ISYE 7661: Linear Inequalities
Homework 4 (Due 11/21)

You are free to discuss the homework with one other person but write your solutions on your own.

1. (a) Show that a polar of *any* set is convex.
   (b) Show that if $0 \notin P$, then $(P^*)^* = conv(P \cup \{0\})$ where 0 is the origin and P^* is the polar of P.
   (c) Show that the polar of a cone $C = \{x : Ax \leq 0\}$ is a cone. Can you describe its inequality description or its extreme rays?

2. Read Section 11.2 and 11.3 in the book. Solve the following linear programs by simplex method in tableau form.

   (a)

$$\begin{aligned} \max\ 2x_1 + 4x_2 - 4x_3 \\ \text{subject to} \\ 3x_1 + 2x_2 + 4x_3 \geq 1 \\ 4x_1 - 3x_2 = 2 \\ 2x_1 + x_2 + 6x_3 \leq 3 \\ x_1, x_2 \geq 0 \end{aligned}$$

   (b)

$$\begin{aligned} \min\ 2x_1 + 2x_2 - 4x_3 \\ \text{subject to} \\ 2x_1 + 2x_2 + 2x_3 = 10 \\ -2x_1 + 6x_2 - x_3 \leq -10 \\ -x_1 + 3x_2 \geq 3 \\ x_1 \leq 0, x_2, x_3 \geq 0 \end{aligned}$$

   (c) Starting from vertex $(1, 0)$ run the simplex algorithm.

$$\begin{aligned} \max\ x_1 + 3x_2 \\ \text{subject to} \\ x_1 + x_2 \leq 2 \\ x_1 \leq 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{aligned}$$

3. (a) Install gurobipy (Google for instructions).

   (b) Let $G = (U, V, E)$ be a complete bipartite graph with $|U| = |V| = n$ with weights $w : E \to \mathbb{R}_+$. A subset of edges $F \subseteq E$ is a perfect matching if $|F \cap \delta(v)| = 1$ for each $v \in V$. Consider the following linear program for the minimum weight perfect matching.

$$\begin{aligned} \min \sum_{i\in U, j\in V} & w_{ij}x_{ij} \\ \text{subject to} & \\ \sum_{j\in V} x_{ij} &= 1 \qquad \forall i \in U \\ \sum_{i\in U} x_{ij} &= 1 \qquad \forall j \in V \\ x_{ij} &\geq 0 \qquad \forall i \in U, j \in V \end{aligned}$$

   Implement the above linear program and solve it using simplex method. Observe that the solution is always integral! We will prove this fact in class.

   (c) Pick different values of n and select the weights for each edge in G be the i.i.d. from uniform distribution $[0, 1]$. For each n, take enough samples to obtain good estimates on the expected size of the minimum weight perfect matching by solving the above linear program. Plot the expected weight of the matching (or equivalently the LP solution) as a function of n. What do you observe?

4. Let $P = \{x : Ax \leq b\}$ be a polyhedron. Show that $\bar{x}$ is a vertex of P if and only there exists a cost vector c such that $\bar{x}$ is the unique optimum solution of $\max\{c^T x : x \in P\}$.

5. If $P = \{x : Ax \leq b\}$ is a polyhedron. Show that $P' = \{x : Ax \leq b + \epsilon\mathbf{1}\}$ is full dimensional for $\epsilon > 0$ if $P \neq \emptyset$ where $\mathbf{1}$ is the vector with all ones.