ISYE 7661: Linear Inequalities
Homework 5 (Due 12/5)

You are free to discuss the homework with one other person but write your solutions on your own.

1. Recall that the simultaneous diophantine approximation problem is as follows: Given rational numbers $\alpha_1, \ldots, \alpha_d \in [0,1]$ and a bound $Q \in \mathbb{Z}_+$, find the denominator $q \in \{1, \ldots, Q\}$ and numerators $p_1, \ldots, p_d \in \mathbb{Z}$ such that

$$\max_{1 \le i \le d} \{|q\alpha_i - p_i|\}$$

is minimized. Show that if d is a fixed constant, this problem can be solved in polynomial time.

2. Let $K \subseteq \mathbb{R}^n$ be a full-dimensional, symmetric convex body with $vol(K) \ge 1$. Show that one can compute a vector $x \in \mathbb{Z}^n \setminus \{0\}$ such that $x \in 2^{O(n)}K$ in polynomial time.

3. Let G be a bipartite graph. Use the integer decomposition property of its matching polytope to show that it can be edge-colored using at most Δ colors where Δ is the maximum degree.

4. Let $G = (V, E)$ be a complete graph. Consider the polytope $P_I = conv(\chi(M) \in \mathbb{R}^E :$ M is a matching). Here for any $F \subseteq E$, $\chi(F)$ denotes the vector whose e^{th} coordinate is 1 if $e \in F$ and 0 otherwise. Let $P = \{x \in \mathbb{R}^E : x(\delta(v)) \le 1 \;\; \forall v \in V, x \ge 0\}$ where $\delta(v)$ denotes the set of edges incident at v and $x(F) := \sum_{e \in F} x_e$ for any subset $F \subseteq E$.

   (a) Show $P_I \subsetneq P$, i.e., P_I is strictly contained in P.

   (b) (Bonus) Consider $Q = \{x \in \mathbb{R}^E : x(\delta(v)) \le 1 \quad \forall v \in V, x \ge 0, x(E(S)) \le \lceil \frac{|S|-1}{2} \rceil \; \forall S \subseteq V, |S| \ge 3, |S| \text{ odd}\}$. Here $E(S)$ is the set of edges with both end-points in S. Show that $P_I \subseteq P' \subseteq Q \subseteq P$ where P' is the Chvátal closure of P. It also turns out that the description of Q is TDI and thus Q is integral and we have $P_I = Q = P'$.

5. A set family $\mathcal{L} \subseteq 2^U$ is a chain if for each $A, B \in \mathcal{L}$ we have $A \subseteq B$ or $B \subseteq A$. It is laminar if for each $A, B \in \mathcal{L}$ we have $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \emptyset$. Let $M(\mathcal{L})$ be a $|\mathcal{L}| \times |U|$ incidence matrix of $\mathcal{L}$. Thus $M(\mathcal{L})_{A,e} = 1$ if $e \in A$ and 0 otherwise.

   (a) Show $M(\mathcal{L})$ is totally unimodular if $\mathcal{L}$ is a chain or laminar family.

   (b) Show $M(\mathcal{L})$ is totally unimodular if $\mathcal{L}$ is a union of two chains. What about the case when it is a union of two laminar families.