Approximation Algorithm for Subset k-Connectivity

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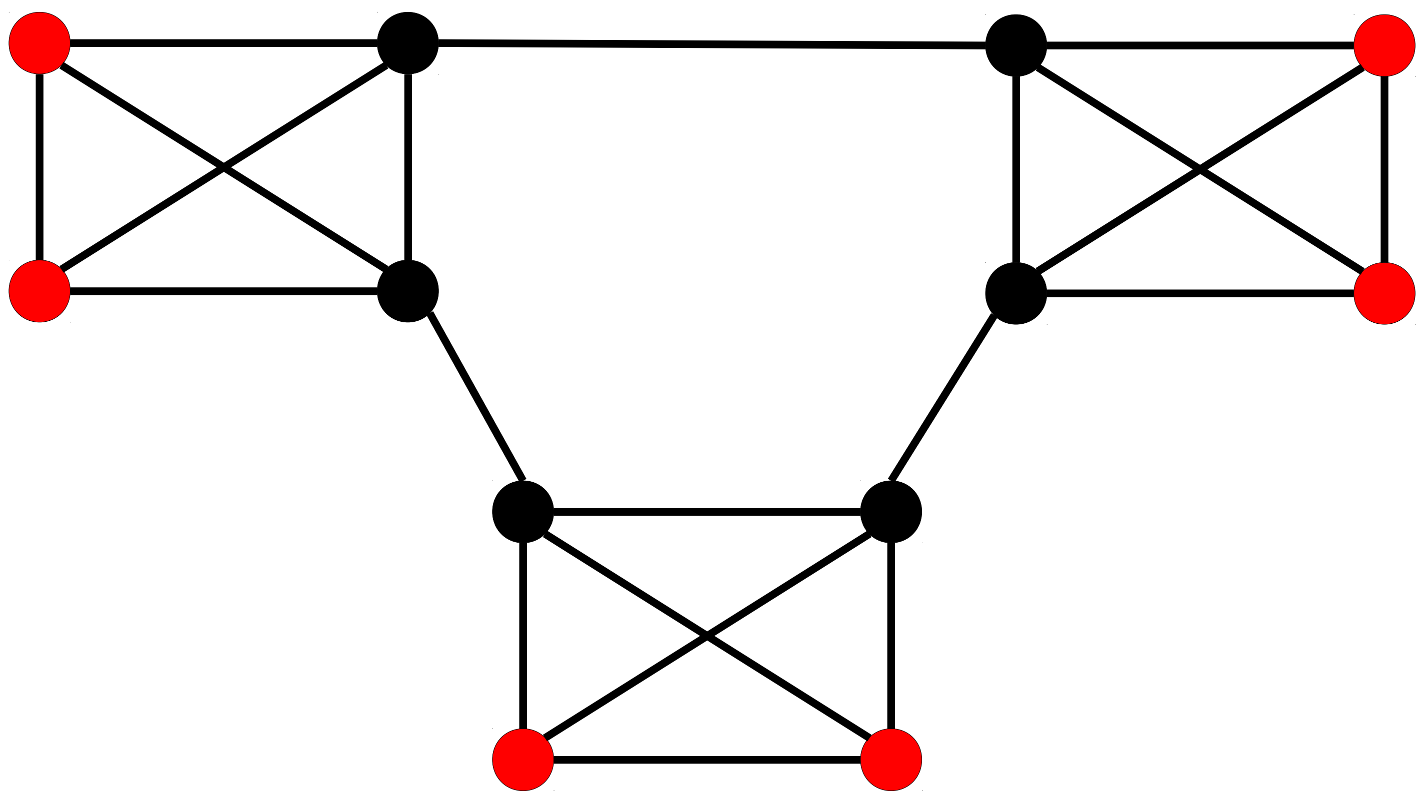
Outline of This Talk

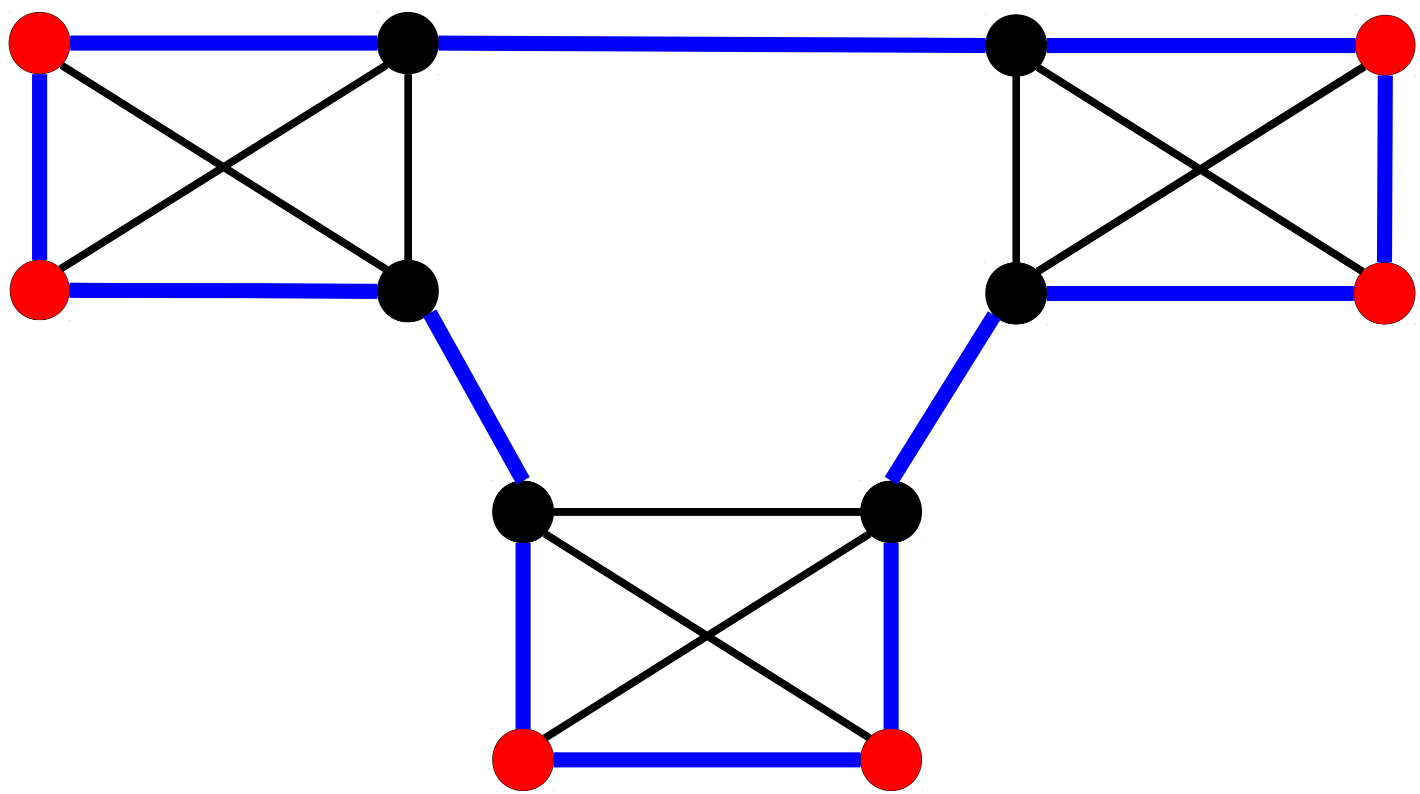
- Problem Formulation
- Structural Properties
- Main Algorithm

Min-cost subset k-connected subgraph problem (Subset k-Conn)

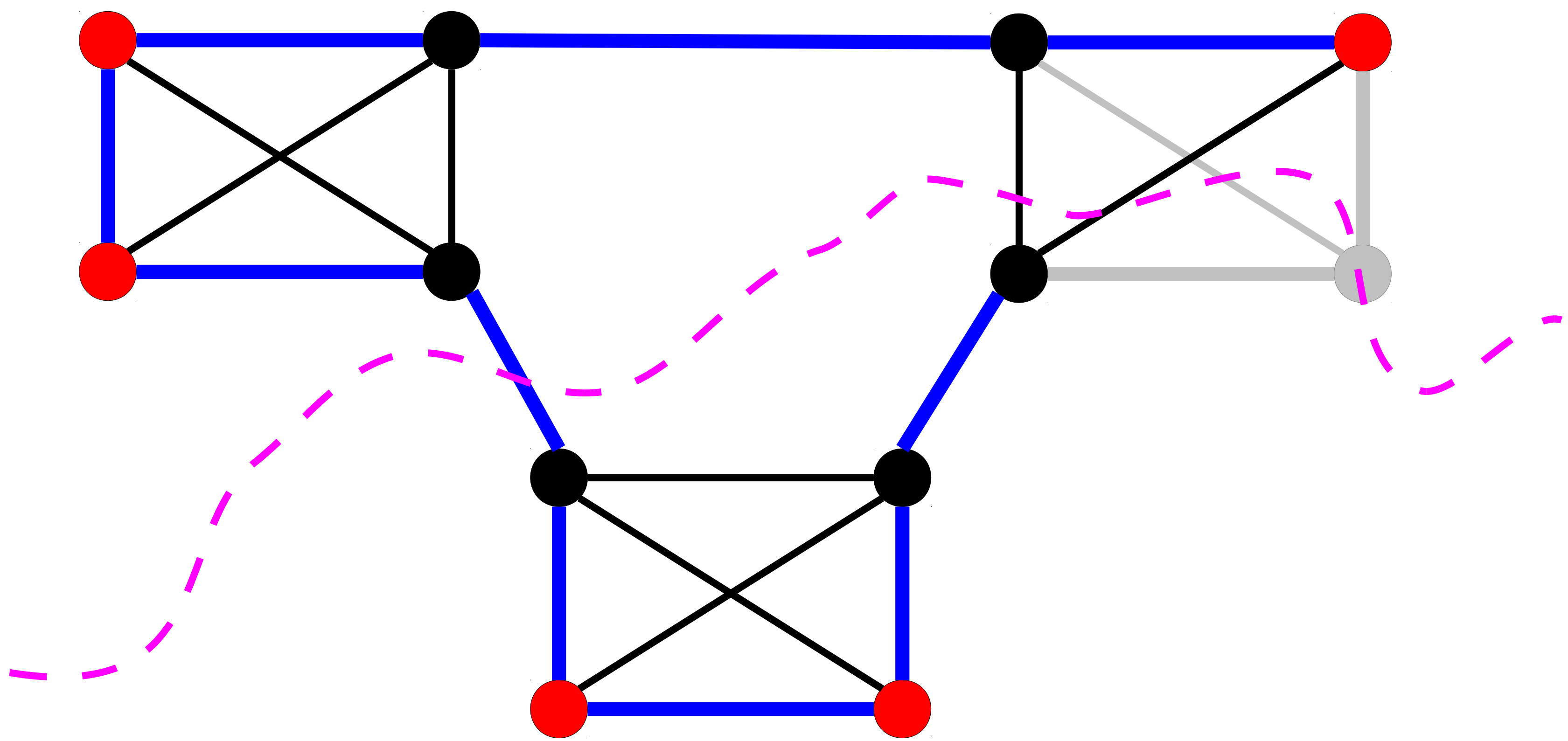
Given a graph with edge-cost, a set of terminals, and requirement k



We want to pay cheap cost to make a graph k-connected on terminals.



E.g, ($k=2$) all terminals remain connected after removing one vertex



Min-cost Subset k-Connected Subgraph Problem (subset k-conn)

Input:

- Graph $G=(V, E)$ with non-negative cost on edges
- A set of terminals T.
- An integer k, a requirement

Goal:

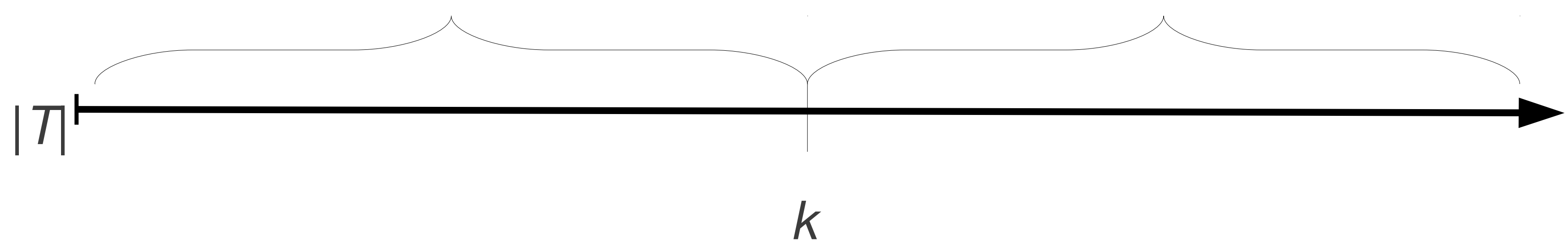
- Find a min-cost subgraph $H=(V, E')$.
- H has k-vertex disjoint paths connecting each pair s,t of terminals.

$k=1$: Steiner tree problem $\Rightarrow$ NP-Hard

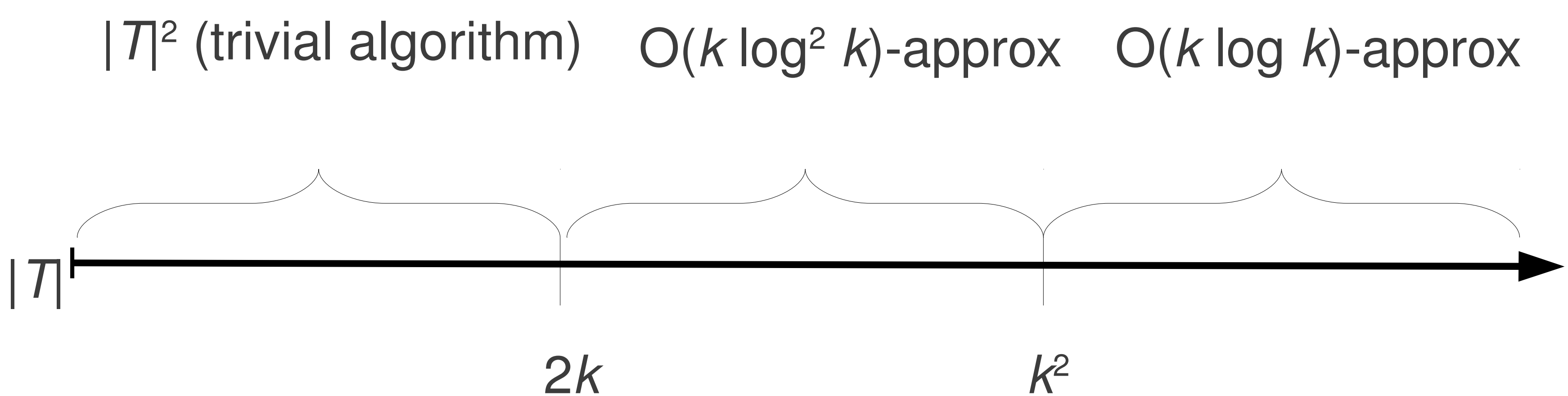
Current Status in terms of $|T|$ and k

UB: $|T|^2$ (trivial algorithm)
LB: LabelCover $\approx \Omega(k^\varepsilon)$

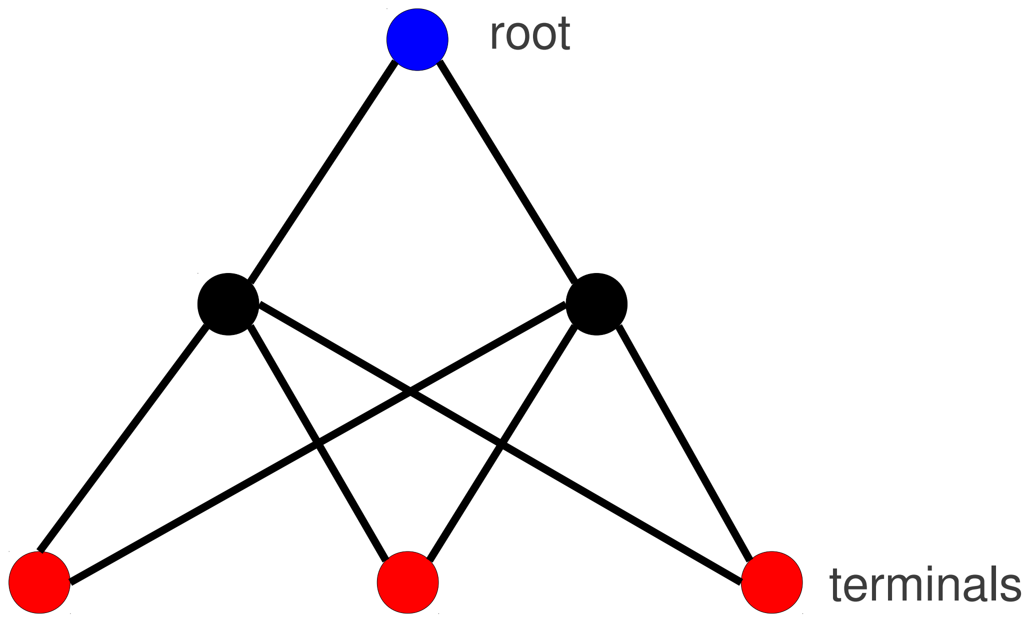
UB: $O(k^2 \log k)$-approx
LB: APX-Hard



Our Results

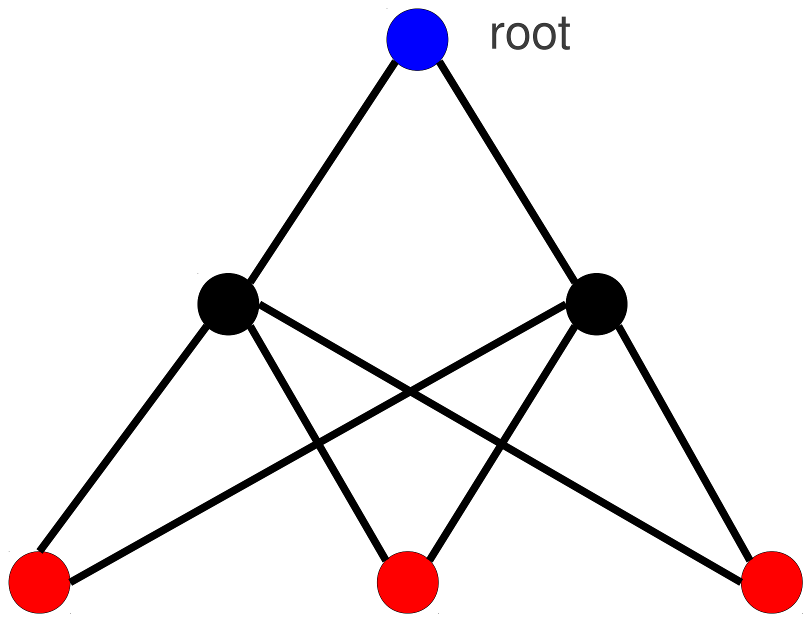


Closely related problem
rooted subset k-connectivity



Given: G=(V,E), edge-cost, root r, terminal set T
Goal: Find min-cost subgraph having k-disjoint r,t path for each terminal t

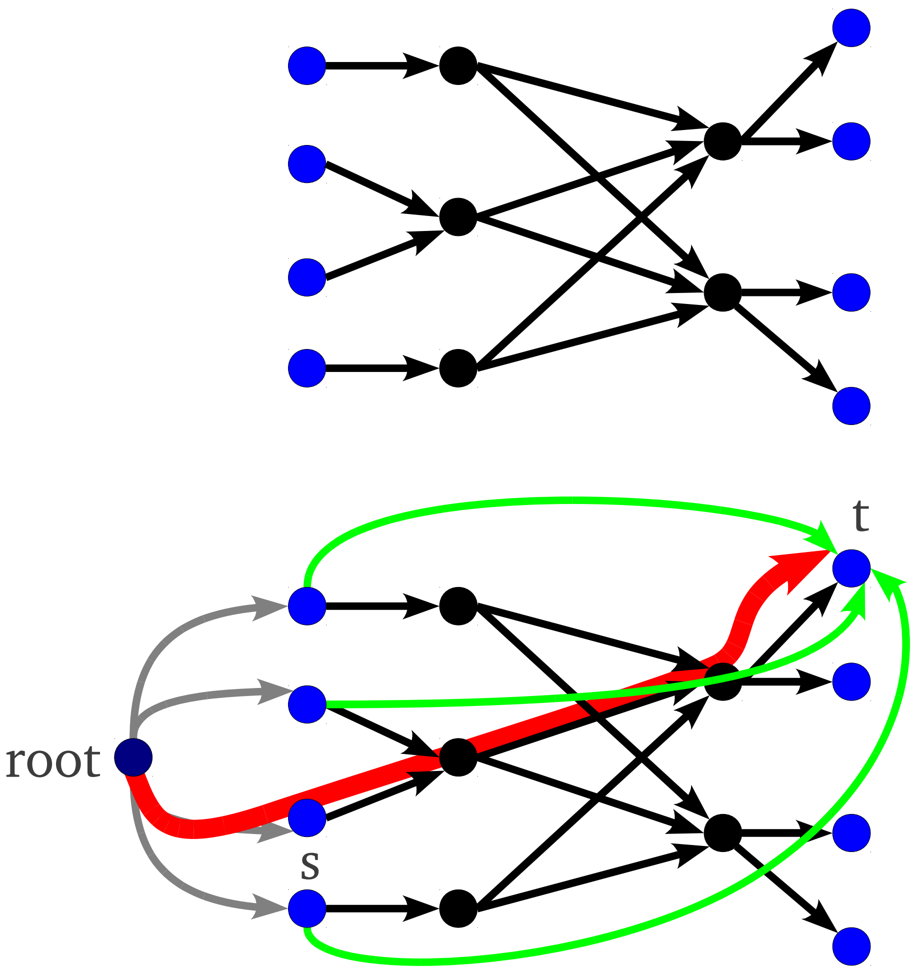
Closely related problem
rooted subset k-connectivity



Current Status:

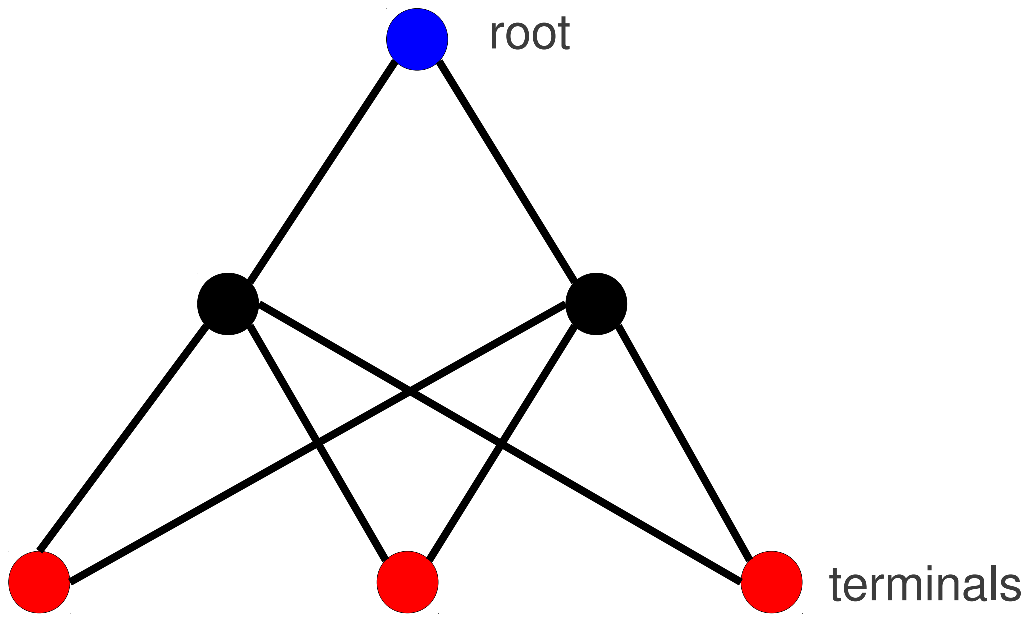
UB : O(k log k)-approx,   LB : O(k^{ε})-hardness [Cheriyan, L. '11]

Hardness of rooted k-conn



- Reduce Directed Steiner Forest to Rooted k-Conn on Directed Graphs
- Apply Lando-Nutov's Thm to reduce the problem to undirected graphs (with connectivity $k'=k+|V|$)
- Hardness can be tighten to k^{ε} by reducing it directly from LabelCover.

Reducing subset k-connectivity to rooted (subset) k-connectivity



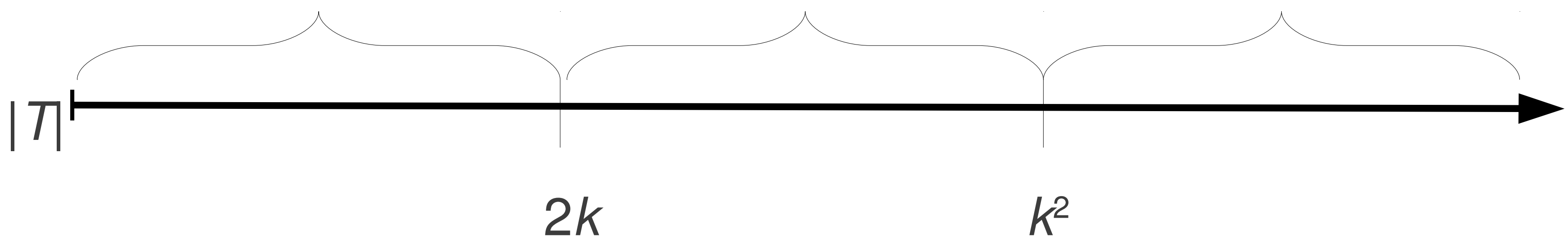
All previous algorithms solve subset k-connectivity by applying rooted k-connectivity algorithm to k terminals (pay a factor of k)

Comparison of Approx Ratio

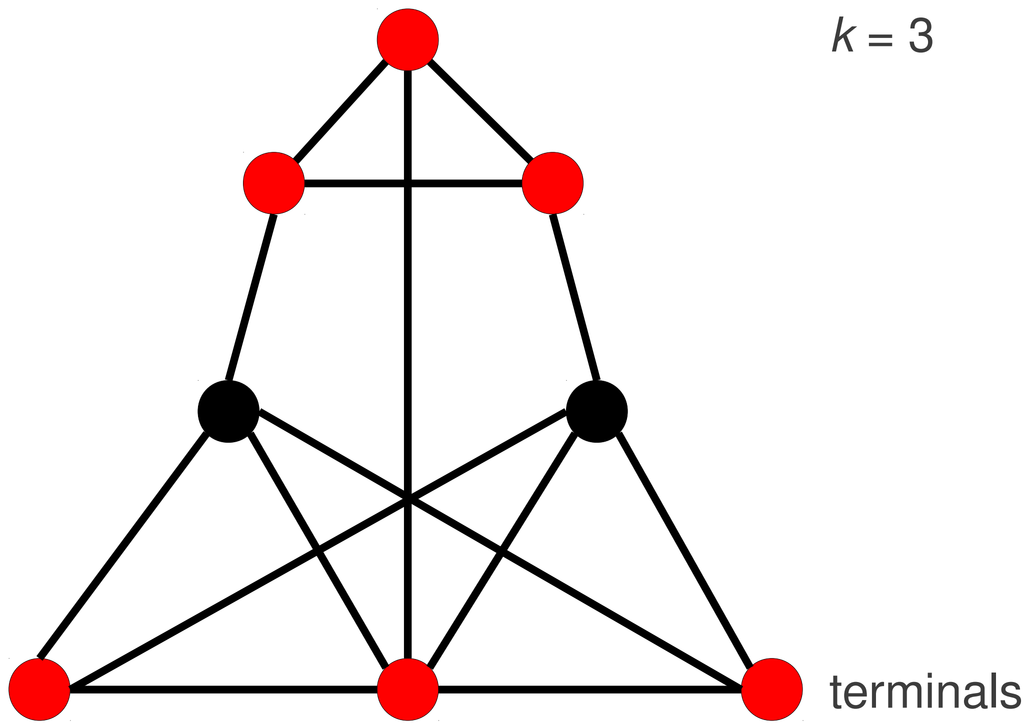
Subset: $|T|^2$
Rooted: $|T|$
(trivial algorithm)

Subset: $O(k \log^2 k)$
Rooted: $O(k \log k)$

Subset & Rooted:
$O(k \log k)$



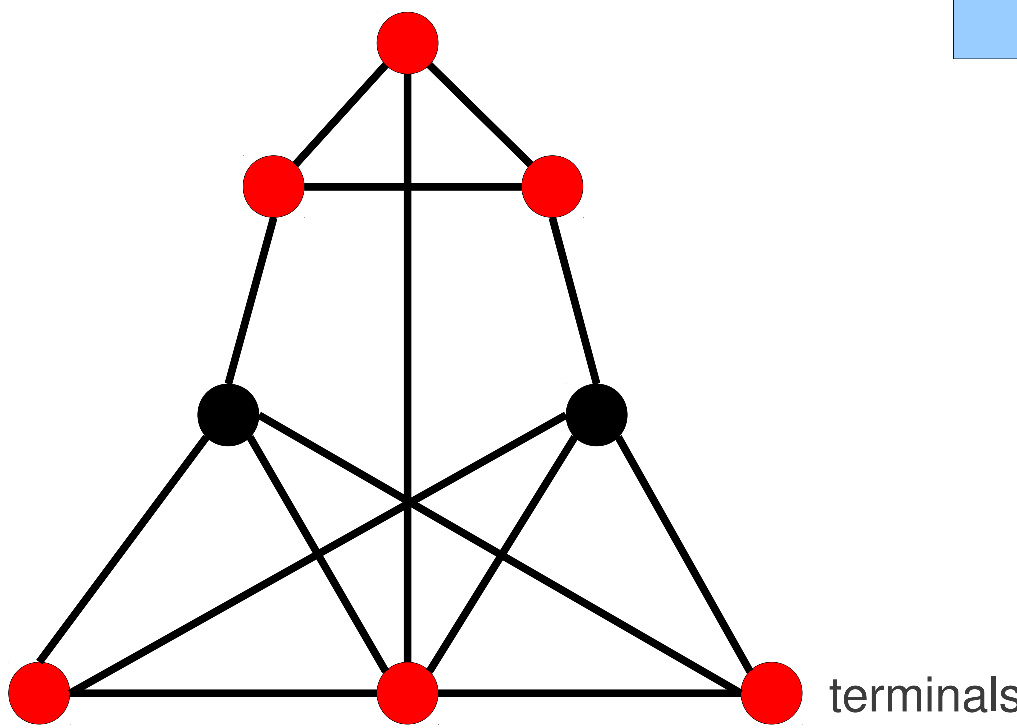
Subset k-conn is hardest when $|T|=k$



Subset k-conn is hardest when $|T|=k$

G(k)-aprx algo for **rooted** k-conn

F(k)-aprx algo for **subset** k-conn for $|T| = k$



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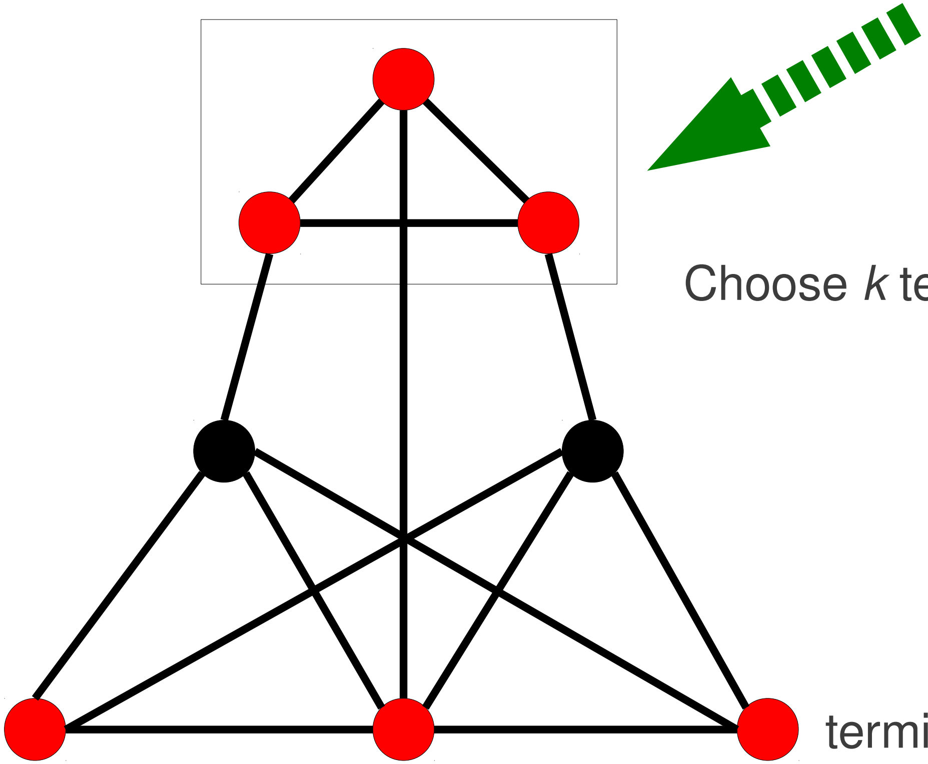
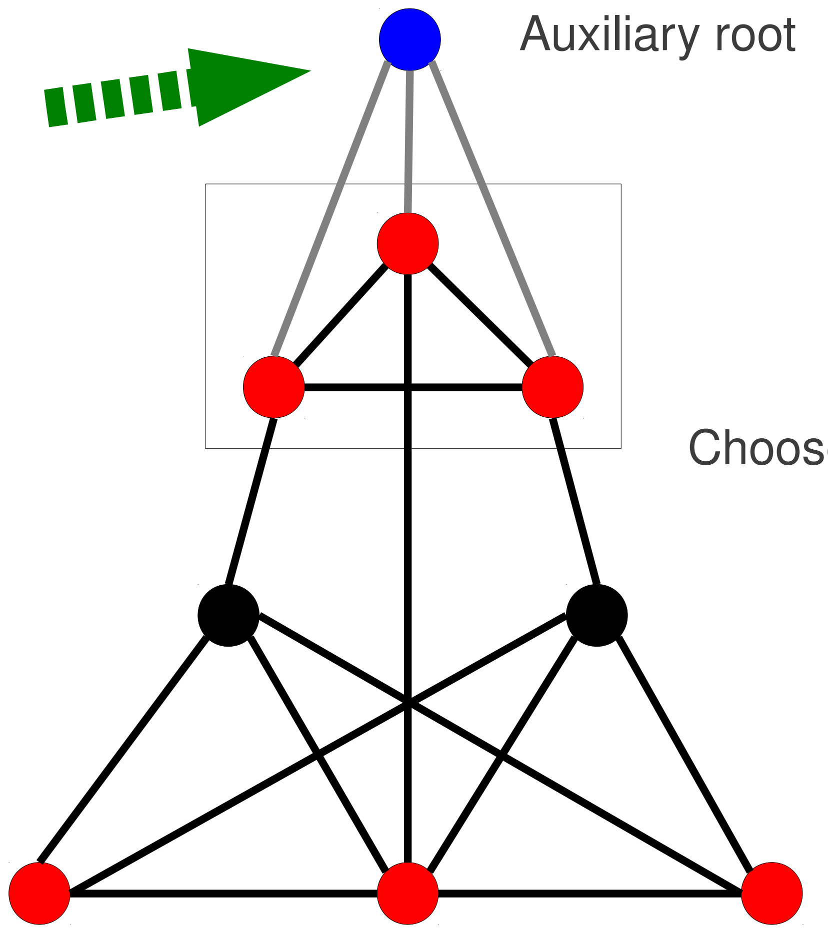


Subset k-conn is hardest when $|T|=k$

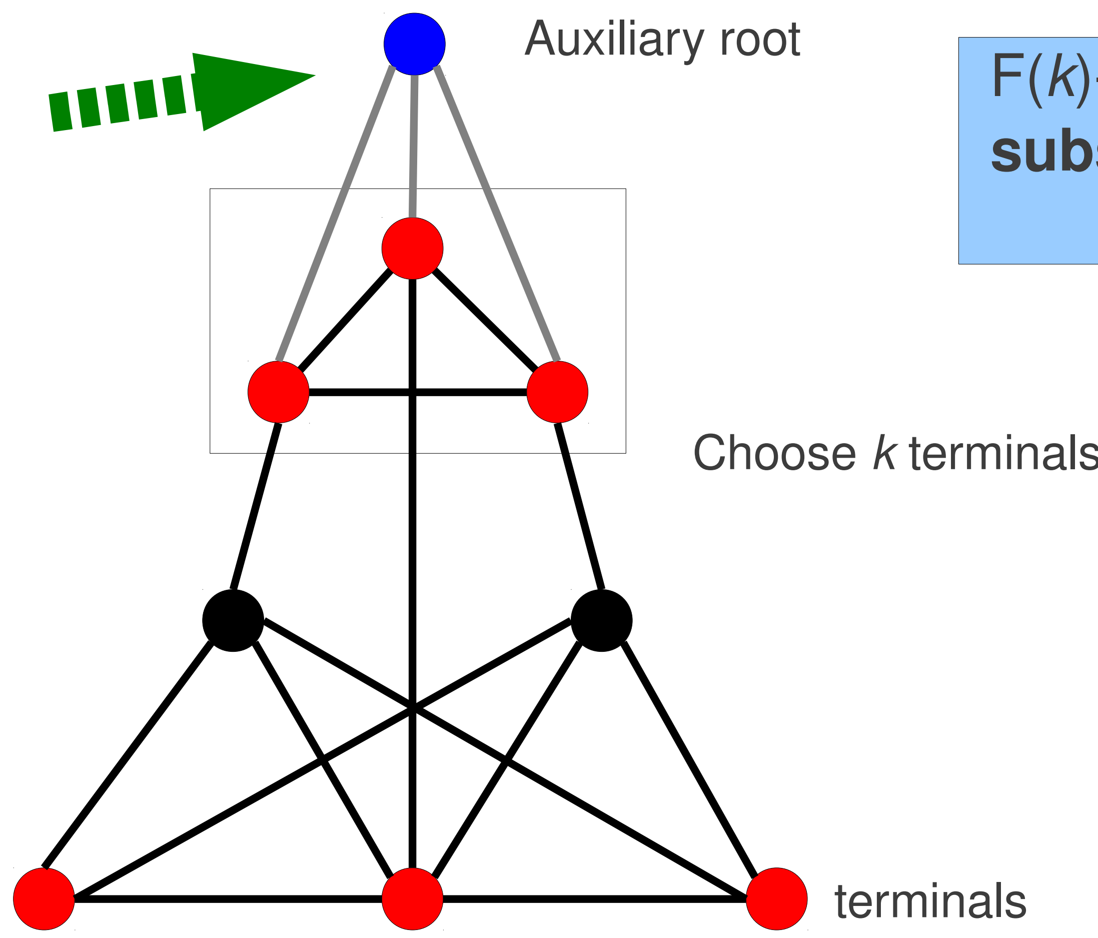
G(k)-aprx algo for **rooted** k-conn



F(k)-aprx algo for **subset** k-conn for $|T| = k$

Subset k-conn is hardest when $|T|=k$

G(k)-aprx algo for **rooted** k-conn



F(k)-aprx algo for **subset** k-conn for $|T| = k$

All terminals k connected, Approx Ratio = G(k) + F(k)

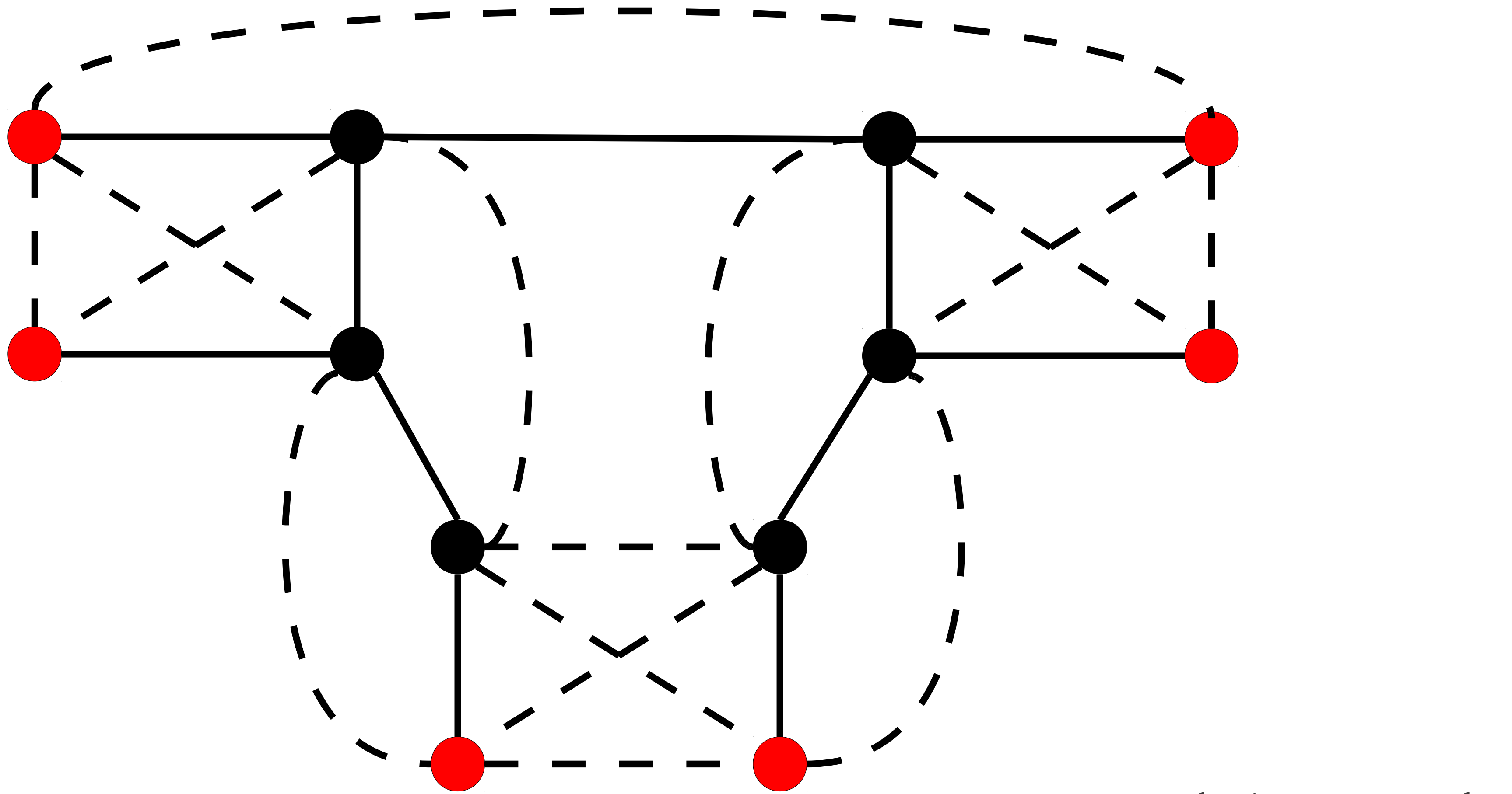
Outline of our technique

- Use Connectivity Augmentation Framework
- Halo-set Method: Apply rooted k-connectivity algorithm to covering deficient sets

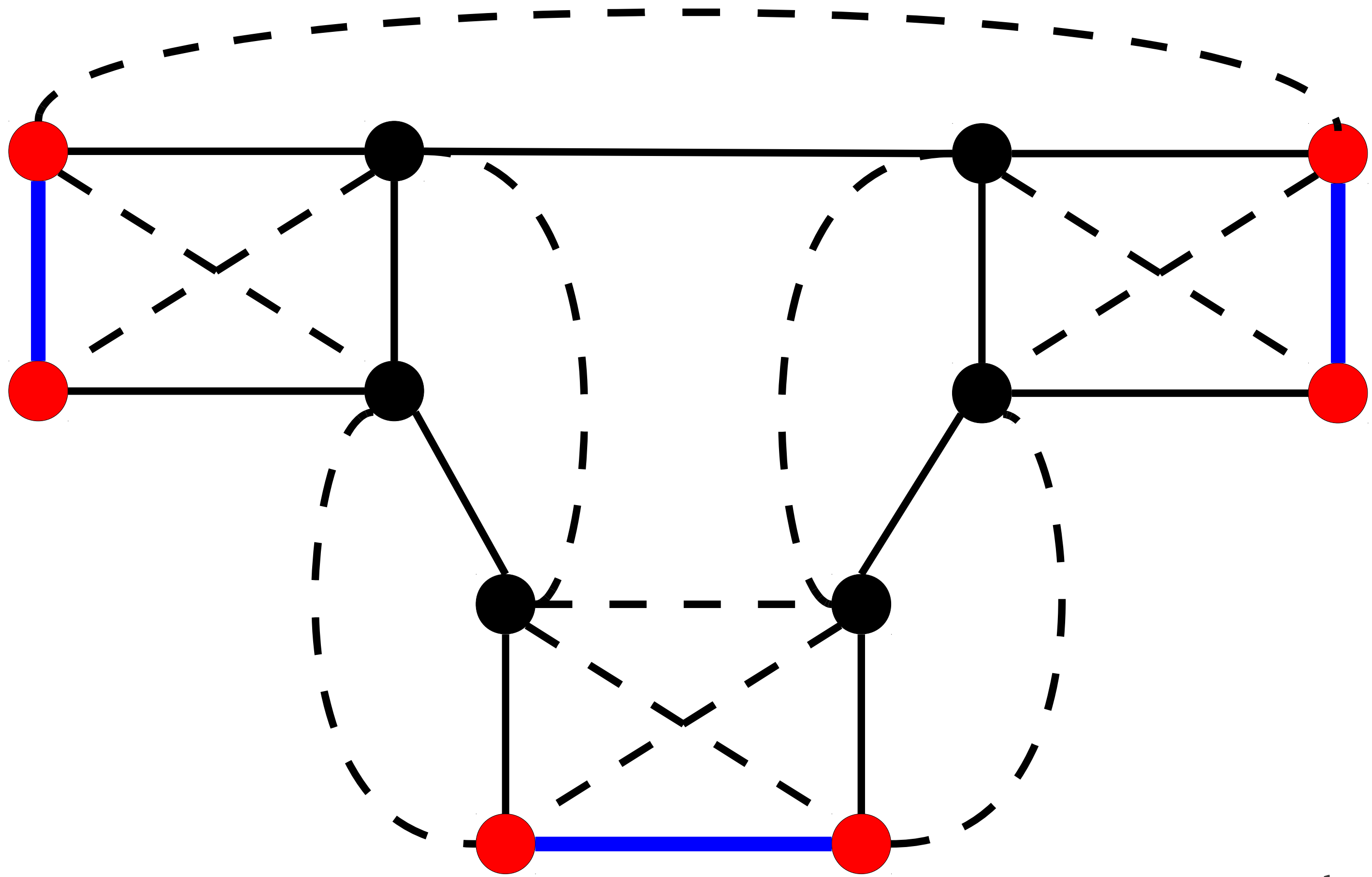
Connectivity Augmentation framework.

Increase connectivity from $L=1, 2,.., k$

Start from connectivity = 1

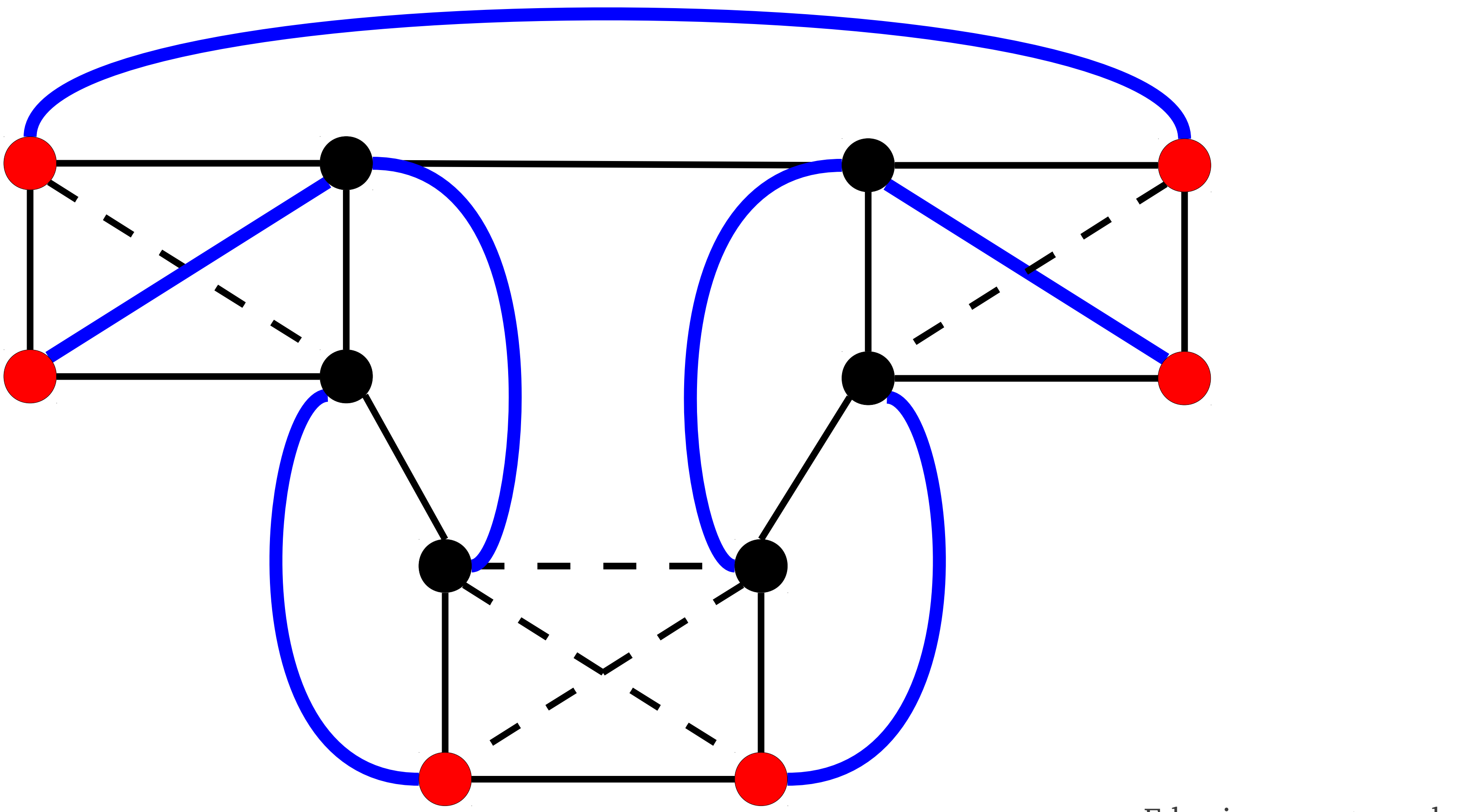


Increase connectivity to 2

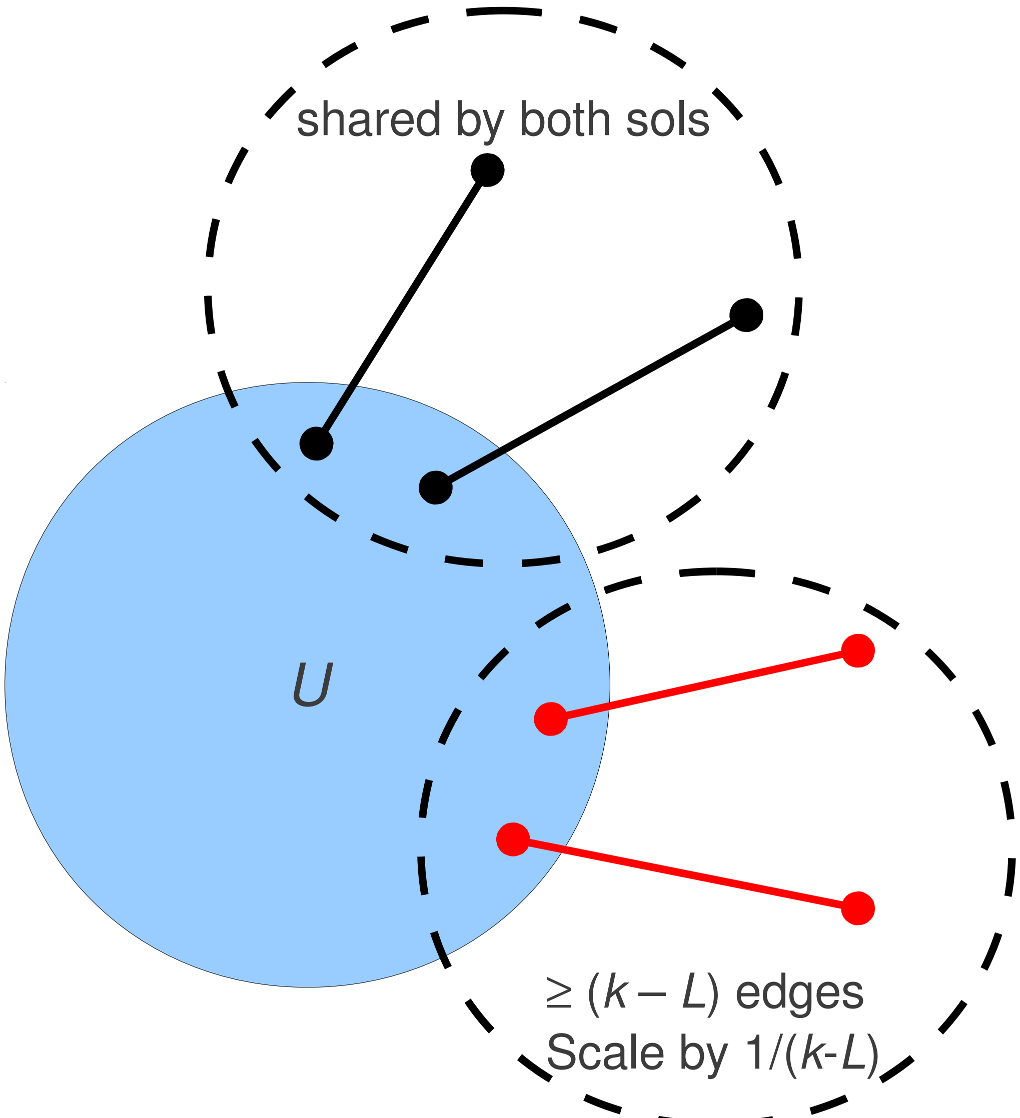


Edge in current graph
Edge that can be added

Increase connectivity to 3



Pay a factor of O(log k)



- LP for Conn. Aug., $L \Rightarrow (L+1)$, asks for one edge covering cut U
- Take integral OPT.
- Scale edges by $1/(k - L)$

  $\Rightarrow$ Sol feasible to LP

- Run k times pay O(log k)

Assume a graph is subset L-connected on T,
and we want to increase connectivity to $L+1$

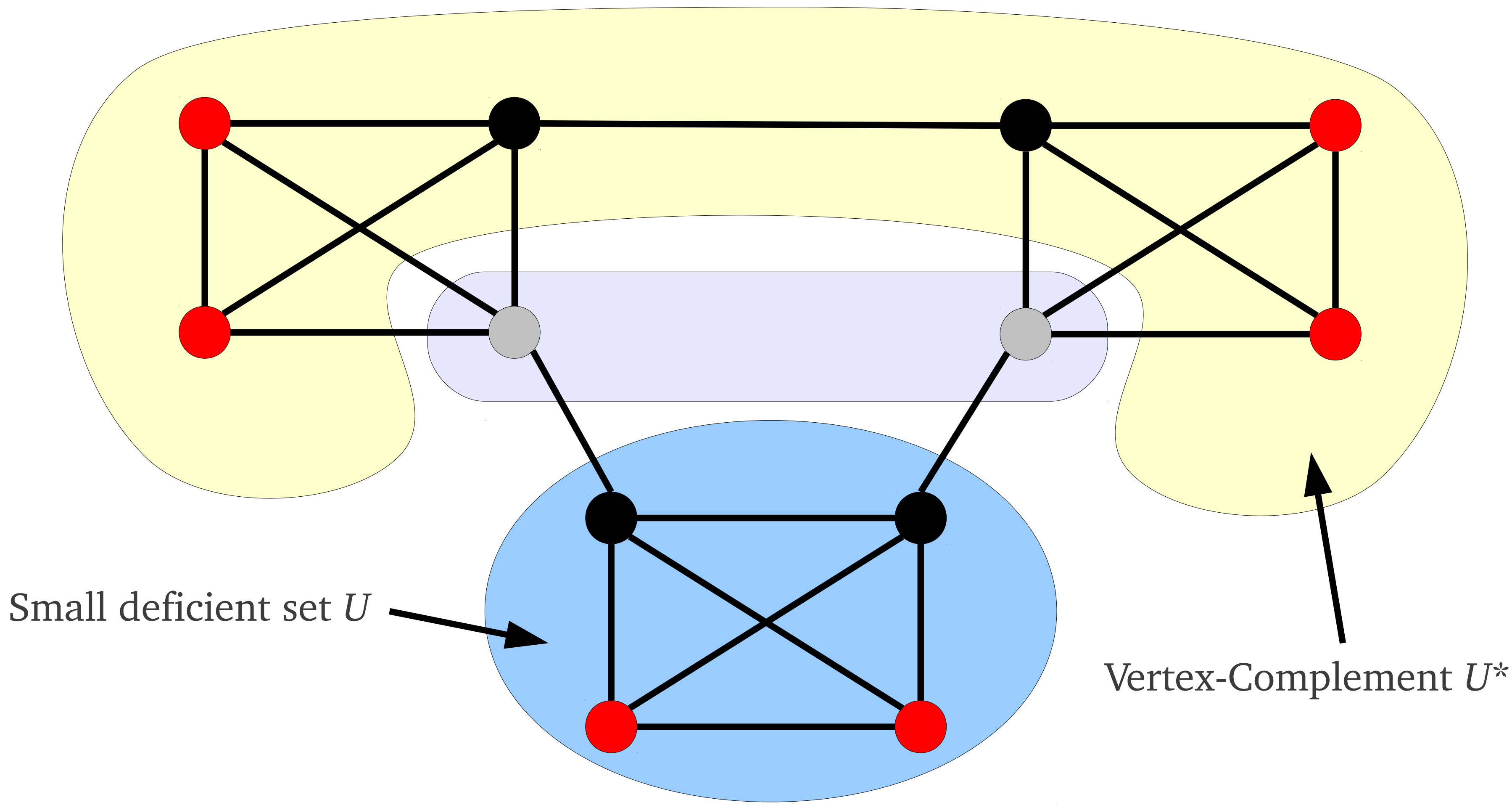
Also, our goal is to attack the case $|T| \geq 2k$

So, we assume $|T| \geq 2L$

Structure of subset L-connected graph

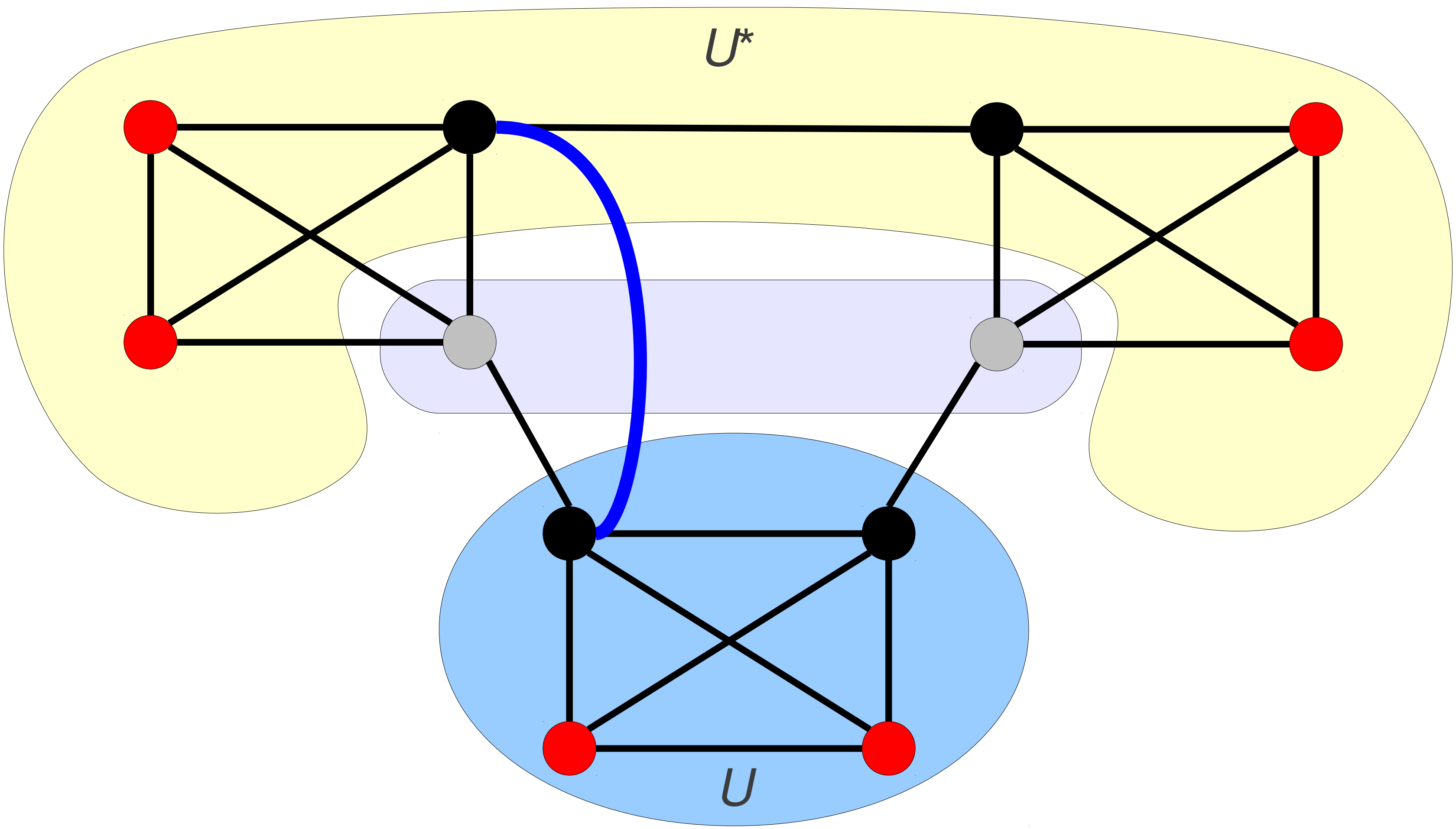
Deficient Set

- Set of veritces U with L neighbors, removing neighbors separates some pair of terminals.
- $|U \cap T| \leq |U^{*} \cap T| \Rightarrow$ call small deficient set.



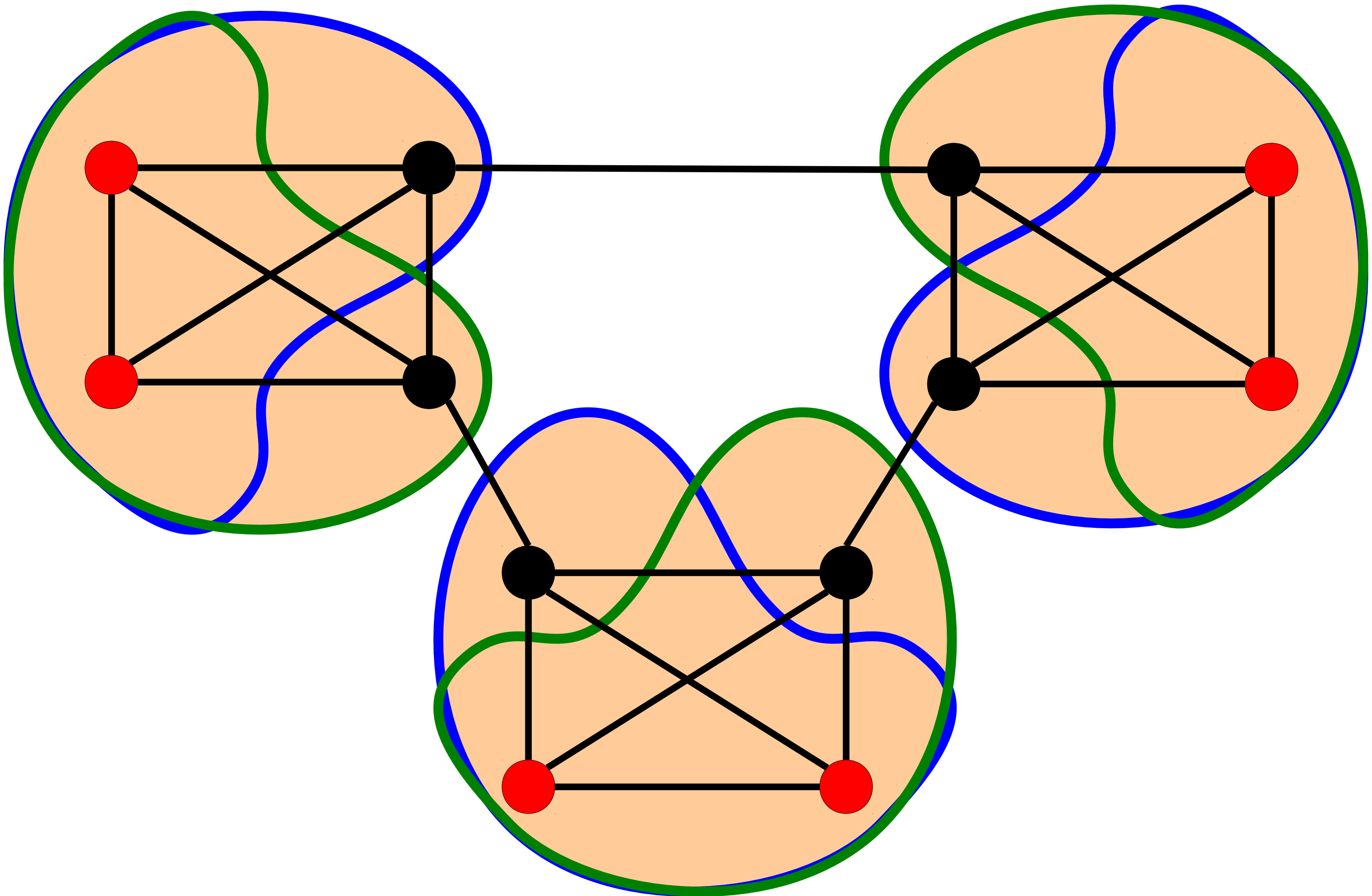
Cover Deficient Set

- Add an edge between *U* and *U**



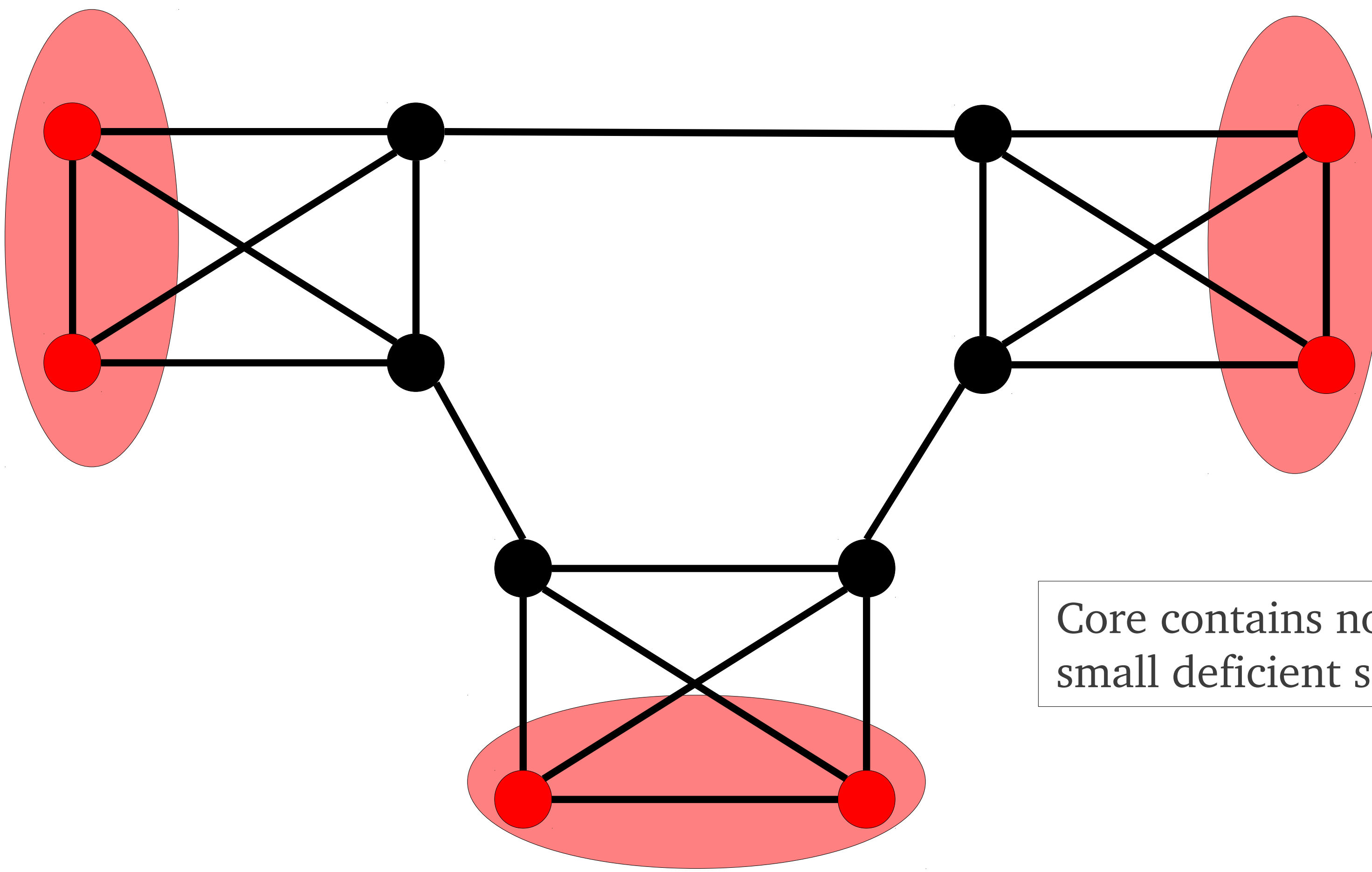
Key Idea (Halo-Set method)

- Group deficient sets by notion of halo-families
- Goal: Pay cheap cost to dec #halo-fami by $\times 1/2$



Core

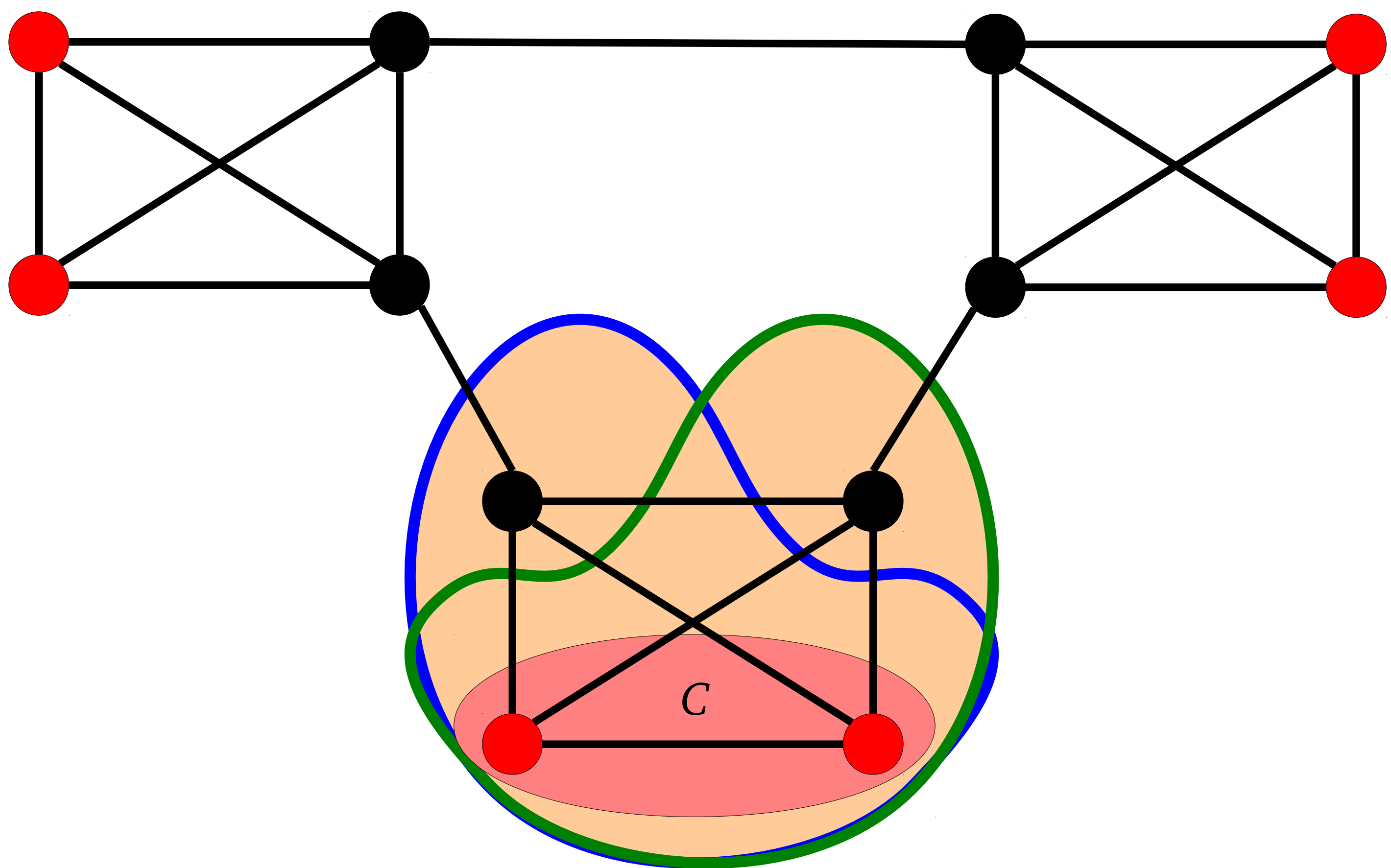
- An inclusionwise minimal small deficient set



Core contains no other small deficient sets

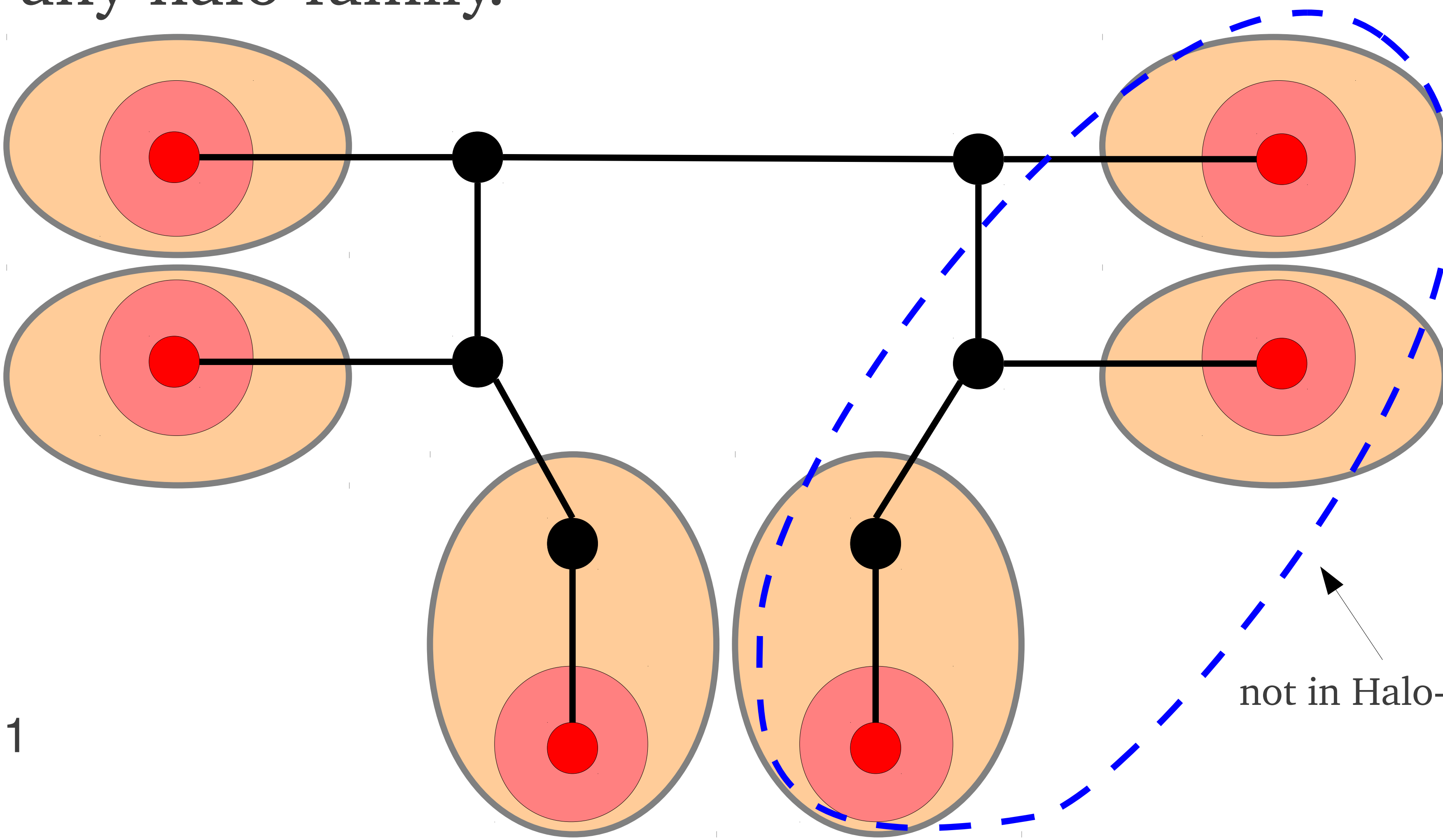
Halo-family of a core C

- $Halo(C) = \{U : U$ is a small deficient set, U contains C and contains no core $D \neq C\}$



Some small deficient sets are not in any halo-family

- Small deficient sets contain ≥ 2 cores are not in any halo-family.



L=1

Halo-set Method: Framework

- **For** $L=0,1, \ldots, k$

  [connectivity augmentation]

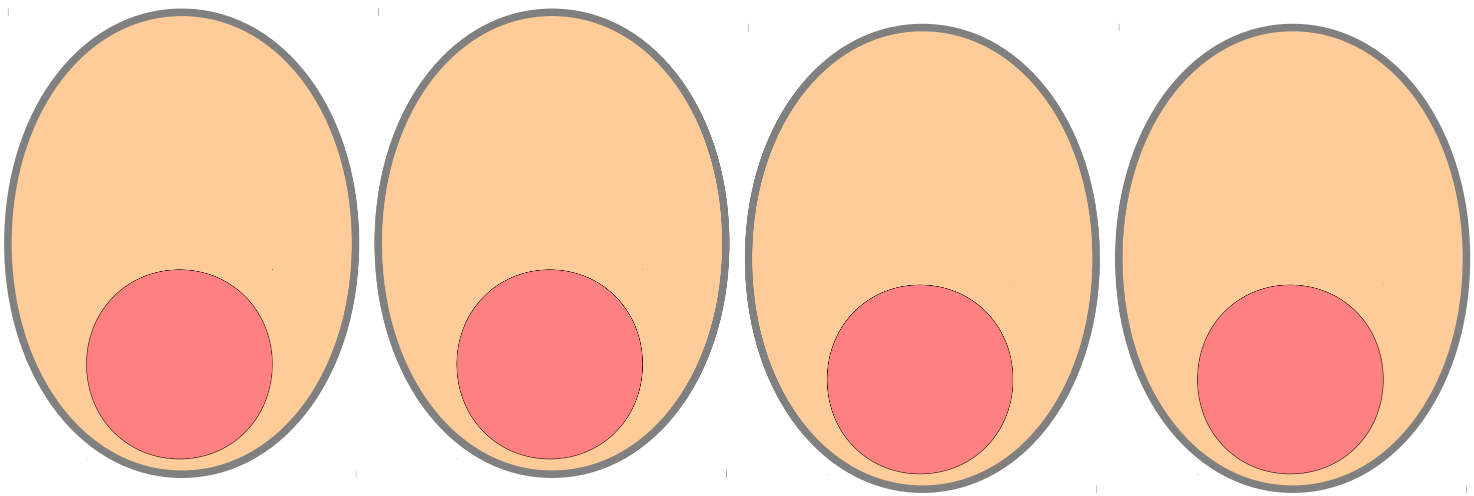
  - **While** # of cores > 0
    - Compute “cores” and “halo-families”
    - Find edges covering all halo-families
  - **End While**

  [end connectivity augmentation]

- **End For**

Halo-set Method

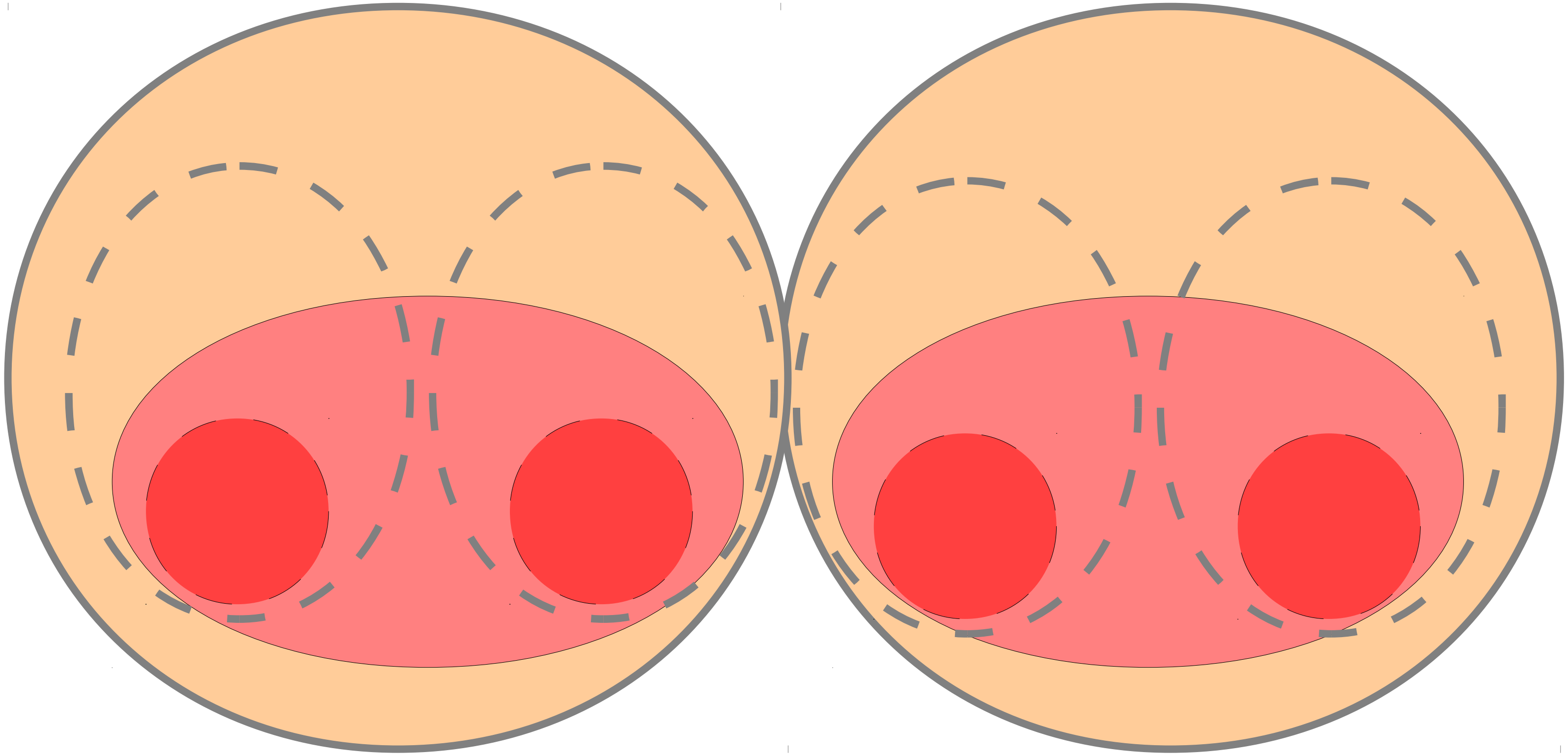
- In each round, we cover all the halo-families.



Round 1

Halo-set Method

- Recompute and cover (new) halo-families.

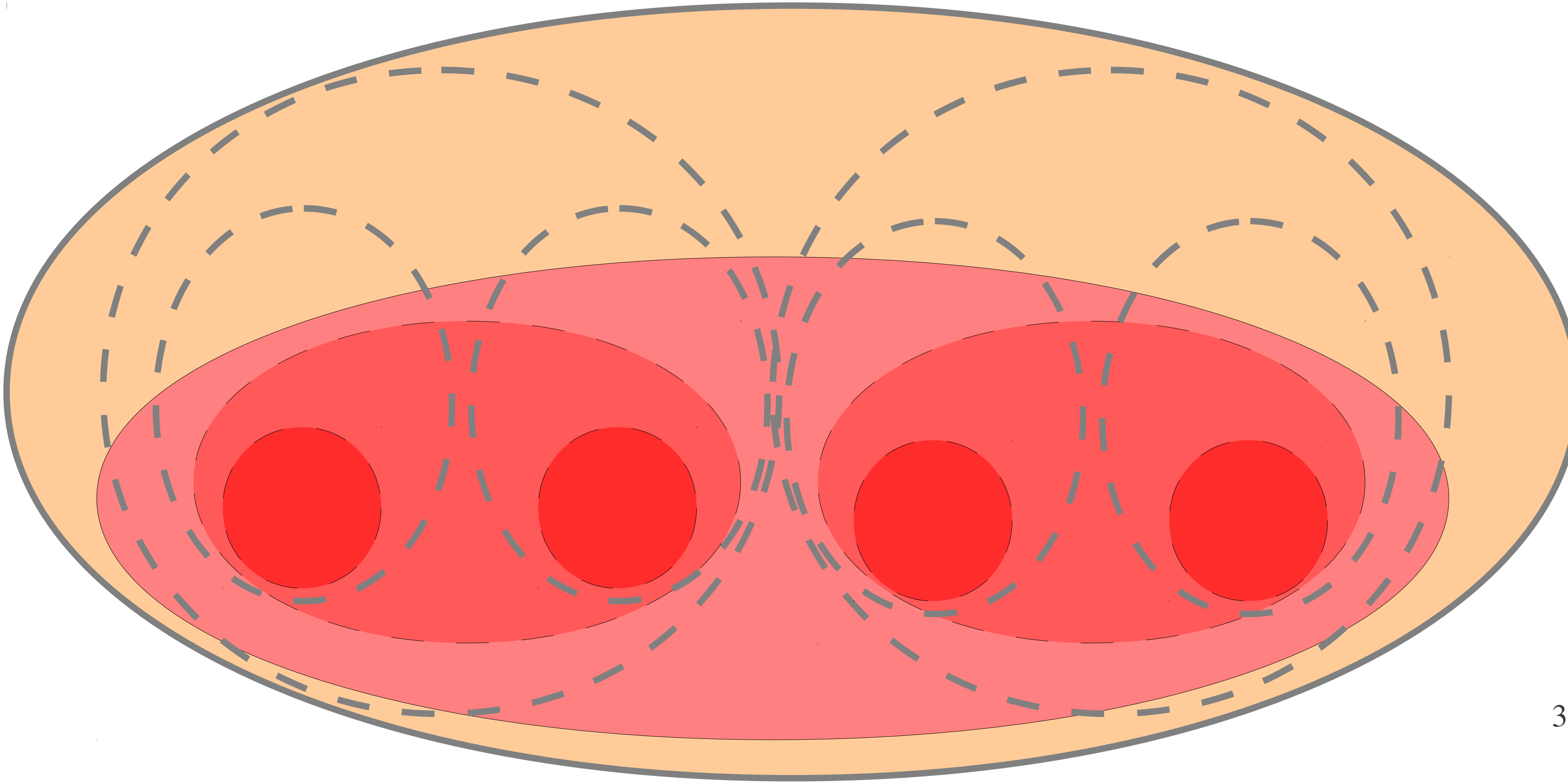


Round 2

Halo-set Method

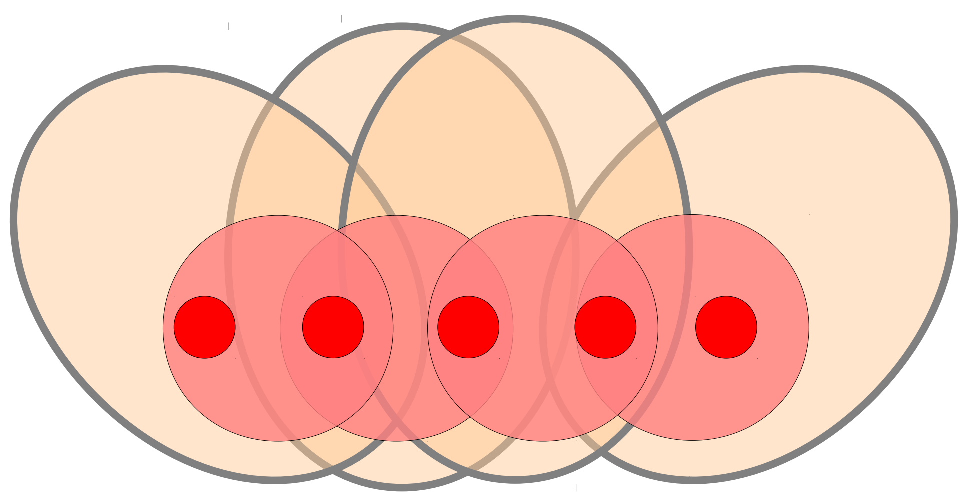
- Repeat it again until no cores/halo-families left.

  $\Rightarrow$ terminates in $O(\log q)$ round, $q = \#$ of cores



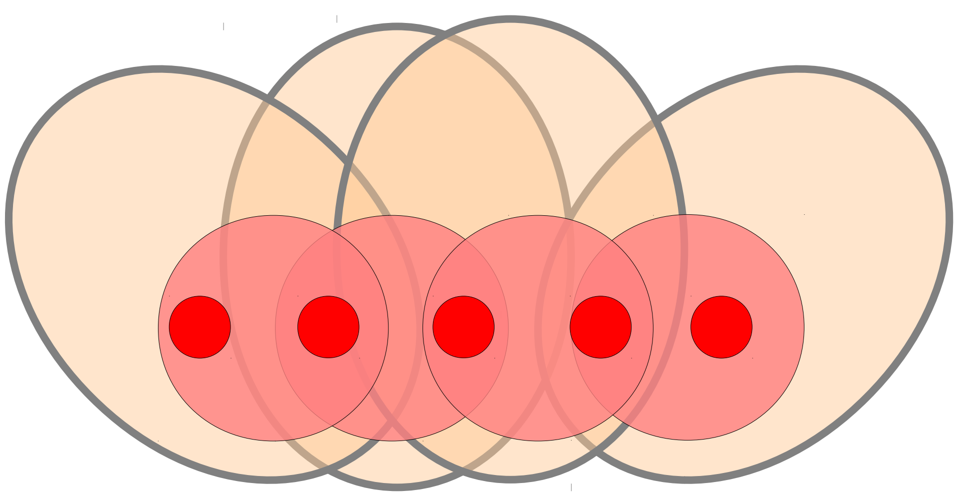
Some difficulties

- Cores/Halo-families can intersect on terminals
- # of Halo-families can be $O(|T|^2)$
- # of Halo-families can increase (after adding edges)
- No known algo for covering all halo-families



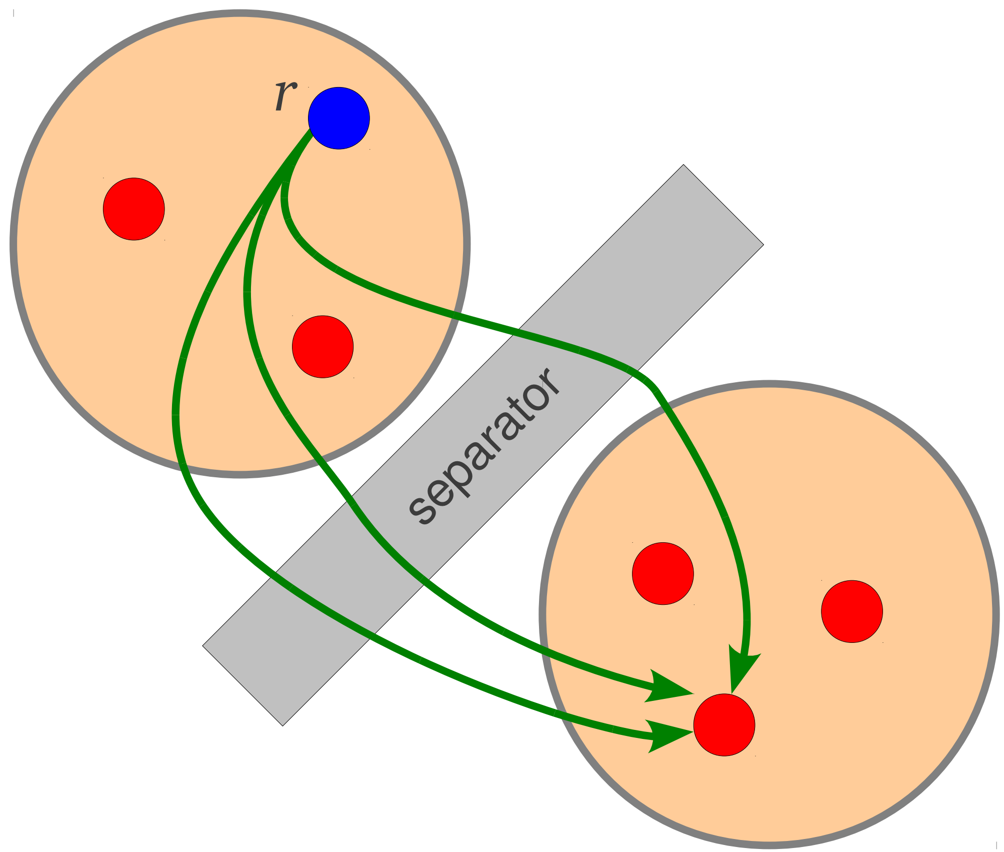
Solve the difficulties (for $|T| \geq 2L$)

- Each terminal is in $\leq$ O(1) Halo-families
- Preprocessing $\Rightarrow$ decreases # of Halo-families to O(L)
- Use max #of terminals-disjoint cores as notion of progress
- **Main Algo**: Cover Halo-families by rooted ($L+1$)-conn algo.



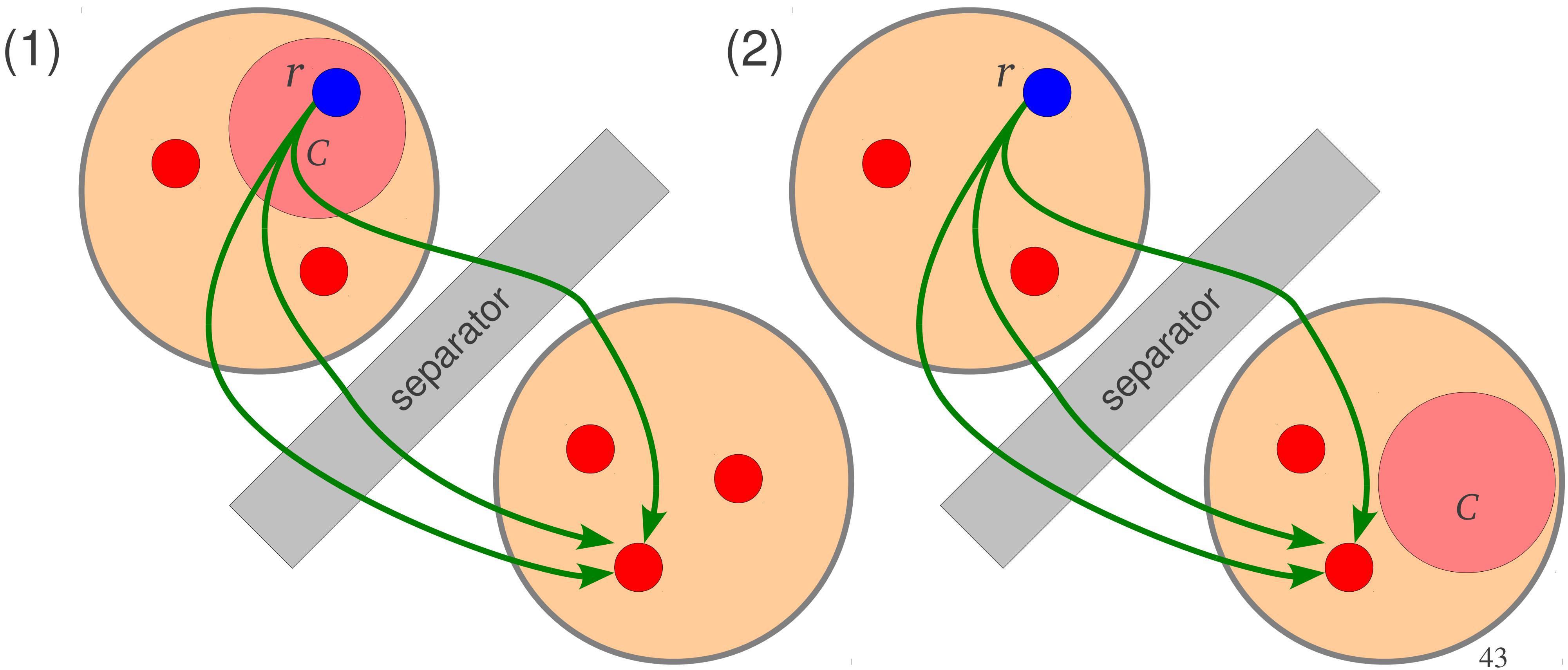
Main Algorithm: Observation

- Running rooted $(L+1)$-conn algorithm with a root r covers all deficient set containing r.



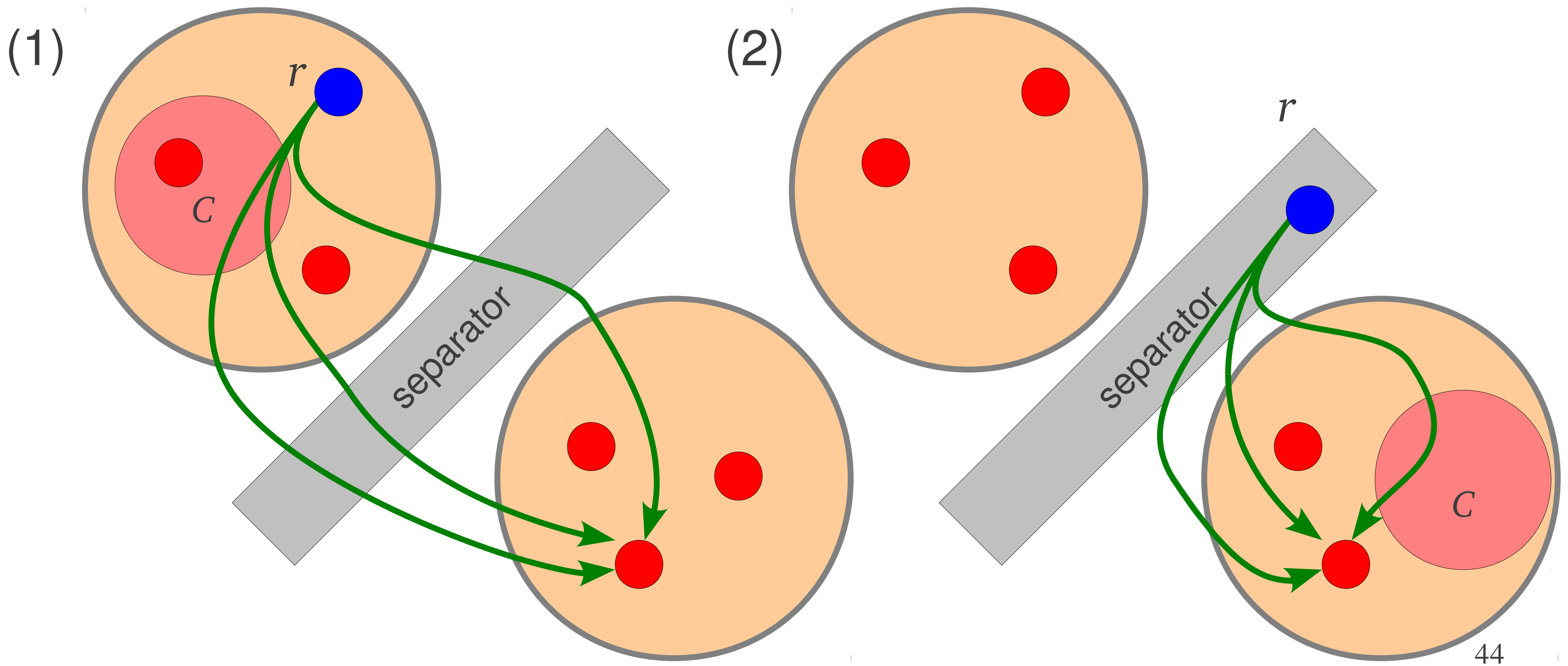
Main Algorithm: Observation

- If chosen root r is (1) in a core C or (2) in vertex-complement of $Halo(C)$, then rooted algo covers $Halo(C)$.



Main Algorithm: Observation

- $Halo(C)$ is NOT covered if chosen root r is (1) in $Halo(C)$ but not in C or (2) is a neighbor $Halo(C)$.

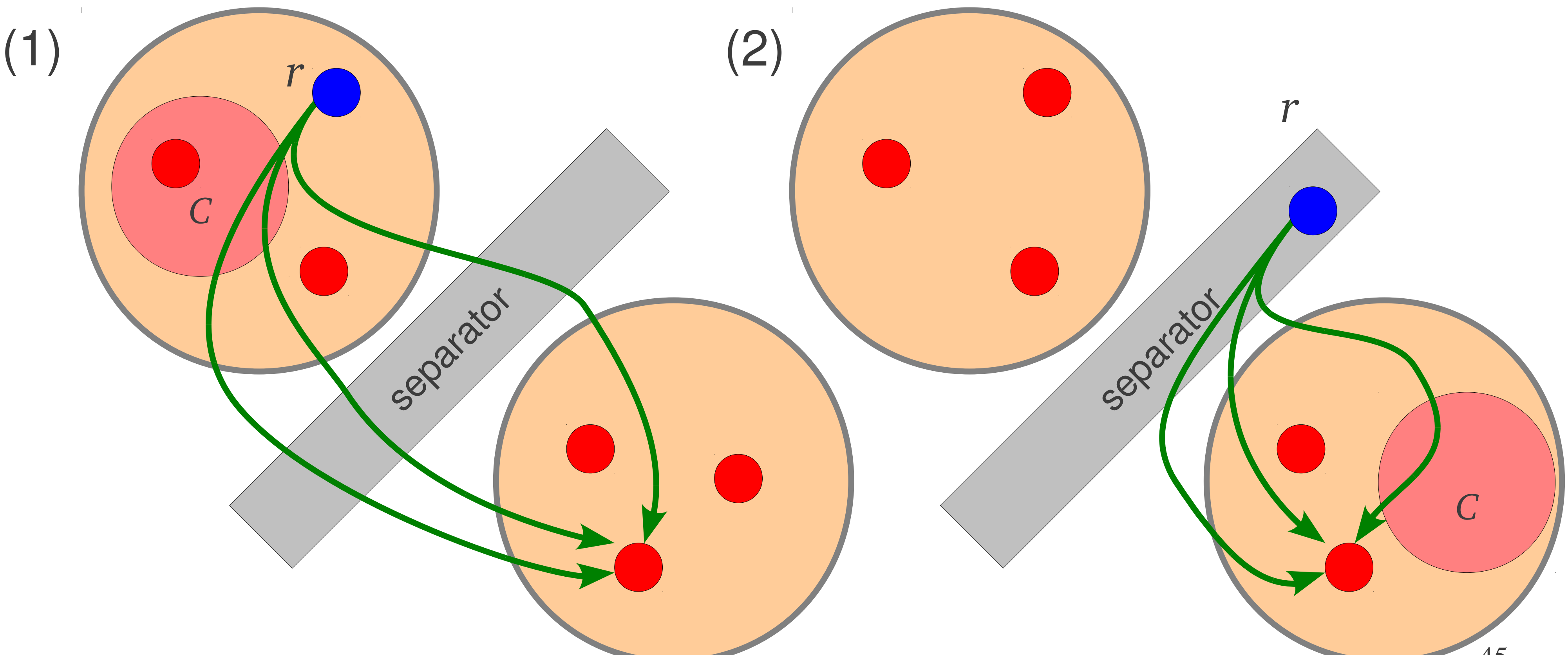


Main Algorithm: Key Lemma

For $|T| \geq 2L$, there is a terminal r such that

1) There are O(1) halo-families of Case 1.

2) There are $O(q/2)$ halo-families of Case 2, where q is # of halo-families.



REMARK: Bound for (1) is $O(L / (|T| - L))$, Bound for (2) is $(q / (|T| / L))$

Main Algorithm: Description

Covering Procedure:

- **Repeat**
  - Find a terminal r satisfying Key Lemma
  - Apply rooted $(L+1)$-conn from r
  - For each $Halo(C)$ of Case 1, choose any terminal t in C and apply rooted $(L+1)$-conn from t
- **Until** all halo-families are covered

- **REMARK:** We consider only the # of halo-families computed at the beginning of the round (of Halo-set Method). So, we have to repeat everything again.

Sketch of Analysis

Covering Procedure:

- **Repeat**
  - Find a terminal r satisfying Key Lemma
  - Apply rooted $(L+1)$-conn from r
  - For each $Halo(C)$ of Case 1, choose any terminal t in C and apply rooted $(L+1)$-conn from t
- **Until** all halo-families are covered

- Key Lemma

$\Rightarrow$ Cover O(1) halo-fam (Case 1) $\Rightarrow$ Remaining are $\leq q/2$ halo-fam (Case 2)

$\Rightarrow$ # of halo-families decrease geometrically $\Rightarrow$ Need $O(\log q)$ rounds

Combine Everything

- Covering-Procedure solves $O(\log k)$ rooted k-conn
- Halo-set method terminates in $O(\log k)$ rounds

$\Rightarrow$ Solves $O(\log^2 k)$ rooted k-conn for conn aug

- Pay $O(\log k)$ factor for increase subset conn to k.

$\Rightarrow$ $O(F(k)\log^3 k)$ for subset k-conn, where $F(k)$ is approx ratio for rooted conn augmentation.

(There is O(k)-aprx algo for rooted conn aug by Nutov [FOCS'09].)

More Analysis

$\Rightarrow$ approx ratio $= O(k \log^2 k)$ for $|T| \geq 2k$

$= O(k \log k)$ for $|T| \geq k^2$

Conclusion

- We presented a LabelCover hardness for rooted subset k-connectivity problem.
- We showed that, for $|T| \geq 2k$, $\mathrm{F}(k)$-approx for rooted connectivity augmentation implies $\mathrm{F}(k)\log^3 k$ for subset k-connectivity.

Questions?

Thank you.