ISYE 7661: Linear Inequalities
Homework 3 (Due 10/31)

You are free to discuss the homework with one other person but write your solutions on your own.

1. Find the lineality space, characteristic cone, minimal faces and decomposition as in Section 8.9 of the following polyhedron.

   (a) $P = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 1, x_1 \geq 0, x_2 \geq 0\}$.

   (b) $P = \{(x_1, x_2, x_3) : x_1 - x_2 = 1, x_1 \geq 0\}$.

   (c) $P = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1 \quad \forall\, 1 \leq i \leq n\}$.

   (d) Let $G = (U \cup V, E)$ be the complete bipartite graph where $|U| = |V| = n$.

   $$P = \{x \in \mathbb{R}^{|E|} : \sum_{j \in V} x_{ij} = 1 \forall i \in U, \sum_{i \in U} x_{ij} = 1 \forall j \in V, x_{ij} \geq 0\}.$$

2. Let $P = \{x : Ax \leq b\}$ be a full dimensional polytope with dimension d. For every inequality $a_i^T x \leq b_i$, we let V_i denote the vertex set in face defined by the inequality $\{x \in P : a_i^T x = b_i\}$. Show that the following are a criterion for checking whether the inequality is redundant (i.e., removing it does not change the polytope).

   (a) The inequality $a_i^T x \leq b_i$ is redundant if and only if $V_i \subseteq V_j$ for some $j \neq i$.

   (b) If $V_i = V_j$ then either both inequalities are multiples of each other or they can both be deleted from the system.

   (c) An inequality is irredundant if and only it defines a facet and no multiple of it is contained in the system.

   (d) If $|V_i| < d$, then the inequality $a_i^T x \leq b_i$ is redundant.

   (e) The inequality $a_i^T x \leq b_i$ is irredundant if and only if there is no multiple of it in the system and rank of matrix given by $\left\{ \begin{pmatrix} 1 \\ v \end{pmatrix} : v \in V_i \right\}$ is d.

3. Use the Farkas Lemma to show that for every unbounded pointed polyhedron P there is an inequality $a^T x \leq 1$ such that

   $$P' = \{x \in P : a^T x \leq 1\}$$

   is a polytope with a facet $F' = \{x \in P : a^T x = 1\}$, such that k-dimensional faces of F' correspond to $k+1$-dimensional faces of P and k-dimensional faces of P' that are not faces of F' are in bijection with the k-dimensional faces of P.