7686 - Adv. Combinatorial Optimization
Homework 5(due 4/24)

1. Let $G = (V, E)$ be a graph where nodes are colored red or blue. Every red node v comes with a profit $p_v \geq 0$ and a blue node with a loss $q_v \geq 0$. If we tell a rumor to a red node, then it tells all its neighbors (red or blue). A blue node does not tell the rumor to any of its neighbors. Moreover, if a red node gets to know the rumor from his neighbors we obtain a profit of p_v for that node and if a blue nodes gets to hear the rumor, we obtain a loss of q_v. Our aim to find a subset $S \subseteq V$ to tell a rumor that achieves best revenue (Profit-loss). Observe that, we will node obtain profit for the nodes that we initially tell the rumor. Give a polynomial time algorithm to find the optimal set.

2. Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$ of nodes, define the density of the induced graph $G[S]$ as $|E[S]|/|S|$ where $E[S]$ is the set of edges with both end points in S. The goal is to find the set of nodes that maximizes the density of the induced graph. Suppose we are given a number λ and wish to check if there is a set S with density at least λ. Use submodular function minimization to solve this problem.

3. A function $f : 2^S \to \mathbb{R}$ is supermodular if $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq S$. Let $A_1, \ldots A_n$ be random events and $S = \{A_1, \ldots, A_n\}$ and let for any $X = \{A_{i_1}, \ldots, A_{i_k}\} \subseteq S$, we let $f(X) = Prob(A_{i_1}, \ldots, A_{i_k})$ be the probability that all events in X occur. Show f is a supermodular function.

4. Let T be a tree. Consider the graph $G = (V, E)$ such that every node of G corresponds to a subtree of T. Moreover, there is an edge connecting two subtrees if they have a common vertex. Show G is perfect.