7686 - Adv. Combinatorial Optimization
Homework 3 (due March 13)

Note: You can discuss the homework with other students. But you have to write your own solutions. Also write the name of everyone you discussed with.

1. Let $G = (V, E)$ be an undirected graph with non-negative edge lengths $w : E \to \mathbb{R}_+$. Given nodes s, t we wish to find a shortest s-t path with an odd number of edges. Show that this problem can be solved via matching techniques by following the hint below. Suppose s, t have degree 1. Consider the reduction of the maximum weight matching problem to the maximum weight perfect matching problem. Adapt this reduction and show how a perfect matching can be used to obtain a shortest odd length path from s to t. How can you get rid of the assumption that s, t have degree 1? Extend the ideas for finding a shortest even length s-t path.

2. Petersens theorem states that every bridgeless (no cut-edge) cubic graph has a perfect matching. Show that this follows (easily) from Tutte-Berge formula. Moreover, show that every bridgeless cubic-graph has a T-join of size at most $\frac{|V|}{2}$ for any even $T \subseteq V$.

3. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and $Y, Z \subseteq S$. Show the following.
   (a) $(\mathcal{M}^*)^* = \mathcal{M}$.
   (b) $\mathcal{M}/Z = (\mathcal{M}^* \setminus Z)^*$ where $\mathcal{M}/Z$ is the contracted matroid.
   (c) $(\mathcal{M} \setminus Y)/Z = (\mathcal{M}/Z) \setminus Y$, i.e., deletion and contraction commute.

4. Let $G = (V, E)$ be a graph. Let $\mathcal{I}$ be the collection of those subsets Y of E so that Y has at most one circuit. Show that $(E, \mathcal{I})$ is a matroid.

5. Let $G = (V, E)$ be a connected graph. For each subset E_0 of E, let $\kappa(V, E_0)$ denote the number of components of the graph (V, E_0). Let $\mathcal{M}$ denote the graphic matroid for graph G. Show that for each $E_0 \subseteq E$:
   (a) $r_{\mathcal{M}}(E_0) = |V| - \kappa(V, E_0)$.
   (b) $r_{\mathcal{M}^*}(E_0) = |E'| - \kappa(V, E \setminus E') + 1$.

6. Let $G = (V, E)$ be a hypergraph, that is each $e \in E$ is a hyperedge, in other words $e \subseteq V$. We say that $X \subseteq E$ a forest-representable if one can choose for each $e \in X$ two nodes in e such that the chosen pairs when viewed as edges form a forest on V. Prove that $(E, \mathcal{I})$ is a matroid where $\mathcal{I} = \{X \subseteq E | X \text{is forest-representable}\}$. This is called the hypergraphic matroid.