On detecting harmonic oscillations

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In this paper, we focus on the following testing problem: assume that we are given observations of a real-valued signal along the grid $0, 1, \ldots, N - 1$, corrupted by white Gaussian noise. We want to distinguish between two hypotheses: (a) the signal is a *nuisance* – a linear combination of d_n harmonic oscillations of known frequencies, and (b) signal is the sum of a nuisance and a linear combination of a given number d_s of harmonic oscillations with *unknown* frequencies, and such that the distance (measured in the uniform norm on the grid) between the signal and the set of nuisances is at least $\rho > 0$. We propose a computationally efficient test for distinguishing between (a) and (b) and show that its "resolution" (the smallest value of ρ for which (a) and (b) are distinguished with a given confidence $1 - \alpha$) is $O(\sqrt{\ln(N/\alpha)/N})$, with the hidden factor depending solely on d_n and d_s and independent of the frequencies in question. We show that this resolution, up to a factor which is polynomial in d_n, d_s and *logarithmic in N*, is the best possible under circumstances. We further extend the outlined results to the case of nuisances and signals *close* to linear combinations of harmonic oscillations, and provide illustrative numerical results.

Keywords: detection by convex optimization; detection in the presence of nuisance; harmonic oscillations detection; multiple hypothesis testing

1. Introduction

In this paper, we address the following detection problem. A signal – a two-sided sequence of reals $x = \{x_t, t = 0, \pm 1, \pm 2, \ldots\}$ – is observed on the time horizon $0, 1, \ldots, N - 1$ according to

$$y = x_0^{N-1} + \xi,$$

where $\xi \sim \mathcal{N}(0, I_N)$ is the white Gaussian noise and $z_0^{N-1} = [z_0; \ldots; z_{N-1}]$. Given y we want to distinguish between two hypotheses:

- *Nuisance hypothesis*: $x \in H_0$, where H_0 is comprised of all *nuisances* – linear combinations of d_n harmonic oscillations of known frequencies.
- *Signal hypothesis*: $x \in H_1(\rho)$, where $H_1(\rho)$ is the set of all sequences x representable as $s + u$ with the "nuisance component" u belonging to H_0 and the "signal component" s being a sum of at most d_s harmonic oscillations (of whatever frequencies) such that the *uniform* distance, on the time horizon in question, from x to all nuisance signals is at least ρ:

$$\min_{z \in H_0} \|x_0^{N-1} - z_0^{N-1}\|_\infty \geq \rho.$$

We are interested in a test which allows to distinguish, with a given confidence $1 - \alpha$, between the above two hypotheses for as small "resolution" ρ as possible.

An approach to this problem which is generally advocated in the signal processing literature is based on the generalized likelihood ratio test [3,7,10,11]. This test seems to enjoy the optimal detection performance in the problem of distinguishing between the "pure noise" hypothesis H_0 (no nuisance is present) and the "signal hypothesis," H_1, that a signal which is a sum of d_s sinusoids is present. For instance, under certain assumptions on the signal frequencies, this test is claimed [11] to distinguish $1 - \alpha$-reliably[1] between H_0 and H_1 if the ℓ_2-norm of the signal is $\geq c\sqrt{d_s \ln(N/\alpha)}$. However, implementation of this detector requires computing minimal log-likelihood – the global optimal value of the optimization problem

$$\min_{a,\omega,\phi} \sum_{k=0}^{N-1} \left(y_k - \sum_{j=1}^{d_s} a_j \sin(\omega_j k + \phi_j) \right)^2$$

(here the minimum is taken with respect to all problem parameters (a_j, ω_j, ϕ_j), $j = 1, \ldots, d_s$), and becomes numerically challenging already for very moderate problem dimension d_s. To circumvent numerical problems, under certain assumptions on the signal frequencies (i.e., they are not "too close to each other") the test can be applied "sequentially" [11], when "fitting one frequency at a time." A second family of tests, which does not require estimation of unknown frequencies and amplitudes, relies upon noise subspace methods, such as multiple signal classification (MUSIC) [14,18] (see also [15,19] for detailed presentation of these techniques). To the best of our knowledge, no theoretical bounds for the resolution of such tests are available. A different test for the case when no nuisance is present, based on the normalized periodogram, has been proposed in [4]. The properties of this test and of its various modifications were extensively studied in the statistical literature (see, e.g., [1,6,15–17,20]). However, few theoretical results on the power of this test are available for the case of sequence x not being a linear combination of Fourier harmonics $e^{2\pi i kt/N}$, $k = 0, 1, \ldots, N - 1$ under signal hypothesis. For instance, in the paper [2], brought to our attention by the referee, the properties of the periodogram test are analysed in the problem of detection of one sinusoid, where it is shown that asymptotically (when $N \to \infty$, the reliability of the test is not "too high," and the signal frequency is "not too close" to 0 or 1 but is otherwise arbitrary) the sensitivity of the test can be improved by the factor up to $\frac{\pi}{2}$, if maximization of the periodogram is carried over all frequencies in $[0, 1]$ instead of the set of "Fourier frequencies" $\frac{k}{N}$, $k = 0, \ldots, N - 1$.

In this paper, we show that a good solution to the outlined problem is offered by an extremely simple test as follows.

Let $F_N u = \{\frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} u_t \exp\{2\pi i kt/N\}\}_{k=0}^{N-1} : \mathbf{C}^N \to \mathbf{C}^N$ be the Discrete Fourier Transform. Given the observation y, we solve the convex optimization problem

$$\text{Opt}(y) = \min_z \{ \| F_N(y - z_0^{N-1}) \|_\infty : z \in H_0 \}$$

and compare the optimal value with a threshold $q_N(\alpha)$ which is a valid upper bound on the $1 - \alpha$-quantile of $\| F_N \xi \|_\infty$, $\alpha \in (0, 1)$ being a given tolerance:

$$\text{Prob}_{\xi \sim \mathcal{N}(0, I_N)} \{ \| F_N \xi \|_\infty > q_N(\alpha) \} \leq \alpha.$$

If $\text{Opt}(y) \leq q_N(\alpha)$, we accept the nuisance hypothesis, otherwise we claim that a signal is present.

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[1] We say that a test distinguishes $1 - \alpha$-reliably between the hypotheses, say, H_0 and H_1 if its risk which is, in our case, the maximal probability of rejecting the hypothesis when it is true, is bounded by α.

It is immediately seen that the outlined test rejects the nuisance hypothesis when it is true with probability at most α.[2] Our main result (Theorem 3.1) states that *the probability to reject the signal hypothesis when it is true is $\leq \alpha$, provided that the resolution ρ is not too small*, specifically, for an appropriately chosen universal function $\mathcal{C}(\cdot)$,

$$\rho \geq \mathcal{C}(d_n + d_s)\sqrt{\ln(N/\alpha)/N}. \tag{!}$$

In the simplest case when the zero hypothesis just states that $x = 0$, the above test is similar to the standard tests based on the maximum of the periodogram; when H_0 is nontrivial, our test can be seen as a natural modification of the classical construction.

Some comments are in order.

A. Our principal contributions, as we see them, are in
    – deriving theoretical upper (and close to them lower) bounds on the resolution of the test under quite general assumptions on nuisances and signals of interest; it should be stressed that these bounds depend solely on the cardinalities of the sets of "participating" frequencies. Moreover, we allow for "multiplicities of frequencies," so that our "sum of harmonic oscillations" can be an algebraic polynomial or a sum of products of algebraic polynomials and harmonic oscillations. We are not aware of comparable, in terms of generality, existing theoretical results on the performance of spectral analysis tests;
    – demonstrating that in order to achieve nearly optimal resolution, one can restrict the periodogram to the frequencies participating in the Discrete Fourier Transform (the traditional recommendation is to consider denser frequency sets).
B. We show that the power of our test when applied to the detection problem in question is nearly as good as it can be: precisely, for every pair d_n, d_s and properly selected H_0, no test can distinguish $(1 - \alpha)$-reliably between H_0 and $H_1(\rho)$ when $\rho < O(1)d_s\sqrt{\ln(1/\alpha)/N}$. Here and from now on, $O(1)$'s are appropriately chosen positive absolute constants.
C. We are measuring the resolution in the "weakest" of all natural scales, namely, via the *uniform* distance from the signal to the set of nuisances. When passing from the uniform norm to the normalized Euclidean norm $|x_0^{N-1}|_2 := \|x_0^{N-1}\|_2/\sqrt{N} \leq \|x_0^{N-1}\|_\infty$, an immediate lower bound on the resolution which allows for reliable detection becomes $O(1)\sqrt{\ln(1/\alpha)/N}$. In the case when, as in our setting, signals obeying H_0 and $H_1(\rho)$ admit parametric description involving K parameters, this lower bound, up to a factor logarithmic in N and linear in K, is also an upper resolution bound, and the associated test is based on estimating the Euclidean distance from the signal underlying the observations to the nuisance set. Note that, in general, the $|\cdot|_2$-norm can be smaller than $\|\cdot\|_\infty$ by a factor as large as $\sqrt{N}$, and the fact that "energy-based" detection allows to distinguish well between parametric hypotheses "separated" by $O(\sqrt{\ln(N/\alpha)/N})$ in $|\cdot|_2$ norm does *not* automatically imply the possibility to distinguish between hypotheses separated by $O(\sqrt{\ln(N/\alpha)/N})$ in the uniform norm.[3] The latter possibility exists in the situation we are interested in due to

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[2]This fact is completely independent of what the nuisance hypothesis is – it remains true when H_0 is an arbitrary set in the space of signals.

[3]Indeed, let H_0 state that the signal is 0, and $H_1(\rho)$ state that the signal is $\geq \rho$ at $t = 0$ and is zero for all other t's. These two hypotheses cannot be reliably distinguished unless $\rho \geq O(1)$, that is, the $\|\cdot\|_\infty$ resolution in this case is much larger than $O(\sqrt{\ln(N/\alpha)/N})$.

the particular structure of the specific nuisance and signal hypotheses; this structure allows also for a dedicated *non-energy-based* test.

D. For the sake of definiteness, throughout the paper we assume that the observation noise is the standard white Gaussian one. This assumption is by no means critical: on a straightforward inspection of what follows, *whatever be the observation noise, with $q_N(\alpha)$ defined as (an upper bound on) the $(1-\alpha)$-quantile of $\|F_N\xi\|_\infty$, the above test $(1-\alpha)$-reliably distinguishes between the hypotheses H_0 and $H_1(\rho)$, provided that $\rho \geq \mathcal{C}(d_n + d_s)q_N(\alpha)/\sqrt{N}$.* For example, the results of Theorems 3.1 and 3.2 remain valid when the observation noise is of the form $\xi = \{\xi_t = \sum_{\tau=-\infty}^{\infty} \gamma_\tau \eta_{t-\tau}\}_{t=0}^{N-1}$ with deterministic γ_τ, $\sum_\tau |\gamma_\tau| \leq 1$, and independent $\eta_t \sim \mathcal{N}(0, 1)$.

E. The main observation underlying the results on the resolution of the above test is as follows: *when x is the sum of at most d harmonic oscillations, $\|F_N x\|_\infty \geq \mathcal{C}(d)\sqrt{N}\|x_0^{N-1}\|_\infty$ with some universal positive function $\mathcal{C}(d)$.*[4] This observation originates from [12] and, along with its modifications and extensions, was utilized, for the time being in the denoising setting, in [5,8,9,13]. It is worth mentioning that it also allows to extend, albeit with degraded constants, the results of Theorems 3.1 and 3.2 to multi-dimensional setting.

The rest of this paper is organized as follows. In Section 2, we give a detailed description of the detection problems (P_1), (P_2), (N_1) and (N_2), we are interested in (where (P_2) is the problem we have discussed so far). Our test is presented in Section 3 where we also provide associated resolution bounds for these problems. Next, in Section 4, we present lower bounds on "good" (allowing for $(1-\alpha)$-reliable hypotheses testing) resolutions, while in Section 5 we describe some numerical illustrations. The proofs of results of Sections 3 and 4 are put into Section 6.

2. Problem description

Let $\mathcal{S}$ stand for the space of all two-sided real sequences $z = \{z_t \in \mathbf{R}\}_{t=-\infty}^\infty$. Assume that a discrete time signal $x \in \mathcal{S}$ is observed on the time horizon $0 \leq t < N$ according to

$$y = x_0^{N-1} + \xi, \qquad \xi \sim \mathcal{N}(0, I_N), \tag{1}$$

where (and from now on) for $z \in \mathcal{S}$ and integers $p \leq q$, z_p^q stands for the vector $[z_p; z_{p+1}; \ldots; z_q]$.

In the sequel, we are interested in the case when the signal is a linear combination of a given number of harmonic oscillations. Specifically, let Δ stand for the shift operator on $\mathcal{S}$:

$$(\Delta z)_t = z_{t-1}, \qquad z \in \mathcal{S}.$$

Let Ω_d be the set of all unordered collections $\mathbf{w} = \{\omega_1, \ldots, \omega_d\}$ of d reals which are "symmetric mod 2π," meaning that for every a, the number of ω_i's equal, modulus 2π, to a is exactly the same as the number of ω_i's equal, modulus 2π, to $-a$. We associate with $\mathbf{w} \in \Omega_d$ the real

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[4]Since $\|x\|_2 = \|F_N x\|_2 \geq \|F_N x\|_\infty$, it follows that for the aforementioned x one has $\mathcal{C}(d)^{-1}\|x\|_\infty \leq N^{-1/2}\|x\|_2 \leq \|x\|_\infty$ – an important by its own right fact which we were unable to find in mathematical literature.

algebraic polynomial

$$p_{\mathbf{w}}(\zeta) = \prod_{\ell=1}^{d} \left(1 - \exp\{\iota\omega_\ell\}\zeta\right)$$

and the subspace $\mathcal{S}[\mathbf{w}]$ of $\mathcal{S}$, comprised of $x \in \mathcal{S}$ satisfying the homogeneous finite-difference equation

$$p_{\mathbf{w}}(\Delta)x \equiv 0. \tag{2}$$

In other words, $\mathcal{S}[\{\omega_1, \ldots, \omega_d\}]$ is comprised of all real two-sided sequences of the form

$$x_t = \sum_{\ell=1}^{d} \left[p_\ell(t)\cos(\omega_\ell t) + q_\ell(t)\sin(\omega_\ell t) \right]$$

with real algebraic polynomials $p_\ell(\cdot)$, $q_\ell(\cdot)$ of degree $< m_\ell$, where m_ℓ is the multiplicity, $\mod 2\pi$, of ω_ℓ in $\mathbf{w}$. We set

$$\mathcal{S}_d = \bigcup_{\mathbf{w} \in \Omega_d} \mathcal{S}[\mathbf{w}].$$

Remark 2.1. In what follows, we refer to the reals ω_i constituting $\mathbf{w} \in \Omega_d$ as the *frequencies* of a signal from $\mathcal{S}[\mathbf{w}]$. A reader would keep in mind that the number of "actual frequencies" in such a signal can be less than d: for instance, frequencies in $\mathbf{w}$ different from $0 \mod 2\pi$ and $\pi \mod 2\pi$ go in "symmetric pairs" $(\omega, \omega' = -\omega \mod 2\pi)$, and such a pair gives rise to a single "actual frequency."

Given a positive integer N, real $\epsilon \geq 0$, and $\mathbf{w} \in \Omega_d$, we set

$$\mathcal{S}^{N,\epsilon}[\mathbf{w}] = \left\{ x \in \mathcal{S} : \left\| \left[p_{\mathbf{w}}(\Delta)x \right]_0^{N-1} \right\|_\infty \leq \epsilon \right\}.$$

Finally, we denote by $\mathcal{S}_d^{N,\epsilon}$ the set

$$\mathcal{S}_d^{N,\epsilon} = \bigcup_{\mathbf{w} \in \Omega_d} \mathcal{S}^{N,\epsilon}[\mathbf{w}].$$

When N is clear from the context, we shorten the notations $\mathcal{S}^{N,\epsilon}[\mathbf{w}]$, $\mathcal{S}_d^{N,\epsilon}$ to $\mathcal{S}^\epsilon[\mathbf{w}]$ and $\mathcal{S}_d^\epsilon$, respectively.

In the definitions above, it was tacitly assumed that d is a positive integer. It makes sense to allow also for the case of $d = 0$. By definition, Ω_0 is comprised of the empty collection $\mathbf{w} = \varnothing$ and $p_\varnothing(\zeta) \equiv 1$. With this convention, $\mathcal{S}^{N,\epsilon}[\varnothing] = \{ x \in \mathcal{S} : \|x_0^{N-1}\|_\infty \leq \epsilon \}$.

Observe that the family $\mathcal{S}_d^{N,\epsilon}$, $d \geq 2$, is quite rich. For instance, it contains "smoothly varying signals" (case of $w_i = 0 \mod 2\pi$), along with "fast varying" – amplitude-modulated and frequency-modulated signals (see [5,8] for more examples).

We detail now the hypothesis testing problems about the sequence x via observation y given by (1). In what follows d_s and d_n are given positive integers, and ρ, ϵ_n, ϵ_s are given positive reals.

(P_1) The "basic" hypothesis testing problem we consider is that of testing of a simple nuisance hypothesis $\{x = 0\}$ against the alternative that a signal $x \in \mathcal{S}_{d_s}$ "is present," meaning that the uniform norm of the signal on the observation window $[0, \ldots, N-1]$ exceeds certain threshold $\rho > 0$. In other words, we consider the following set of hypotheses:

$$H_0 = \{x = 0\},$$
$$H_1(\rho) = \left\{x \in \mathcal{S}_{d_s} : \left\|x_0^{N-1}\right\|_\infty \geq \rho\right\}.$$

(P_2) We suppose that $x \in \mathcal{S}$ decomposes into "signal" and "nuisance":

$$x = s + u,$$

where s is the signal of interest and a *nuisance* u belongs to a subspace $\mathcal{S}[\overline{\mathbf{w}}]$, assumed to be known a priori. We consider a composite nuisance hypothesis that x is a "pure nuisance," and the alternative (signal hypothesis) that useful signal s does not vanish, and the deviation, when measured in the uniform norm on the observation window, of "signal + nuisance" from the nuisance subspace is at least $\rho > 0$. Thus, we arrive at the testing problem: given $\overline{\mathbf{w}} \in \Omega_{d_n}$ decide between the hypotheses

$$H_0 = \left\{x = u \in \mathcal{S}[\overline{\mathbf{w}}]\right\},$$
$$H_1(\rho) = \left\{x = u + s : u \in \mathcal{S}[\overline{\mathbf{w}}], s \in \mathcal{S}_{d_s}, \right.$$
$$\left. \text{such that } \min_z \left\{\left\|[x - z]_0^{N-1}\right\|_\infty : z \in \mathcal{S}[\overline{\mathbf{w}}]\right\} \geq \rho\right\}.$$

Clearly, problem (P_1) is a particular case of (P_2) with $d_n = 0$ (and thus $\mathcal{S}_{d_n} = \{0\}$ is a singleton).

(N_1) Given $\epsilon_n > 0$ and $\overline{\mathbf{w}} \in \Omega_{d_n}$, consider the nonparametric nuisance hypothesis that the nuisance $u \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}]$ with some known $\overline{\mathbf{w}}$. The signal hypothesis is that the useful signal $s \in \mathcal{S}_{d_s}$ is present, and $x = s + u$ deviates from the nuisance set on the observation window by at least $\rho > 0$ in the uniform norm:

$$H_0 = \left\{x = u \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}]\right\},$$
$$H_1(\rho) = \left\{x = u + s : u \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}], s \in \mathcal{S}_{d_s}, \right.$$
$$\left. \text{such that } \min_z \left\{\left\|[x - z]_0^{N-1}\right\|_\infty : z \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}]\right\} \geq \rho\right\}.$$

(N_2) The last decision problem is a natural extension of (N_1): we consider the problem of testing a nonparametric nuisance hypothesis against a nonparametric signal alternative

that the useful signal $s \in \mathcal{S}_{d_s}^{N, \epsilon_s}$ is present:

$$H_0 = \left\{ x = u \in \mathcal{S}^{N, \epsilon_n}[\overline{\mathbf{w}}] \right\},$$

$$H_1(\rho) = \left\{ x = u + s : u \in \mathcal{S}^{N, \epsilon_n}[\overline{\mathbf{w}}], s \in \mathcal{S}_{d_s}^{N, \epsilon_s}, \right.$$

$$\text{and such that } \min_{z} \left\{ \left\| [x - z]_0^{N-1} \right\|_\infty : z \in \mathcal{S}^{N, \epsilon_n}[\overline{\mathbf{w}}] \right\} \geq \rho \right\}.$$

Note that problem (N_1) is a particular case of (N_2) with $\epsilon_s = 0$.

In the sequel, we refer to the sequences obeying H_0 (resp., $H_1 = H_1(\rho)$) as *nuisance* (resp., *signal*) sequences.

Let $\varphi(\cdot)$ be a test, that is, a Borel function on $\mathbf{R}^N$ taking values in $\{0, 1\}$, which receives on input observation (1) (along with parameters describing H_0 and H_1). The event $\{\varphi(y) = 1\}$ corresponds to rejecting the hypothesis H_0, while $\{\varphi(y) = 0\}$ implies that H_1 is rejected. The quality of the test is characterized by the error probabilities – the probabilities of rejecting erroneously each of the hypotheses:

$$\varepsilon_0(\varphi; H_0) = \sup_{x \in H_0} \text{Prob}_x \{\varphi(y) = 1\}, \qquad \varepsilon_1(\varphi; H_1(\rho)) = \sup_{x \in H_1(\rho)} \text{Prob}_x \{\varphi(y) = 0\}.$$

We define the *risk of the test* as

$$\text{Risk}(\varphi, \rho) = \max \left\{ \varepsilon_0(\varphi; H_0), \varepsilon_1(\varphi; H_1(\rho)) \right\}.$$

Let $\alpha \in (0, 1/2)$ be given. In this paper, we address the following question: *for the testing problems above, what is the smallest possible ρ such that one can distinguish $(1 - \alpha)$-reliably between the hypotheses H_0 and $H_1 = H_1(\rho)$ via observation* (1) (i.e., is such that $\text{Risk}(\varphi, \rho) \leq \alpha$). In the sequel, we refer to such ρ as to α-*resolution* in the testing problem in question, and our goal is to find reasonably tight upper and lower bounds on this resolution along with the test underlying the upper bound.

3. Basic test and upper resolution bounds

In this section, we present a simple test which provides some upper bounds on the α-resolutions in problems (P_1)–(N_2).

Let $\Gamma_N = \{\mu_\tau = \exp\{2\pi\iota\tau/N\} : 0 \leq \tau < N - 1\}$ be the set of all roots $\mu \in \mathbf{C}$ of unity of degree N, and let $F_N : \mathbf{C}^N \to \mathbf{C}(\Gamma_N)$ be the normalized Fourier transform:

$$[F_N f](\mu) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} f_t \mu^t, \qquad \mu \in \Gamma_N. \tag{3}$$

Note that (3) can also be seen as a mapping from $\mathcal{S}$ to $\mathbf{C}(\Gamma_N)$.

Given a tolerance $\alpha \in (0, 1/2)$, let $q_N(\alpha)$ be the $(1 - \alpha)$-quantile of $\|F_N \xi\|_\infty$, so that

$$\text{Prob}_{\xi \sim \mathcal{N}(0, I_N)} \left\{ \|F_N \xi\|_\infty \geq q_N(\alpha) \right\} \leq \alpha.$$

Let $Q_{\mathcal{N}}(\alpha)$ be the $(1-\alpha)$-quantile of the standard normal distribution:

$$\frac{1}{\sqrt{2\pi}} \int_{Q_{\mathcal{N}(\alpha)}}^{\infty} e^{-s^2/2} \, ds = \alpha.$$

In the sequel, we use the following immediate bound for $q_N(\cdot)$:[5]

$$
q_N(\alpha) \leq
\begin{cases}
\displaystyle\inf_{0 \leq s \leq 1} \max\left[Q_{\mathcal{N}}\left(\frac{s\alpha}{4}\right), \sqrt{\ln\left(\frac{N-2}{2(1-s)\alpha}\right)}\, \right] & \text{for } N \text{ even,} \\[4mm]
\displaystyle\inf_{0 \leq s \leq 1} \max\left[Q_{\mathcal{N}}\left(\frac{s\alpha}{2}\right), \sqrt{\ln\left(\frac{N-1}{2(1-s)\alpha}\right)}\, \right] & \text{for } N \text{ odd}
\end{cases}
$$

$$\asymp \sqrt{\ln(N/\alpha)}, \tag{4}$$

where $a \asymp b$ means that the ratio a/b is in-between absolute positive constants.

The test we are about to consider (and which we refer to as *basic test* in the sequel) is as follows:

1. Given y, we solve the convex optimization problem

$$\mathrm{Opt}_{\mathcal{Z}}(y) = \min_{z \in \mathcal{Z}} \left\| F_N\left(y - z_0^{N-1}\right) \right\|_{\infty}, \tag{5}$$

where the set $\mathcal{Z}$ is defined according to

$$
\mathcal{Z} =
\begin{cases}
\{0\} & \text{for problem } (P_1), \\
\mathcal{S}[\overline{\mathbf{w}}] & \text{for problem } (P_2), \\
\mathcal{S}^{\epsilon}[\overline{\mathbf{w}}] & \text{for problems } (N_1) \text{ and } (N_2).
\end{cases}
\tag{6}
$$

2. We compare $\mathrm{Opt}_{\mathcal{Z}}(y)$ to $q_N(\alpha)$, where α is a given tolerance: if $\mathrm{Opt}_{\mathcal{Z}}(y) \leq q_N(\alpha)$, we accept H_0, otherwise we accept H_1.

We describe now the properties of the basic test as applied to problems (P_1), (P_2), (N_1) and (N_2).

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[5]For instance, for even N, $[F_N\xi](1)$ and $[F_N\xi](-1)$ are (real) standard normal variables, and $|[F_N\xi](\mu)|^2$ follows exponential distribution when $\mu \neq \pm 1$. Indeed, in the latter case the joint distribution of real and imaginary parts of $[F_N\xi](\mu)$ is $\mathcal{N}(0, \frac{1}{2}I_2)$, so that $2|[F_N\xi](\mu)|^2 \sim \chi_2^2$ (χ^2 distribution with 2 degrees of freedom). As a result, $\|F_N\xi\|_{\infty}$ can be bounded with $\max(|[F_N\xi](1)|, |[F_N\xi](-1)|, \sqrt{\eta_{N-2}})$, where η_{N-2} is the maximum of $N-2$ independent exponential random variables, what implies the bound (4) in this case.

It is worth to mention that the same argument leads to the bound for $q_N(\alpha)$ which is equivalent to (4) in the case when the observation noise is of the form $\xi = \{\xi_t = \sum_{\tau=-\infty}^{\infty} \gamma_\tau \eta_{t-\tau}\}_{t=0}^{N-1}$ with deterministic γ_τ, $\sum_\tau |\gamma_\tau| \leq 1$, and independent $\eta_t \sim \mathcal{N}(0,1)$. Indeed, in this case we have $[F_N\xi](\mu) = \sum_{\tau=-\infty}^{\infty} \overline{\gamma}_{\mu,\tau} \eta_\tau$, with $\sum_\tau |\overline{\gamma}_{\mu,\tau}|^2 \leq 1$. In other words, the joint distribution of real and imaginary parts of $[F_N\xi](\mu)$ is $\mathcal{N}(0, \Sigma_2)$, with $\mathrm{Trace}(\Sigma_2) \leq 1$. Then for any $\rho > 0$,

$$\mathrm{Prob}\{\|F_N\xi\|_{\infty} \geq \rho\} \leq N \max_{\mu \in \Gamma_N} \mathrm{Prob}\{|[F_N\xi](\mu)| \geq \rho\} \leq 2N \, \mathrm{Prob}\{|\zeta| \geq \rho\},$$

where $\zeta \sim \mathcal{N}(0,1)$, and $\mathrm{Prob}\{\|F_N\xi\|_{\infty} \geq Q_{\mathcal{N}}(\frac{\alpha}{4N})\} \leq \alpha$.

Theorem 3.1. *The risk of the basic test as applied to problems* (P_1), (P_2) *is bounded by* α, *provided that* $d_* = d_n + d_s > 0$ *and*

$$\rho \geq O(1)d_*^3 \ln(2d_*)q_N(\alpha)N^{-1/2} = O(1)d_*^3 \ln(2d_*)\sqrt{N^{-1}\ln(N/\alpha)} \tag{7}$$

with properly chosen positive absolute constants $O(1)$.[6]

The result for the nonparametric problems (N_1) and (N_2) is similar.

Theorem 3.2. *The risk of the basic test as applied to problems* (N_1), (N_2) *is bounded by* α, *provided that* $d_* = d_n + d_s > 0$, ρ *satisfies* (7) *with properly selected* $O(1)$'s *and, in addition,* ϵ_n *and* ϵ_s *are small enough, specifically,*

$$N^{d_n+1/2}\epsilon_n + N^{d_s+1/2}\epsilon_s \leq O(1)q_N(\alpha) \tag{8}$$

with properly selected positive absolute constant $O(1)$.

The proofs of Theorems 3.1 and 3.2 are relegated to Section 6.

Theorems 3.1 and 3.2 provide us with upper resolution bounds independent of the frequencies constituting $\overline{\mathbf{w}}$ and $\mathbf{w}$. When ϵ_n, ϵ_s are "small enough," so that (8) holds true (we refer to the corresponding range of problems' parameters as the *parametric zone*), our upper bound on α-resolution in all testing problems of interest is essentially the same as in the case of $\epsilon_n = \epsilon_s = 0$ – it is $\mathcal{C}(d_n + d_s)\sqrt{\ln(N/\alpha)/N}$ with the factor $\mathcal{C}(d) = O(1)d^3\ln(2d)$ depending solely on d.

On the other hand, when ϵ_n and ϵ_s are *not* "small enough," that is, when $(N, \epsilon_n, \epsilon_s)$ are not in the range described by (8), and $\rho = O(1)\sqrt{\ln(1/\alpha)/N}$, $(1 - \alpha)$-reliable decision between the hypotheses participating in (N_1), (N_2) becomes impossible, whatever be the test (see items (ii) in Propositions 4.2, 4.3 below). This phenomenon is by no means surprising: indeed, allowing for ϵ_n and/or ϵ_s to be positive means allowing for *nonparametric* components in the nuisance and/or nonnuisance signals.[7] When these components are not "small enough," their presence changes significantly the resolution level at which reliable test is possible. The study of problems (N_1) and (N_2) in the nonparametric range (i.e., with $(N, \epsilon_n, \epsilon_s)$ beyond the range given by (8)) goes beyond the scope of this paper[8] and deserves, we believe, a dedicated study. Our "educated guess" is that correct nonparametric version of the basic test in problems (N_1), (N_2) should be as follows: given $(N, \epsilon_n, \epsilon_s)$ *not* satisfying (8), find the largest integer $\overline{N}$ such that $(\overline{N}, \epsilon_n, \epsilon_s)$ does satisfy (8), and then run the basic test on, say, all $\overline{N}$-point subintervals of the N-point observation horizon, inferring the validity of the nuisance hypothesis if and only if it was accepted by the basic test on every subinterval.[9]

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[6] Note that the signal $s \in \mathcal{S}_{d_s}$ is assumed to be at a distance $\rho > 0$ from the nuisance subspace (and thus from the origin). In other words, under the premise of the Theorem 3.1 $d_s > 0$.

[7] Think, for example, of a "plain nonparametric" signal x_0^{N-1}, which is a restriction on the grid $\{i/N\}_{i=0}^{N-1}$ of a smooth function on [0, 1]. It is immediately seen that such a signal belongs to $\mathcal{S}^{N,\epsilon}[0]$ with $\epsilon \leq \|f'\|_\infty/N$.

[8] For "de-noising" analogies of our testing problems in the nonparametric range, see [5,8,9,13].

[9] With this approach, the basic test on a subinterval should be associated with confidence $O(\alpha/N)$ instead of α, in order to account for the possibility of "large deviations" in at least one of $N - \overline{N} + 1$ runs of the basic test.

4. Lower resolution bounds

The lower resolution bounds of this section complement the upper bounds of Section 3. We start with the parametric setting (P_1) and (P_2). Through this section, $c_i(d_n, d_s)$ are properly selected positive and monotone functions of their arguments.

Proposition 4.1. *Given integers $d_n \geq 0$, $d_s \geq 1$, and a real $\alpha \in (0, 1/2)$, consider problems (P_1) and (P_2) with parameters d_n ($d_n = 0$ in the case of problem (P_1)), d_s, α and $\overline{\mathbf{w}} = \{\overbrace{0, \ldots, 0}^{d_n}\}$. Then for properly selected $c_0(d_n, d_s)$ and all $N \geq c_0(d_n, d_s)$ the α-resolution ρ in the problems (P_1) and (P_2) admits the lower bound*

$$O(1)d_s\sqrt{\ln(1/\alpha)/N}.$$

We see that in the problem (P_1) α-resolution grows with d_s at least linearly. Note that by Theorem 3.1, this growth is at most cubic (more precisely, it is not faster than $O(1)d_s^3 \ln(d_s)$). Besides this, we see that the upper bounds on α-resolution for problems (P_1) and (P_2) stemming from Theorem 3.1 and associated with the basic test coincide, within a factor depending solely on d_n, d_s, N and logarithmic in N, with lower bounds on α-resolution.

We have the following lower bound on the α-resolution in the problem (N_1).

Proposition 4.2. *Given integers $d_n > 0$, $d_s \geq 2$ and reals $\alpha \in (0, 1/2)$, $\epsilon_n \geq 0$, consider problem (N_1) with parameters d_n, d_s, N, ϵ_n, α and $\overline{\mathbf{w}} = \{\overbrace{0, \ldots, 0}^{d_n}\}$. Then for properly selected $c_i(d_n, d_s) > 0$ depending solely on d_n, d_s and for all N satisfying*

$$N \geq c_0(d_n, d_s), \tag{9}$$

the α-resolution $\rho_*(\alpha)$ in the problem (N_1) satisfies:

(i) *in the range $0 \leq \epsilon_n \leq c_1(d_n, d_s)N^{-d_n-1/2}\sqrt{\ln(1/\alpha)}$,*

$$\rho_*(\alpha) \geq c_2(d_n, d_s)\sqrt{\ln(1/\alpha)/N};$$

(ii) *in the range*

$$c_3(d_n, d_s)N^{-d_n-1/2}\sqrt{\ln(1/\alpha)} \leq \epsilon_n \leq c_1(d_n, d_s)N\sqrt{\ln(1/\alpha)}, \tag{10}$$

$$\rho_*(\alpha) \geq c_4(d_n, d_s)\big[\epsilon_n N^{d_n+1/2}[\ln(1/\alpha)]^{-1/2}\big]^{1/(2d_n+3)}\sqrt{N^{-1}\ln(1/\alpha)}.$$

In the case of the problem (N_2), we have a similar lower bound on α-resolution when $\epsilon_n \leq \epsilon_s$.

Proposition 4.3. *Given integers $d_n > 0$, $d_s \geq 2$ and reals $\alpha \in (0, 1/2)$, $\epsilon_n \geq 0$, consider problem (N_2) with parameters d_n, d_s, N, ϵ_n, ϵ_s, α and $\overline{\mathbf{w}} = \{\overbrace{0, \ldots, 0}^{d_n}\}$.*
Assume that $0 \leq \epsilon_n \leq \epsilon_s$. Then for properly selected $c_i(d_n, d_s) > 0$ depending solely on d_n, d_s and all $N \geq c_0(d_n, d_s)$, the α-resolution $\rho_*(\alpha)$ in the problem (N_2) satisfies:

(i) *in the range* $0 \le \epsilon_s \le c_1(d_n, d_s) N^{-d_s - 1/2} \sqrt{\ln(1/\alpha)}$,

$$\rho_*(\alpha) \ge c_2(d_n, d_s)\sqrt{\ln(1/\alpha)/N};$$

(ii) *in the range*

$$c_3(d_n, d_s) N^{-d_s - 1/2}\sqrt{\ln(1/\alpha)} \le \epsilon_s \le c_1(d_n, d_s)\sqrt{\ln(1/\alpha)}, \tag{11}$$

$$\rho_*(\alpha) \ge c_4(d_n, d_s)\big[\epsilon_s N^{d_s + 1/2}\big[\ln(1/\alpha)\big]^{-1/2}\big]^{1/(2d_s + 1)} \sqrt{N^{-1}\ln(1/\alpha)}$$
$$\ge c_5(d_n, d_s)\epsilon_s^{1/(2d_s + 1)}\big(\ln(1/\alpha)\big)^{d_s/(2d_s + 1)}. \tag{12}$$

The results of items (i) in Propositions 4.2 and 4.3 say that when d_n, d_s are fixed, N is large, and ϵ_n, ϵ_s are small enough so that the problem parameters are in the parametric zone (i.e., (8) holds), Theorem 3.2 describes "nearly correctly" (i.e., up to factors depending solely on d_n, d_s, N and logarithmic in N) the α-resolution in problems (N_1) and (N_2): within such a factor, the α-resolution for problems (N_1), (N_2), same as for problems (P_1), (P_2), is $\sqrt{\ln(1/\alpha)/N}$. Besides, items (ii) in Propositions 4.2 and 4.3 show that when (ϵ_n, ϵ_s) goes "far beyond" the range (8), the α-resolution in problems (N_1), (N_2) becomes "much worse" than $\sqrt{\ln(1/\alpha)/N}$.

5. Numerical results

Below we report on some numerical experiments with the basic test.

5.1. Problem (N_1)

The goal of the first series of simulations was to quantify "practical performance" of the basic test as applied to problem (N_1).

Organization of experiments

We consider problem (N_1) on the time horizon $0 \le t < N$ for $N \in \{128, 512, 1024\}$ with reliability threshold $\alpha = 0.01$. In these simulations $d_n = 4$, the frequencies in $\overline{\mathbf{w}}$ are selected at random, $\epsilon_n = 0.01$, and $d_s = 4$ (note that N and ϵ_n are deliberately chosen *not* to satisfy (8)). As explained in Section 3, the above setup specifies the basic test for problem (N_1), and our goal is to find the "empirical resolution" of this test. To this end, we ran 10 experiments as follows. In a particular experiment:

- We draw at random $\mathbf{w} \in \mathcal{S}_4$, a *shift* $\bar{s} \in \mathcal{S}[\mathbf{w}]$ and *basic nuisance* $u \in \mathcal{S}^{\epsilon_n}[\overline{\mathbf{w}}]$.
- We generate a "true signal" x according to $x_\lambda = \lambda \bar{s} + u$, where $\lambda > 0$ is (nearly) as small as possible under the restriction that with $x = x_\lambda$, the basic test "rejects reliably" the hypothesis

Table 1. Problem (N_1) with $d = d_s = 4$, $\alpha = \epsilon = 0.01$

	Experiment #										Mean	Mean$\times N^{1/2}$
	1	2	3	4	5	6	7	8	9	10		
Experiments with $N = 128$:												
Resolution	1.10	1.58	1.52	1.51	2.28	1.85	1.12	1.92	1.12	1.82	1.58	17.9
Signal/noise	0.70	1.09	0.94	0.95	1.36	1.03	0.77	1.06	0.69	1.10	0.97	11.0
Experiments with $N = 512$:												
Resolution	0.79	1.30	1.31	0.79	0.79	0.79	0.48	0.79	0.81	0.48	0.83	18.8
Signal/noise	0.44	0.71	0.74	0.43	0.43	0.44	0.30	0.50	0.44	0.34	0.48	10.8
Experiments with $N = 1024$:												
Resolution	0.60	0.92	0.36	0.58	0.46	0.35	0.36	0.56	0.42	0.59	0.52	16.6
Signal/noise	0.32	0.56	0.24	0.41	0.31	0.27	0.26	0.43	0.26	0.39	0.35	11.0

H_0, namely, rejects it in every one of 15 trials with $x = x_\lambda$ and different realizations of the observation noise ξ, see (1).[10]

- For the resulting λ, we compute $\rho = \min_{u \in \mathcal{S}^{\epsilon_n}[\overline{\mathbf{w}}]} \|x_\lambda - u\|_\infty$, which is the output of the experiment. We believe that the collection of 10 outputs of this type gives a good impression on the "true resolution" of the basic test. As a byproduct of an experiment, we get also the $\|\cdot\|_\infty$-closest to x_λ point $u_x \in \mathcal{S}^{\epsilon_n}[\overline{\mathbf{w}}]$; the quantity $r = \|x_\lambda - u_x\|_2/\sqrt{N}$ can be thought of as a natural in our context "signal-to-noise ratio."

The results are presented in Table 1. We would qualify them as quite compatible with the theory we have developed: both empirical resolution and empirical signal-to-noise ratio decreases with N as $N^{-1/2}$. The "empirically observed" resolution ρ for which the basic test $(1 - \alpha)$-reliably, $\alpha = 0.01$, distinguishes between the hypotheses H_0 and $H_1(\rho)$ associated with problem (N_1) is $\approx 6\sqrt{\ln(N/\alpha)/N}$.

Comparison with MUSIC

An evident alternative to the basic test is (a) to apply the standard MUSIC algorithm [18] in order to recover the spectrum of the observed signal, (b) to delete from this spectrum the "nuisance frequencies," and (c) to decide from the remaining data if the signal of interest is present. Our related numerical results are, to the best of our understanding, strongly in favor of the basic test. Let us look at Figure 1 where we present four MUSIC pseudospectra (we use `pmusic` function from MATLAB Signal Processing Toolbox) of the observations associated with signals x obeying the hypothesis $H_1(\rho)$ (magenta) and of the observations coming from the $\|\cdot\|_\infty$-closest to x nuisance (i.e., obeying the hypotheses H_0) u_x (blue). ρ was chosen large enough for

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[10]Since a run of the test requires solving a nontrivial convex program, it would be too time-consuming to replace 15 trials with few hundreds of them required to check reliably that the probability to reject H_0, the signal being x_λ, is at least the desired $1 - \alpha = 0.99$.
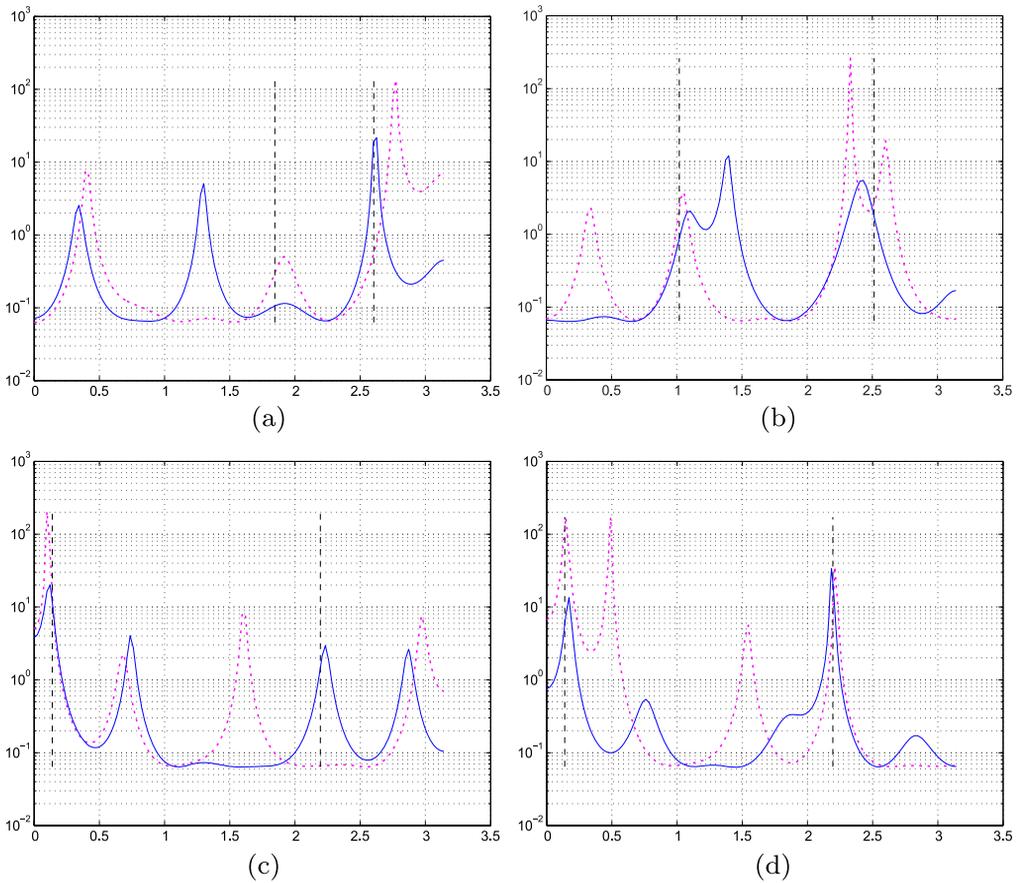
Figure 1. MUSIC pseudospectra as built by MATLAB function `pmusic`$(\cdot,8)$.[†] Dot (magenta): signal plus nuisance; solid (blue): pure nuisance; dash vertical bars: nuisance frequencies ($d = 4$ elements in $\overline{\mathbf{w}}$ correspond to 2 "actual" frequencies). (a), (b) $N = 128$; (c), (d) $N = 1024$.
[†]In the present setup, $\overline{\mathbf{w}} \cup \mathbf{w} = \{\pm w_j, 1 \leq j \leq 4\}$, which requires the `pmusic` parameter `p` to be set to 8.

the basic test to accept reliably the hypothesis $H_1(\rho)$ when it is true. We see that while sometimes MUSIC pseudospectrum indeed allows to understand which one of the hypotheses takes place (as it is the case in the example (d)), "MUSIC abilities" in our context are rather limited.[11] For example, it is hard to imagine a routine which would attribute magenta curves in the examples (a)–(c) to signals, and the blue curves – to the nuisances.

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[11]It should be noted that MUSIC is designed for a problem different from (and more complex than) the detection we are interested in, and thus its weakness relative to a dedicated detection test does not harm algorithm's well-established reputation.

5.2. Comparison with energy test

Our objective here is to compare the resolution of the basic test to that of the test which implements the straightforward idea of how to discover if the signal x underlying observations (1) does not belong to a known nuisance set $\mathcal{U} \subset \mathcal{S}$. The test in question, which we refer to as *energy test*, is as follows: given a tolerance α and an observation y, we solve the optimization problem

$$\mathrm{Opt}(y) = \inf_{u \in \mathcal{U}} \|y - u_0^{N-1}\|_2^2$$

and compare the optimal value with the $(1 - \alpha)$-quantile

$$p_N(\alpha) : \mathrm{Prob}_{\xi \sim \mathcal{N}(0, I_N)} \{\|\xi\|_2^2 > p_N(\alpha)\} = \alpha$$

of the χ^2-distribution with N degrees of freedom. If $\mathrm{Opt}(y) > p_N(\alpha)$, we reject the nuisance hypothesis H_0 stating that $x \in \mathcal{U}$, otherwise we accept the hypothesis. Note that the basic test is of a completely similar structure, with $\|F_N(y - u_0^{N-1})\|_\infty^2$ in the role of $\|y - u_0^{N-1}\|_2^2$ and $q_N^2(\alpha)$ in the role of $p_N(\alpha)$. It is clear that the energy test rejects H_0 when the hypothesis is true with probability at most α (cf. item 1^0 in Section 6.2). In order to simplify the presentation, we restrict this test comparison to the simplest case of $\mathcal{U} = \{0\}$, i.e., the case of problem (P_1). Let us start with some theoretical analysis. Given a natural $d_s > 0$ and a real $\rho > 0$, consider the signal hypothesis $\overline{H}_1(\rho)$ stating that the signal x underlying observations (1) satisfies $\|x_0^{N-1}\|_\infty \geq \rho$ and that x_t is a real algebraic polynomial of degree $\leq d - 1$ of $t \in \mathbf{Z}$, meaning that $x \in \mathcal{S}[\overbrace{0, \ldots, 0}^{d}]$. Observe that with our $\mathcal{U} = \{0\}$, $\mathrm{Opt}(y)$ is nothing but

$$\|y\|_2^2 = \|x_0^{N-1} + \xi\|_2^2 = \|\xi\|_2^2 + 2\xi^T x_0^{N-1} + \|x_0^{N-1}\|_2^2.$$

It follows that the hypothesis H_0 is accepted whenever the event

$$\|\xi\|_2^2 + 2\xi^T x_0^{N-1} + \|x_0^{N-1}\|_2^2 \leq p_N(\alpha)$$

takes place. Now, from the standard results on the χ^2 distribution it follows that for every $\alpha \in (0, 1)$, for all large enough values of N with properly chosen absolute constants it holds

$$\mathrm{Prob}_{\xi \sim \mathcal{N}(0, I_N)} \{p_N(\alpha) - \|\xi\|_2^2 \geq O(1)\sqrt{N \ln(1/\alpha)}\} \geq O(1),$$

whence also

$$\mathrm{Prob}_{\xi \sim \mathcal{N}(0, I_N)} \{\{p_N(\alpha) - \|\xi\|_2^2 \geq O(1)\sqrt{N \ln(1/\alpha)}\} \cap \{\xi^T x_0^{N-1} \leq 0\}\} \geq O(1).$$

As a result, whenever $x \in \mathcal{S}$ satisfies $\|x_0^{N-1}\|_2^2 \leq O(1)\sqrt{N \ln(1/\alpha)}$, the probability to accept H_0, the true signal being x, is at least $O(1)$, provided that N is large enough. On the other hand,

for a given d_s and large N there exists a polynomial x of degree $d_s - 1$ such that $\|x_0^{N-1}\|_2 \leq d_s^{-1} N^{1/2} \|x_0^{N-1}\|_\infty$, see the proof of Proposition 4.1. It immediately follows that with $d_s \geq 1$ and (small enough) $\alpha > 0$ fixed, the energy test cannot distinguish $(1 - \alpha)$-reliably between the hypotheses H_0 and $\overline{H}_1(\rho)$, provided that

$$\rho = O(1) d_s \left[N^{-1} \ln(1/\alpha) \right]^{1/4} \tag{13}$$

and N is large enough. In other words, with d_s and (small enough) α fixed, the resolution of the energy test in problem (P_1) admits, for large N, the lower bound (13). Note that as N grows, this bound goes to 0 as $N^{-1/4}$, while the resolution of the basic test goes to 0 as $N^{-1/2}\sqrt{\ln(N)}$ (Theorem 3.1). We conclude that the basic test provably outperforms the energy test as $N \to \infty$. The goal of the simulation experiments we are about to report is to investigate this phenomenon numerically.

Organization of experiments

In the simulations to follow, the basic test and the energy test were tuned to 0.99-reliability ($\alpha = 0.01$) and used on time horizons $N \in \{256, 1024, 4096\}$. For a fixed N, and every value of the "resolution parameter" ρ from the equidistant grid on $[0, 4]$ with the grid step of 0.05, we run 10 000 simulations as follows:

- we generate $z \in \mathcal{S}[\mathbf{w}]$, and specify signal x as $\rho z / \|z_0^{N-1}\|_\infty$;
- we generate y according to (1) and run on the observations y the basic test and the energy test.

For each test, the outcome of a series of 10 000 simulations is the empirical probability p of rejecting the nuisance hypothesis H_0 (which states that the signal underlying the observations is identically zero). For $\rho = 0$, p is the (empirical) probability of false alarm (rejecting H_0 when it is true), and we want it to be small (about $\alpha = 0.01$). For $\rho > 0$, p is the empirical probability of successful detection of an actually present signal, and we want it to be close to 1 (about $1 - \alpha = 0.99$). Given that $p \leq \alpha$ when $\rho = 0$, the performance of a test can be quantified as the smallest value ρ_* of ρ for which p is at least $1 - \alpha$ (the less is ρ_*, the better).

We use 4-element collections $\mathbf{w}$ (i.e., $d_s = 4$), and for every N and ρ run two 10 000-element series of simulations differing in how we select $\mathbf{w}$ and z. In the first series ("random signals"), $\mathbf{w}$ is selected at random, and z is a random combination of the corresponding harmonic oscillations. In the second series ("bad signal") we use $\mathbf{w} = \{0, 0, 0, 0\}$, and z is the algebraic polynomial of degree 3 with the largest, among these polynomials, ratio of $\|z_0^{N_1}\|_\infty / \|z_0^{N-1}\|_2$. In the latter case, only the realisation of noise varied from one experiment to another.

The results of our experiments are presented in Table 2. They are in full accordance to what is suggested by our theoretical analysis; for $N = 256$, both tests exhibit nearly the same empirical performance. As N grows, the empirical performances of both tests improve, and the "performance gap" (which, as expected, is in favor of the basic test) grows.

Table 2. Basic test vs. Energy test, problem (P_1) with $d_s = 4$. $p(B)$, $p(E)$: empirical probabilities, taken over 10 000 trials, of detecting signal using the basic test (B) and the energy test (E). $\rho_*(\cdot)$ is the smallest ρ for which $p(\cdot) \geq 1 - \alpha = 0.99$

$N = 256$, random signals ($\rho_*(B) \approx 1.10$, $\rho_*(E) \approx 1.35$):

ρ													
	0.00	0.95	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50	1.55
$p(B)$	0.010	0.960	0.977	0.987	0.993	0.998	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000
$p(E)$	0.011	0.710	0.779	0.842	0.887	0.933	0.956	0.974	0.987	0.994	0.997	0.999	1.000	1.000

$N = 256$, "bad" signal ($\rho_*(B) \approx 2.65$, $\rho_*(E) \approx 2.75$):

ρ													
	0.00	2.50	2.55	2.60	2.65	2.70	2.75	2.80	2.85	2.90	2.95	3.00	3.05	3.10
$p(B)$	0.010	0.973	0.984	0.987	0.991	0.994	0.998	0.998	0.999	1.000	1.000	1.000	1.000	1.000
$p(E)$	0.011	0.941	0.956	0.971	0.978	0.984	0.990	0.993	0.995	0.997	0.998	0.998	1.000	1.000

$N = 1024$, random signals ($\rho_*(B) \approx 0.60$, $\rho_*(E) \approx 0.90$):

ρ													
	0.00	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10
$p(B)$	0.010	0.960	0.986	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$p(E)$	0.010	0.303	0.421	0.559	0.686	0.795	0.886	0.938	0.974	0.992	0.997	0.999	1.000	1.000

Table 2. *Continued*

$N = 1024$, "bad" signal ($\rho_*(B) \approx 1.40$, $\rho_*(E) \approx 1.90$):

ρ															
0.00	1.30	1.35	1.40	1.45	1.50	1.55	1.60	1.65	1.70	1.75	1.80	1.85	1.90	1.95	2.00
$p(B)$ 0.011	0.960	0.980	0.990	0.997	0.998	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$p(E)$ 0.011	0.505	0.564	0.633	0.703	0.770	0.819	0.863	0.903	0.932	0.960	0.971	0.987	0.993	0.996	0.997

$N = 4096$, random signals ($\rho_*(B) \approx 0.30$, $\rho_*(E) \approx 0.65$):

ρ										
0.00	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70
$p(B)$ 0.009	0.931	0.993	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$p(E)$ 0.009	0.084	0.165	0.291	0.472	0.667	0.823	0.930	0.980	0.997	1.000

$N = 4096$, "bad" signal ($\rho_*(B) \approx 0.75$, $\rho_*(E) \approx 1.35$):

ρ															
0.00	0.70	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45
$p(B)$ 0.010	0.975	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$p(E)$ 0.012	0.184	0.223	0.290	0.376	0.477	0.567	0.676	0.767	0.843	0.899	0.945	0.975	0.986	0.995	0.999	1.000

6. Proofs

6.1. Preliminaries

Notation

In what follows, for a complex valued polynomial $p(\zeta) = \sum_{k=0}^{m} p_k \zeta^k$, we denote

$$\|p(\cdot)\|_\infty = \max_{\zeta \in \mathbf{C}, |\zeta|=1} |p(\zeta)|$$

and denote by

$$|p|_s = \big\|[p_0; p_1; \ldots; p_m]\big\|_s, \qquad 1 \le s \le \infty,$$

the ℓ_s-norm of the vector of coefficients, so that

$$\|p(\cdot)\|_2^2 := \frac{1}{2\pi} \oint_{|\zeta|=1} |p(\zeta)|^2 |\mathrm{d}\zeta| = |p|_2^2.$$

The key fact underlying Theorems 3.1, 3.2 is the following proposition.

Proposition 6.1. *Let d, N be positive integers and $s \in \mathcal{S}_d$. Then*

$$\|F_N s\|_\infty \ge c(d) N^{1/2} \big\|s_0^{N-1}\big\|_\infty, \tag{14}$$

where $c(d) > 0$ is a universal nonincreasing function of d. One can take

$$c(d) = \mathrm{O}(1)/\big(d^3 \ln(2d)\big) \tag{15}$$

with properly selected positive absolute constant $\mathrm{O}(1)$.

Proof. Let us fix $s \in \mathcal{S}_d$; we intend to prove that s obeys (14). Let $\mathbf{u} = \{\omega_1, \ldots, \omega_d\}$ be a symmetric mod 2π collection such that $s \in \mathcal{S}[\mathbf{u}]$, and let

$$p_\mathbf{u}(\zeta) = \prod_{\ell=1}^{d} \big(1 - \exp\{\iota \omega_\ell\} \zeta\big),$$

so that $p_\mathbf{u}(\Delta)s \equiv 0$. Further, let M be the index of the largest in magnitude of the reals $s_0, s_1, \ldots, s_{N-1}$, so that

$$|s_M| = \big\|s_0^{N-1}\big\|_\infty. \tag{16}$$

We can w.l.o.g. assume that $M \ge (N-1)/2$. Indeed, otherwise we could pass from s to the "reversed" sequence $s' \in \mathcal{S}_d$: $s'_t = s_{N-t-1}$, $t \in \mathbf{Z}$, which would not affect the validity of our target relation (14) and would convert $M < (N-1)/2$ into $M' = N - 1 - M \ge (N-1)/2$.

1^0. We need the following technical lemma.

Lemma 6.1. *Let d be a positive integer, and let $\mathbf{u} = \{v_1, \ldots, v_d\} \in \Omega_d$. For every integer m satisfying*

$$m \geq m(d) := d \operatorname{Ceil}\left(5d \max\left[2, \frac{1}{2}\ln(2d)\right]\right) \tag{17}$$

one can point out real polynomials $q(\zeta) = \sum_{j=1}^{m} q_j \zeta^j$ and $r(\zeta) = 1 + \sum_{j=1}^{m-d} r_j \zeta^j$ such that

$$1 - q(\zeta) = p_{\mathbf{u}}(\zeta) r(\zeta), \tag{18}$$

and

$$|q|_2 \leq C_1(d)/\sqrt{m}, \qquad \text{where } C_1(d) = 3ed^{3/2}\sqrt{\ln(2d)}. \tag{19}$$

The proof of Lemma 6.1 is presented in the Appendix.
2^0. The following statement is immediate:

Lemma 6.2. *Let $m \leq (N-1)/2$, and $g \in \mathbf{C}^N$ be such that $g_i = 0$ for $i > m$. Let $h \in \mathbf{C}^N$ be the discrete autoconvolution of g, that is, the vector with entries $h_k = \sum_{0 \leq i,j \leq m, i+j=k} g_i g_j$, $0 \leq k \leq 2m$ and with zero remaining $N - 2m - 1$ entries. Then*

$$\|F_N h\|_1 = \sqrt{N}\|g\|_2^2.$$

Proof. We have

$$[F_N h](\mu) = N^{-1/2} \sum_{0 \leq t \leq 2m}\left[\sum_{0 \leq j,k \leq m, j+k=t} g_j g_k\right]\mu^t$$

$$= N^{-1/2} \sum_{0 \leq t \leq 2m} \sum_{0 \leq j,k \leq m, j+k=t} (g_j \mu^j)(g_k \mu^k)$$

$$= N^{1/2}\left[N^{-1/2}\sum_{j=0}^{m} g_j \mu^j\right]^2.$$

Invoking the Parseval identity, we conclude that

$$\|F_N h\|_1 = N^{1/2}\|F_N g\|_2^2 = N^{1/2}\|g\|_2^2. \qquad \square$$

3^0. Let

$$N > 60d^2 \ln(2d), \tag{20}$$

and let

$$m = \operatorname{Floor}\left(\frac{N-1}{4}\right). \tag{21}$$

Then (17) is satisfied, and, according to Lemma 6.1, there exists a polynomial $q(\zeta) = \sum_{j=1}^{m} q_j \zeta^j$ such that

$$1 - q(\zeta) = p_{\mathbf{u}}(\zeta) r(\zeta), \qquad |q|_2 \leq C_1(d)/\sqrt{m},$$

with some polynomial r. Setting $q^+(\zeta) = q^2(\zeta) = \sum_{j=1}^{2m} q_j^+ \zeta^j$, we get

$$q_k^+ = \sum_{1 \leq i,j \leq m, i+j=k} q_i q_j,$$

$$\bigl(1 - q^+(\Delta)\bigr)s = \bigl(1 + q(\Delta)\bigr)\bigl(1 - q(\Delta)\bigr)s = \bigl(1 + q(\Delta)\bigr)r(\Delta)p_{\mathbf{u}}(\Delta)s \equiv 0,$$

whence $s_M = \sum_{i=1}^{2m} s_{M-i} q_i^+$ (note that $M \geq (N-1)/2 \geq 2m$ by (21)). Let now $h \in \mathbf{R}^N$ be the vector with coordinates

$$h_i = \begin{cases} q_{M-i}^+, & i = M-1, \dots, M-2m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that by Lemma 6.2 and due to $|q|_2 \leq C_1(d)/\sqrt{m}$ one has

$$\|F_N h\|_1 \leq N^{1/2} |q|_2^2 \leq C_1^2(d)\sqrt{N}/m.$$

We have

$$\left\|s_0^{N-1}\right\|_\infty = |s_M| = \left|\sum_{i=1}^{2m} q_i^+ s_{M-i}\right| = \left|\langle h, s_0^{N-1}\rangle\right| = \left|\langle F_N h, F_N s\rangle\right| \leq \|F_N h\|_1 \|F_N s\|_\infty,$$

where the last equality is given by the fact that F_N is unitary, whence

$$\|F_N s\|_\infty \geq \left\|s_0^{N-1}\right\|_\infty / \|F_N h\|_1 \geq \frac{m}{C_1^2(d)\sqrt{N}} \left\|s_0^{N-1}\right\|_\infty.$$

Invoking (21), (20), and (19), we see that for N satisfying (20) our target relation (14) indeed holds true, provided that

$$c(d) \leq O(1)\bigl[d^3 \ln(2d)\bigr]^{-1} \tag{22}$$

with properly selected positive absolute constant $O(1)$.

4^0. It remains to verify (14) when $N \leq 60d^2 \ln(2d)$. Since F_N is unitary, we have $\|s_0^{N-1}\|_\infty \leq \|s_0^{N-1}\|_2 = \|F_N s\|_2 \leq \|F_N s\|_\infty \sqrt{N}$, whence

$$\|F_N s\|_\infty \geq N^{-1/2}\left\|s_0^{N-1}\right\|_\infty \geq N^{-1}\left[N^{1/2}\left\|s_0^{N-1}\right\|_\infty\right] \geq \frac{1}{60d^2 \ln(2d)}\left[N^{1/2}\left\|s_0^{N-1}\right\|_\infty\right],$$

which completes the proof. $\qquad\square$

6.2. Proof of Theorem 3.1

1^0. Let us prove the result for the basic test, let it be denoted $\widehat{\varphi}$, as applied to the problem (P_2); note that (P_1) is the particular case of (P_2) corresponding to $d_n = 0$.

We have to show that under the premise of the theorem $\varepsilon_0(\widehat{\varphi}; H_0) \leq \alpha$ and $\varepsilon_1(\widehat{\varphi}; H_1(\rho)) \leq \alpha$. The first bound is evident. Indeed, let $\Xi_\alpha = \{\xi : \|F_N \xi\|_\infty \leq q_N(\alpha)\}$, so that $\text{Prob}_{\xi \sim \mathcal{N}(0, I_N)}\{\xi \in \Xi_\alpha\} \geq 1 - \alpha$. Under the hypothesis H_0, the set $\mathcal{Z}$ from (6) contains the true signal x_0^{N-1}, so that the optimal value $\text{Opt}_{\mathcal{Z}}(y)$ in (5) is at most $\|F_N \xi\|_\infty$. It follows that when $\xi \in \Xi_\alpha$ (which happens with probability $\geq 1 - \alpha$) we have $\text{Opt}_{\mathcal{Z}}(y) \leq q_N(\alpha)$, and the basic test will therefore accept H_0. We conclude that $\varepsilon_0(\widehat{\varphi}; H_0) \leq \text{Prob}\{\xi \notin \Xi_\alpha\} \leq \alpha$.

2^0. Now let $x \in H_1(\rho)$, that is, $x = s + u$, where $s \in \mathcal{S}[\mathbf{w}]$ for some $\mathbf{w} \in \Omega_{d_s}$, $u \in \mathcal{S}[\overline{\mathbf{w}}]$, and

$$\left\| [x - z]_0^{N-1} \right\|_\infty \geq \rho \qquad \forall z \in \mathcal{S}[\overline{\mathbf{w}}].$$

Let $z \in \mathcal{S}[\overline{\mathbf{w}}]$, and let $s = x - z$. Then $s \in \mathcal{S}_{d_*}$, $d_* = d_n + d_s$, and $\|s_0^{N-1}\|_\infty \geq \rho$, whence, by Proposition 6.1,

$$\|F_N s\|_\infty \geq c(d_*) N^{1/2} \|s_0^{N-1}\|_\infty \geq c(d_*) N^{1/2} \rho.$$

It follows that the optimal value $\text{Opt}_{\mathcal{Z}}(y)$ in (5) is at least $c(d_*) N^{1/2} \rho - \|F_N \xi\|_\infty$. Recalling the definition of $q_N(\alpha)$, we conclude that

$$\text{Prob}\{\text{Opt}_{\mathcal{Z}}(y) > c(d_*) N^{1/2} \rho - q_N(\alpha)\} \geq 1 - \alpha$$

as soon as

$$\rho > \frac{2 q_N(\alpha)}{c(d_*) \sqrt{N}}, \tag{23}$$

and the probability to reject $H_1(\rho)$ when the hypothesis is true is $\leq \alpha$. We see that for the proper choice of the absolute constant $O(1)$ under the premise of Theorem 3.1 one has $\varepsilon_1(\widehat{\varphi}, H_1(\rho)) \leq \alpha$.

6.3. Proof of Theorem 3.2

1^0. We start with the following simple

Lemma 6.3. *Let d and N be positive integers, let $\epsilon \geq 0$, let $\mathbf{u} = \{\upsilon_1, \ldots, \upsilon_d\}$ be a symmetric mod 2π d-element collection of reals. Whenever $w \in \mathcal{S}^{N,\epsilon}[\mathbf{u}]$, there exists a decomposition $w = s + z$ such that $s \in \mathcal{S}[\mathbf{u}]$ and*

$$\left\| z_0^{N-1} \right\|_\infty \leq N^d \epsilon. \tag{24}$$

Proof. Let $p_{\mathbf{u}}(\zeta) = \prod_{\ell=1}^d (1 - \exp\{\iota \upsilon_\ell\} \zeta)$ and $r = p_{\mathbf{u}}(\Delta) w$, so that $\|r_0^{N-1}\|_\infty \leq \epsilon$ due to $w \in \mathcal{S}^{N,\epsilon}[\mathbf{u}]$. Let, further, δ be the discrete convolution unit (i.e., $\delta \in \mathcal{S}$ is given by $\delta_0 = 1$, $\delta_t = 0$, $t \neq 0$). For $\ell = 1, \ldots, d$, let $\gamma^{(\ell)}$ be a two-sided complex-valued sequence obtained from the

sequence $\{\exp\{\imath\upsilon_\ell t\}\}_{t\in\mathbf{Z}}$ by replacing the terms with negative indexes with zeros, and let r^+ be obtained by similar operation from the sequence r. Let us set

$$\chi = \gamma^{(1)} * \gamma^{(2)} * \cdots * \gamma^{(d)} * r^+,$$

where $*$ stands for discrete time convolution. It is immediately seen that χ is a real-valued two-sided sequence which vanishes for $t < 0$ and satisfies the finite-difference equation $p_{\mathbf{u}}(\Delta)\chi = r^+$ (due to the evident relation $(1 - \exp\{\imath\upsilon_k\}\Delta)\gamma^{(k)} = \delta$). It follows that $(p_{\mathbf{u}}(\Delta)(w - \chi))_t = 0$ for $t = 0, 1, \ldots$, which (along with the fact that all the roots of $p_{\mathbf{u}}(\zeta)$ are nonzero) implies that the sequence $s = w - \chi$ can be modified on the domain $t < 0$ so that $p_{\mathbf{u}}(\Delta)s \equiv 0$. Then $z = w - s$ coincides with χ on the domain $t \geq 0$, and $w = s + z$ with $s \in \mathcal{S}[\mathbf{u}]$ and $z_t = \chi_t$, $t = 0, 1, \ldots$. It remains to note that for two-sided complex-valued sequences μ, ν starting at $t = 0$ we clearly have $\|[\mu * \nu]_0^{N-1}\|_\infty \leq \|\mu_0^{N-1}\|_1 \|\nu_0^{N-1}\|_\infty$. Applying this rule recursively and taking into account that $\|[\gamma^{(\ell)}]_0^{N-1}\|_1 = N$, we get the recurrence

$$\left\|\left[\gamma^{(1)} * \cdots * \gamma^{(\ell+1)} * r^+\right]_0^{N-1}\right\|_\infty \leq N\left\|\left[\gamma^{(1)} * \cdots * \gamma^{(\ell)} * r^+\right]_0^{N-1}\right\|_\infty, \qquad \ell = 0, 1, \ldots, d-1,$$

whence $\|\chi_0^{N-1}\|_\infty \leq N^d\epsilon$. Since $\chi_t = z_t$ for $t = 0, 1, \ldots$, (24) follows. $\qquad\square$

2^0. We are ready to prove Theorem 3.2. It suffices to consider the case of problem (N_2) (problem (N_1) is the particular case of (N_2) corresponding to $\epsilon_s = 0$. The fact that for the basic test $\widehat{\varphi}$ one has $\varepsilon_0(\widehat{\phi}) \leq \alpha$ can be verified exactly as in the case of Theorem 3.1. Let us prove that under the premise of Theorem 3.2 we have $\varepsilon_1(\widehat{\varphi}) \leq \alpha$ as well. To this end let the signal x underlying (1) belong to $H_1(\rho)$, so that $x = r + u$ for some $u \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}]$ and some $r \in \mathcal{S}^{N,\epsilon_n}[\mathbf{w}]$ with d_n-element collection $\overline{\mathbf{w}}$ and d_s-element collection $\mathbf{w}$, both symmetric mod 2π. Let also $z \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}]$. Since $x \in H_1(\rho)$, we have

$$\left\|[x - z]_0^{N-1}\right\|_\infty \geq \rho. \tag{25}$$

Applying Lemma 6.3 to r, u, z, we get the decompositions

$$\begin{aligned}
x = s + s' + v' &: s \in \mathcal{S}[\mathbf{w}], \ s' \in \mathcal{S}[\overline{\mathbf{w}}], \ \left\|[v']_0^{N-1}\right\|_\infty \leq N^{d_n}\epsilon_n + N^{d_s}\epsilon_s, \\
z = s'' + v'' &: s'' \in \mathcal{S}[\overline{\mathbf{w}}], \ \left\|[v'']_0^{N-1}\right\|_\infty \leq N^{d_n}\epsilon_n, \\
\Rightarrow \quad w &:= x - z = \bar{s} + \bar{v}, \\
\bar{s} &= s + s' - s'' \in \mathcal{S}[\overline{\mathbf{w}} \cup \mathbf{w}], \\
\bar{v} &= v' - v'', \ \left\|\bar{v}_0^{N-1}\right\|_\infty \leq \sigma := 2N^{d_n}\epsilon_n + N^{d_s}\epsilon_s.
\end{aligned} \tag{26}$$

Now, (25) implies that $\|w_0^{N-1}\| \geq \rho$, whence, by (26),

$$\left\|\bar{s}_0^{N-1}\right\|_\infty \geq \widehat{\rho} := \rho - \sigma.$$

Assuming that $\widehat{\rho} > 0$, noting that $\bar{s} \in \mathcal{S}[\widetilde{\mathbf{w}}]$ for $(d_* = d_n + d_s)$-element symmetric mod 2π collection $\widetilde{\mathbf{w}}$ and invoking Proposition 6.1, we get

$$\|F_N \bar{s}\|_\infty \geq c(d_*) N^{1/2} \widehat{\rho}.$$

Taking into account that $\|F_N \bar{v}\|_\infty \leq \|\bar{v}_0^{N-1}\|_2 \leq N^{1/2} \|\bar{v}_0^{N-1}\|_\infty$ and (26), we get also $\|F_N \bar{v}\|_\infty \leq N^{1/2}\sigma$. Combining these observations, we get

$$\begin{aligned}
\left\|F_N[x-z]\right\|_\infty &= \|F_N \bar{s} + F_N \bar{v}\|_\infty \geq \|F_N \bar{s}\|_\infty - \|F_N \bar{v}\|_\infty \\
&\geq c(d_*) N^{1/2} \bar{\rho} - N^{1/2}\sigma = c(d_*) N^{1/2}[\rho - \sigma] - N^{1/2}\sigma \\
&= \underbrace{c(d_*) N^{1/2}\left[\rho - \left[1 + c_*^{-1}(d_*)\right]\sigma\right]}_{\vartheta}.
\end{aligned}$$

Since $z \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}]$ is arbitrary, we conclude that the optimal value $\mathrm{Opt}_\mathcal{Z}(y)$ in (5) is at least $\vartheta - \|F_N \xi\|_\infty$, so that

$$\mathrm{Prob}\{\mathrm{Opt}_\mathcal{Z}(y) > \vartheta - q_N(\alpha)\} \geq 1 - \alpha. \tag{27}$$

It remains to note that with properly selected positive absolute constants O(1)'s in (7) and (8), these restrictions on $\rho, \epsilon_n, \epsilon_s$ ensure that $\vartheta > 2q_N(\alpha)$ (see (26), (15)), and therefore (27) implies the desired bound $\varepsilon_1(\widehat{\varphi}) \leq \alpha$.

6.4. Proof of Proposition 4.1

Here we prove the lower resolution bound for problem (P_2). The result of the proposition for the setting (P_1) may be obtained by an immediate modification of the proof below for $d_n = 0$, $p_{\overline{\mathbf{w}}}(\cdot) = 1$, and $z \equiv 0$. Below we use notation κ_i for positive absolute constants.

1^0. Note that for every integer $d_s > 0$ there exists a real polynomial q_{d_s} on $[0, 1]$ of degree $d_s - 1$ such that $\int_0^1 q_{d_s}^2(r)\, dr = 1$ and $\max_{0 \leq r \leq 1} |q_{d_s}(r)| = q_{d_s}(1) = d_s$.[12] Let $q = \{q_t = N^{-1/2}q_{d_s}(t/N)\}_{t=-\infty}^{+\infty}$, $\lambda > 0$, and consider the two-sided sequence

$$\bar{x} = \left\{\bar{x}_t = \theta_N \lambda (-1)^t q_t\right\}_{t=-\infty}^{+\infty},$$

where $\theta_N > 0$ is chosen in such a way that

$$\left\|\bar{x}_0^{N-1}\right\|_2 = \lambda.$$

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[12]Indeed, let a real polynomial p of degree κ satisfy $p(1) = 1$ (and thus $\sum_{i=0}^\kappa p_i = 1$). Then the vector of coefficients p^* of the polynomial p_κ^* of the minimal $L_2(0, 1)$-norm is the optimal solution to the convex quadratic optimization problem

$$\min\left\{\int_0^1 \left(\sum_{i=0}^\kappa p_i t^i\right)^2 dt = \sum_{i,j=0}^\kappa \frac{p_i p_j}{i + j + 1}, \text{ subject to } \sum_{i=0}^\kappa p_i = 1\right\}.$$

The latter problem can be solved explicitly and its optimal value is $(\kappa + 1)^{-2}$.

Note that θ_N given by this requirement does not depend on λ and that $\theta_N \to 1$ as $N \to \infty$ due to $\int_0^1 q_{d_s}^2(r)\,dr = 1$. We have

$$|\bar{x}_{N-1}|/\|\bar{x}_0^{N-1}\|_2 \to \frac{|q_{d_s}(1)|}{\sqrt{\int_0^1 q_{d_s}^2(r)\,dr}} = d_s \qquad \text{as } N \to \infty. \tag{28}$$

The derivative $q'_{d_s}(r)$ of the polynomial q_{d_s} satisfies

$$\max_{0 \le r \le 1} |q'_{d_s}(r)| \le 2(d_s - 1)^2 \max_{0 \le r \le 1} |q_{d_s}(r)| \le 2d_s^3$$

(the first inequality in this chain follows from Markov brothers' inequality). We conclude that for properly selected $\kappa_1 \ge 1$ and all $N \ge \kappa_1 d_s^2(d_n + 1)$ it holds $q_{d_s}(t/N) \ge d_s/2$ whenever $N - d_n - 1 \le t \le N - 1$. Taking into account that $\theta_N \to 1$ as $N \to \infty$, it follows that for properly selected $c_0(d_s, d_n) \ge \kappa_1 d_s^2(d_n + 1)$ and all $N \ge c_0(d_s, d_n)$ it holds

$$N - d_n - 1 \le t \le N - 1 \quad \Rightarrow \quad |\bar{x}_t| = (-1)^t \bar{x}_t,$$
$$\min_{N-d_n-1 \le t \le N-1} |\bar{x}_t| \ge |\bar{x}_{N-1}|/2 \ge \kappa_2 \theta_N \lambda N^{-1/2} d_s \ge \kappa_3 \lambda N^{-1/2} d_s. \tag{29}$$

Beside this, for every d_s and N, we clearly have $\bar{x} \in S[\mathbf{w}]$ with $\mathbf{w} = \{\overbrace{\pi, \dots, \pi}^{d_s}\}$ (indeed, for all $t \in \mathbf{Z}$, $((1 + \Delta)^{d_s} \bar{x})_t = (-1)^t \theta_N \lambda ((1 - \Delta)^{d_s} q)_t = 0$).

2^0. Let $\overline{\mathbf{w}} = \{\overbrace{0, \dots, 0}^{d_n}\}$, with $p_{\overline{\mathbf{w}}}(\zeta) = (1 - \zeta)^{d_n}$. Assuming $N \ge c_0(d_s, d_n)$, we have

$$\left|(p_{\overline{\mathbf{w}}}(\Delta)\bar{x})_{N-1}\right| \ge 2^{d_n} \min_{N-d_n-1 \le t \le N-1} |\bar{x}_t| \ge 2^{d_n} \kappa_3 \lambda N^{-1/2} d_s,$$

so that for every $z \in S[\overline{\mathbf{w}}]$ we have

$$\left|(p_{\overline{\mathbf{w}}}(\Delta)[\bar{x} - z])_{N-1}\right| \ge 2^{d_n} \kappa_3 \lambda N^{-1/2} d_s.$$

Since $N - 1 \ge d_n$, when taking into account that $|p_{\overline{\mathbf{w}}}|_1 = 2^{d_n}$ we get for any $z \in S[\overline{\mathbf{w}}]$

$$2^{d_n} \left\|[\bar{x} - z]_0^{N-1}\right\|_\infty = |p_{\overline{\mathbf{w}}}|_1 \left\|[\bar{x} - z]_0^{N-1}\right\|_\infty \ge \left|(p_{\overline{\mathbf{w}}}(\Delta)[\bar{x} - z])_{N-1}\right| \ge 2^{d_n} \kappa_3 \lambda N^{-1/2} d_s,$$

whence

$$\forall z \in S[\widehat{\mathbf{w}}]: \left\|[\bar{x} - z]_0^{N-1}\right\|_\infty \ge \kappa_3 \lambda N^{-1/2} d_s. \tag{30}$$

Now let us set $\lambda = \kappa_4 \sqrt{\ln(1/\alpha)}$, with the absolute constant κ_4 selected to ensure that $\lambda < 2Q_{\mathcal{N}}(\alpha)$. The latter relation, due to $\lambda = \|x_0^{N-1}\|_2$, ensures that the hypotheses *"observation (1) comes from $x \equiv 0$"* and *"observation (1) comes from $x = \bar{x}$"* cannot be distinguished $(1 - \alpha)$-reliably. Thus, (30) implies the lower resolution bound $\kappa_3 \kappa_4 d_s N^{-1/2} \sqrt{\ln(1/\alpha)}$.

6.5. Proof of Proposition 4.2

In the reasoning below, c_i denote positive quantities depending solely on $d = d_n$, and κ_i denote positive absolute constants. We start with proving the claim (ii).

1^0. Let us set

$$\bar{f}_t = \sin\left(\frac{\pi}{8} + \frac{\pi}{4}\frac{t}{N-1}\right), \qquad t \in \mathbf{Z}. \tag{31}$$

Assuming $c_0 > 40d$, let us fix an integer τ such that

$$d - 1 \leq \tau \leq (N-1) - 20d \tag{32}$$

and set

$$\gamma = \gamma_{\tau,N} = \epsilon \bar{f}_\tau^{-1}, \qquad f = \gamma \bar{f}. \tag{33}$$

By the definition of f we have

$$0 \leq f_t \leq \epsilon \qquad \text{for } 0 \leq t \leq \tau, \tag{34}$$

and

$$\epsilon + \kappa_0 N^{-1}(t-\tau)\epsilon \leq f_t \leq \epsilon + \kappa_1 N^{-1}(t-\tau)\epsilon \qquad \text{for } \tau \leq t \leq N-1. \tag{35}$$

Let $p(\zeta) = (1-\zeta)^d = p_{\overline{\mathbf{w}}}(\zeta)$. We clearly can find a sequence $x = \{x_t = a\cos(\frac{\pi}{4}t + b)\}_{t=-\infty}^{\infty}$ such that $p(\Delta)x = f$, and, due to $d_s \geq 2$, we have $x \in \mathcal{S}_{d_s}$. Further, let $\bar{z} \in \mathcal{S}$ satisfy $\bar{z}_t = x_t$ for $0 \leq t \leq \tau$, and

$$\left(p(\Delta)\bar{z}\right)_t = \begin{cases} f_t, & 0 \leq t \leq \tau, \\ 0, & t < 0, \\ \epsilon, & t \geq \tau. \end{cases}$$

Note that $\bar{z}$ is well defined due to $p(\Delta)x = f$, and, taking into account (33)–(35), we conclude that $\bar{z} \in \mathcal{S}^\epsilon[\overline{\mathbf{w}}]$.

2^0. By the above construction, the sequence $\delta = x - \bar{z}$ is such that $(p(\Delta)\delta)_t = 0$ for $0 \leq t \leq \tau$, and $\kappa_0(t-\tau)N^{-1}\epsilon \leq (p(\Delta)\delta)_t \leq \kappa_1(t-\tau)N^{-1}\epsilon$ when $\tau < t < N$, see (35). Besides, $\delta_t = 0$ when $0 \leq t \leq \tau$. By evident reasons, these two observations combine with the first inequality in (32) to imply that

$$\left\|\delta_0^{N-1}\right\|_\infty \leq c_4(N-\tau)^{d+1}N^{-1}\epsilon \tag{36}$$

for some $c_4 > 0$ depending solely on d. We conclude that

$$\left\|\delta_0^{N-1}\right\|_2 \leq \left\|\delta_0^{N-1}\right\|_\infty \sqrt{N-\tau} \leq c_4(N-\tau)^{d+3/2}N^{-1}\epsilon.$$

Now note that the hypotheses "observation (1) comes from $x = \bar{z}$" and "observation (1) comes from $x = \bar{z} + \delta$" cannot be distinguished $(1-\alpha)$-reliably unless $\|\delta_0^{N-1}\|_2 \geq 2Q_\mathcal{N}(\alpha) > \sqrt{\kappa_3 \ln(1/\alpha)}$. Equipped with this κ_3 and with c_4 participating in (36), let us set

$$\nu = \text{Floor}\left(\left(\kappa_3 \ln(1/\alpha)N^2 c_4^{-2}\epsilon^{-2}\right)^{1/(2d+3)}\right). \tag{37}$$

It is immediately seen that with properly chosen positive c_1, c_3 depending solely on d, and with ϵ satisfying (10), we have $20d < \nu < N - d$, so that setting

$$\tau = N - \nu,$$

we ensure (32). From now on, we assume that c_1, c_3 are as needed in the latter conclusion. With the just defined τ, we have $\|\delta_0^{N-1}\|_2^2 \le \kappa_3 \ln(1/\alpha)$, meaning that x and $\bar{z}$ cannot be distinguished $(1 - \alpha)$-reliably.

3^0. To prove the claim (ii) it now suffices to show that a properly chosen $c_5 > 0$, depending solely on d,

$$\forall z \in \mathcal{S}^{\epsilon}[\overline{\mathbf{w}}] : \left\| [x - z]_0^{N-1} \right\|_{\infty} \ge c_5 (N - \tau)^{d+1} N^{-1} \epsilon = c_5 \nu^{d+1} N^{-1} \epsilon. \tag{38}$$

Indeed, given $z \in \mathcal{S}^{\epsilon}$, we have

$$\tau \le t < N \quad \Rightarrow \quad \left(p(\Delta)[x - z] \right)_t = \left(p(\Delta)x \right)_t - \left(p(\Delta)z \right)_t \ge f_t - \epsilon \ge \kappa_0 N^{-1} \epsilon (t - \tau),$$

with concluding inequality given by (35). Setting $\theta = \lfloor (N - 1 - \tau)/2 \rfloor$, the sequence

$$s_t = (x - z)_{t-\tau}, \qquad t \in \mathbf{Z}$$

satisfies

$$\left(p(\Delta)s \right)_t \ge \kappa_2 \theta N^{-1} \epsilon, \qquad \theta \le t \le 2\theta, \tag{39}$$

and, by the right inequality in (32), $\theta \ge 10d$. Setting $k = \lfloor (\theta - d)/d \rfloor \ge 2$, consider the polynomial

$$q(\zeta) = \left(1 - \zeta^k \right)^d = (1 - \zeta)^d \underbrace{\left(1 + \zeta + \cdots + \zeta^{k-1} \right)^d}_{v(\zeta)} = (1 - \zeta)^d \sum_{j=0}^{d(k-1)} v_j \zeta^j;$$

where, clearly,

$$v_j \ge 0, \qquad \sum_j v_j = k^d. \tag{40}$$

Let now $r = q(\Delta)s$. Taking into account that $|q(\cdot)|_1 = 2^d$, we have

$$r_{2\theta} \le 2^d \left\| s_{\theta}^{2\theta} \right\|_{\infty} \le 2^d \left\| [x - z]_0^{N-1} \right\|_{\infty}, \tag{41}$$

since, by construction, $\|[x - z]_0^{N-1}\|_{\infty} \ge \|s_{\theta}^{2\theta}\|_{\infty}$. On the other hand, $r = v(\Delta)u$, $u = p(\Delta)s$, so that

$$r_{2\theta} = \sum_{j=0}^{d(k-1)} v_j u_{2\theta - j}.$$

By (39), we have $u_{2\theta-j} \geq \kappa_2 \theta N^{-1} \epsilon$ for $0 \leq j \leq d(k-1)$ (note that $d(k-1) < \theta$), and by (40),

$$r_{2\theta} \geq \kappa_2 \theta N^{-1} \epsilon \sum_j v_j = k^d \kappa_2 \theta N^{-1} \epsilon.$$

Combining the latter inequality with (41) we come to $\|[\bar{x} - z]_0^{N-1}\|_\infty \geq \kappa_2 2^{-d} k^d \theta N^{-1} \epsilon$. Recalling that by construction $k > \kappa_{10}(N - \tau)/d$, $\theta \geq \kappa_{11}(N - \tau)$, we arrive at (38). (ii) is proved.

4^0. It remains to prove (i). Note that when $\epsilon < c_{12} N^{-1/2} \sqrt{\ln(1/\alpha)}$, the conclusion in (i) is readily given by a straightforward modification of the reasoning in Section 6.4. Note that for the time being the only restriction on the lower bound c_0 on N, see (9), was that $c_0 \geq 40d$, see the beginning of item 1^0. Now let us also assume that N is large enough to ensure that $c_{12} N^{-1/2} \geq c_3 N^{-d-1/2}$ (which still allows to choose c_0 as a function of $d = d_n$ only). With the resulting c_0, the range of values of ϵ for which we have justified the conclusion in (i) covers the corresponding range of ϵ allowed by the premise of (i).

6.6. Proof of Proposition 4.3

The proof of the statement (i) is given by a straightforward modification of the reasoning in Section 6.4. Let us prove (ii). Let $\mathbf{w} = \{\overbrace{\pi, \dots, \pi}^{d_s}\}$, and $0 \leq \tau < N - d_n - 1$, so that $\mathcal{S}^{N,\epsilon_s}[\mathbf{w}]$ contains the sequence $\bar{x} = \{\bar{x}_t = c_1 \epsilon_s (-1)^t (t - \tau)_+^{d_s}\}_{t=-\infty}^\infty$ with $0 < c_1 = c_1(d_s)$ small enough. Then for $\overline{\mathbf{w}} = \{0, \dots, 0\}$ (d_n zeros), with $p_{\overline{\mathbf{w}}}(\zeta) = (1 - \zeta)^{d_n}$, we have for any $z \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}]$:

$$\left|\left(p_{\overline{\mathbf{w}}}(\Delta)[\bar{x} - z]\right)_{N-1}\right| \geq \left|\left(p_{\overline{\mathbf{w}}}(\Delta)\bar{x}\right)_{N-1}\right| - \left|\left(p_{\overline{\mathbf{w}}}(\Delta)z\right)_{N-1}\right| \geq 2^{d_n}|\bar{x}_{N-d-1}| - \epsilon_n$$

$$\geq c_2 \epsilon_s (N - \tau)^{d_s} - \epsilon_n,$$

where $c_2 > 0$ depends only on d_s and d_n. Therefore, when denoting $v = N - \tau$, for every $z \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}]$ we have

$$2^{d_n} \left\|[\bar{x} - z]_0^{N-1}\right\|_\infty = |p_{\overline{\mathbf{w}}}|_1 \left\|[\bar{x} - z]_0^{N-1}\right\|_\infty \geq \left|\left(p_{\overline{\mathbf{w}}}(\Delta)[\bar{x} - z]\right)_{N-1}\right| \geq c_2 \epsilon_s v^{d_s} - \epsilon_n,$$

whence,

$$\forall z \in \mathcal{S}^{N,\epsilon_n}[\overline{\mathbf{w}}] : \left\|[\bar{x} - z]_0^{N-1}\right\|_\infty \geq 2^{-d_n}\left(c_2 \epsilon_s v^{d_s} - \epsilon_n\right). \tag{42}$$

It is immediately seen that with properly selected positive $c_i = c_i(d_n, d_s)$, $i = 3, 4, 5, 6$, assuming

$$\epsilon_s N^{d_s+1/2} \geq c_3 \sqrt{\ln(1/\alpha)} \quad \text{and} \quad \epsilon_s \leq c_4 \sqrt{\ln(1/\alpha)} \tag{43}$$

and selecting τ according to

$$v = N - \tau = \text{Floor}\left(c_5\left[\epsilon_s^{-2} \ln(1/\alpha)\right]^{1/(2d_s+1)}\right),$$

we ensure that $0 \leq \tau < N - d_n - 1$ and

(a) $\tau < N - d_n - 1$,

(b) $\left\| \bar{x}_0^{N-1} \right\|_2 < 2Q_{\mathcal{N}}(\alpha)$,

(c) $2^{-d_n} \left(c_2 \epsilon_s v^{d_s} - \epsilon_n \right) \geq \bar{\rho} := c_6 \epsilon_s^{1/(2d_s+1)} \left(\ln(1/\alpha) \right)^{d_s/(2d_s+1)}$

(when verifying (c), take into account that $0 \leq \epsilon_n \leq \epsilon_s$). By (b), the hypotheses "observation (1) comes from $x \equiv 0$" and "observation (1) comes from $x = \bar{x}$" cannot be distinguished $(1 - \alpha)$-reliably, while (42), the already established inclusion $\bar{x} \in \mathcal{S}^{N,\epsilon_s}[\mathbf{w}]$ and (c) imply that $\bar{x}$ obeys the hypothesis $H_1(\bar{\rho})$ associated with the problem (N_2). The bottom line is that in the case of (43), $\bar{\rho}$ defined in (c) is a lower bound on $\rho_*(\alpha)$.

Appendix: Proof of Lemma 6.1

1^0. Let $\lambda \in \mathbf{C}$, $0 < |\lambda| \leq 1$, let $\epsilon \in (0, 1)$, and let $n \geq 1$ be an integer. Setting $\delta = 1 - \epsilon$, consider the polynomials

$$f_k(\zeta) = f_k(\zeta; \lambda, \epsilon) := (1 - \lambda\zeta) \sum_{\ell=0}^{k} (\delta\lambda\zeta)^\ell = \frac{1 - \lambda\zeta}{1 - \delta\lambda\zeta} \left[1 - [\delta\lambda\zeta]^{k+1} \right], \qquad k = 0, 1, \dots$$

$$p_n(\zeta) = p_n(\zeta; \lambda, \epsilon) := \frac{1}{n} \sum_{k=0}^{n-1} f_k(\zeta) = (1 - \lambda\zeta) \sum_{\ell=0}^{n-1} [\delta\lambda\zeta]^\ell [1 - \ell/n]. \tag{44}$$

Observe that

$$f_k(0) = 1, \qquad f_k(1/\lambda) = 0,$$

whence

$$p_n(0) = 1, \qquad p_n(1/\lambda) = 0. \tag{45}$$

1.1^0. Let us bound from above the uniform norm $\|p_n(\cdot)\|_\infty$ of p_n on the unit circle. We have $p_n(\zeta) = r_n(\lambda\zeta)$, where

$$r_n(\zeta) = \frac{1}{n}(1 - \zeta) \sum_{k=0}^{n-1} \frac{1 - (\delta\zeta)^{k+1}}{1 - \delta\zeta} = (1 - \zeta) \frac{n - \delta\zeta((1 - [\delta\zeta]^n)/(1 - \delta\zeta))}{n(1 - \delta\zeta)}$$

$$= \underbrace{\frac{1 - \zeta}{1 - \delta\zeta}}_{g_n(\zeta)} \underbrace{\frac{n(1 - \delta\zeta) - \delta\zeta + [\delta\zeta]^{n+1}}{n(1 - \delta\zeta)}}_{h_n(\zeta)}.$$

Since $|\lambda| \leq 1$, we have $\|p_n(\cdot)\|_\infty \leq \|g_n(\cdot)\|_\infty \|h_n(\cdot)\|_\infty$. When $|\zeta| = 1$, we have

$$\left| g_n(\zeta) \right| = \frac{1}{\delta} \frac{|1 - \zeta|}{|\delta^{-1} - \zeta|} \leq \frac{1}{\delta}. \tag{46}$$

If we set $\zeta = \cos(\phi) + \iota \sin(\phi)$, and $\epsilon = \theta/n$ and $\delta = 1 - \epsilon = 1 - \theta/n$ with some $\theta \in (0, n)$, we obtain

$$
\begin{aligned}
\left| h_n(\zeta) \right| &\leq \frac{|n(1 - \delta\zeta) - \delta\zeta| + \delta^{n+1}}{n|1 - \delta\lambda|} \\
&= \frac{\sqrt{[n - (n+1)\delta \cos(\phi)]^2 + (n+1)^2\delta^2 \sin^2(\phi)} + \delta^{n+1}}{n\sqrt{[1 - \delta \cos(\phi)]^2 + \delta^2 \sin^2(\phi)}} \\
&= \frac{\sqrt{n^2 + (n+1)^2\delta^2 \cos^2(\phi) - 2n(n+1)\delta \cos(\phi) + (n+1)^2\delta^2 \sin^2(\phi)} + \delta^{n+1}}{n\sqrt{1 + \delta^2 \cos^2(\phi) - 2\delta \cos(\phi) + \delta^2 \sin^2(\phi)}} \\
&= \frac{\sqrt{n^2 + (n+1)^2 - 2n(n+1)\delta \cos(\phi)} + \delta^{n+1}}{n\sqrt{1 + \delta^2 - 2\delta \cos(\phi)}} \\
&= \frac{\sqrt{[n - (n+1)\delta)]^2 + 2n(n+1)\delta[1 - \cos(\phi)]} + \delta^{n+1}}{n\sqrt{[1 - \delta]^2 + 2\delta[1 - \cos(\phi)]}} \\
&= \frac{\sqrt{[\theta((n+1)/n) - 1]^2 + 2(n+1)(n - \theta)[1 - \cos(\phi)]} + \delta^{n+1}}{n\sqrt{\theta^2/n^2 + 2n^{-1}(n - \theta)[1 - \cos(\phi)]}} \\
&= \frac{\sqrt{[\theta((n+1)/n) - 1]^2 + 2(n+1)(n - \theta)[1 - \cos(\phi)]} + \delta^{n+1}}{\sqrt{\theta^2 + 2n(n - \theta)[1 - \cos(\phi)]}} = \frac{\sqrt{\alpha + \beta t} + \gamma}{\sqrt{\mu + \nu t}},
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= \left[\theta\big((n+1)/n\big) - 1\right]^2, \qquad \beta = 2(n+1)(n - \theta), \qquad \gamma = \delta^{n+1}, \\
\mu &= \theta^2, \qquad \nu = 2n(n - \theta), \qquad t = 1 - \cos(\phi).
\end{aligned} \tag{47}
$$

It is immediately seen that with positive $\alpha, \beta, \gamma, \mu$ and ν such that

$$
\mu/\nu > \alpha/\beta, \tag{48}
$$

the maximum of the function $\frac{\sqrt{\alpha + \beta t} + \gamma}{\sqrt{\mu + \nu t}}$ over t such that $\alpha + \beta t \geq 0$ is achieved when $\alpha + \beta t = \sqrt{\frac{\beta\mu - \alpha\nu}{\gamma\nu}}$ and is equal to $\sqrt{\frac{\beta}{\nu}}\sqrt{1 + \frac{\gamma^2\nu}{\beta\mu - \alpha\nu}}$. Now, assume that

$$
1 < \theta < n. \tag{49}
$$

Then $\frac{n+1}{n}\theta^2 > [\theta\frac{n+1}{n} - 1]^2$, so that the parameters $\alpha, \ldots, \nu$ defined in (47) satisfy (48). We conclude that

$$
\left\| h_n(\cdot) \right\|_\infty \leq \sqrt{\frac{n+1}{n}}\sqrt{1 + \frac{[1 - \theta/n]^{2(n+1)}}{((n+1)/n)\theta^2 - [\theta((n+1)/n) - 1]^2}}. \tag{50}
$$

Note that

$$\frac{n+1}{n}\theta^2 - \left[\theta\frac{n+1}{n} - 1\right]^2 = 2\theta\frac{n+1}{n} - \theta^2\frac{n+1}{n^2} - 1 \geq 1$$

when θ satisfies

$$\frac{2n}{n+1} \leq \theta < n, \tag{51}$$

and (50) implies in this case that

$$\left\|h_n(\cdot)\right\|_\infty \leq \sqrt{\frac{n+1}{n}}\sqrt{1 + [1 - \theta/n]^{2(n+1)}} \leq \exp\left\{\frac{1}{2n} + \frac{1}{2}e^{-2\theta}\right\}.$$

The latter bound combines with (46) to imply that in the case of (51) (recall that $1 - \delta = \epsilon = \theta/n$) we get for all $|\lambda| \leq 1$:

$$\begin{aligned}\max_{|z|\leq 1}\left|p_n(z;\lambda,\epsilon)\right| &\leq \frac{1}{1-\theta/n}\exp\left\{\frac{1}{2n} + \frac{1}{2}e^{-2\theta}\right\} \\ &\leq \exp\left\{\frac{\theta}{n-\theta} + \frac{1}{2n} + \frac{1}{2}e^{-2\theta}\right\}.\end{aligned} \tag{52}$$

1.2^0. Now let us bound from above $\|1 - p_n(\cdot)\|_2$. We have

$$\begin{aligned}p_n(\zeta) &= 1 + \sum_{\ell=1}^{n-1}\left(\delta^\ell\left[1 - \frac{\ell}{n}\right] - \delta^{\ell-1}\left[1 - \frac{\ell-1}{n}\right]\right)[\lambda\zeta]^\ell - \frac{1}{n}\delta^{n-1}[\lambda\zeta]^n \\ &= 1 + \sum_{\ell=1}^{n-1}\delta^{\ell-1}\left([1-\epsilon]\left[1 - \frac{\ell}{n}\right] - \left[1 - \frac{\ell-1}{n}\right]\right)[\lambda\zeta]^\ell - \frac{1}{n}\delta^{n-1}[\lambda\zeta]^n \\ &= 1 - \left(\sum_{\ell=1}^{n-1}\delta^{\ell-1}\left[\epsilon\left[1 - \frac{\ell}{n}\right] + \frac{1}{n}\right][\lambda\zeta]^\ell + \frac{1}{n}\delta^{n-1}[\lambda\zeta]^n\right).\end{aligned}$$

Taking into account that $|\lambda| \leq 1$, we conclude that

$$\begin{aligned}\left\|1 - p_n(\cdot)\right\|_2 &\leq \left\|\sum_{\ell=1}^{n-1}(1-\epsilon)^{\ell-1}\epsilon\left[1 - \frac{\ell}{n}\right][\lambda\zeta]^\ell\right\|_2 + \frac{1}{n}\left\|\sum_{\ell=1}^{n}(1-\epsilon)^{\ell-1}[\lambda\zeta]^\ell\right\|_2 \\ &\leq \sqrt{\sum_{\ell=1}^{n}\epsilon^2(1-\epsilon)^{2(\ell-1)}} + \sqrt{1/n} \\ &\leq \sqrt{\frac{\epsilon^2}{1-(1-\epsilon)^2}} + \sqrt{1/n} \leq \sqrt{\epsilon} + \sqrt{\frac{1}{n}} \leq 2\sqrt{\max\left[\epsilon,\frac{1}{n}\right]}.\end{aligned} \tag{53}$$

2^0. Let $n = \lfloor m/d \rfloor$ and $\lambda_\ell = \exp\{-\iota \upsilon_\ell\}$, $1 \leq \ell \leq d$, where m, d and $\upsilon_1, \ldots, \upsilon_d$ are as described in the premise of the lemma. Let us set

$$\theta = \max\left[2, \frac{1}{2}\ln(2d)\right] \leq 3\ln(2d), \qquad \epsilon = \frac{\theta}{n},$$

$$q(\zeta) = 1 - p_n(\zeta; \lambda_1, \epsilon) \cdot p_n(\zeta; \lambda_2, \epsilon) \cdot \ldots \cdot p_n(\zeta; \lambda_d, \epsilon).$$

2.1^0. Observe that by (17) we have

$$n \geq n(d) := \mathrm{Ceil}(5d\theta) = \mathrm{Ceil}\left(5d \max\left[2, \frac{1}{2}\ln(2d)\right]\right). \tag{54}$$

Our choice of θ and n ensures (51), so that $0 < \epsilon < 1$, and by (52) we also have

$$\max_{|z|\leq 1}\left|p_n(z; \lambda_\ell, \epsilon)\right| \leq \exp\left\{\frac{\theta}{n - \theta} + \frac{1}{2n} + \frac{1}{2}e^{-2\theta}\right\} \leq e^{1/d}, \tag{55}$$

where the concluding inequality is readily given by the choice of θ and by (54). Recall that by (44), $p_n(\zeta; \lambda_\ell, \zeta)$ is divisible by $(1 - \lambda_\ell \zeta)$; when setting

$$r_\ell(\zeta) = \frac{p_n(\zeta; \lambda_\ell, \epsilon)}{1 - \lambda_\ell \zeta} = \sum_{\ell=0}^{n-1}\left[(1 - \epsilon)\lambda_\ell \zeta\right]^\ell[1 - \ell/n],$$

and $r(\zeta) = \prod_{\ell=1}^d r_\ell(\zeta)$, we clearly have $r(0) = 1$, $\deg r \leq d(n-1) \leq m - d$, and

$$1 - q(\zeta) = p_{\mathbf{u}}(\zeta) \prod_{\ell=1}^d r_\ell(\zeta) = p_{\mathbf{u}}(\zeta)r(\zeta),$$

as required in (18) (note that q is a real polynomial due to $\mathbf{u} \in \Omega_d$).

2.2^0. By (55) we have $\|p_n(\cdot; \epsilon, \lambda_k)\|_\infty \leq e^{1/d}$, while (53) says that

$$\left\|p_n(\cdot; \lambda_k, \epsilon) - 1\right\|_2 \leq 3\sqrt{\ln(2d)/n}.$$

We have

$$1 - \prod_{k=1}^{\ell+1} p_n(\cdot; \lambda_k, \epsilon) = \left[1 - \prod_{k=1}^{\ell} p_n(\cdot; \lambda_k, \epsilon)\right]p_n(\cdot; \lambda_{\ell+1}, \epsilon) + \left[1 - p_n(\cdot; \lambda_{\ell+1}, \epsilon)\right].$$

When denoting $\alpha_\ell = \|1 - \prod_{k=1}^\ell p_n(\cdot; \lambda_k, \epsilon)\|_2$ for $\ell = 1, 2, \ldots, d$ and setting $\alpha_0 = 0$, we get

$$\alpha_{\ell+1} \leq \alpha_\ell\left\|p_n(\cdot; \lambda_{\ell+1}, \epsilon)\right\|_\infty + \left\|p_n(\cdot; \lambda_{\ell+1}, \epsilon) - 1\right\|_2 \leq \alpha_\ell e^{1/d} + 3\sqrt{\ln(2d)/n}, \qquad 0 \leq \ell < d.$$

It follows that $\alpha_\ell \leq 3\ell\sqrt{\frac{\ln(2d)}{n}}\mathrm{e}^{\ell/d}$ when $\ell \leq d$, which implies

$$\left\|q(\cdot)\right\|_2 \leq 3\mathrm{e}d\sqrt{\ln(2d)/n} \leq 3\mathrm{e}d^{3/2}\sqrt{\frac{\ln(2d)}{m}},$$

and (19) follows.

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