What can be expressed via Conic Quadratic and Semidefinite Programming?

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Abstract

Tremendous recent progress in the performance of optimization techniques for Linear, Conic Quadratic and Semidefinite Programming makes it important to know how to recognize optimization problems lending themselves to these advanced techniques. To this end, we present in the talk a kind of simple, powerful and "completely algorithmic" calculus of problems reducible to CQP and SDP. We demonstrate also that problems representable via Conic Quadratic Programming admit polynomial time Linear Programming approximations.

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What the story is about

♣ **Consider the following optimization program:**

minimize $c^T x$ subject to
$(a)\ \begin{cases} Ax = b \\ x \geq 0 \end{cases}$
$(b)\ \begin{cases} \left(\sum_{i=1}^{8} \lvert x_i\rvert^3\right)^{1/3} \leq x_2^{1/7} x_3^{2/7} x_4^{3/7} + 2 x_1^{1/5} x_5^{2/5} x_6^{1/5} \\ 5x_2 \geq \frac{1}{x_1^{1/2} x_2^2} + \frac{2}{x_2^{1/3} x_3^3 x_4^{5/8}} \\ \begin{pmatrix} x_2 & x_1 & & \\ x_1 & x_4 & x_3 & \\ & x_3 & x_6 & x_3 \\ & & x_3 & x_8 \end{pmatrix} \preceq 5I \end{cases}$
$(c)\ \begin{cases} \begin{pmatrix} x_1 & x_2 - x_1 & x_3 - x_2 & x_4 - x_3 \\ x_2 - x_1 & x_2 & x_3 - x_2 & x_4 - x_3 \\ x_3 - x_2 & x_3 - x_2 & x_3 & x_4 - x_3 \\ x_4 - x_3 & x_4 - x_3 & x_4 - x_3 & x_4 \end{pmatrix} \succeq 0 \\ x_1 + x_2 \sin(\phi) + x_3 \sin(2\phi) + x_4 \sin(4\phi) \geq 0 \quad \forall \phi \in \left[0, \frac{\pi}{2}\right] \end{cases}$

♠ **The problem can be converted,** *in a systematic way*, **into an equivalent semidefinite program**

$$d^T z \to \min \mid\ P_0 + \sum_{i=1}^{\dim z} z_i P_i \succeq 0. \qquad \textbf{(SDP)}$$

♠ **Removing constraints** (c), **the resulting problem can be converted,** *in a systematic way*, **into an equivalent conic quadratic program**

$$d^T z \to \min \mid\ \|P_i z + p_i\|_2 \leq q_i^T z + r_i,\ i = 1, ..., m. \qquad \textbf{(CQP)}$$

♠ **The resulting problem (CQP) can be approximated,** *in a polynomial time fashion*, **by a linear programming program**

$$d^T z \to \min \mid\ Pz + p \geq 0. \qquad \textbf{(LP)}$$

minimize $c^T x$ subject to

$$(a)\quad \begin{cases} Ax = b \\ x \geq 0 \end{cases}$$

$$(b)\quad \begin{cases}
\begin{pmatrix} x_i & v_i \\ v_i & v_{8+i} \end{pmatrix} \succeq 0,\ \begin{pmatrix} v_i & x_i \\ x_i & v_{17} \end{pmatrix} \succeq 0,\ i = 1, ..., 8, \\
\sum_{i=9}^{16} v_i \leq v_{17}; \\
\begin{pmatrix} v_{18} & v_{19} \\ v_{19} & x_2 \end{pmatrix} \succeq 0,\ \begin{pmatrix} x_4 & v_{20} \\ v_{20} & 1 \end{pmatrix} \succeq 0,\ \begin{pmatrix} v_{19} & v_{21} \\ v_{21} & x_3 \end{pmatrix} \succeq 0, \\
\begin{pmatrix} x_4 & v_{22} \\ v_{22} & v_{20} \end{pmatrix} \succeq 0,\ \begin{pmatrix} v_{21} & v_{18} \\ v_{18} & v_{22} \end{pmatrix} \succeq 0,\ \begin{pmatrix} x_1 & v_{24} \\ v_{24} & v_{23} \end{pmatrix} \succeq 0, \\
\begin{pmatrix} x_6 & v_{25} \\ v_{25} & 1 \end{pmatrix} \succeq 0,\ \begin{pmatrix} v_{23} & v_{26} \\ v_{26} & v_{24} \end{pmatrix} \succeq 0,\ \begin{pmatrix} x_5 & v_{27} \\ v_{27} & v_{25} \end{pmatrix} \succeq 0, \\
\begin{pmatrix} v_{26} & v_{23} \\ v_{23} & v_{27} \end{pmatrix} \succeq 0, \\
v_{17} \leq v_{18} + 2v_{23}; \\
\begin{pmatrix} x_1 & v_{29} \\ v_{29} & 1 \end{pmatrix} \succeq 0,\ \begin{pmatrix} v_{28} & v_{30} \\ v_{30} & v_{29} \end{pmatrix} \succeq 0,\ \begin{pmatrix} v_{30} & 1 \\ 1 & x_2 \end{pmatrix} \succeq 0, \\
\begin{pmatrix} x_4 & v_{32} \\ v_{32} & 1 \end{pmatrix} \succeq 0,\ \begin{pmatrix} x_4 & v_{33} \\ v_{33} & v_{32} \end{pmatrix} \succeq 0,\ \begin{pmatrix} x_4 & v_{34} \\ v_{34} & v_{33} \end{pmatrix} \succeq 0, \\
\begin{pmatrix} v_{31} & v_{35} \\ v_{35} & 1 \end{pmatrix} \succeq 0,\ \begin{pmatrix} x_2 & v_{36} \\ v_{36} & x_3 \end{pmatrix} \succeq 0,\ \begin{pmatrix} x_4 & v_{37} \\ v_{37} & v_{34} \end{pmatrix} \succeq 0, \\
\begin{pmatrix} v_{35} & v_{38} \\ v_{38} & v_{31} \end{pmatrix} \succeq 0,\ \begin{pmatrix} v_{36} & v_{39} \\ v_{39} & v_{37} \end{pmatrix} \succeq 0,\ \begin{pmatrix} v_{38} & v_{40} \\ v_{40} & v_{39} \end{pmatrix} \succeq 0, \\
\begin{pmatrix} x_3 & 1 \\ 1 & v_{40} \end{pmatrix} \succeq 0, \\
v_{28} + 2v_{31} \leq 5x_2; \\
\begin{pmatrix} x_2 & x_1 & & \\ x_1 & x_4 & x_3 & \\ & x_3 & x_6 & x_3 \\ & & x_3 & x_8 \end{pmatrix} \preceq 5I
\end{cases}$$

$$
(c)\quad \left\{
\begin{array}{l}
\begin{pmatrix}
x_1 & x_2-x_1 & x_3-x_2 & x_4-x_3\\
x_2-x_1 & x_2 & x_3-x_2 & x_4-x_3\\
x_3-x_2 & x_3-x_2 & x_3 & x_4-x_3\\
x_4-x_3 & x_4-x_3 & x_4-x_3 & x_4
\end{pmatrix} \succeq 0\\
\begin{pmatrix}
x_1 & 0 & v_{41} & v_{42} & v_{43} & v_{44} & v_{45} & v_{46} & v_{47}\\
0 & v_{48} & -v_{42} & v_{49} & v_{50} & v_{51} & v_{52} & v_{53} & v_{54}\\
v_{41} & -v_{42} & v_{55} & v_{56} & v_{57} & v_{58} & v_{59} & v_{60} & v_{61}\\
v_{42} & v_{49} & v_{56} & v_{62} & v_{63} & v_{64} & v_{65} & v_{66} & v_{67}\\
v_{43} & v_{50} & v_{57} & v_{63} & v_{68} & v_{69} & v_{70} & v_{71} & v_{72}\\
v_{44} & v_{51} & v_{58} & v_{64} & v_{69} & v_{73} & v_{74} & v_{75} & v_{76}\\
v_{45} & v_{52} & v_{59} & v_{65} & v_{70} & v_{74} & v_{77} & -v_{76} & v_{78}\\
v_{46} & v_{53} & v_{60} & v_{66} & v_{71} & v_{75} & -v_{76} & v_{79} & 0\\
v_{47} & v_{54} & v_{61} & v_{67} & v_{72} & v_{76} & v_{78} & 0 & v_{80}
\end{pmatrix} \succeq 0,\\
\begin{aligned}
2v_{41}+v_{48}-8x_1-2x_2-4x_3-8x_4 &= 0\\
2v_{43}+2v_{49}+v_{55}-32x_1-14x_2-28x_3-56x_4 &= 0\\
v_{44}+v_{50}+v_{56} &= 0\\
2v_{45}+2v_{51}+2v_{57}+v_{62}-80x_1-48x_2-88x_3-112x_4 &= 0\\
v_{46}+v_{52}+v_{58}+v_{63} &= 0\\
2v_{47}+2v_{53}+2v_{59}+2v_{64}+v_{68}-136x_1-100x_2-160x_3 &= 0\\
v_{54}+v_{60}+v_{65}+v_{69} &= 0\\
2v_{61}+2v_{66}+2v_{70}+v_{73}-160x_1-136x_2-176x_3+224x_4 &= 0\\
v_{67}+v_{71}+v_{74} &= 0\\
2v_{72}+2v_{75}+v_{77}-128x_1-120x_2-112x_3+224x_4 &= 0\\
2v_{78}+v_{79}-64x_1-64x_2-32x_3+64x_4 &= 0\\
v_{80}-16x_1-16x_2 &= 0
\end{aligned}
\end{array}\right.
$$

What can be expressed via CQP and SDP?

♣ **Let us look at three generic families of convex programs:**

♠ **Linear Programming:**

$$c^T x \to \min \mid Ax + b \geq 0 \qquad \textbf{(LP)}$$

♠ **Conic Quadratic Programming:**

$$c^T x \to \min \mid \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, ..., m \qquad \textbf{(CQP)}$$

♠ **Semidefinite Programming:**

$$c^T x \to \min \mid \sum_{i=1}^{n} x_i A_i + B \succeq 0 \qquad \textbf{(SDP)}$$

♣ **Geometrically, all these problems are of the form**

$$c^T x \to \min \mid Ax + b \in \mathbf{K}, \qquad \textbf{(CP)}$$

where K is a closed pointed convex cone with a nonempty interior belonging to a *specific for the generic problem in question* **family** $\mathcal{K}$ **of convex cones.**

$$c^T x \to \min \mid \; Ax + b \in \mathbf{K} \quad [\mathbf{K} \in \mathcal{K}] \qquad \textbf{(CP)}$$

LP: $\mathcal{K}$ is comprised of direct products of rays $\mathbf{R}_+$

CQP: $\mathcal{K}$ is comprised of direct products of the Lorentz cones

$$\mathbf{L}^n = \{(x,t) \in \mathbf{R}^{n+1} : \|x\|_2 \le t\}$$

SDP: $\mathcal{K}$ is comprised of direct products of the semidefinite cones

$$\mathbf{S}^n_+ = \{A \in \mathbf{S}^n : A \succeq 0\}$$

Note: *The above families of cones are closed w.r.t.*

(1) *taking (finite) direct products of cones;*
(2) *passing from a cone* $\mathbf{K}$ *to its dual cone*

$$\mathbf{K}_* = \{\eta \mid \; \eta^T x \ge 0 \quad \forall x \in \mathbf{K}\}.$$

♣ **Assume that we are given a family $\mathcal{K}$ of finite-dimensional convex cones (closed, pointed and with a nonempty interior) and know how to solve problems of the form**

$$c^T y \to \min \mid Ay + b \in \mathbf{K} \qquad (*)$$

with all possible data (c, A, b) and all $\mathbf{K} \in \mathcal{K}$.

Question: *What is the family of problems we can actually solve? When an optimization problem*

$$e^T x \to \min \mid x \in X \subset \mathbf{R}^n \qquad \mathbf{(P)}$$

can be equivalently reformulated in the form of $(*)$?

An answer: This is the case when X is a $\mathcal{K}$-representable set ($\mathcal{K}$-r.s.), i.e., there exists a $\mathcal{K}$*-representation* **($\mathcal{K}$-r.) of** X

$$X = \left\{ x \in \mathbf{R}^n \mid \exists u \in \mathbf{R}^k : \quad Px + Qu + b \in \mathbf{K} \right\} \qquad \mathbf{(R)}$$
$$[\mathbf{K} \in \mathcal{K}]$$

Indeed, given the data $P, Q, b, \mathbf{K}$ of (R), we can rewrite (P) equivalently in the form of $(*)$:

$$d^T x \to \min \mid x \in X$$

$$\Updownarrow$$

$$c^T y \equiv d^T x \to \min \mid Ay + b \equiv Px + Qu + b \in \mathbf{K}$$
$$\left[y = \begin{pmatrix} x \\ u \end{pmatrix} \right]$$

<u>**An example:**</u> **Consider the optimization problem**

$$t \to \min \mid xyzt \geq 1,\; A\begin{pmatrix} x \\ y \\ z \end{pmatrix} + b \geq 0,\; x,y,z \geq 0$$
$$\Updownarrow \qquad (*)$$
$$\frac{1}{xyz} \to \min \mid x,y,z \geq 0,\; A\begin{pmatrix} x \\ y \\ z \end{pmatrix} + g \geq 0.$$

It turns out that the feasible set of this problem is semidefinite representable:

$$xyzt \geq 1 \;\&\; x,y,z \geq 0 \;\&\; A\begin{pmatrix} x \\ y \\ z \end{pmatrix} + b \geq 0$$
$$\Updownarrow$$
$$\left\{\begin{array}{ll} (1) & \left[\begin{array}{ll} \exists u_1, u_2 : & \\ (a) & \underbrace{\begin{pmatrix} x & u_1 \\ u_1 & y \end{pmatrix} \succeq 0, \begin{pmatrix} z & u_2 \\ u_2 & t \end{pmatrix} \succeq 0}_{\text{say that } xy \geq u_1^2,\ zt \geq u_2^2 \text{ and } x,y,z,t \geq 0} \\ (b) & \underbrace{\begin{pmatrix} u_1 & 1 \\ 1 & u_2 \end{pmatrix} \succeq 0}_{\text{says that } u_1u_2 \geq 1 \text{ and } u_1,u_2 \geq 0} \end{array}\right. \\ & [(1) \textbf{ says exactly that } xyzt \geq 1 \;\&\; x,y,z \geq 0] \\ (2) & A\begin{pmatrix} x \\ y \\ z \end{pmatrix} + b \geq 0 \end{array}\right.$$

Thus, (*) is equivalent to the semidefinite problem

$$t \to \min \mid \begin{pmatrix} x & u_1 \\ u_1 & y \end{pmatrix} \succeq 0, \begin{pmatrix} z & u_2 \\ u_2 & t \end{pmatrix} \succeq 0, \begin{pmatrix} u_1 & 1 \\ 1 & u_2 \end{pmatrix} \succeq 0, A\begin{pmatrix} x \\ y \\ z \end{pmatrix} + b \geq 0.$$

♣ **We see that a natural interpretation of the question**

Given a possibility to solve problems

$$c^T x \to \min \mid \; Ax + b \in \mathbf{K} \in \mathcal{K}$$

what can we actually solve?

is

♠ *What are $\mathcal{K}$-representable sets?*

♠ *How to recognize $\mathcal{K}$-representability?*

♣ **<u>Claim:</u> Consider a family $\mathcal{K}$ of finite-dimensional closed pointed cones with a nonempty interior, and let this family be closed w.r.t.**

♠ **taking direct products**

♠ **passing from a cone to its dual.**

There exists a simple and powerful "calculus" of $\mathcal{K}$-representable sets: essentially,

> *Every standard convexity-preserving operation as applied to (finitely many) $\mathcal{K}$-representable sets $X_1, ..., X_k$, yields a $\mathcal{K}$-representable result X.*
>
> *Moreover, a $\mathcal{K}$-representation of X is readily given by $\mathcal{K}$-representations of $X_1, ..., X_k$.*

♣ **Applying the "calculus machinery" to (specific for a family $\mathcal{K}$) collection of "raw materials" – simple $\mathcal{K}$-representable sets – we get a possibility to recognize complicated $\mathcal{K}$-representable sets and thus to pose various optimization problems in the "$\mathcal{K}$-form".**

Calculus of $\mathcal{K}$-representable sets

I. The intersection of finitely many $\mathcal{K}$-r.s.'s is a $\mathcal{K}$-r.s.:

$$X_i = \{x \mid \exists u_i : P_i x + Q_i u_i + b_i \in \mathbf{K}_i\}, \; i = 1, ..., k$$

$$\Downarrow$$

$$\bigcap_{i=1}^{k} X = \Big\{u \mid \exists u = \begin{pmatrix} u_1 \\ ... \\ u_k \end{pmatrix} : \\ \begin{pmatrix} P_1 x + Q_i u_1 + b_1 \\ ... \\ P_k x + Q_k u_k + b_k \end{pmatrix} \in \mathbf{K}_1 \times ... \times \mathbf{K}_k \Big\}$$

II. The direct product of finitely many $\mathcal{K}$-r.s.'s is a $\mathcal{K}$-r.s.

III. The image of a $\mathcal{K}$-r.s. under an affine mapping is a $\mathcal{K}$-r.s.

IV. The inverse image of a $\mathcal{K}$-r.s. under an affine mapping is a $\mathcal{K}$-r.s.

V. The arithmetic sum of finitely many $\mathcal{K}$-r.s.'s is a $\mathcal{K}$-r.s.

Calculus of $\mathcal{K}$-representable sets (cont.)

VI. A polyhedral set

$$X = \{x \mid a_i^T x + b_i \geq 0,\ i = 1, ..., k\}$$

is a $\mathcal{K}$-r.s.

Indeed, let $\mathbf{K} \in \mathcal{K}$ and $0 \neq e \in \mathbf{K}$. Then

$$X = \left\{ x \mid \begin{pmatrix} [a_1^T x + b_1]e \\ ... \\ [a_k^T x + b_k]e \end{pmatrix} \in \mathbf{K} \times ... \times \mathbf{K} \right\}.$$

♣ Intersection of infinitely many half-spaces not necessarily is $\mathcal{K}$-representable. It, however, is so when the half-spaces are "well-organized".

VIa. Assume that the data (α, β) defining a half-space

$$X_{\alpha,\beta} = \{x \mid \alpha^T x + \beta \geq 0\}$$

vary in a $\mathcal{K}$-r.s. $\mathcal{U}$:

$$\mathcal{U} = \{(\alpha, \beta) \mid \exists u : P\alpha + \beta p + Qu + b \in \mathbf{K}\}$$

and let the representation be *strictly feasible***:**

$$\exists \bar{\alpha}, \bar{\beta}, \bar{u} : P\bar{\alpha} + \bar{\beta}p + Q\bar{u} + b \in \text{int } \mathbf{K}.$$

Then the set

$$X = \bigcap_{(\alpha,\beta)\in\mathcal{U}} X_{\alpha,\beta} = \left\{x \mid \alpha^T x + \beta \geq 0 \quad \forall (\alpha, \beta) \in \mathcal{U}\right\}$$

is a $\mathcal{K}$-r.s.:

$$X = \left\{ x \mid \exists \eta : \begin{array}{r} \eta \in \mathbf{K}_* \\ P^T\eta = x \\ p^T\eta = 1 \\ Q^T\eta = 0 \\ b^T\eta \leq 0 \end{array} \right\}$$

Thus, X is the projection of the intersection of the $\mathcal{K}$-r.s. $\mathbf{K}_*$ and a polyhedral set and thus is a $\mathcal{K}$-r.s.

Calculus of $\mathcal{K}$-representable sets (cont.)

♣ **Several operations with sets "nearly preserve" $\mathcal{K}$-representability.**

Definition. A set $X \subset \mathbf{R}$ is called *nearly $\mathcal{K}$-representable*, **if there exists a $\mathcal{K}$-r.s. X' such that**

$$X \subset X' \subset \operatorname{cl} X. \qquad (*)$$

A *nearly $\mathcal{K}$-representation* **of X is a $\mathcal{K}$-representation of a set X' satisfying $(*)$.**

Note: Nearly $\mathcal{K}$-representable *closed* **set X is $\mathcal{K}$-representable, and every nearly $\mathcal{K}$-representation of such a set is its $\mathcal{K}$-representation.**

VII. The conic hull

$$\operatorname{Cone}(X) = \{0\} \bigcup \left\{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid t > 0, t^{-1}x \in X\right\}$$

of a nonempty $\mathcal{K}$-r.s. X is a nearly $\mathcal{K}$-r.s.

VIII. Convex hull of the union of finitely many $\mathcal{K}$-r.s.'s is a nearly $\mathcal{K}$-r.s.

IX. Let

$$X = \{x \mid \exists u : Px + Qu + b \in \mathbf{K}\} \qquad (*)$$

be a $\mathcal{K}$-r.s., and let the representation $(*)$ be *strictly feasible*:

$$\exists(\bar{x}, \bar{u}) : P\bar{x} + Q\bar{u} + b \in \operatorname{int} \mathbf{K}.$$

Then the *polar cone* **of X – the set**

$$X_* = \{(\eta, s) \mid \eta^T x \geq s \quad \forall x \in X\}$$

– is a $\mathcal{K}$-r.s.

$\mathcal{K}$-representable functions

♣ **In many applications, sets are represented by (systems of) inequalities**

$$f(x) \le 0 \qquad (*)$$
$$[f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}]$$

Definition: f is called a $\mathcal{K}$-representable function ($\mathcal{K}$-r.f.), if the epigraph of f

$$\mathrm{Epi}(f) = \{(x,t) \mid t \ge f(x)\}$$

is a $\mathcal{K}$-r.s.

A $\mathcal{K}$-representation of $\mathrm{Epi}(f)$ **is called a $\mathcal{K}$-representation ($\mathcal{K}$-r.) of f.**

Observation: If f is $\mathcal{K}$-representable, then so are all level sets

$$\{x \mid f(x) \le a\} \quad [a \in \mathbf{R}]$$

of f.

Indeed,

$$f(x) \le t \Leftrightarrow \exists u : Px + tp + Qu + b \in \mathbf{K}$$
$$\Downarrow$$
$$\{x \mid f(x) \le a\} = \{x \mid \exists u : Px + Qu + [ap + b] \in \mathbf{K}\}$$

$\mathcal{K}$-representable functions (cont.)

♣ **Calculus of $\mathcal{K}$-representable sets can be straightforwardly converted into a "calculus of $\mathcal{K}$-representable functions". The latter, essentially, can be summarized in the following statements:**

(i). [Conic combinations and taking maximum] A linear combination, with nonnegative coefficients, and the maximum of finitely many $\mathcal{K}$-r.f.'s are themselves $\mathcal{K}$-r.f.'s.

(ii). [Adding affine form] The sum of a $\mathcal{K}$-representable and an affine function is $\mathcal{K}$-representable

(iii). [Affine substitution of argument] The superposition of a $\mathcal{K}$-r.f. and an affine mapping is a $\mathcal{K}$-r.f.

(iv). [Taking superpositions] Let functions

$$f_1, ..., f_k : \mathbf{R}^n \to \mathbf{R} \bigcup \{+\infty\}$$

and a function

$$g : \mathbf{R}^k \to \mathbf{R} \bigcup \{+\infty\}$$

be $\mathcal{K}$-representable. Let also g be monotone:

$$z \leq z' \Rightarrow g(z) \leq g(z').$$

Then the superposition

$$h(x) = g(f_1(x), ..., f_k(x))$$

is a $\mathcal{K}$-r.f.

$\mathcal{K}$-representable functions (cont.)

(v). [Legendre transformation] The Legendre transformation

$$f_*(\eta) = \sup_x [\eta^T x - f(x)]$$

of a $\mathcal{K}$-r.f. f *with strictly feasible $\mathcal{K}$-r.* **is itself a $\mathcal{K}$-r.f.**

Example: $f(x) = \|Ax\|_2^2 + \|Bx - c\|_2^4$

$f(x) = \|Ax\|_2^2 + \|Bx - c\|_2^4$	$f_*(\xi) = \sup_x \left[\xi^T x - f(x)\right] = ???$
$t \geq f(x)$ $\Updownarrow$ $\exists s_1, s_2, s_3 :$ $\|Ax\|_2 \leq s_1$ $\|Bx - c\|_2 \leq s_2$ $\left\\|\begin{pmatrix} 2s_2 \\ 1 - s_3 \end{pmatrix}\right\\|_2 \leq 1 + s_3$ $\left\\|\begin{pmatrix} 2s_1 \\ 2s_3 \\ 1 - t \end{pmatrix}\right\\|_2 \leq 1 + t$	$\tau \geq f_*(\xi)$ $\Updownarrow$ $\exists \eta_1, \eta_2, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 :$ $\|\eta_1\|_2 \leq \sigma_1$ $\|\eta_2\|_2 \leq \sigma_2$ $\left\\|\begin{pmatrix} \sigma_2 \\ \sigma_3 \end{pmatrix}\right\\|_2 \leq \sigma_4$ $\left\\|\begin{pmatrix} 2\sigma_1 \\ \sigma_3 + \sigma_4 \\ 4 - 2\sigma_5 \end{pmatrix}\right\\|_2 \leq 2\sigma_5$ $2c^T \eta_2 - \sigma_3 + \sigma_4 + 2\sigma_5 - 2 \leq 2\tau$

What can be expressed via LP?

$$\boxed{\mathcal{K} = \mathcal{LP} \equiv \{\mathbf{R}^n_+\}_{n=1}^{\infty}}$$

♣ **Of course,**

♠ $\mathcal{LP}$**-representable sets are exactly the polyhedral sets,**

♠ $\mathcal{LP}$**-representable functions are exactly convex piecewise linear functions.**

E.g., the function

$$f(x_1, ..., x_n) = \prod_{i=1}^{n} x_i^{-\frac{i}{n}} : \mathrm{int}\, \mathbf{R}^n_+ \to \mathbf{R}$$

is not $\mathcal{LP}$**-representable. But...**

What can be expressed via CQP?

$$\mathcal{K} = \mathcal{CQ} \equiv \left\{\prod_i \mathbf{L}^{n_i}\right\}$$
$$\mathbf{L}^n = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|_2 \leq t\}$$

Elementary $\mathcal{CQ}$-representable functions/sets

♣ **The following functions/sets are $\mathcal{CQ}$-representable with explicit $\mathcal{CQ}$-r.'s:**

1. **An affine function** $f(x) = a^T x + b$
2. **The Euclidean norm** $\|x\|_2 : \mathbf{R}^n \to \mathbf{R}$
3. **The set**

$$\{(x,s,t) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} : s,t \geq 0, x^T x \leq st\}. \qquad (*)$$

4. **The squared Euclidean norm** $f(x) = x^T x : \mathbf{R}^n \to \mathbf{R}$

Note: by Calculus it follows from 4. that every convex quadratic form is $\mathcal{CQ}$-representable.

5. **A simple fractional-quadratic function**

$$f(x,s) = \frac{x^2}{s} : \{(x,s) \in \mathbf{R}^2 \mid s \geq 0\} \to \mathbf{R} \bigcup \{+\infty\}$$

is $\mathcal{CQ}$-representable.

Extension: The set of triples $(\tau \in \mathbf{R}, x \in \mathbf{R}^n, y \in \mathbf{R}^m_+)$ **for which**

$$\begin{pmatrix} \tau & x^T \\ x & A^T \operatorname{Diag}(y) A \end{pmatrix} \succeq 0$$

is CQ-representable.

6. Let $\pi_1, ..., \pi_k$ be positive rationals with $\sum\limits_{i=1}^{k} \pi_i \le 1$. Then the function

$$f(x) = -\prod_{i=1}^{k} x_i^{\pi_i} : \mathbf{R}_+^k \to \mathbf{R}$$

admits an explicit $\mathcal{CQ}$-r. The "size" of the representation is proportional to the magnitude of the common denominator of the fractions $\pi_1, ..., \pi_k$.

<u>Example:</u> A $\mathcal{CQ}$-r. of the set

$$t \ge -x_1^{\frac{1}{7}} x_2^{\frac{2}{7}} x_3^{\frac{3}{7}},\ x \ge 0$$

is

$$\begin{array}{c}
\exists s, u_1, u_2, u_3, u_4 : \\
\begin{pmatrix} 2u_1 \\ x_1 - x_3 \\ x_1 + x_3 \end{pmatrix} \in \mathbf{L}^2 \quad \begin{pmatrix} 2u_2 \\ s - t - 1 \\ s - t + 1 \end{pmatrix} \in \mathbf{L}^2 \\
\begin{pmatrix} 2u_3 \\ u_1 - u_2 \\ u_1 + u_2 \end{pmatrix} \in \mathbf{L}^2 \quad \begin{pmatrix} 2u_4 \\ x_2 - x_3 \\ x_2 + x_3 \end{pmatrix} \in \mathbf{L}^2 \\
\begin{pmatrix} 2s - 2t \\ u_3 - u_4 \\ u_3 + u_4 \end{pmatrix} \in \mathbf{L}^2 \\
s \ge 0
\end{array}$$

7. Let $\pi_1, ..., \pi_k$ be positive rationals. Then the function

$$f(x) = \prod_{i=1}^{k} x_i^{-\pi_i} : \text{int } \mathbf{R}_+^k \to \mathbf{R}$$

admits an explicit $\mathcal{CQ}$-r. The "size" of the representation is proportional to the magnitude of the common denominator of the fractions $\pi_1, ..., \pi_k$.

Example: A $\mathcal{CQ}$-r. of the set

$$t \geq x_1^{-\frac{1}{7}} x_2^{-\frac{2}{7}} x_3^{-\frac{3}{7}}$$

is

$$\exists u_1, u_2, u_3, u_4, u_5, u_6 :$$

$$\begin{pmatrix} 2u_1 \\ x_2 - x_3 \\ x_2 + x_3 \end{pmatrix} \in \mathbf{L}^2 \quad \begin{pmatrix} 2u_2 \\ t - 1 \\ t + 1 \end{pmatrix} \in \mathbf{L}^2 \quad \begin{pmatrix} 2u_3 \\ x_2 - x_3 \\ x_2 + x_3 \end{pmatrix} \in \mathbf{L}^2$$

$$\begin{pmatrix} 2u_4 \\ u_1 - u_2 \\ u_1 + u_2 \end{pmatrix} \in \mathbf{L}^2 \quad \begin{pmatrix} 2u_5 \\ u_3 - u_4 \\ u_3 + u_4 \end{pmatrix} \in \mathbf{L}^2 \quad \begin{pmatrix} 2u_6 \\ u_2 - t \\ u_2 + t \end{pmatrix} \in \mathbf{L}^2$$

$$\begin{pmatrix} 2 \\ u_3 - u_6 \\ u_3 + u_6 \end{pmatrix} \in \mathbf{L}^2$$

Elementary $\mathcal{CQ}$-representable functions/sets (cont.)

8. Let $\pi \geq 1$ be a rational. Then the function

$$f_+(x) = (\max[x, 0])^\pi$$

admits an explicit $\mathcal{CQ}$-r.

So does the function

$$g(x) = |x|^\pi \quad [\equiv f(x) + f(-x)]$$

9. For rational $\pi \geq 1$, the π-norm $f(x) = \|x\|_\pi : \mathbf{R}^n \to \mathbf{R}$ is $\mathcal{CQ}$-representable.

10. The cone of symmetric 3-diagonal positive semidefinite matrices is $\mathcal{CQ}$-representable. In particular, the maximum eigenvalue $\lambda_{\max}(X)$ of a 3-diagonal symmetric matrix X is a $\mathcal{CQ}$-representable function of X.

♣ **There are, of course, nice convex functions which are** *not* $\mathcal{CQ}$**-representable, e.g., the exponent**

$$f(x) = \exp\{x\}.$$

<u>But</u>: for every $p \geq 1$**,**

$$\exp\{x\} = \lim_{r\to\infty} \Big(\underbrace{1 + \frac{x}{2^r} + \frac{1}{2}\left(\frac{x}{2^r}\right)^2 + ... + \frac{1}{p!}\left(\frac{x}{2^r}\right)^p}_{h_p\left(\frac{x}{2^r}\right)}\Big)^{(2^r)}$$

It follows that if p **is such that** $h_p(\cdot)$ **is** $\mathcal{CQ}$**-representable and nonnegative, then** $\exp\{x\}$ **can be** *approximated* **by** $\mathcal{CQ}$**-r.f.**

$$g_{p,r}(x) = h_p^{(2^r)}\left(\frac{x}{2^r}\right).$$

For p **fixed and for a given** *bounded* **range** $|x| \leq T$ **of values of** x**, the quality of this approximation grows rapidly with** r**:**

$$r \geq O\left(\ln\left(\tfrac{1}{\varepsilon}\right) + T\right),\ \ |x| \leq T$$

$$\Downarrow$$

$$(1-\varepsilon)\exp\{x\} \leq g_{p,r}(x) \leq (1+\varepsilon)\exp\{x\}.$$

E.g., the system of 24 conic quadratic constraints with 23 variables:

$$
\boxed{\begin{array}{c}
\exists u_1,...,u_{21}: \\
\begin{pmatrix} 2+\frac{x}{2^{17}} \\ 1-u_1 \\ 1+u_1 \end{pmatrix} \in \mathbf{L}^2 \quad \begin{pmatrix} \frac{5}{3}+\frac{x}{2^{18}} \\ 1-u_2 \\ 1+u_2 \end{pmatrix} \in \mathbf{L}^2 \quad \begin{pmatrix} 2u_1 \\ 1-u_3 \\ 1+u_3 \end{pmatrix} \in \mathbf{L}^2 \\
\frac{19}{72}+u_2+\frac{1}{24}u_3 \leq u_4 \\
\begin{pmatrix} 2u_{\ell-1} \\ 1-u_\ell \\ 1+u_\ell \end{pmatrix} \in \mathbf{L}^2,\ \ \ell=5,...,21 \\
\begin{pmatrix} 2u_{21} \\ 1-t \\ 1+t \end{pmatrix} \in \mathbf{L}^2 \\
-512 \leq x \leq 512
\end{array}} \qquad (*)
$$

is a conic representation of the univariate function

$$g(x) = \min\{t \mid \exists u_1,...,u_{21} : (t,x,u) \textbf{ satisfies } (*)\}.$$

The function g **approximates** $\exp\{x\}$ **in the segment**

$\Delta = \{x \mid |x| \leq 512\} = \{x \mid$ `4.38e-223` $\leq \exp\{x\} \leq$ `2.28e222`$\}$

within relative accuracy `1.e-10`**.**

♠ **For any practical purpose** $\exp\{x\}$ **is** $\mathcal{CQ}$**-representable!**

“Whether Conic Quadratic Programming does exist?”

♣ **Surprisingly, Lorentz cones (and thus – all $\mathcal{CQ}$-representable sets!) are “nearly polyhedrally representable”.**

<u>**Theorem**</u> [**BT-N, ’98**] *For every $n > 0$ and every $\varepsilon \in (0, 1/2)$, there exist (and can be explicitly written down) matrices $P_{n,\varepsilon}$, $Q_{n,\varepsilon}$ and vector $b_{n,\varepsilon}$ such that*

(i) *The row and the column sizes of $P_{n,\varepsilon}, Q_{n,\varepsilon}$ do not exceed $O\left(n \ln \frac{1}{\varepsilon}\right)$*

(ii) *If*

$$(x, t) \in \mathbf{L}^n = \{(x, t) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|_2 \le t\},$$

then there exists a vector u such that

$$P_{n,\varepsilon} x + t b_{n,\varepsilon} + Q_{n,\varepsilon} u \ge 0, \qquad (*)$$

and “nearly vice versa”: if (x, t) can be extended by some u to a solution of $(*)$, then

$$(x, t) \in \mathbf{L}^n_\varepsilon = \{(x, t) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|_2 \le (1 + \varepsilon) t\}.$$

Thus, $\mathbf{L}^n$ “can be approximated within accuracy ε” by a cone $\mathbf{M}^n_\varepsilon$:

$$\mathbf{L}^n \subset \mathbf{N}^n_\varepsilon \subset \mathbf{L}^n_\varepsilon$$

which is the projection onto the (x, t)-space of a “simple” polyhedral cone

$$\mathbf{N}^n_\varepsilon = \{(x, t) \mid \exists u : P_{n,\varepsilon} x + t b_{n,\varepsilon} + Q_{n,\varepsilon} u \ge 0\}.$$

given by just $O\left(n \ln \frac{1}{\varepsilon}\right)$ linear inequalities involving $O\left(n \ln \frac{1}{\varepsilon}\right)$ variables.

Corollary. Consider a conic quadratic program

$$c^T x \to \min \mid Ax+b \geq 0,\ Px+q \in \mathbf{K} = \mathbf{L}^{n_1} \times \ldots \times \mathbf{L}^{n_k} \quad \text{(CQP)}$$

and let the problem be both
r-strictly feasible for some $r > 0$:

$$\exists \bar{x} : A\bar{x} + b \geq 0 \ \&\ \forall(\xi : \|\xi\|_2 \leq r) : P\bar{x} + q + \xi \in \mathbf{K}$$

R-semibounded:

$$Ax + b \geq 0, Px + q \in \mathbf{K} \Rightarrow \|Px + q\|_2 \leq R.$$

For every $\varepsilon \in (0,1)$, there exists an LP program

$$c^T x \to \min \mid Bx + Cv + d \geq 0 \qquad \text{(LP)}$$

which is an ε-relaxation of (CQP) **of polynomial complexity:**

♠ **If x is feasible for** (CQP)**, then x can be extended to a feasible solution (x, v) of** (LP)**;**

♠ **If (x, v) is feasible for** (LP)**, then the vector**

$$(1 - \varepsilon)x + \varepsilon\bar{x}$$

is feasible for (CQP)**.**

♠ **The size** $\dim\, x + \dim\, v + \dim\, d$ **of** (LP) **does not exceed**

$$\dim\, x + \dim\, b + O\left((\dim\, q) \ln \frac{2R}{\varepsilon r}\right)$$

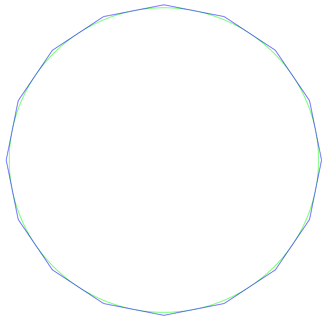
and the data (c, B, C, d) of (LP) is readily given by the data (c, A, b, P, q) of (CQP).

♣ **“Fast polyhedral approximations” of the Lorentz cone are based on the fact that if $\mathbf{P}^\nu$ is the polytope defined as**

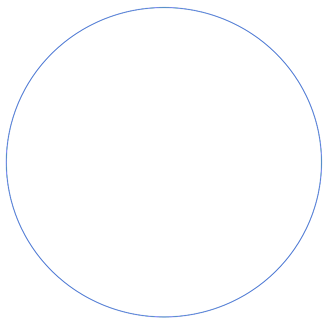
$$\left\{(x,u)\in\mathbf{R}^2\times\mathbf{R}^{2\nu+4}:\left\{\begin{array}{rcl} u_0 & \geq & |x_1|;\\ u_1 & \geq & |x_2|;\\ \\ u_{2j} & = & \cos\left(\frac{\pi}{2^{j+1}}\right)u_{2j-2}\\ & & +\sin\left(\frac{\pi}{2^{j+1}}\right)u_{2j-1},\\ & & \qquad j=1,...,\nu;\\ u_{2j+1} & \geq & \Big|-\sin\left(\frac{\pi}{2^{j+1}}\right)u_{2j-2}\\ & & +\cos\left(\frac{\pi}{2^{j+1}}\right)u_{2j-1}\Big|,\\ & & \qquad j=1,...,\nu;\\ \\ u_{2\nu+3} & \leq & 1;\\ u_{2\nu+4} & \leq & \tan\left(\frac{\pi}{2^{\nu+1}}\right)u_{2\nu+3} \end{array}\right.\right\}$$

then the projection $\mathbf{P}^\nu_x$ of the polytope on the x-plane approximates the unit 2D disk $\mathbf{D}_2$ within accuracy

$$\frac{1}{\cos\left(\frac{\pi}{2^{\nu+1}}\right)}-1\approx\frac{\pi^2}{2^{2\nu+3}}.$$



♠ P^3 "lives" in R^9 and is given by 12 linear inequalities.
P^3_x approximates D_2 within accuracy `5.e-3`
(as good as the 16-side perfect polygon)



♠ P^6 "lives" in R^{12} and is given by 18 linear inequalities.
P^6_x approximates D_2 within accuracy `3.e-4`
(as good as the 127-side perfect polygon)

♠ P^{12} "lives" in R^{18} and is given by 30 linear inequalities.
P^{12}_x approximates D_2 within accuracy `7.e-8`
(as good as the 8,192-side perfect polygon)

♠ P^{24} "lives" in R^{30} and is given by 54 linear inequalities.
P^{24}_x approximates D_2 within accuracy `4.e-15`
(as good as the 34,200,933-side perfect polygon)

What can be expressed via SDP?

$$\mathcal{K} = \mathcal{SD} \equiv \left\{ \begin{array}{l} \textbf{Direct products of the cones of} \\ \textbf{positive semidefinite matrices} \end{array} \right\}$$

Examples of $\mathcal{SD}$-representable functions/sets

1-10. Every $\mathcal{CQ}$-r. function/set is $\mathcal{SD}$-r. as well.

Indeed, the Lorentz cone $\mathbf{L}^n$ is $\mathcal{SD}$-r.:

$$(x,t) \in \mathbf{L}^n$$

$$\Updownarrow$$

$$\Theta_n(x) \equiv \begin{pmatrix} t & x_1 & x_2 & \ldots & x_n \\ x_1 & t & & & \\ x_2 & & t & & \\ \vdots & & & \ddots & \\ x_n & & & & t \end{pmatrix} \succeq 0$$

Consequently, a $\mathcal{CQ}$-r. of a set X:

$$X = \{x \mid \exists u : P_i x + Q_i u + b_i \in \mathbf{L}^{n_i},\ i = 1, ..., k\}$$

yields an $\mathcal{SD}$-r. of X:

$$X = \{x \mid \exists u : \Theta_{n_i}(P_i x + Q_i u + b_i) \succeq 0,\ i = 1, ..., k\}.$$

Elementary $\mathcal{SD}$-representable functions/sets (cont.)

Functions of eigenvalues of symmetric matrices, I

♣ **For a symmetric $n \times n$ matrix X, let**

$$\lambda_1(X) \geq \lambda_2(X) \geq ... \geq \lambda_n(X)$$

be the eigenvalues of X taken with their multiplicities in the non-ascending order.

11. Maximum eigenvalue $\lambda_{\max}(X)$ of a symmetric matrix X is $\mathcal{SD}$-representable:

$$t \geq \lambda_{\max}(X) \Leftrightarrow tI - X \succeq 0.$$

12. The sum

$$S_k(X) = \lambda_1(X) + ... + \lambda_k(X)$$

of k largest eigenvalues of a symmetric matrix X is $\mathcal{SD}$-representable:

$$S_k(X) \leq t \Leftrightarrow \exists s, Z : \begin{cases} t - ks - \mathrm{Tr}(Z) \geq 0 \\ Z \succeq 0 \\ Z - X + sI \succeq 0 \end{cases}$$

Elementary $\mathcal{SD}$-representable functions/sets (cont.)
Functions of eigenvalues of symmetric matrices, II

13. Let a function f on $\mathbf{R}^n$ $\mathcal{SD}$-representable and symmetric (i.e., invariant w.r.t. permutations of coordinates). Then the function

$$F(X) = f(\lambda(X))$$

of a symmetric $n\times n$ matrix X also is $\mathcal{SD}$-representable.

E.g., the following functions of a symmetric $n \times n$ matrix X are $\mathcal{SD}$-representable explicit $\mathcal{SD}$-r.'s:

- $-(\mathrm{Det}\ X)^{\pi}$, $X \succeq 0$, $0 < \pi \leq \frac{1}{n}$ **is rational**
  $[f(x) = -(\prod_{i=1}^{n} x_i)^{\pi},\ x \geq 0]$
- $\|X\|_{\pi} \equiv \|\lambda(X)\|_{\pi}$, $\pi \geq 1$ **is rational**
  $[f(x) = \|x\|_{\pi}]$
- $\|X_+\|_{\pi} \equiv \|\{\max[\lambda_i(X), 0]\}_{i=1}^{n}\|_{\pi}$, $\pi \geq 1$ **is rational**
  $[f(x) = \|\{\max[x_i, 0]\}_{i=1}^{n}\|_{\pi}]$

Elementary $\mathcal{SD}$-representable functions/sets (cont.)
Functions of singular values of rectangular matrices

♣ **For a $n \times m$ matrix X $[n \leq m]$, let**

$$\sigma(X) = \lambda\left(\sqrt{XX^T}\right) \in \mathbf{R}^n$$

be the vector of singular values of X taken in the non-ascending order.

14. Let a function f on $\mathbf{R}^n_+$ be symmetric, monotone and $\mathcal{SD}$-representable. Then the function

$$F(X) = f(\sigma(X)) : \mathbf{R}^{n\times m} \to \mathbf{R} \cup \{+\infty\}$$

is $\mathcal{SD}$-representable.

E.g., the following functions of rectangular $n \times m$ matrix X are $\mathcal{SD}$-representable with explicit $\mathcal{SD}$-r.'s:

- $\|X\| = \sigma_{\max}(X)$ $\left[\|X\| \leq t \Leftrightarrow \begin{pmatrix} tI_n & X \\ X^T & tI_m \end{pmatrix} \succeq 0\right]$
- $\Sigma_k(X) = \sum\limits_{i=1}^{k} \sigma_i(X)$, $1 \leq k \leq n$
- $\|X\|_\pi = \|\sigma(X)\|_\pi$, $\pi \geq 1$ **is rational**

Convex hulls of some sets of matrices

15. Let $S \subset \{x \in \mathbf{R}^n \mid x_1 \geq x_2 \geq ... \geq x_n\}$ be an $\mathcal{SD}$-representable set.

♠ **The convex hull of the set of symmetric matrices X such that $\lambda(X) \in S$ is $\mathcal{SD}$-representable.**

♠ **Let, in addition, S be monotone:**

$$(y \geq 0, y_1 \geq y_2 \geq ... \geq y_n, \exists x \in S : y \leq x) \Rightarrow y \in S.$$

Then the convex hull of the set of rectangular $n \times m$ matrices X such that $\sigma(X) \in S$ is $\mathcal{SD}$-representable.

Elementary $\mathcal{SD}$-representable functions/sets (cont.)

"$\succeq$-epigraphs" of matrix-valued functions, I

♣ **The following sets are $\mathcal{SD}$-representable:**

16. The set of solutions of the "$\succeq$-convex quadratic matrix inequality"

$$(AXB)(AXB)^T + CXD + (CXD)^T + E \preceq Y$$

in variable matrices $X \in \mathbf{R}^{k\times\ell}$ and $Y \in \mathbf{S}^m$;

17. The set of solutions of the "fractional-quadratic matrix inequality"

$$F(X,Y) \equiv XY^{-1}X^T \preceq Z$$

in variable matrices $X \in \mathbf{R}^{n\times m}$, $0 \preceq Y \in \mathbf{S}^m$, $Z \in \mathbf{S}^n$;

18. The set of solutions of the "double fractional" matrix inequality

$$Y \preceq (A^T X^{-1} A)^{-1}$$

in symmetric matrices $Y \in \mathbf{S}^n$ and $0 \preceq X \in \mathbf{S}^m$;

19. The "$\succeq$-hypograph" of the matrix square root, i.e., the set of solutions of the matrix inequality

$$Y \preceq X^{1/2}$$

in symmetric $n \times n$ matrices $X \succeq 0$ and Y.

Note: Solution sets of matrix inequalities

$$\begin{array}{ll} -X^{1/2} \preceq Y \preceq X^{1/2} & (a) \\ Y^2 \preceq X & (b) \end{array}$$

differ from each other! (And both are $\mathcal{SD}$-r....)

Elementary $\mathcal{SD}$-representable functions/sets (cont.)
Univariate polynomials, I

♣ **Let Δ be a convex subset of the real axis and k be a positive integer.**

20. [Yu. Nesterov, '97] The following sets are $\mathcal{SD}$-representable with explicit $\mathcal{SD}$-representations:

♠ **The set of coefficients of all nonnegative on Δ algebraic polynomials**

$$p(x) = \sum_{\ell=0}^{k} p_\ell x^\ell$$

of degree $\leq k$;

> **Example: The cone of all nonnegative on the entire axis algebraic polynomials of degree not exceeding $2m$ is the image of $\mathbf{S}_+^{m+1}$ under the linear mapping**
>
> $$Z \mapsto p_Z(x) \equiv \sum_{i,j=1}^{m+1} Z_{ij} x^{i+j-2}.$$

♠ **The set of coefficients of all nonnegative on Δ trigonometric polynomials**

$$p(x) = p_0 + \sum_{\ell=1}^{k} [p_{2\ell-1}\cos(\ell x) + p_{2\ell}\sin(\ell x)]$$

of degree $\leq k$.

Elementary $\mathcal{SD}$-representable functions/sets (cont.)
Univariate polynomials, II

Corollary of 20: The functions

$$\begin{array}{rcl} f(p_0,p_1,...,p_k) & = & \sup\limits_{x\in\Delta} \sum\limits_{\ell=1}^{k} p_\ell x^\ell \\ g(p_0,p_1,...,p_{2k}) & = & \sup\limits_{x\in\Delta} \left[p_0 + \sum\limits_{\ell=1}^{k} [p_{2\ell-1}\cos(\ell x) + p_{2\ell}\sin(\ell x)]\right] \end{array}$$

are $\mathcal{SD}$-representable.

21. Let $p(x) = \sum\limits_{\ell=0}^{k} p_\ell x^\ell$ be an algebraic polynomial. The closed convex hull of the epigraph of $p\Big|_\Delta$ is $\mathcal{SD}$-representable.

In particular, if p is convex on Δ, then the epigraph of $p\Big|_\Delta$ is $\mathcal{SD}$-representable.

Elementary $\mathcal{SD}$-representable functions/sets (cont.)
Families of ellipsoids

♣ **Consider two parameterizations of ellipsoids in $\mathbf{R}^n$**

$V(X,x) = \{u \in \mathbf{R}^n \mid u = Xv + x, v^Tv \leq 1\}$	$[X \in \mathbf{R}^{n\times n}]$
"ellipsoid is an affine image of Euclidean ball"	
$W(Y,y) = \{u \in \mathbf{R}^n \mid u^TY^TYu - 2u^TY^Ty + y^Ty \leq 1\}$	$[Y \in \mathbf{R}^{n\times n}]$
"ellipsoid is a level set of a below bounded quadratic function"	

Proposition [Boyd et al.] $V(X,x) \subset W(Y,y)$ **if and only if there exists** λ **such that**

$$\begin{pmatrix} I & Yx-y & YX \\ x^TY^T - y^T & 1-\lambda & \\ X^TY^T & & \lambda I \end{pmatrix} \succeq 0 \qquad (*)$$

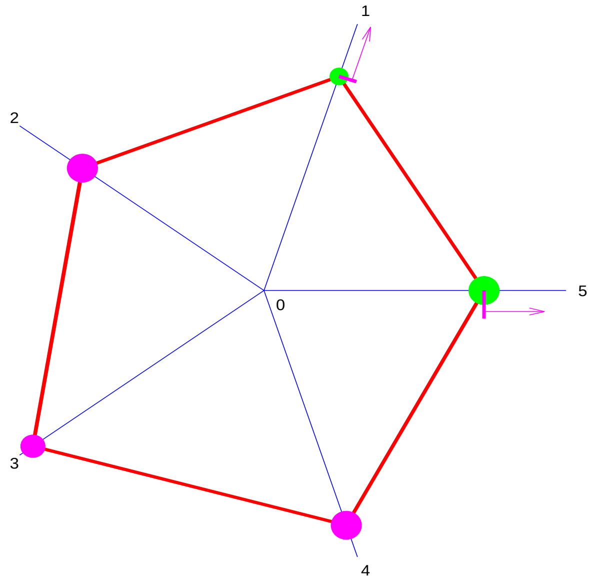
Observation: When one of the ellipsoids is fixed, $(*)$ **is a linear matrix inequality in** λ **and in the (parameters of) the other ellipsoid.**
Thus, the set of (parameters of) all ellipsoids contained in/containing a fixed ellipsoid is $\mathcal{SD}$**-r.**

Corollary. The problems of

(1) finding the *largest volume* **ellipsoid** *contained in the intersection* **of finitely many given ellipsoids,**

(2) finding the *smallest volume* **ellipsoid** *containing the union* **of finitely many given ellipsoids**

can be posed as semidefinite programs.

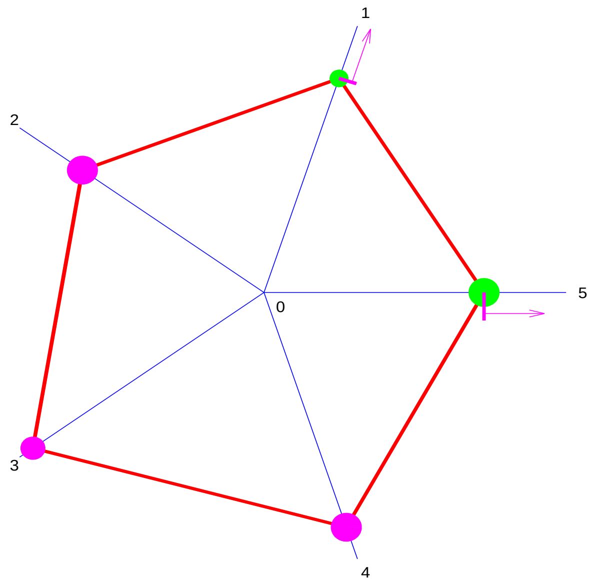


• **<u>Plant:</u> 5 material points linked by elastic springs and sliding without friction along given axes**
• **<u>Input:</u> the forces u_i applied at "green" points**
• **<u>Output:</u> velocities $\dot{x}_i$ of the points**
• **<u>Control law:</u> linear feedback which brings the nominal plant to the origin while minimizing the cost**

$$\int_0^\infty \left[\sum_{i=1}^{5} (\dot{x}_i)^2(t) + u_1^2(t) + u_5^2(t)\right] dt,$$

• **<u>Uncertainty:</u> ρ%-perturbations in rigidities of springs, masses of points, and coefficients of the feedback matrix**

♣ *What is the largest level of (dynamical) perturbations $\rho^\star$ preserving the stability of the closed loop system?*



♣ **The computable upper bound on $\rho^\star$ turns out to be 4.1%. Thus, one can be sure that the closed loop system remains stable at least up to 4.1% perturbations in the parameters of the plant and the coefficients in the feedback matrix.**

♣ **Simulation demonstrates that there exist dynamical 6.5% perturbations which make the closed loop system unstable.**

Thus, we have specified the stability margin of the system up to factor $\frac{6.5}{4.1} \approx 1.6$.