbation blocks”}] \\ \{\Theta^j \in \mathbb{C}^{p_j \times q_j} : \|\Theta^j\|_{2,2} \le 1\}, \quad j \in I^{\mathrm{c}}_{\mathrm{f}} \\ \quad [\text{“full complex perturbation blocks”}] \ . \end{array} \right. \tag{A.4.2}$$

Given $\rho \ge 0$, check whether

$$\mathcal{U}[\rho] \subset \mathbf{H}^m_+. \tag{$\mathcal{A}(\rho)$}$$

Remark A.1 *We always assume that $p_j = q_j > 1$ for $j \in I^{\mathrm{c}}_{\mathrm{s}}$. Indeed, one-dimensional complex scalar perturbations can always be regarded as full complex perturbations.*

Our main result is as follows:

Theorem A.6 [The Complex Matrix Cube Theorem [3, section B.4]] *Consider, along with predicate $\mathcal{A}(\rho)$, the predicate*

$$\begin{array}{ll} \exists Y_j \in \mathbf{H}^m, \ j = 1, ..., L \ \textit{such that}: & \\ (a) \quad Y_j \succeq L_j^H \Theta^j R_j + R_j^H [\Theta^j]^H L_j \quad \forall (\Theta^j \in \mathcal{Z}^j, 1 \le j \le L) & \\ (b) \quad A - \rho \sum_{j=1}^{L} Y_j \succeq 0. & \end{array} \tag{$\mathcal{B}(\rho)$}$$

Then:

(i) *Predicate $\mathcal{B}(\rho)$ is stronger than $\mathcal{A}(\rho)$ — the validity of the former predicate implies the validity of the latter one.*

(ii) *$\mathcal{B}(\rho)$ is computationally tractable — the validity of the predicate is equivalent to the solvability of the system of LMIs*

$$\begin{array}{lrcl}
(s.\mathbb{R}) & Y_j \pm \left[L_j^H R_j + R_j^H L_j\right] & \succeq & 0,\ j\in I_{\rm s}^{\rm r}, \\
(s.\mathbb{C}) & \left[\begin{array}{cc} Y_j - V_j & L_j^H R_j \\ R_j^H L_j & V_j \end{array}\right] & \succeq & 0,\ j\in I_{\rm s}^{\rm c}, \\
(f.\mathbb{C}) & \left[\begin{array}{cc} Y_j - \lambda_j L_j^H L_j & R_j^H \\ R_j & \lambda_j I_{p_j} \end{array}\right] & \succeq & 0,\ j\in I_{\rm f}^{\rm c} \\
(*) & A - \rho\sum\limits_{j=1}^{L} Y_j & \succeq & 0.
\end{array} \tag{A.4.3}$$

in the matrix variables $Y_j \in \mathbf{H}^m$, $j = 1,...,k$, $V_j \in \mathbf{H}^m$, $j \in I_{\rm s}^{\rm c}$, and the real variables λ_j, $j \in I_{\rm f}^{\rm c}$.

(iii) *"The gap" between $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$ can be bounded solely in terms of the maximal size*

$$p^{\rm s} = \max\left\{p_j : j \in I_{\rm s}^{\rm r} \cup I_{\rm s}^{\rm c}\right\} \tag{A.4.4}$$

of the scalar perturbations (here the maximum over an empty set by definition is 0). Specifically, there exists a universal function $\vartheta_{\mathbb{C}}(\cdot)$ such that

$$\vartheta_{\mathbb{C}}(\nu) \le 4\pi\sqrt{\nu},\ \nu \ge 1, \tag{A.4.5}$$

and

$$\text{if } \mathcal{B}(\rho) \text{ is not valid, then } \mathcal{A}(\vartheta_{\mathbb{C}}(p^{\rm s})\rho) \text{ is not valid.} \tag{A.4.6}$$

(iv) *Finally, in the case $L = 1$ of single perturbation block $\mathcal{A}(\rho)$ is equivalent to $\mathcal{B}(\rho)$.*

Remark A.2 *From the proof of Theorem A.6 it follows that $\vartheta_{\mathbb{C}}(0) = \frac{4}{\pi}$, $\vartheta_{\mathbb{C}}(1) = 2$. Thus,*

- *when there are no scalar perturbations: $I_{\rm s}^{\rm r} = I_{\rm s}^{\rm c} = \emptyset$, the factor ϑ in the implication*

$$\neg\mathcal{B}(\rho) \Rightarrow \neg\mathcal{A}(\vartheta\rho) \tag{A.4.7}$$

  can be set to $\frac{4}{\pi} = 1.27...$

- *when there are no complex scalar perturbations (cf. Remark A.1) and all real scalar perturbations are non-repeated ($I_{\rm s}^{\rm c} = \emptyset$, $p_j = 1$ for all $j \in I_{\rm s}^{\rm r}$), the factor ϑ in* (A.4.7) *can be set to* 2.

The following simple observation is crucial when applying Theorem A.6.

Remark A.3 *Assume that the data A, $R_1,...,R_L$ of the Matrix Cube problem are affine in a vector of parameters y, while the data $L_1,...,L_L$ are independent of y. Then* (A.4.3) *is a system of LMIs in the variables Y_j, V_j, λ_j* and *y.*

Matrix Cube Theorem, Real Case

The Real Matrix Cube problem is as follows:

RMC: *Let* m, $p_1, q_1, ..., p_L, q_L$ *be positive integers, and* $A \in \mathbf{S}^m$, $L_j \in \mathbb{R}^{p_j \times m}$, $R_j \in \mathbb{R}^{q_j \times m}$ *be given matrices,* $L_j \neq 0$*. Let also a partition* $\{1, 2, ..., L\} = I_{\rm s}^{\rm r} \cup I_{\rm f}^{\rm r}$ *of the index set* $\{1, ..., L\}$ *into two non-overlapping sets be given. With these data, we associate a parametric family of "matrix boxes"*

$$\mathcal{U}[\rho] = \left\{ A + \rho \sum_{j=1}^{L} [L_j^T \Theta^j R_j + R_j^T [\Theta^j]^T L_j] : \Theta^j \in \mathcal{Z}^j, 1 \leq j \leq L \right\} \subset \mathbf{S}^m, \tag{A.4.8}$$

where $\rho \geq 0$ *is the parameter and*

$$\mathcal{Z}^j = \left\{ \begin{array}{l} \{\theta I_{p_j} : \theta \in \mathbb{R}, |\theta| \leq 1\}, j \in I_{\rm s}^{\rm r} \\ \qquad [\text{“scalar perturbation blocks”}] \\ \{\Theta^j \in \mathbb{R}^{p_j \times q_j} : \|\Theta^j\|_{2,2} \leq 1\}, j \in I_{\rm f}^{\rm r} \\ \qquad [\text{“full perturbation blocks”}] \end{array} \right. . \tag{A.4.9}$$

Given $\rho \geq 0$*, check whether*

$$\mathcal{U}[\rho] \subset \mathbf{S}^m_+ \tag{$\mathcal{A}(\rho)$}$$

Remark A.4 *We always assume that* $p_j > 1$ *for* $j \in I_{\rm s}^{\rm r}$*. Indeed, non-repeated (*$p_j = 1$*) scalar perturbations always can be regarded as full perturbations.*

Consider, along with predicate $\mathcal{A}(\rho)$, the predicate

$$\begin{array}{ll} \exists Y_j \in \mathbf{S}^m, \; j = 1, ..., L : & \\ (a) \quad Y_j \succeq L_j^T \Theta^j R_j + R_j^T [\Theta^j]^T L_j \; \forall \left(\Theta^j \in \mathcal{Z}^j, 1 \leq j \leq L\right) & \\ (b) \quad A - \rho \sum_{j=1}^{L} Y_j \succeq 0. & \end{array} \tag{$\mathcal{B}(\rho)$}$$

The Real case version of Theorem A.6 is as follows:

Theorem A.7 [The Real Matrix Cube Theorem [3, section B.4]] *One has:*

(i) *Predicate* $\mathcal{B}(\rho)$ *is stronger than* $\mathcal{A}(\rho)$ *— the validity of the former predicate implies the validity of the latter one.*

(ii) $\mathcal{B}(\rho)$ *is computationally tractable — the validity of the predicate is equivalent to the solvability of the system of LMIs*

$$\begin{array}{lccl} (s) & Y_j \pm \left[L_j^T R_j + R_j^T L_j\right] & \succeq & 0, \; j \in I_{\rm s}^{\rm r}, \\ (f) & \left[\begin{array}{cc} Y_j - \lambda_j L_j^T L_j & R_j^T \\ R_j & \lambda_j I_{p_j} \end{array} \right] & \succeq & 0, \; j \in I_{\rm f}^{\rm r} \\ (*) & A - \rho \sum_{j=1}^{L} Y_j & \succeq & 0. \end{array} \tag{A.4.10}$$

in matrix variables $Y_j \in \mathbf{S}^m$, $j = 1, ..., L$*, and real variables* λ_j, $j \in I_{\rm f}^{\rm r}$*.*

(iii) *"The gap" between $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$ can be bounded solely in terms of the maximal rank*

$$p^{\rm s} = \max_{j \in I_{\rm S}^{\rm r}} \mathrm{Rank}(L_j^T R_j + R_j^T L_j)$$

of the scalar perturbations. Specifically, there exists a universal function $\vartheta_{\mathbb{R}}(\cdot)$ satisfying the relations

$$\vartheta_{\mathbb{R}}(2) = \frac{\pi}{2}; \vartheta_{\mathbb{R}}(4) = 2; \vartheta_{\mathbb{R}}(\mu) \leq \pi\sqrt{\mu}/2 \, \forall \mu \geq 1$$

such that with $\mu = \max[2, p^{\rm s}]$ one has

$$\text{if } \mathcal{B}(\rho) \text{ is not valid, then } \mathcal{A}(\vartheta_{\mathbb{R}}(\mu)\rho) \text{ is not valid.} \tag{A.4.11}$$

(iv) *Finally, in the case $L = 1$ of single perturbation block $\mathcal{A}(\rho)$ is equivalent to $\mathcal{B}(\rho)$.*

A.4.2 Approximate $\mathcal{S}$-Lemma

Theorem A.8 [Approximate $\mathcal{S}$-Lemma [3, section B.3]] *Let $\rho > 0$, $A, B, B_1, ..., B_J$ be symmetric $m \times m$ matrices such that $B = bb^T$, $B_j \succeq 0$, $j = 1, ..., J \geq 1$, and $B + \sum_{j=1}^{J} B_j \succ 0$.*

Consider the optimization problem

$$\mathrm{Opt}(\rho) = \max_x \left\{ x^T A x : x^T B x \leq 1, x^T B_j x \leq \rho^2, \, j = 1, ..., J \right\} \tag{A.4.12}$$

along with its semidefinite relaxation

$$\begin{array}{rcl} \mathrm{SDP}(\rho) & = & \max\limits_X \left\{ \mathrm{Tr}(AX) : \mathrm{Tr}(BX) \leq 1, \mathrm{Tr}(B_j X) \leq \rho^2, \right. \\ & & \quad \left. j = 1, ..., J, X \succeq 0 \right\} \\ & = & \min\limits_{\lambda, \{\lambda_j\}} \left\{ \lambda + \rho^2 \sum\limits_{j=1}^{J} \lambda_j : \lambda \geq 0, \lambda_j \geq 0, \, j = 1, ..., J, \right. \\ & & \quad \left. \lambda B + \sum\limits_{j=1}^{J} \lambda_j B_j \succeq A \right\}. \end{array} \tag{A.4.13}$$

Then there exists $\bar{x}$ such that

$$\begin{array}{lrcl} (a) & \bar{x}^T B \bar{x} & \leq & 1 \\ (b) & \bar{x}^T B_j \bar{x} & \leq & \Omega^2(J)\rho^2, j = 1, ..., J \\ (c) & \bar{x}^T A \bar{x} & = & \mathrm{SDP}(\rho), \end{array} \tag{A.4.14}$$

where $\Omega(J)$ is a universal function of J such that $\Omega(1) = 1$ and

$$\Omega(J) \leq 9.19\sqrt{\ln(J)}, \, J \geq 2. \tag{A.4.15}$$

In particular,

$$\mathrm{Opt}(\rho) \leq \mathrm{SDP}(\rho) \leq \mathrm{Opt}(\Omega(J)\rho). \tag{A.4.16}$$

A.4.3 Talagrand Inequality

Theorem A.9 [Talagrand Inequality] *Let $\eta_1, ..., \eta_m$ be independent random vectors taking values in unit balls of the respective finite-dimensional vector spaces $(E_1, \|\cdot\|_{(1)})$,...,$(E_m, \|\cdot\|_{(m)})$, and let $\eta = (\eta_1, ..., \eta_m) \in E = E_1 \times ... \times E_m$. Let us equip E with the norm $\|(z^1, ..., z^m)\| = \sqrt{\sum\limits_{i=1}^m \|z^i\|_{(i)}^2}$, and let Q be a closed convex subset of E. Then*

$$\mathbf{E}\left\{\exp\{\frac{\mathrm{dist}^2_{\|\cdot\|}(\eta, Q)}{16}\}\right\} \le \frac{1}{\mathrm{Prob}\{\eta \in Q\}}.$$

For proof, see, e.g., [60].

A.4.4 A Concentration Result for Gaussian Random Vector

Theorem A.10 [3, Theorem B.5.1] *Let $\zeta \sim \mathcal{N}(0, I_m)$, and let Q be a closed convex set in $\mathbb{R}^m$ such that*

$$\mathrm{Prob}\{\zeta \in Q\} \ge \chi > \frac{1}{2}. \tag{A.4.17}$$

Then

(i) *Q contains the centered at the origin $\|\cdot\|_2$-ball of the radius*

$$r(\chi) = \mathrm{ErfInv}(1-\chi) > 0. \tag{A.4.18}$$

(ii) *If Q contains the centered at the origin $\|\cdot\|_2$-ball of a radius $r \ge r(\chi)$, then*

$$\begin{array}{l}\forall \alpha \in [1, \infty) : \mathrm{Prob}\{\zeta \notin \alpha Q\} \le \mathrm{Erf}\left(\mathrm{ErfInv}(1-\chi) + (\alpha - 1)r\right) \\ \quad \le \mathrm{Erf}\left(\alpha \mathrm{ErfInv}(1-\chi)\right) \le \frac{1}{2}\exp\left\{-\frac{\alpha^2 \mathrm{ErfInv}^2(1-\chi)}{2}\right\}.\end{array} \tag{A.4.19}$$

In particular, for a closed and convex set Q, $\zeta \sim \mathcal{N}(0, \Sigma)$ and $\alpha \ge 1$ one has

$$\begin{array}{l}\mathrm{Prob}\left\{\zeta \notin Q\right\} \le \delta < \frac{1}{2} \Rightarrow \\ \mathrm{Prob}\left\{\zeta \notin \alpha Q\right\} \le \mathrm{Erf}(\alpha \mathrm{ErfInv}(\delta)) \le \frac{1}{2}\exp\{-\frac{\alpha^2 \mathrm{ErfInv}^2(\delta)}{2}\}.\end{array} \tag{A.4.20}$$

Appendix B

Solutions to Exercises

B.1 Exercises from Lecture 1

Exercise 1.1. We should prove that x is robust feasible if and only if it can be extended, by properly chosen $u, v \geq 0$ such that $u - v = x$, to a feasible solution to (1.6.1). First, let x be robust feasible, and let $u_i = \max[x_i, 0]$, $v_i = \max[-x_i, 0]$. Then $u, v \geq 0$ and $u - v = x$. Besides this, since x is robust feasible and uncertainty is element-wise, we have for every i

$$\sum_j \max_{\underline{A}_{ij} \leq a_{ij} \leq \overline{A}_{ij}} a_{ij} x_j \leq \underline{b}_i. \qquad (*)$$

With our u, v we clearly have $\max_{\underline{A}_{ij} \leq a_{ij} \leq \overline{A}_{ij}} a_{ij} x_j = [\overline{A}_{ij} u_j - \underline{A}_{ij} v_j]$, so that by $(*)$ we have $\overline{A}u - \underline{A}v \leq \underline{b}$, so that (x, u, v) is a feasible solution of (1.6.1).

Vice versa, let (x, u, v) be feasible for (1.6.1), and let us prove that x is robust feasible for the original uncertain problem, that is, that the relations $(*)$ take place. This is immediate, since from $u, v \geq 0$ and $x = u - v$ it clearly follows that $\max_{\underline{A}_{ij} \leq a_{ij} \leq \overline{A}_{ij}} a_{ij} x_j \leq \overline{A}_{ij} u_j - \underline{A}_{ij} v_j$, so that the validity of $(*)$ is ensured by the constraints of (1.6.1). The respective RCs are (equivalent to)

$$\begin{array}{ll} [a^{\mathrm{n}}; b^{\mathrm{n}}]^T [x; -1] + \rho \| P^T [x; -1] \|_q \leq 0,\ q = \frac{p}{p-1} & (a) \\ [a^{\mathrm{n}}; b^{\mathrm{n}}]^T [x; -1] + \rho \| (P^T [x; -1])_+ \|_q \leq 0,\ q = \frac{p}{p-1} & (b) \\ [a^{\mathrm{n}}; b^{\mathrm{n}}]^T [x; -1] + \rho \| P^T [x; -1] \|_\infty \leq 0 & (c) \end{array}$$

where for a vector $u = [u_1; ...; u_k]$ the vector $(u)_+$ has the coordinates $\max[u_i, 0]$, $i = 1, ..., k$.
Comment to (c): The uncertainty set in question is nonconvex; since the RC remains intact when a given uncertainty set is replaced with its convex hull, we can replace the restriction $\|\zeta\|_p \leq \rho$ in (c) with the restriction $\zeta \in \mathrm{Conv}\{\zeta : \|\zeta\|_p \leq \rho\} = \{\|\zeta\|_1 \leq \rho\}$, where the concluding equality is due to the following reasons: on one hand, with $p \in (0, 1)$ we have

$$\begin{array}{l} \|\zeta\|_p \leq \rho \Leftrightarrow \sum_i (|\zeta_i|/\rho)^p \leq 1 \Rightarrow |\zeta_i|/\rho \leq 1 \forall i \Rightarrow |\zeta_i|/\rho \leq (|\zeta_i|/\rho)^p \\ \Rightarrow \sum_i |\zeta_i|/\rho \leq \sum_i (|\zeta_i|/\rho)^p \leq 1, \end{array}$$

whence $\mathrm{Conv}\{\|\zeta\|_p \leq \rho\} \subset \{\|\zeta\|_1 \leq \rho\}$. To prove the inverse inclusion, note that all extreme points of the latter set (that is, vectors with all but one coordinates equal to 0 and the remaining coordinate equal $\pm\rho$) satisfy $\|\zeta\|_p \leq 1$.

Exercise 1.3: The RC can be represented by the system of conic quadratic constraints

$$\begin{array}{l} [a^{\rm n};b^{\rm n}]^T[x;-1]+\rho\sum_j\|u_j\|_2\leq 0\\ \sum_j Q_j^{1/2}u_j=P^T[x;-1]\end{array}$$

in variables $x, \{u_j\}_{j=1}^J$.

Exercise 1.4: • The RC of i-th problem is

$$\min_x\left\{-x_1-x_2:0\leq x_1\leq\min_{b\in\mathcal{U}_i}b_1, 0\leq x_2\leq\min_{b\in\mathcal{U}_i}b_2, x_1+x_2\geq p\right\}$$

We see that both RC's are identical to each other and form the program

$$\min_x\{-x_1-x_2:0\leq x_1,x_2\leq 1/3, x_1+x_2\geq p\}$$

• When $p=3/4$, all instances of $\mathcal{P}_1$ are feasible (one can set $x_1=b_1, x_2=b_2$), while the RC of $\mathcal{P}_2$ is not so, so that there is a gap. In contrast to this, $\mathcal{P}_2$ has infeasible instances, and its RC is infeasible; in this case, there is no gap.
• When $p=2/3$, the (common) RC of the two problems $\mathcal{P}_1$, $\mathcal{P}_2$ is feasible with the unique feasible solution $x_1=x_2=1/3$ and the optimal value $-2/3$. Since every instance of $\mathcal{P}_1$ has a feasible solution $x_1=b_1, x_2=b_2$, the optimal value of the instance is $\leq -b_1-b_2\leq -1$, so that there is a gap. In contrast to this, the RC is an instance of $\mathcal{P}_2$, so that in this case there is no gap.

Exercise 1.5: • Problem $\mathcal{P}_2$ has a constraint-wise uncertainty and is the constraint-wise envelope of $\mathcal{P}_1$.
• Proof for item 2: Let us prove first that if all the instances are feasible, then so is the RC. Assume that the RC is infeasible. Then for every $x\in X$ there exists $i=i_x$ and a realization $[a_{i_x,x}^T, b_{i_x,x}]\in\mathcal{U}_{i_x}$ of the uncertain data of i-th constraint such that

$$a_{i_x,x}^T x'-b_{i_x,x}>0$$

when $x'=x$ and consequently when x' belongs to a small enough neighborhood U_x of x. Since X is a convex compact set, we can find a finite collection of $x_j\in X$ such that the corresponding neighborhoods U_{x_j} cover the entire X. In other words, we can points out finitely many linear forms

$$f_\ell(x)=a_{i_\ell}^T x-b_{i_\ell},\ \ell=1,...,L,$$

such that $[a_{i_\ell}^T, b_{i_\ell}]\in\mathcal{U}_{i_\ell}$ and the maximum of the forms over $\ell=1,...,L$ is positive at every point $x\in X$. By standard facts on convexity it follows that there exists a convex combination of our forms

$$f(x)=\sum_{\ell=1}^L\lambda_\ell[a_{i_\ell}^T x-b_{i_\ell}]$$

which is positive everywhere on X. Now let $I_i=\{\ell: i_\ell=i\}$ and $\mu_i=\sum_{\ell\in I_i}\lambda_\ell$. For i with $\mu_i>0$, let us set

$$[a_i^T, b_i]=\sum_{\ell\in I_i}\frac{\lambda_\ell}{\mu_i}[a_{i_\ell}, b_{i_\ell}],$$

so that $[a_i^T, b_i] \in \mathcal{U}_i$ (since the latter set is convex). For i with $\mu_i = 0$, let $[a_i^T, b_i]$ be a whatever point of $\mathcal{U}_i$. Observe that by construction

$$f(x) \equiv \sum_{i=1}^{m} \mu_i [a_i^T x - b_i].$$

Now, *since the uncertainty is constraint-wise*, the matrix $[A, b]$ with the rows $[a_i^T, b_i]$ belongs to $\mathcal{U}$ and thus corresponds to an instance of $\mathcal{P}$. For this instance, we have

$$\mu^T [Ax - b] = f(x) > 0 \ \forall x \in X,$$

so that no $x \in X$ can be feasible for the instance; due to the origin of X, this means that the instance we have built is infeasible, same as the RC.

Now let us prove that if all instances are feasible, then the optimal value of the RC is the supremum, let it be called τ, of the optimal values of instances (this supremum clearly is achieved and thus is the maximum of optimal values of instances). Consider the uncertain problem $\mathcal{P}'$ which is obtained from $\mathcal{P}$ by adding to every instance the (certain!) constraint $c^T x \leq \tau$. The resulting problem still is with constraint-wise uncertainty, has feasible instances, and feasible solutions of an instance belong to X. By what we have already proved, the RC of $\mathcal{P}'$ is feasible; but a feasible solution to the latter RC is a robust feasible solution of $\mathcal{P}$ with the value of the objective $\leq \tau$, meaning that the optimal value in the RC of $\mathcal{P}$ is $\leq \tau$. Since the strict inequality here is impossible due to the origin of τ, we conclude that the optimal value of the RC of $\mathcal{P}$ is equal to the maximum of optimal values of instances of $\mathcal{P}$, as claimed.

B.2 Exercises from Lecture 2

Exercise 2.1: W.l.o.g., we may assume $t > 0$. Setting $\phi(s) = \cosh(ts) - [\cosh(t) - 1]s^2$, we get an even function such that $\phi(-1) = \phi(0) = \phi(1) = 1$. We claim that $\phi(s) \leq 1$ when $-1 \leq s \leq 1$.

> Indeed, otherwise ϕ attains its maximum on $[-1, 1]$ at a point $\bar{s} \in (0, 1)$, and $\phi''(\bar{s}) \leq 0$. The function $g(s) = \phi'(s)$ is convex on $[0, 1]$ and $g(0) = g(\bar{s}) = 0$. The latter, due to $g'(\bar{s}) \leq 0$, implies that $g(s) = 0$, $0 \leq s \leq \bar{s}$. Thus, ϕ is constant on a nontrivial segment, which is not the case.

For a symmetric P supported on $[-1, 1]$ with $\int s^2 dP(s) \equiv \bar{\nu}^2 \leq \nu^2$ we have, due to $\phi(s) \leq 1$, $-1 \leq s \leq 1$:

$$\begin{array}{l}
\int \exp\{ts\} dP(s) = \int_{-1}^{1} \cosh(ts) dP(s) \\
= \int_{-1}^{1} [\cosh(ts) - (\cosh(t) - 1)s^2] dP(s) + (\cosh(t) - 1) \int_{-1}^{1} s^2 dP(s) \\
\leq \int_{-1}^{1} dP(s) + (\cosh(t) - 1)\bar{\nu}^2 \leq 1 + (\cosh(t) - 1)\nu^2,
\end{array}$$

as claimed in example 8. Setting $h(t) = \ln(\nu^2 \cosh(t) + 1 - \nu^2)$, we have $h(0) = h'(0) = 0$, $h''(t) = \frac{\nu^2(\nu^2 + (1-\nu^2)\cosh(t))}{(\nu^2 \cosh(t) + 1 - \nu^2)^2}$, $\max_t h''(t) = \left\{ \begin{array}{ll} \nu^2, & \nu^2 \geq \frac{1}{3} \\ \frac{1}{4}\left[1 + \frac{\nu^4}{1 - 2\nu^2}\right] \leq \frac{1}{3}, & \nu^2 \leq \frac{1}{3} \end{array} \right.$, whence $\Sigma_{(3)}(\nu) \leq 1$.

Exercise 2.2: Here are the results:

n	ϵ	$t_{\rm tru}$	$t_{\rm Nrm}$	$t_{\rm Bll}$	$t_{\rm BllBx}$	$t_{\rm Bdg}$
16	5.e-2	3.802	3.799	9.791	9.791	9.791
16	5.e-4	7.406	7.599	15.596	15.596	15.596
16	5.e-6	9.642	10.201	19.764	16.000	16.000
256	5.e-2	15.195	15.195	39.164	39.164	39.164
256	5.e-4	30.350	30.396	62.383	62.383	62.383
256	5.e-6	40.672	40.804	79.054	79.054	79.054

n	ϵ	$t_{\rm tru}$	$t_{\rm E.7}$	$t_{\rm E.8}$	$t_{\rm E.9}$	$t_{\rm Unim}$
16	5.e-2	3.802	6.228	5.653	5.653	10.826
16	5.e-4	7.406	9.920	9.004	9.004	12.502
16	5.e-6	9.642	12.570	11.410	11.410	13.705
256	5.e-2	15.195	24.910	22.611	22.611	139.306
256	5.e-4	30.350	39.678	36.017	36.017	146.009
256	5.e-6	40.672	50.282	45.682	45.682	150.821

Exercise 2.3: Here are the results:

n	ϵ	$t_{\rm tru}$	$t_{\rm Nrm}$	$t_{\rm Bll}$	$t_{\rm BllBx}$	$t_{\rm Bdg}$	$t_{\rm E.7}$	$t_{\rm E.8}$
16	5.e-2	4.000	6.579	9.791	9.791	9.791	9.791	9.791
16	5.e-4	10.000	13.162	15.596	15.596	15.596	15.596	15.596
16	5.e-6	14.000	17.669	19.764	16.000	16.000	19.764	19.764
256	5.e-2	24.000	26.318	39.164	39.164	39.164	39.164	39.164
256	5.e-4	50.000	52.649	63.383	62.383	62.383	62.383	62.383
256	5.e-6	68.000	70.674	79.054	79.054	79.054	79.053	79.053

Exercise 2.4: In the case of (a), the optimal value is $t_a = \sqrt{n}{\rm ErfInv}(\epsilon)$, since for a feasible x we have $\xi^n[x] \sim \mathcal{N}(0,n)$. In the case of (b), the optimal value is $t_b = n{\rm ErfInv}(n\epsilon)$. Indeed, the rows in B_n are of the same Euclidean length and are orthogonal to each other, whence the columns are orthogonal to each other as well. Since the first column of B_n is the all-one vector, the conditional on η distribution of $\xi = \sum_j \widehat{\zeta}_j$ has the mass $1/n$ at the point $n\eta$ and the mass $(n-1)/n$ at the origin. It follows that the distribution of ξ is the convex combination of the Gaussian distribution $\mathcal{N}(0,n^2)$ and the unit mass, sitting at the origin, with the weights $1/n$ and $(n-1)/n$, respectively, and the claim follows.

The numerical results are as follows:

n	ϵ	t_a	t_b	t_b/t_a
10	1.e-2	7.357	12.816	1.74
100	1.e-3	30.902	128.155	4.15
1000	1.e-4	117.606	1281.548	10.90

Exercise 2.5: In the notation of section 2.4.2, we have

$$\begin{array}{rcl}
\Phi(w) & \equiv & \ln\left(\mathbf{E}\{\exp\{\sum_\ell w_\ell\zeta_\ell\}\}\right) = \sum_\ell \lambda_\ell(\exp\{w_\ell\}-1)\\
& = & \max_u[w^Tu - \phi(u)],\\
\phi(u) & = & \max_w[u^Tw - \Phi(w)] = \left\{\begin{array}{ll} \sum_\ell[u_\ell\ln(u_\ell/\lambda_\ell) - u_\ell + \lambda_\ell], & u \geq 0\\ +\infty, & \text{otherwise.}\end{array}\right.
\end{array}$$

Consequently, the Bernstein approximation is

$$\inf_{\beta>0}\left[z_0 + \beta\sum_\ell \lambda_\ell(\exp\{w_\ell/\beta\}-1) + \beta\ln(1/\epsilon)\right] \leq 0,$$

or, in the RC form,

$$z_0 + \max_u \left\{ w^T u : u \in \mathcal{Z}_\epsilon = \{u \geq 0, \sum_\ell [u_\ell \ln(u_\ell/\lambda_\ell) - u_\ell + \lambda_\ell] \leq \ln(1/\epsilon)\} \right\} \leq 0.$$

Exercise 2.6: $w(\epsilon)$ is the optimal value in the chance constrained optimization problem

$$\min_{w_0} \left\{ w_0 : \mathrm{Prob}\{-w_0 + \sum_{\ell=1}^L c_\ell \zeta_\ell \leq 0\} \geq 1 - \epsilon \right\},$$

where ζ_ℓ are independent Poisson random variables with parameters λ_ℓ.

When all c_ℓ are integral in certain scale, the random variable $\zeta^L = \sum_{\ell=1}^L c_\ell \zeta_\ell$ is also integral in the same scale, and we can compute its distribution recursively in L:

$$p_0(i) = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases}, p_k(i) = \sum_{j=0}^\infty p_{k-1}(i - c_\ell j) \frac{\lambda_k^j}{j!} \exp\{-\lambda_k\};$$

(in computations, $\sum_{j=0}^\infty$ should be replaced with $\sum_{j=0}^N$ with appropriately large N).
With the numerical data in question, the expected value of per day requested cash is $c^T \lambda = 7,000$, and the remaining requested quantities are listed below:

	ϵ					
	1.e-1	1.e-2	1.e-3	1.e-4	1.e-5	1.e-6
$w(\epsilon)$	8,900	10,800	12,320	13,680	14,900	16,060
CVaR	9,732 +9.3%	11,451 +6.0%	12,897 +4.7%	14,193 +3.7%	15,390 +3.3%	16,516 +2.8%
BCV	9,836 +10.5%	11,578 +7.2%	13,047 +5.9%	14,361 +5.0%	15,572 +4.5%	16,709 +4.0%
B	10,555 +18.6%	12,313 +14.0%	13,770 +11.8%	15,071 +10.2%	16,270 +9.2%	17,397 +8.3%
E	8,900 +0.0%	10,800 +0.0%	12,520 +1.6%	17,100 +25.0%	—	—

"BCV" stands for the bridged Bernstein-CVaR, "B" — for the Bernstein, and "E" — for the $(1-\epsilon)$-reliable empirical bound on $w(\epsilon)$. The BCV bound corresponds to the generating function $\gamma_{16,10}(\cdot)$, see p. 63. The percents represent the relative differences between the bounds and $w(\epsilon)$. All bounds are right-rounded to the closest integers.

Exercise 2.7: The results of computations are as follows (as a benchmark, we display also the results of Exercise 2.6 related to the case of independent $\zeta_1, ..., \zeta_L$):

	ϵ					
	1.e-1	1.e-2	1.e-3	1.e-4	1.e-5	1.e-6
Exer. 2.6	8,900	10,800	12,320	13,680	14,900	16,060
Exer. 2.7, lower bound	11,000 +23.6%	15,680 +45.2%	19,120 +55.2%	21,960 +60.5%	26,140 +75.4%	28,520 +77.6%
Exer. 2.7, upper bound	13,124 +47.5%	17,063 +58.8%	20,507 +66.5%	23,582 +72.4%	26,588 +78.5%	29,173 +81.7%

Percents display relative differences between the bounds and $w(\epsilon)$

Exercise 2.8. Part 1: By Exercise 2.5, the Bernstein upper bound on $w(\epsilon)$ is

$$\begin{array}{rcl} B_\lambda(\epsilon) & = & \inf\{w_0 : \inf_{\beta>0}[-w_0 + \beta\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}-1) + \beta\ln(1/\epsilon)] \le 0\} \\ & = & \inf_{\beta>0}[\beta\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}-1) + \beta\ln(1/\epsilon)] \end{array}$$

The "ambiguous" Bernstein upper bound on $w(\epsilon)$ is therefore

$$\begin{array}{rcl} B_\Lambda(\epsilon) & = & \max_{\lambda\in\Lambda}\inf_{\beta>0}[\beta\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}-1) + \beta\ln(1/\epsilon)] \\ & = & \inf_{\beta>0}\beta[\max_{\lambda\in\Lambda}\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}-1) + \ln(1/\epsilon)] \end{array} \quad (*)$$

where the swap of $\inf_{\beta>0}$ and $\max_{\lambda\in\Lambda}$ is justified by the fact that the function $\beta\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}- 1) + \beta\ln(1/\epsilon)$ is concave in λ, convex in β and by the compactness and convexity of Λ.

Part 2: We should prove that if Λ is a convex compact set in the domain $\lambda \ge 0$ such that for every affine form $f(\lambda) = f_0 + e^T\lambda$ one has

$$\max_{\lambda\in\Lambda} f(\lambda) \le 0 \Rightarrow \mathrm{Prob}_{\lambda\sim P}\{f(\lambda)\le 0\} \ge 1-\delta, \quad (!)$$

then, setting $w_0 = B_\Lambda(\epsilon)$, one has

$$\mathrm{Prob}_{\lambda\sim P}\left\{\lambda : \mathrm{Prob}_{\zeta\sim P_{\lambda_1}\times\ldots\times P_{\lambda_L}}\left\{\sum_\ell \zeta_\ell c_\ell > w_0\right\} > \epsilon\right\} \le \delta. \quad (?)$$

It suffices to prove that under our assumptions on Λ inequality (?) is valid for all $w_0 > B_\Lambda(\epsilon)$. Given $w_0 > B_\Lambda(\epsilon)$ and invoking the second relation in $(*)$, we can find $\bar\beta > 0$ such that

$$\bar\beta\left[\max_{\lambda\in\Lambda}\sum_\ell \lambda_\ell(\exp\{c_\ell/\bar\beta\}-1) + \ln(1/\epsilon)\right] \le w_0,$$

or, which is the same,

$$[-w_0 + \bar\beta\ln(1/\epsilon)] + \max_{\lambda\in\Lambda}\sum_\ell \lambda_\ell[\bar\beta(\exp\{c_\ell/\bar\beta\}-1)] \le 0,$$

which, by (!) as applied to the affine form

$$f(\lambda) = [-w_0 + \bar\beta\ln(1/\epsilon)] + \sum_\ell \lambda_\ell[\bar\beta(\exp\{c_\ell/\bar\beta\}-1)],$$

implies that

$$\mathrm{Prob}_{\lambda\sim P}\{f(\lambda) > 0\} \le \delta. \quad (**)$$

It remains to note that when $\lambda \ge 0$ is such that $f(\lambda) \le 0$, the result of Exercise 2.5 states that

$$\mathrm{Prob}_{\zeta\sim P_{\lambda_1}\times\ldots\times P_{\lambda_m}}\left\{-w_0 + \sum_\ell \zeta_\ell c_\ell > 0\right\} \le \epsilon.$$

Thus, when $\omega_0 > B_\Lambda(\epsilon)$, the set of λ's in the left hand side of (?) is contained in the set $\{\lambda \ge 0 : f(\lambda) > 0\}$, and therefore (?) is readily given by $(**)$.

B.3 Exercises from Lecture 3

Exercise 3.1: Let $\mathcal{S}[\cdot]$ be a safe tractable approximation of $(C_{\mathcal{Z}_*}[\cdot])$ tight within the factor ϑ. Let us verify that $\mathcal{S}[\lambda\gamma\rho]$ is a safe tractable approximation of $(C_{\mathcal{Z}}[\rho])$ tight within the factor $\lambda\vartheta$. All we should prove is that (a) if x can be extended to a feasible solution to $\mathcal{S}[\lambda\gamma\rho]$, then x is feasible for $(C_{\mathcal{Z}}[\rho])$, and that (b) if x cannot be extended to a feasible solution to $\mathcal{S}[\lambda\gamma\rho]$, then x is not feasible for $(C_{\mathcal{Z}}[\lambda\vartheta\rho])$. When x can be extended to a feasible solution of $\mathcal{S}[\lambda\gamma\rho]$, x is feasible for $(C_{\mathcal{Z}_*}[\lambda\gamma\rho])$, and since $\rho\mathcal{Z} \subset \lambda\gamma\rho\mathcal{Z}_*$, x is feasible for $(C_{\mathcal{Z}}[\rho])$ as well, as required in (a). Now assume that x cannot be extended to a feasible solution of $\mathcal{S}[\lambda\gamma\rho]$. Then x is not feasible for $(C_{\mathcal{Z}_*}[\vartheta\lambda\gamma\rho])$, and since the set $\vartheta\lambda\gamma\rho\mathcal{Z}_*$ is contained in $\vartheta\lambda\rho\mathcal{Z}$, x is not feasible for $(C_{\mathcal{Z}}[(\vartheta\lambda)\rho])$, as required in (b). □

Exercise 3.2: 1) Consider the ellipsoid

$$\mathcal{Z}_* = \{\zeta : \zeta^T[\sum_i Q_i]\zeta \leq M\}.$$

We clearly have $M^{-1/2}\mathcal{Z}_* \subset \mathcal{Z} \subset \mathcal{Z}_*$; by assumption, $(C_{\mathcal{Z}_*}[\cdot])$ admits a safe tractable approximation tight within the factor ϑ, and it remains to apply the result of Exercise 3.1.

2) This is a particular case of 1) corresponding to $\zeta^T Q_i \zeta = \zeta_i^2$, $1 \leq i \leq M = \dim\zeta$.

3) Let $\mathcal{Z} = \bigcap_{i=1}^{M} E_i$, where E_i are ellipsoids. Since $\mathcal{Z}$ is symmetric w.r.t. the origin, we also have $\mathcal{Z} = \bigcap_{i=1}^{M} [E_i \cap (-E_i)]$. We claim that for every i, the set $E_i \cap (-E_i)$ contains an ellipsoid F_i centered at the origin and such that $E_i \cap (-E_i) \subset \sqrt{2}F_i$, and that this ellipsoid F_i can be easily found. Believing in the claim, we have

$$\mathcal{Z}_* \equiv \bigcap_{i=1}^{M} F_i \subset \mathcal{Z} \subset \sqrt{2}\bigcap_{i=1}^{M} F_i.$$

By 1), $(C_{\mathcal{Z}_*}[\cdot])$ admits a safe tractable approximation with the tightness factor $\vartheta\sqrt{M}$; by Exercise 3.1, $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation with the tightness factor $\vartheta\sqrt{2M}$.

It remains to support our claim. For a given i, applying nonsingular linear transformation of variables, we can reduce the situation to the one where $E_i = B + e$, where B is the unit Euclidean ball, centered at the origin, and $\|e\|_2 < 1$ (the latter inequality follows from $0 \in \mathrm{int}\mathcal{Z} \subset \mathrm{int}(E_i \cap (-E_i))$). The intersection $G = E_i \cap (-E_i)$ is a set that is invariant w.r.t. rotations around the axis $\mathbb{R}e$; a 2-D cross-section H of G by a 2D plane Π containing the axis is a 2-D solid symmetric w.r.t. the origin. It is well known that for every symmetric w.r.t. 0 solid Q in $\mathbb{R}^d$ there exists a centered at 0 ellipsoid E such that $E \subset Q \subset \sqrt{d}E$. Therefore there exists (and in fact can easily be found) an ellipsis I, centered at the origin, that is contained in H and is such that $\sqrt{2}I$ contains H. Now, the ellipsis I is the intersection of Π and an ellipsoid F_i that is invariant w.r.t. rotations around the axis $\mathbb{R}e$, and F_i clearly satisfies the required relations $F_i \subset E_i \cap (-E_i) \subset \sqrt{2}F_i$. [1]

[1]In fact, the factor $\sqrt{2}$ in the latter relation can be reduced to $2/\sqrt{3} < \sqrt{2}$, see Solution to Exercise 3.4.

Exercise 3.3: With y given, all we know about x is that there exists $\Delta \in \mathbb{R}^{p\times q}$ with $\|\Delta\|_{2,2} \leq \rho$ such that $y = B_{\rm n}[x;1] + L^T\Delta R[x;1]$, or, denoting $w = \Delta R[x;1]$, that there exists $w \in \mathbb{R}^p$ with $w^Tw \leq \rho^2[x;1]^TR^TR[x;1]$ such that $y = B_{\rm n}[x;1] + L^Tw$. Denoting $z = [x;w]$, all we know about the vector z is that it belongs to a given affine plane $\mathcal{A}z = a$ and satisfies the quadratic inequality $z^T\mathcal{C}z + 2c^Tz + d \leq 0$, where $\mathcal{A} = [A_{\rm n}, L^T]$, $a = y - b_{\rm n}$, and

$$[\xi;\omega]^T\mathcal{C}[\xi;\omega] + 2c^T[\xi;\omega] + d \equiv \omega^T\omega - \rho^2[\xi;1]^TR^TR[\xi;1],\ [\xi;\omega] \in \mathbb{R}^{n+p}.$$

Using the equations $\mathcal{A}z = a$, we can express the $n+p$ z-variables via $k \leq n+p$ u-variables:

$$\mathcal{A}z = a \Leftrightarrow \exists u \in \mathbb{R}^k : z = Eu + e.$$

Plugging $z = Eu + e$ into the quadratic constraint $z^T\mathcal{C}z + 2c^Tz + d \leq 0$, we get a quadratic constraint $u^TFu + 2f^Tu + g \leq 0$ on u. Finally, the vector Qx we want to estimate can be represented as Pu with easily computable matrix P. The summary of our developments is as follows:

> (!) *Given y and the data describing $\mathcal{B}$, we can build k, a matrix P and a quadratic form $u^TFu + 2f^Tu + g \leq 0$ on $\mathbb{R}^k$ such that the problem of interest becomes the problem of the best, in the worst case, $\|\cdot\|_2$-approximation of Pu, where unknown vector $u \in \mathbb{R}^k$ is known to satisfy the inequality $u^TFu + 2f^Tu + g \leq 0$.*

By (!), our goal is to solve the semi-infinite optimization program

$$\min_{t,v}\left\{t : \|Pu - v\|_2 \leq t\,\forall(u : u^TFu + 2f^Tu + g \leq 0)\right\}. \qquad (*)$$

Assuming that $\inf_u\left[u^TFu + 2f^Tu + g\right] < 0$ and applying the inhomogeneous version of $\mathcal{S}$-Lemma, the problem becomes

$$\min_{t,v,\lambda}\left\{t \geq 0 : \left[\begin{array}{c|c}\lambda F - P^TP & \lambda f - P^Tv \\ \hline \lambda f^T - v^TP & \lambda g + t^2 - v^Tv\end{array}\right] \succeq 0, \lambda \geq 0\right\}.$$

Passing from minimization of t to minimization of $\tau = t^2$, the latter problem becomes the semidefinite program

$$\min_{\tau,v,\lambda,s}\left\{\tau : \begin{array}{c} v^Tv \leq s, \lambda \geq 0 \\ \left[\begin{array}{c|c}\lambda F - P^TP & \lambda f - P^Tv \\ \hline \lambda f^T - v^TP & \lambda g + \tau - s\end{array}\right] \succeq 0\end{array}\right\}.$$

In fact, the problem of interest can be solved by pure Linear Algebra tools, without Semidefinite optimization. Indeed, assume for a moment that P has trivial kernel. Then $(*)$ is feasible if and only if the solution set S of the quadratic inequality $\phi(u) \equiv u^TFu + 2f^Tu + g \leq 0$ in variables u is nonempty and bounded, which is the case if and only if this set is an ellipsoid $(u-c)^TQ(u-c) \leq r^2$ with $Q \succ 0$ and $r \geq 0$; whether this indeed is the case and what are c, Q, r, if any, can be easily found out by Linear Algebra tools. The image PS of S under the mapping P also is an ellipsoid (perhaps "flat") centered at $v_* = Pc$, and the optimal solution to $(*)$ is (t_*, v_*), where t_* is the largest half-axis of the ellipsoid PS. In the case when P has a kernel, let E be the orthogonal complement to Ker$P$, and $\widehat{P}$ be the restriction of P onto E; this mapping has a trivial kernel. Problem $(*)$ clearly is equivalent to

$$\min_{t,v}\left\{t : \|\widehat{P}\widehat{u} - v\|_2 \leq t\,\forall(\widehat{u} \in E : \exists w \in \mathrm{Ker}P : \phi(\widehat{u} + w) \leq 0\right\}.$$

The set

$$\widehat{U} = \{\widehat{u} \in E : \exists w \in \mathrm{Ker} P : \phi(\widehat{u} + w) \le 0\}$$

clearly is given by a single quadratic inequality in variables $\widehat{u} \in E$, and $(*)$ reduces to a similar problem with E in the role of the space where u lives and $\widehat{P}$ in the role of P, and we already know how to solve the resulting problem.

Exercise 3.4: In view of Theorem 3.9, all we need to verify is that $\mathcal{Z}$ can be "safely approximated" within an $O(1)$ factor by an intersection $\widehat{\mathcal{Z}}$ of $O(1)J$ ellipsoids *centered at the origin*: there exists $\widehat{\mathcal{Z}} = \{\eta : \eta^T \widehat{Q}_j \eta \le 1, 1 \le j \le \widehat{J}\}$ with $\widehat{Q}_j \succeq 0$, $\sum_j \widehat{Q}_j \succ 0$ such that

$$\theta^{-1}\widehat{\mathcal{Z}} \subset \mathcal{Z} \subset \widehat{\mathcal{Z}},$$

with an absolute constant θ and $\widehat{J} \le O(1)J$. Let us prove that the just formulated statement holds true with $\widehat{J} = J$ and $\theta = \sqrt{3}/2$. Indeed, since $\mathcal{Z}$ is symmetric w.r.t. the origin, setting $E_j = \{\eta : (\eta - a_j)^T Q_j (\eta - a_j) \le 1\}$, we have

$$\mathcal{Z} = \bigcap_{j=1}^{J} E_j = \bigcap_{j=1}^{J} (-E_j) = \bigcap_{j=1}^{J} (E_j \cap [-E_j]);$$

all we need is to demonstrate that every one of the sets $E_j \cap [-E_j]$ is in between two proportional ellipsoids centered at the origin with the larger one being at most $2/\sqrt{3}$ multiple of the smaller one. After an appropriate linear one-to-one transformation of the space, all we need to prove is that if $E = \{\eta \in \mathbb{R}^d : (\eta_1 - r)^2 + \sum_{j=2}^k \eta_j^2 \le 1\}$ with $0 \le r < 1$, then we can point out the set $F = \{\eta : \eta_1^2/a^2 + \sum_{j=2}^k \eta_j^2/b^2 \le 1\}$ such that

$$\frac{\sqrt{3}}{2} F \subset E \cap [-E] \subset F.$$

When proving the latter statement, we lose nothing when assuming $k = 2$. Renaming η_1 as y, η_2 as x and setting $h = 1 - r \in (0, 1]$ we should prove that the "loop" $\mathcal{L} = \{[x; y] : [|y| + (1 - h)]^2 + x^2 \le 1\}$ is in between two proportional ellipses centered at the origin with the ratio of linear sizes $\theta \le 2/\sqrt{3}$. Let us verify that we can take as the smaller of these ellipses the ellipsis

$$\mathcal{E} = \{[x; y] : y^2/h^2 + x^2/(2h - h^2) \le \mu^2\}, \mu = \sqrt{\frac{3 - h}{4 - 2h}},$$

and to choose $\theta = \mu^{-1}$ (so that $\theta \le 2/\sqrt{3}$ due to $0 < h \le 1$). First, let us prove that $\mathcal{E} \subset \mathcal{L}$. This inclusion is evident when $h = 1$, so that we can assume that $0 < h < 1$. Let $[x; y] \in \mathcal{E}$, and let $\lambda = \frac{2(1-h)}{h}$. We have

$$\begin{array}{l}
y^2/h^2 + x^2/(2h - h^2) \le \mu^2 \Rightarrow \left\{ \begin{array}{ll} y^2 \le h^2[\mu^2 - x^2/(2h - h^2)] & (a) \\ x^2 \le \mu^2 h(2 - h) & (b) \end{array} \right. ; \\
(|y| + (1 - h))^2 + x^2 = y^2 + 2|y|(1 - h) + (1 - h)^2 \le y^2 + \left[\lambda y^2 + \frac{1}{\lambda}(1 - h)^2\right] \\
+(1 - h)^2 = y^2 \frac{2-h}{h} + \frac{(2-h)(1-h)}{2} + x^2 \\
\le \left[\mu^2 - \frac{x^2}{h(2-h)}\right](2h - h^2) + \frac{(2-h)(1-h)}{2} + x^2 \equiv q(x^2),
\end{array}$$

where the concluding $\le$ is due to (a). Since $0 \le x^2 \le \mu^2(2h - h^2)$ by (b), $q(x^2)$ is in-between its values for $x^2 = 0$ and $x^2 = \mu^2(2h - h^2)$, and both these values with our μ are equal to 1. Thus, $[x; y] \in \mathcal{L}$.

It remains to prove that $\mu^{-1}\mathcal{E} \supset \mathcal{L}$, or, which is the same, that when $[x;y] \in \mathcal{L}$, we have $[\mu x; \mu y] \in \mathcal{E}$. Indeed, we have

$$\begin{array}{l}
[|y| + (1-h)]^2 + x^2 \leq 1 \Rightarrow |y| \leq h \ \&\ x^2 \leq 1 - y^2 - 2|y|(1-h) - (1-h)^2 \\
\Rightarrow x^2 \leq 2h - h^2 - y^2 - 2|y|(1-h) \\
\Rightarrow \mu^2 \left[\frac{y^2}{h^2} + \frac{x^2}{2h-h^2}\right] = \mu^2 \frac{y^2(2-h)+hx^2}{h^2(2-h)} \leq \mu^2 \frac{h^2(2-h)+2(1-h)[\overbrace{y^2 - |y|h}^{\leq 0}]}{h^2(2-h)} \leq \mu^2 \\
\Rightarrow [x;y] \in \mathcal{E},
\end{array}$$

as claimed.

Exercise 3.5: 1) We have

$$\begin{array}{l}
\text{EstErr} = \sup\limits_{v \in V, A \in \mathcal{A}} \sqrt{v^T (GA-I)^T (GA-I) v + \text{Tr}(G^T \Sigma G)} \\
= \sup\limits_{A \in \mathcal{A}} \sup\limits_{u: u^T u \leq 1} \sqrt{u^T Q^{-1/2} (GA-I)^T (GA-I) Q^{-1/2} u + \text{Tr}(G^T \Sigma G)} \\
\qquad\qquad [\text{substitution } v = Q^{-1/2} u] \\
= \sqrt{\sup\limits_{A \in \mathcal{A}} \|(GA-I)Q^{-1/2}\|_{2,2}^2 + \text{Tr}(G^T \Sigma G)}.
\end{array}$$

By the Schur Complement Lemma, the relation $\|(GA-I)Q^{-1/2}\|_{2,2} \leq \tau$ is equivalent to the LMI $\left[\begin{array}{c|c} \tau I & [(GA-I)Q^{-1/2}]^T \\ \hline (GA-I)Q^{-1/2} & \tau I \end{array}\right]$, and therefore the problem of interest can be posed as the semi-infinite semidefinite program

$$\min_{t,\tau,\delta,G} \left\{ t : \begin{array}{l} \sqrt{\tau^2 + \delta^2} \leq t, \ \sqrt{\text{Tr}(G^T \Sigma G)} \leq \delta \\ \left[\begin{array}{c|c} \tau I & [(GA-I)Q^{-1/2}]^T \\ \hline (GA-I)Q^{-1/2} & \tau I \end{array}\right] \succeq 0 \ \forall A \in \mathcal{A} \end{array} \right\},$$

which is nothing but the RC of the uncertain semidefinite program

$$\left\{ \min_{t,\tau,\delta,G} \left\{ t : \begin{array}{l} \sqrt{\tau^2 + \delta^2} \leq t, \ \sqrt{\text{Tr}(G^T \Sigma G)} \leq \delta \\ \left[\begin{array}{c|c} \tau I & [(GA-I)Q^{-1/2}]^T \\ \hline (GA-I)Q^{-1/2} & \tau I \end{array}\right] \succeq 0 \end{array} \right\} : A \in \mathcal{A} \right\}.$$

In order to reformulate the only semi-infinite constraint in the problem in a tractable form, note that with $A = A_{\rm n} + L^T \Delta R$ we have

$$\begin{array}{l}
\mathcal{N}(A) := \left[\begin{array}{c|c} \tau I & [(GA-I)Q^{-1/2}]^T \\ \hline (GA-I)Q^{-1/2} & \tau I \end{array}\right] \\
= \underbrace{\left[\begin{array}{c|c} \tau I & [(GA_{\rm n}-I)Q^{-1/2}]^T \\ \hline (GA_{\rm n}-I)Q^{-1/2} & \tau I \end{array}\right]}_{\mathcal{B}_{\rm n}(G)} + \mathcal{L}^T(G) \Delta \mathcal{R} + \mathcal{R}^T \Delta^T \mathcal{L}(G), \\
\mathcal{L}(G) = \left[0_{p \times n}, LG^T\right], \mathcal{R} = \left[RQ^{-1/2}, 0_{q \times n}\right].
\end{array}$$

Invoking Theorem 3.12, the semi-infinite LMI $\mathcal{N}(A) \succeq 0 \ \forall A \in \mathcal{A}$ is equivalent to

$$\exists \lambda : \left[\begin{array}{c|c} \lambda I_p & \rho \mathcal{L}(G) \\ \hline \rho \mathcal{L}^T(G) & \mathcal{B}_{\rm n}(G) - \lambda \mathcal{R}^T \mathcal{R} \end{array}\right] \succeq 0,$$

and thus the RC is equivalent to the semidefinite program

$$\min_{\substack{t,\tau,\\ \delta,\lambda,G}} \left\{ t : \begin{array}{l} \sqrt{\tau^2+\delta^2} \leq t,\ \sqrt{\mathrm{Tr}(G^T\Sigma G)} \leq \delta \\ \left[\begin{array}{c|c|c} \lambda I_p & & \rho L G^T \\ \hline & \tau I_n - \lambda Q^{-1/2}R^TRQ^{-1/2} & Q^{-1/2}(A_{\rm n}^TG^T - I_n) \\ \hline \rho G L^T & (GA_{\rm n} - I_n)Q^{-1/2} & \tau I_n \end{array}\right] \succeq 0 \end{array} \right\}.$$

2): Setting $v = U^T\widehat{v}$, $\widehat{y} = W^Ty$, $\widehat{\xi} = W^T\xi$, our estimation problem reduces to the exactly the same problem, but with $\mathrm{Diag}\{a\}$ in the role of $A_{\rm n}$ and the diagonal matrix $\mathrm{Diag}\{q\}$ in the role of Q; a linear estimate $\widehat{G}\widehat{y}$ of $\widehat{v}$ in the new problem corresponds to the linear estimate $U^T\widehat{G}W^Ty$, of exactly the same quality, in the original problem. In other words, the situation reduces to the one where $A_{\rm n}$ and Q are diagonal positive semidefinite, respectively, positive definite matrices; all we need is to prove that in this special case we lose nothing when restricting G to be diagonal. Indeed, in the case in question the RC reads

$$\min_{\substack{t,\tau,\\ \delta,\lambda,G}} \left\{ t : \begin{array}{l} \sqrt{\tau^2+\delta^2} \leq t,\ \sigma\sqrt{\mathrm{Tr}(G^TG)} \leq \delta \\ \left[\begin{array}{c|c|c} \lambda I_n & & \rho G^T \\ \hline & \tau I_n - \lambda \mathrm{Diag}\{\mu\} & \mathrm{Diag}\{\nu\}G^T - \mathrm{Diag}\{\eta\} \\ \hline \rho G & G\mathrm{Diag}\{\nu\} - \mathrm{Diag}\{\eta\} & \tau I_n \end{array}\right] \succeq 0 \end{array} \right\} \qquad (*)$$

where $\mu_i = q_i^{-1}$, $\nu_i = a_i/\sqrt{q_i}$ and $\eta_i = 1/\sqrt{q_i}$. Replacing the G-component in a feasible solution with EGE, where E is a diagonal matrix with diagonal entries ± 1, we preserve feasibility (look what happens when you multiply the matrix in the LMI from the left and from the right by $\mathrm{Diag}\{I, I, E\}$). Since the problem is convex, it follows that whenever a collection $(t, \tau, \delta, \lambda, G)$ is feasible for the RC, so is the collection obtained by replacing the original G with the average of the matrices E^TGE taken over all 2^n diagonal $n \times n$ matrices with diagonal entries ± 1, and this average is the diagonal matrix with the same diagonal as the one of G. Thus, when $A_{\rm n}$ and Q are diagonal and $L = R = I_n$ (or, which is the same in our situation, L and R are orthogonal), we lose nothing when restricting G to be diagonal.

Restricted to diagonal matrices $G = \mathrm{Diag}\{g\}$, the LMI constraint in $(*)$ becomes a bunch of 3×3 LMIs

$$\left[\begin{array}{c|c|c} \lambda & 0 & \rho g_i \\ \hline 0 & \tau - \lambda\mu_i & \nu_i g_i - \eta_i \\ \hline \rho g_i & \nu_i g_i - \eta_i & \tau \end{array}\right] \succeq 0,\ i = 1, ..., n,$$

in variables λ, τ, g_i. Assuming w.l.o.g. that $\lambda > 0$ and applying the Schur Complement Lemma, these 3×3 LMIs reduce to 2×2 matrix inequalities

$$\left[\begin{array}{c|c} \tau - \lambda\mu_i & \nu_i g_i - \eta_i \\ \hline \nu_i g_i - \eta_i & \tau - \rho^2 g_i^2/\lambda \end{array}\right] \succeq 0,\ i = 1, ..., n.$$

For given τ, λ, every one of these inequalities specifies a segment $\Delta_i(\tau, \lambda)$ of possible value of g_i, and the best choice of g_i in this segment is the point $g_i(\tau, \lambda)$ of the segment closest to 0 (when the segment is empty, we set $g_i(\tau, \lambda) = \infty$). Note that $g_i(\tau, \lambda) \geq 0$ (why?). It follows that $(*)$ reduces to the convex (due to its origin) problem

$$\min_{\tau,\lambda \geq 0} \left\{ \sqrt{\tau^2 + \sigma^2 \sum_i g_i^2(\tau, \lambda)} \right\}$$

with easily computable convex nonnegative functions $g_i(\tau, \lambda)$.

Exercise 3.6: 1) Let $\lambda > 0$. For every $\xi \in \mathbb{R}^n$ we have $\xi^T[pq^T + qp^T]\xi = 2(\xi^T p)(\xi^T q) \leq \lambda(\xi^T p)^2 + \frac{1}{\lambda}(\xi^T q)^2 = \xi^T[\lambda pp^T + \frac{1}{\lambda}qq^T]\xi$, whence $pq^T + qp^T \preceq \lambda pp^T + \frac{1}{\lambda}qq^T$. By similar argument, $-[pq^T + qp^T] \preceq \lambda pp^T + \frac{1}{\lambda}qq^T$. 1) is proved.

2) Observe, first, that if $\lambda(A)$ is the vector of eigenvalues of a symmetric matrix A, then $\|\lambda(pq^T + qp^T)\|_1 = 2\|p\|_2\|q\|_2$. Indeed, there is nothing to verify when $p = 0$ or $q = 0$; when $p, q \neq 0$, we can normalize the situation to make p a unit vector and then to choose the orthogonal coordinates in $\mathbb{R}^n$ in such a way that p is the first basic orth, and q is in the linear span of the first two basic orths. With this normalization, the nonzero eigenvalues of A are exactly the same as the eigenvalues of the 2×2 matrix $\begin{bmatrix} 2\alpha & \beta \\ \beta & 0 \end{bmatrix}$, where α and β are the first two coordinates of q in our new orthonormal basis. The eigenvalues of the 2×2 matrix in question are $\alpha \pm \sqrt{\alpha^2 + \beta^2}$, and the sum of their absolute values is $2\sqrt{\alpha^2 + \beta^2} = 2\|q\|_2 = 2\|p\|_2\|q\|_2$, as claimed.

To prove 2), let us lead to a contradiction the assumption that $Y, p, q \neq 0$ are such that $Y \succeq \pm[pq^T + qp^T]$ and there is no $\lambda > 0$ such that $Y - \lambda pp^T - \frac{1}{\lambda}qq^T \succeq 0$, or, which is the same by the Schur Complement Lemma, the LMI

$$\begin{bmatrix} Y - \lambda pp^T & q \\ q^T & \lambda \end{bmatrix} \succeq 0$$

in variable λ has no solution, or, equivalently, the optimal value in the (clearly strictly feasible) SDO program

$$\min_{t,\lambda} \left\{ t : \begin{bmatrix} tI + Y - \lambda pp^T & q \\ q^T & \lambda \end{bmatrix} \succeq 0 \right\}$$

is positive. By semidefinite duality, the latter is equivalent to the dual problem possessing a feasible solution with a positive value of the dual objective. Looking at the dual, this is equivalent to the existence of a matrix $Z \in \mathbf{S}^n$ and a vector $z \in \mathbb{R}^n$ such that

$$\begin{bmatrix} Z & z \\ z^T & p^T Z p \end{bmatrix} \succeq 0, \quad \mathrm{Tr}(ZY) < 2q^T z.$$

Adding, if necessary, to Z a small positive multiple of the unit matrix, we can assume w.l.o.g. that $Z \succ 0$. Setting $\bar{Y} = Z^{1/2} Y Z^{1/2}$, $\bar{p} = Z^{1/2}p$, $\bar{q} = Z^{1/2}q$, $\bar{z} = Z^{-1/2}z$, the above relations become

$$\begin{bmatrix} I & \bar{z} \\ \bar{z}^T & \bar{p}^T\bar{p} \end{bmatrix} \succeq 0, \mathrm{Tr}(\bar{Y}) < 2\bar{q}^T\bar{z}. \qquad (*)$$

Observe that from $Y \succeq \pm[pq^T + qp^T]$ it follows that $\bar{Y} \succeq \pm[\bar{p}\bar{q}^T + \bar{q}\bar{p}^T]$. Looking at what happens in the eigenbasis of the matrix $[\bar{p}\bar{q}^T + \bar{q}\bar{p}^T]$, we conclude from this relation that $\mathrm{Tr}(\bar{Y}) \geq \|\lambda(\bar{p}\bar{q}^T + \bar{q}\bar{p}^T)\|_1 = 2\|\bar{p}\|_2\|\bar{q}\|_2$. On the other hand, the matrix inequality in $(*)$ implies that $\|\bar{z}\|_2 \leq \|\bar{p}\|_2$, and thus $\mathrm{Tr}(\bar{Y}) < 2\|\bar{p}\|_2\|\bar{q}\|_2$ by the second inequality in $(*)$. We have arrived at a desired contradiction.

3) Assume that x is such that all $L_\ell(x)$ are nonzero. Assume that x can be extended to a feasible solution $Y_1, ..., Y_L, x$ of (3.7.2). Invoking 2), we can find $\lambda_\ell > 0$ such that $Y_\ell \succeq \lambda_\ell R_\ell^T R_\ell + \frac{1}{\lambda_\ell} L_\ell^T(x) L_\ell(x)$. Since $\mathcal{A}_{\mathrm{n}}(x) - \rho \sum_\ell Y_\ell \succeq 0$, we have $[\mathcal{A}_{\mathrm{n}}(x) - \rho \sum_\ell \lambda_\ell R_\ell^T R_\ell] - \sum_\ell \frac{\rho}{\lambda_\ell} L_\ell^T(x) L_\ell(x) \succeq 0$, whence, by the Schur Complement Lemma, $\lambda_1, ..., \lambda_L, x$ are feasible for (3.7.3). Vice versa, if $\lambda_1, ..., \lambda_L, x$ are feasible for (3.7.3), then $\lambda_\ell > 0$ for all ℓ due to $L_\ell(x) \neq 0$, and, by the same Schur Complement Lemma, setting $Y_\ell = \lambda_\ell R_\ell^T R_\ell + \frac{1}{\lambda_\ell} L_\ell^T(x) L_\ell(x)$, we have

$$\mathcal{A}_{\mathrm{n}}(x) - \rho \sum_\ell Y_\ell \succeq 0,$$

while $Y_\ell \succeq \pm\left[L_\ell^T(x)R_\ell + R_\ell^T L_\ell(x)\right]$, that is, $Y_1, ..., Y_L, x$ are feasible for (3.7.2).

We have proved the equivalence of (3.7.2) and (3.7.3) in the case when $L_\ell(x) \neq 0$ for all ℓ. The case when some of $L_\ell(x)$ vanish is left to the reader.

Exercise 3.7: *A* solution *might* be as follows. The problem of interest is

$$\min_{G,t}\{t : t \geq \|(GA-I)v + G\xi\|_2\, \forall (v \in V, \xi \in \Xi, A \in \mathcal{A})\}$$
$$\Updownarrow$$
$$\min_{G,t}\left\{t : u^T(GA-I)v + u^T G\xi \leq t \forall \left(u, v, \xi : \begin{array}{c} u^T u \leq 1 \\ v^T P_i v \leq 1, \\ 1 \leq i \leq I \\ \xi^T Q_j \xi \leq \rho_\xi^2, \\ 1 \leq j \leq J \end{array}\right) \forall A \in \mathcal{A}\right\}. \qquad (*)$$

Observing that

$$u^T[GA-I]v + u^T G\xi = [u; v; \xi]^T \left[\begin{array}{c|c|c} & \frac{1}{2}[GA-I] & \frac{1}{2}G \\ \hline \frac{1}{2}[GA-I]^T & & \\ \hline \frac{1}{2}G^T & & \end{array}\right][u; v; \xi],$$

for A fixed, a sufficient condition for the validity of the semi-infinite constraint in $(*)$ is the existence of nonnegative μ, ν_i, ω_j such that

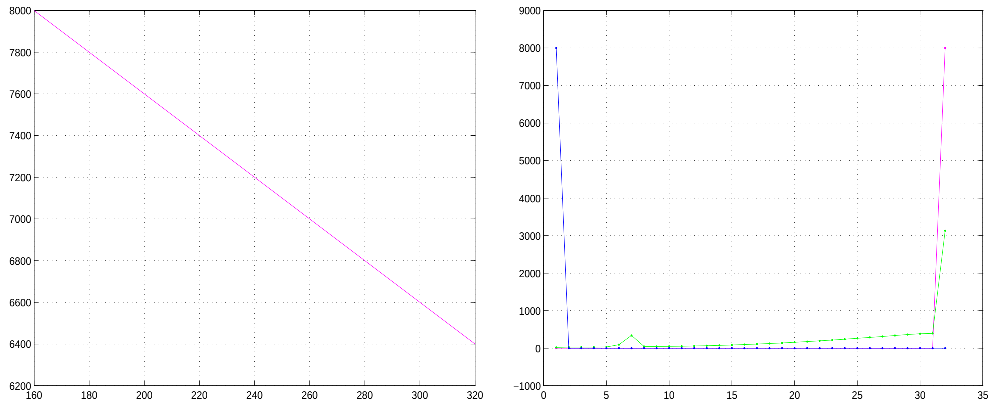
$$\left[\begin{array}{c|c|c} \mu I & & \\ \hline & \sum_i \nu_i P_i & \\ \hline & & \sum_j \omega_j Q_j \end{array}\right] \succeq \left[\begin{array}{c|c|c} & \frac{1}{2}[GA-I] & \frac{1}{2}G \\ \hline \frac{1}{2}[GA-I]^T & & \\ \hline \frac{1}{2}G^T & & \end{array}\right]$$

and $\mu + \sum_i \nu_i + \rho_\xi^2 \sum_j \omega_j \leq t$. It follows that the validity of the semi-infinite system of constraints

$$\begin{array}{l} \mu + \sum_i \nu_i + \rho_\xi^2 \sum_j \omega_j \leq t,\ \ \mu \geq 0, \nu_i \geq 0, \omega_j \geq 0 \\ \left[\begin{array}{c|c|c} \mu I & & \\ \hline & \sum_i \nu_i P_i & \\ \hline & & \sum_j \omega_j Q_j \end{array}\right] \succeq \left[\begin{array}{c|c|c} & \frac{1}{2}[GA-I] & \frac{1}{2}G \\ \hline \frac{1}{2}[GA-I]^T & & \\ \hline \frac{1}{2}G^T & & \end{array}\right] \\ \hfill \forall A \in \mathcal{A} \end{array} \qquad (!)$$

in variables $t, G, \mu, \nu_i, \omega_j$ is a sufficient condition for (G,t) to be feasible for $(*)$. The only semi-infinite constraint in (!) is in fact an LMI with structured norm-bounded uncertainty:

$$\begin{array}{l} \left[\begin{array}{c|c|c} \mu I & & \\ \hline & \sum_i \nu_i P_i & \\ \hline & & \sum_j \omega_j Q_j \end{array}\right] \\ \quad - \left[\begin{array}{c|c|c} & \frac{1}{2}[GA-I] & \frac{1}{2}G \\ \hline \frac{1}{2}[GA-I]^T & & \\ \hline \frac{1}{2}G^T & & \end{array}\right] \ \succeq\ 0\ \forall A \in \mathcal{A} \\ \Updownarrow \\ \underbrace{\left[\begin{array}{c|c|c} \mu I & -\frac{1}{2}[GA_{\rm n}-I] & -\frac{1}{2}G \\ \hline -\frac{1}{2}[GA_{\rm n}-I]^T & \sum_i \nu_i P_i & \\ \hline -\frac{1}{2}G^T & & \sum_j \omega_j Q_i \end{array}\right]}_{\mathcal{B}(\mu,\nu,\omega,G)} \\ \quad + \sum_{\ell=1}^L [\mathcal{L}_\ell(G)^T \Delta_\ell \mathcal{R}_\ell + \mathcal{R}_\ell^T \Delta_\ell^T \mathcal{L}_\ell(G)]\ \succeq\ 0 \\ \hfill \forall\left(\|\Delta_\ell\|_{2,2} \leq \rho_A, 1 \leq \ell \leq L\right), \\ \mathcal{L}_\ell(G) = \frac{1}{2}\left[L_\ell G^T, 0_{p_\ell \times n}, 0_{p_\ell \times m}\right],\ \ \mathcal{R}_\ell = [0_{q_\ell \times n}, R_\ell, 0_{q_\ell \times m}]. \end{array}$$



Opt(α) vs. α — solutions for $\alpha = 160, 180, 320$

Figure B.1: Results for Exercise 4.1.

Invoking Theorem 3.13, we end up with the following safe tractable approximation of $(*)$:

$$
\begin{array}{ll}
\min\limits_{t,G,\mu,\nu_i,\omega_j,\lambda_\ell,Y_\ell} & t \\
\text{s.t.} & \\
& \mu + \sum_i \nu_i + \rho_\xi^2 \sum_j \omega_j \leq t,\ \ \mu \geq 0, \nu_i \geq 0, \omega_j \geq 0 \\
& \left[\begin{array}{c|c} \lambda_\ell I & \mathcal{L}_\ell(G) \\ \hline \mathcal{L}_\ell^T(G) & Y_\ell - \lambda_\ell \mathcal{R}_\ell^T \mathcal{R}_\ell \end{array}\right] \succeq 0,\ 1 \leq \ell \leq L \\
& \mathcal{B}(\mu,\nu,\omega,G) - \rho_A \sum_{\ell=1}^L Y_\ell \succeq 0.
\end{array}
$$

B.4 Exercises from Lecture 4

Exercise 4.1: A solution *might be* as follows. We define the normal range of the uncertain cost vector as the box $\mathcal{Z} = \{c' : 0 \leq c' \leq c\}$, where c is the current cost, the cone $\mathcal{L}$ as

$$\mathcal{L} = \{\zeta \in \mathbb{R}^n : \zeta \geq 0, \zeta_j = 0 \text{ whenever } v_j = 0\}$$

and equip $\mathbb{R}^n$ with the norm

$$\|\zeta\|_v = \max_j |\zeta|_j / \bar{v}_j,\ \bar{v}_j = \left\{ \begin{array}{ll} v_j, & v_j > 0 \\ 1, & v_j = 0 \end{array} \right.$$

With this setup, the model becomes

$$\mathrm{Opt}(\alpha) = \min_x \left\{ c^T x : Px \geq b,\ x \geq 0, v^T x \leq \alpha \right\}.$$

With the data of the Exercise, computation says that the minimal value of α for which the problem is feasible is $\underline{\alpha} = 160$ and that the bound on the sensitivity becomes redundant when $\alpha \geq \overline{\alpha} = 320$. The tradeoff between $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ is shown on the left plot in figure B.1; the right plot depicts the solutions for $\alpha = \underline{\alpha} = 160$ (magenta), $\alpha = \overline{\alpha} = 320$ (blue) and $\alpha = 180$ (green).

Exercise 4.2: 1) With $\phi(\rho) = tau + \alpha\rho$ problem (!) does not make sense, meaning that it is always infeasible, unless $E = \{0\}$. Indeed, otherwise g contains a nonzero vector g, and assuming (!) feasible, we should have for certain τ and α

$$\left[\begin{array}{c|c} 2(\tau+\alpha\rho) & f^T+\rho g^T \\ \hline f+\rho g & A(t) \end{array}\right] \succeq 0 \ \forall \rho > 0$$

or, which is the same,

$$\left[\begin{array}{c|c} 2\rho^{-1}\tau + 2\rho^{-1}\alpha & \rho^{-1}f^T+g^T \\ \hline \rho^{-1}f+g & A(t) \end{array}\right] \succeq 0 \ \forall \rho > 0.$$

passing to limit as $\rho \to +\infty$, the matrix $\left[\begin{array}{c|c} 0 & g^T \\ \hline g & A(t) \end{array}\right]$ should be $\succeq 0$, which is not the case when $g \neq 0$.

The reason why the GRC methodology does not work in our case is pretty simple: we are *not* applying this methodology, we are doing something else. Indeed, with the GRC approach, we would require the validity of the semidefinite constraints

$$\left[\begin{array}{c|c} \tau & f^T+\rho g^T \\ \hline f+\rho g & A(t) \end{array}\right] \geq 0, \ f \in \mathcal{F}, g \in E$$

for all $f \in \mathcal{F}$ *in the case of* $\rho = 0$ and were allowing "controlled deterioration" of these constraints when $\rho > 0$:

$$\mathrm{dist}\left(\left[\begin{array}{c|c} \tau & f^T+\rho g^T \\ \hline f+\rho g & A(t) \end{array}\right], \mathbf{S}^{m+1}_+\right) \leq \alpha\rho \ \forall (f \in \mathcal{F}, g \in E).$$

When τ and α are large enough, this goal clearly is feasible. In the situation described in item 1) of Exercise, our desire is completely different: we want to keep the semidefinite constraints feasible, compensating for perturbations ρg by replacing the compliance τ with $\tau + \alpha\rho$. As it is shown by our analysis, this goal is infeasible – the "compensation in the value of compliance" should be at least quadratic in ρ.

2) With $\phi(\rho) = (\sqrt{\tau} + \sqrt{\alpha}\rho)^2$, problem (!) makes perfect sense; moreover, given $\tau \geq 0$, $\alpha \geq 0$, $t \in \mathcal{T}$ is feasible for (!) if and only if the system of relations

$$\begin{array}{ll} (a) & \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \ \forall f \in \mathcal{F} \\ (b) & \left[\begin{array}{c|c} 2\alpha & g^T \\ \hline g & A(t) \end{array}\right] \succeq 0 \ \forall g \in E \end{array}$$

Since $\mathcal{F}$ is finite, (a) is just a finite collection of LMIs in t, τ; and since E is a centered at the origin ellipsoid, the results of section 3.4.2 allow to convert the semi-infinite LMI (b) into an equivalent tractable system of LMIs, so that $(a), (b)$ is computationally tractable.

The claim that t is feasible for the semi-infinite constraint

$$\left[\begin{array}{c|c} 2(\sqrt{\tau}+\sqrt{\alpha}\rho)^2 & f^T+\rho g^T \\ \hline f+\rho g & A(t) \end{array}\right] \geq 0 \, \forall (f \in \mathcal{F}, g \in E, \rho \geq 0) \qquad (*)$$

if and only if t satisfies (a) and (b) is evident. Indeed, if t is feasible for the latter semi-infinite LMI, t indeed satisfies (a) and (b) – look what happens when $\rho = 0$ and when $\rho \to \infty$. Vice versa, assume that t satisfies (a) and $()b)$, and let us prove that t satisfies $(*)$ as well. Indeed, given $\rho > 0$, let us set $\mu = \frac{\sqrt{\tau}}{\sqrt{\tau}+\rho\sqrt{\alpha}}$, $\nu = 1-\mu = \frac{\rho\sqrt{\alpha}}{\sqrt{\tau}+\rho\sqrt{\alpha}}$ and $s = \frac{\mu}{\nu}\rho$. For $f \in \mathcal{F}$ and $g \in E$ from (a), (b) it follows that

$$\left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0, \left[\begin{array}{c|c} 2s^2\alpha & sg^T \\ \hline sg & A(t) \end{array}\right] \succeq 0;$$

or, which is the same by the Schur Complement Lemma, for every $\epsilon > 0$ one has

$$\|[A(t)+\epsilon I]^{-1/2} f\|_2 \leq \sqrt{2\tau},\ \|[A(t)+\epsilon I]^{-1/2} g\|_2 \leq \sqrt{2\alpha},$$

whence, by the triangle inequality,

$$\|[A(t)+\epsilon I]^{-1/2}[f+\rho g]\|_2 \leq \sqrt{2\tau}+\sqrt{2\alpha}\rho,$$

meaning that

$$\left[\begin{array}{c|c} [\sqrt{2\tau}+\sqrt{2\alpha}\rho]^2 & f^T+\rho g^T \\ \hline f+\rho g & A(t)+\epsilon I \end{array}\right] \succeq 0.$$

The latter relation holds true for all $(\epsilon > 0, f \in \mathcal{F}, g \in E)$, and thus t is feasible for $(*)$.

Exercise 4.3: 1) The worst-case error of a candidate linear estimate $g^T y$ is

$$\max_{z,\xi:\|z\|_2\leq 1, \|\xi_2\|\leq 1} \|(Az+\xi)^T g - f^T z\|_2,$$

so that the problem of building the best, in the minimax sense, estimate reads

$$\min_{\tau,G} \left\{\tau : |(Az+\xi)^T g - f^T z| \leq \tau\ \forall([z;\xi] : \|z\|_2 \leq 1, \|\xi\|_2 \leq 1)\right\},$$

which is nothing but the RC of the uncertain Least Squares problem

$$\left\{\min_{\tau,g}\left\{\tau : (Az+\xi)^T g - f^T z \leq \tau, f^T z - (Az+\xi)^T g \leq \tau\right\} : \zeta := [z;\xi] \in \mathcal{Z} = B \times \Xi\right\} \qquad (*)$$

in variables g, τ with certain objective and two constraints affinely perturbed by $\zeta = [z;\xi] \in B \times \Xi$. The equivalent tractable reformulation of this RC clearly is

$$\min_{\tau,g}\left\{\tau : \|A^T g - f\|_2 + \|g\| \leq \tau\right\}.$$

2) Now we want of our estimate to satisfy the relations

$$\forall(\rho_z \geq 0, \rho_\xi \geq 0) : |(Az+\xi)^T g - f^T z| \leq \tau + \alpha_z\rho_z + \alpha_\xi\rho_\xi,\ \forall(z : \|z\|_2 \leq 1+\rho_z, \xi : \|\xi\|_2 \leq 1+\rho_\xi),$$

or, which is the same,

$$\forall[z;\xi] : |(Az+\xi)^T g - f^T z| \leq \tau + \alpha_z \mathrm{dist}_{\|\cdot\|_2}(z,B) + \alpha_\xi \mathrm{dist}_{\|\cdot|_2}(\xi,\Xi).$$

This is exactly the same as to say that g should be feasible for the GRC of the uncertainty-affected inclusion

$$(Az+\xi)^T g - f^T z \in \mathbf{Q} = [-\tau,\tau].$$

in the case where the uncertain perturbations are $[z;\xi]$, the perturbation structure for z is given by $\mathcal{Z}_z = B, \mathcal{L}_z = \mathbb{R}^n$ and the norm on $\mathbb{R}^n$ is $\|\cdot\|_2$, and the perturbation structure for ξ is given by $\mathcal{Z}_\xi = \Xi$, $\mathcal{L}_\xi = \mathbb{R}^m$ and the norm on $\mathbb{R}^m$ is $\|\cdot\|_2$. Invoking Proposition 4.2, g is feasible for our GRC if and only if

$$\begin{array}{ll}(a) & (Az+\xi)^T g - f^T z \in \mathbf{Q}\ \forall (z\in B, \xi\in\Xi)\\ (b.1) & |(A^T g - f)^T z| \leq \alpha_z\ \forall (z : \|z\|_2 \leq 1)\\ (b.2) & |\xi^T g| \leq \alpha_\xi\ \forall(\xi : \|\xi\|_2 \leq 1)\end{array}$$

or, which is the same, g meets the requirements if and only if

$$\|A^T g - f\|_2 + \|g\|_2 \leq \tau,\ \|A^T g - f\|_2 \leq \alpha_z,\ \|g\|_2 \leq \alpha_\xi.$$

Exercise 4.4: 1) The worst-case error of a candidate linear estimate Gy is

$$\max_{z,\xi:\|z\|_2\leq 1, \|\xi_2\|\leq 1} \|G(Az+\xi) - Cz\|_2,$$

so that the problem of building the best, in the minimax sense, estimate reads

$$\min_{\tau,G} \left\{\tau : \|G(Az+\xi) - Cz\|_2 \leq \tau\ \forall ([z;\xi] : \|z\|_2 \leq 1, \|\xi\|_2 \leq 1)\right\},$$

which is nothing but the RC of the uncertain Least Squares problem

$$\left\{\min_{\tau,g}\left\{\tau : \|G(Az+\xi) - Cz\|_2 \leq \tau\right\} : \zeta := [z;\xi] \in \mathcal{Z} = B\times\Xi\right\} \qquad (*)$$

in variables G, τ. The body of the left hand side of the uncertain constraint is

$$G(Az+\xi) - Cz = L_1^T(G)zR_1 + L_2^T(G)\xi R_2,\ L_1(G) = A^T G^T - C^T, R_1 = 1, L_2(G) = G^T, R_2 = 1$$

that is, we deal with structured norm-bounded uncertainty with two full uncertain blocks: $n\times 1$ block z and $m\times 1$ block ξ, the uncertainty level ρ being 1 (see section 3.3.1). Invoking Theorem 3.4, the system of LMIs

$$\begin{array}{l}\left[\begin{array}{c|c|c} u_0-\lambda & u^T & \\ \hline u & U & GA-C \\ \hline & [GA-C]^T & \lambda I\end{array}\right] \succeq 0,\\ \left[\begin{array}{c|c|c} v_0-\mu & v^T & \\ \hline v & V & G \\ \hline & G^T & \mu I\end{array}\right] \succeq 0\\ \left[\begin{array}{c|c} \tau-u_0-v_0 & -u^T-v^T \\ \hline -u-v & \tau I - U - V\end{array}\right] \succeq 0\end{array} \qquad (S)$$

in variables $G, \tau, \lambda, \mu, u_0, u, U, v_0, v, V$ is a tight within the factor $\pi/2$ safe tractable approximation of the RC.

2) Now we want of our estimate to satisfy the relations

$$\forall(\rho_z \geq 0, \rho_\xi \geq 0) : \|G(Az+\xi) - Cz\|_2 \leq \tau + \alpha_z\rho_z + \alpha_\xi\rho_\xi,\ \forall (z : \|z\|_2 \leq 1+\rho_z, \xi : \|\xi\|_2 \leq 1+\rho_\xi),$$

or, which is the same,

$$\forall [z;\xi] : \|G(Az+\xi) - Cz\|_2 \leq \tau + \alpha_z \mathrm{dist}_{\|\cdot\|_2}(z, B) + \alpha_\xi \mathrm{dist}_{\|\cdot\|_2}(\xi, \Xi).$$

This is exactly the same as to say that G should be feasible for the GRC of the uncertainty-affected inclusion

$$G(Az+\xi)-Cz\in\mathbf{Q}=\{w:\|w\|_2\leq\tau\}$$

in the case where the uncertain perturbations are $[z;\xi]$, the perturbation structure for z is given by $\mathcal{Z}_z=B,\mathcal{L}_z=\mathbb{R}^n$, the perturbation structure for ξ is given by $\mathcal{Z}_\xi=\Xi,\mathcal{L}_\xi=\mathbb{R}^m$, the global sensitivities w.r.t. z,ξ are, respectively, α_z, α_ξ, and all norms in the GRC setup are the standard Euclidean norms on the corresponding spaces. Invoking Proposition 4.2, G is feasible for our GRC if and only if

$$\begin{array}{ll}(a) & \|G(Az+\xi)-Cz\|_2\in\mathbf{Q}\ \forall(z\in B,\xi\in\Xi)\\ (b) & \|(GA-C)z\|_2\leq\alpha_z\ \forall(z:\|z\|_2\leq 1)\\ (c) & \|G\xi\|_2\leq\alpha_\xi\ \forall(\xi:\|\xi\|_2\leq 1).\end{array}$$

(b), (c) merely say that

$$\|GA-I\|_{2,2}\leq\alpha_z,\|G\|_{2,2}\leq\alpha_\xi,$$

while (a) admits the safe tractable approximation (S).

B.5 Exercises from Lecture 5

Exercise 5.1: From state equations (5.5.1) coupled with control law (5.5.3) it follows that

$$w^N=W_N[\Xi]\zeta+w_N[\Xi],$$

where $\Xi=\{U_t^z,U_t^d,u_t^0\}_{t=0}^N$ is the "parameter" of the control law (5.5.3), and $W_N[\Xi]$, $w_N[\Xi]$ are matrix and vector affinely depending on Ξ. Rewriting (5.5.2) as the system of linear constraints

$$e_j^Tw^N-f_j\leq 0,\ j=1,...,J,$$

and invoking Proposition 4.1, the GRC in question is the semi-infinite optimization problem

$$\begin{array}{lll}\min_{\Xi,\alpha} & \alpha & \\ \text{subject to} & & \\ & e_j^T[W_N[\Xi]\zeta+w_N[\Xi]]-f_j\leq 0\ \forall(\zeta:\|\zeta-\bar{\zeta}\|_s\leq R) & (a_j)\\ & e_j^TW_N[\Xi]\zeta\leq\alpha\ \forall(\zeta:\|\zeta\|_r\leq 1) & (b_j)\\ & \qquad\qquad 1\leq j\leq J. & \end{array}$$

This problem clearly can be rewritten as

$$\begin{array}{ll}\min_{\Xi,\alpha} & \alpha\\ \text{subject to} & \\ & R\|W_N^T[\Xi]e_j\|_{s_*}+e_j^T[W_N[\Xi]\bar{\zeta}+w_N[\Xi]]-f_j\leq 0,\ 1\leq j\leq J\\ & \|W_N^T[\Xi]e_j\|_{r_*}\leq\alpha,\ 1\leq j\leq J\end{array}$$

where

$$s_*=\frac{s}{s-1},\ r_*=\frac{r}{r-1}.$$

Exercise 5.2: The AAGRC is equivalent to the convex program

$$\begin{array}{ll}\min_{\Xi,\alpha} & \alpha\\ \text{subject to} & \\ & R\|W_N^T[\Xi]e_j\|_{s*}+e_j^T[W_N[\Xi]\bar{\zeta}+w_N[\Xi]]-f_j\leq 0,\ 1\leq j\leq J\\ & \|[W_N^T[\Xi]e_j]_{d,+}\|_{r*}\leq\alpha,\ 1\leq j\leq J\end{array}$$

where

$$s_*=\frac{s}{s-1},\ r_*=\frac{r}{r-1}$$

and for a vector $\zeta=[z;d_0;...;d_N]\in\mathbb{R}^K$, $[\zeta]_{d,+}$ is the vector obtained from ζ by replacing the z-component with 0, and replacing every one of the d-components with the vector of positive parts of its coordinates, the positive part of a real a being defined as $\max[a,0]$.

Exercise 5.3: 1) For $\zeta=[z;d_0;...;d_{15}]\in\mathcal{Z}+\mathcal{L}$, a control law of the form (5.5.3) can be written down as

$$u_t=u_t^0+\sum_{\tau=0}^{t}u_{t\tau}d_\tau,$$

and we have

$$x_{t+1}=\sum_{\tau=0}^{t}\left[u_\tau^0-d_\tau+\sum_{s=0}^{\tau}u_{\tau s}d_s\right]=\sum_{\tau=0}^{t}u_\tau^0+\sum_{s=0}^{t}\left[\sum_{\tau=s}^{t}u_{\tau s}-1\right]d_s.$$

Invoking Proposition 4.1, the AAGRC in question is the semi-infinite problem

$$\begin{array}{ll}\min_{\{u_t^0,u_{t\tau}\},\alpha} & \alpha\\ \text{subject to} & \\ (a_x) & |\theta\left[\sum_{\tau=0}^{t}u_\tau^0\right]|\leq 0,\ 0\leq t\leq 15\\ (a_u) & |u_t^0|\leq 0,\ 0\leq t\leq 15\\ (b_x) & |\theta\sum_{s=0}^{t}\left[\sum_{\tau=s}^{t}u_{\tau s}-1\right]d_s|\leq\alpha\\ & \quad\forall(0\leq t\leq 15,[d_0;...;d_{15}]:\|[d_0;...;d_{15}]\|_2\leq 1)\\ (b_u) & |\sum_{\tau=0}^{t}u_{t\tau}d_\tau|\leq\alpha\\ & \quad\forall(0\leq t\leq 15,[d_0;...;d_{15}]:\|[d_0;...;d_{15}]\|_2\leq 1)\end{array}$$

We see that the desired control law is linear ($u_t^0=0$ for all t), and the AAGRC is equivalent to the conic quadratic problem

$$\min_{\{u_{t\tau}\},\alpha}\left\{\alpha:\begin{array}{l}\sqrt{\sum_{s=0}^{t}\left[\sum_{\tau=s}^{t}u_{\tau s}-1\right]^2}\leq\theta^{-1}\alpha,\ 0\leq t\leq 15\\ \sqrt{\sum_{\tau=0}^{t}u_{\tau t}^2}\leq\alpha,\ 0\leq t\leq 15\end{array}\right\}.$$

2) In control terms, we want to "close" our toy linear dynamical system, where the initial state is once and for ever set to 0, by a linear state-based non-anticipative control law in such a way that the states $x_1,...,x_{16}$ and the controls $u_1,...,u_{15}$ in the closed loop system are "as insensitive to the perturbations $d_0,...,d_{15}$ as possible," while measuring the changes in the state-control trajectory

$$w^{15}=[0;x_1;...;x_{16};u_1,...,u_{15}]$$

in the weighted uniform norm $\|w^{15}\|_{\infty,\theta}=\max[\theta\|x\|_\infty,\|u\|_\infty]$, and measuring the changes in the sequence of disturbances $[d_0;...;d_{15}]$ in the "energy" norm $\|[d_0;...;d_{15}]\|_2$. Specifically, we are interested to find a linear non-anticipating state-based control law that results in the smallest possible constant α satisfying the relation

$$\forall \Delta d^{15} : \|\Delta w^{15}\|_{\infty,\theta} \leq \alpha \|\Delta d^{15}\|_2,$$

where Δd^{15} is a shift of the sequence of disturbances, and Δw^{15} is the induced shift in the state-control trajectory.
3) The numerical results are as follows:

θ	α
1.e6	4.0000
10	3.6515
2	2.8284
1	2.3094

Exercise 5.4: 1) Denoting by x^{ij}_γ the amount of information in the traffic from i to j travelling through γ, by q_γ the increase in the capacity of arc γ, and by $O(k)$, $I(k)$ — the sets of outgoing, resp., incoming, arcs for node k, the problem in question becomes

$$\min_{\substack{\{x^{ij}_\gamma\},\\ \{q_\gamma\}}} \left\{ \sum_{\gamma\in\Gamma} c_\gamma q_\gamma : \begin{array}{c} \sum_{(i,j)\in\mathcal{J}} x^{ij}_\gamma \leq p_\gamma + q_\gamma \,\forall\gamma \\ \sum_{\gamma\in O(k)} x^{ij}_\gamma - \sum_{\gamma\in I(k)} x^{ij}_\gamma = \left\{\begin{array}{ll} d_{ij}, & k=i \\ -d_{ij}, & k=j \\ 0, & k\notin\{i,j\}\end{array}\right. \\ \forall((i,j)\in\mathcal{J}, k\in V) \\ q_\gamma \geq 0, x^{ij}_\gamma \geq 0 \ \forall((i,j)\in\mathcal{J}, k\in V) \end{array}\right\}. \qquad (*)$$

2) To build the AARC of $(*)$ in the case of uncertain traffics d_{ij}, it suffices to plug into $(*)$, instead of decision variables x^{ij}_γ, affine functions $X^{ij}_\gamma(d) = \xi^{ij,0}_\gamma + \sum_{(\mu,\nu)\in\mathcal{J}} \xi^{ij\mu\nu}_\gamma d_{\mu\nu}$ of $d = \{d_{ij} : (i,j) \in \mathcal{J}\}$ (in the case of (a), the functions should be restricted to be of the form $X^{ij}_\gamma(d) = \xi^{ij,0}_\gamma + \xi^{ij}_\gamma d_{ij}$) and to require the resulting constraints in variables $q_\gamma, \xi^{ij\mu\nu}_\gamma$ to be valid for all realizations of $d \in \mathcal{Z}$. The resulting semi-infinite LO program is computationally tractable (as the AARC of an uncertain LO problem with fixed recourse, see section 5.3.1).

3) Plugging into $(*)$, instead of variables x^{ij}_γ, affine decision rules $X^{ij}_\gamma(d)$ of the just indicated type, the constraints of the resulting problem can be split into 3 groups:

$$\begin{array}{ll} (a) & \sum_{(i,j)\in\mathcal{J}} X^{ij}_\gamma(d) \leq p_\gamma + q_\gamma \,\forall\gamma\in\Gamma \\ (b) & \sum_{\substack{(i,j)\in\mathcal{J}\\ \gamma\in\Gamma}} \mathcal{R}^{ij}_\gamma X^{ij}_\gamma(d) = r(d) \\ (c) & q_\gamma \geq 0, X^{ij}_\gamma(d) \geq 0 \,\forall((i,j)\in\mathcal{J}, \gamma\in\Gamma). \end{array}$$

In order to ensure the feasibility of a given candidate solution for this system with probability at least $1-\epsilon$, $\epsilon < 1$, when d is uniformly distributed in a box, the linear equalities (b) *must* be satisfied for all d's, that is, (b) induces a system $A\xi = b$ of linear equality constraints on the vector ξ of coefficients of the affine decision rules $X^{ij}_\gamma(\cdot)$. We can use this system of linear equations, if it is feasible, in order to express ξ as an affine function of a shorter vector η of "free" decision variables, that is, we can easily find H and h in such a way that $A\xi = b$ is equivalent to the existence of η such that $\xi = H\eta + h$. We can now plug $\xi = H\eta + h$ into $(a), (c)$ and forget about (b), thus ending up with a system of constraints of the form

$$\begin{array}{ll} (a') & a_\ell(\eta, q) + \alpha^T_\ell(\eta, q) d \leq 0,\ 1 \leq \ell \leq L = \mathrm{Card}(\Gamma)(\mathrm{Card}(\mathcal{J})+1), \\ (b') & q \geq 0 \end{array}$$

with a_ℓ, α_ℓ affine in $[\eta; q]$ (the constraints in (a') come from the Card(Γ) constraints in (a) and the Card(Γ)Card($\mathcal{J}$) constraints $X_\gamma^{ij}(d) \geq 0$ in (c)).

In order to ensure the validity of the uncertainty-affected constraints (a'), as evaluated at a candidate solution $[\eta; q]$, with probability at least $1 - \epsilon$, we can use either the techniques from Lecture 2, or the techniques from section 3.6.4.

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