7686 - Adv. Combinatorial Optimization
Homework 14(due 4/10)

1. Derive the min-max relationship for the common independent set of two matroids from the matroid union theorem.

2. Let A be a full-rank $n \times n$ matrix over the reals. Let R and C be the index sets of the rows and columns. Given $I \subseteq R$ show that there exists $J \subseteq C$ such that $|I| = |J|$ and both $A(I, J)$ and $A(R \setminus I, C \setminus J)$ are of full rank. Use matroid intersection.

3. Given a graph $G = (V, E)$, let
$$\mathcal{I} = \{F \subseteq E | |E(S) \cap F| \leq 2|S| - 3 \ \ \forall\, S \subseteq V, |S| > 1\}.$$
Show that $(E, \mathcal{I})$ defines a matroid.

4. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and $r : 2^S \to \mathbb{Z}$ be its rank function. Show that
   (a) $r(\emptyset) = 0$ and monotone submodular.
   (b) r is integer valued.
   (c) $r(X \cup \{e\}) \leq r(X) + 1$ for all $X \subseteq S$ and $e \in S$.

   Also show the converse, i.e. if $r : 2^S \to \mathbb{Z}$ is a submodular function satisfying above properties, then it is a rank function of some matroid.

5. Let $G = (V, E)$ be an undirected graph, $S \subseteq V$ an independent set, and let $u : S \to \mathbb{Z}_+$, and $k \geq 1$. Give a polynomial algorithm to decide if the graph contains k edge-disjoint spanning trees, such that the total degree in these trees is at most $u(s)$ for any $s \in S$.

6. Suppose we are given an undirected graph $G = (V, E)$, and additional vertex $s \notin V$, an integer k, and we would like to add the minimum number of edges between s and vertices of V (multiple edges are allowed) such that the resulting graph H on $V + s$ has k edge-disjoint paths between any two vertices of V (i.e. the only cut that could possibly have fewer than k edges is the cut separating s from V).
   (a) Argue that this problem is equivalent to finding $x : V \to \mathbb{Z}_+$ minimizing $x(V) := \sum_{v \in V} x(v)$ such that $\forall \emptyset \neq S \subset V$:
   $$x(S) \geq k - d_E(S),$$
   where $d_E(S) = |\delta_E(S)|$ corresponds to the number of edges between S and $V \setminus S$ in G and $x(S) := \sum_{v \in S} x(v)$.
   (b) Add k edges between s and each vertex of V. Let A be these $k|V|$ newly added edges. Say that $F \subseteq A$ is feasible if the graph $(V + s, E \cup (A \setminus F))$ has at least k edge-disjoint paths between any two vertices of V. Prove that the feasible sets form the independence sets of a matroid.
   (c) How would you efficiently solve the original problem (with k part of the input)?