ISYE 7661: Linear Inequalities
Homework 2 Solution Hints

You are free to discuss the homework with one other person but write your solutions on your own.

1. . Let $\Lambda = \Lambda(B)$ with $B \in \mathbb{R}^{n \times n}$ be a lattice. Show that for any $\epsilon > 0$ there is a radius $R := R(\epsilon, n, B)$ so that

$$(1-\epsilon) \cdot \frac{vol(B(0,R))}{det(\Lambda)} \leq |B(0,R) \cap \Lambda| \leq (1+\epsilon) \cdot \frac{vol(B(0,R))}{det(\Lambda)}.$$

   Here $B(0,R)$ is the ball of radius R around the origin.

   Solution: Let r denote the diameter of parallelopiped P defined by columns of B. Let $S = \Lambda \cap B(0,R)$. Then $x + P$ for $x \in S$ are disjoint and all lie within $B(0, R+r)$ and moreover cover $B(0, R-r)$. Thus we get

$$\frac{B(0,R-r)}{det(\Lambda)} \leq |S| \leq \frac{B(0,R+r)}{det(\Lambda)}.$$

   But $\frac{B(0,R-r)}{B(0,R)} \geq 1-\epsilon$ for R large enough and similarly $\frac{B(0,R+r)}{B(0,R)}$ is at most $1+\epsilon$ for R large enough.

2. This is an application of Dirichlet's Theorem: Let $a \in]0,1]^n$ be a real vector and consider the hyperplane $H := \{x \in \mathbb{R}^n : \langle a, x \rangle = 0\}$. Then there is a rational vector $\tilde{a} \in \frac{\mathbb{Z}^n}{q}$ with $q \leq (2nR)^n$ so that $\tilde{H} := \{x \in \mathbb{R}^n : \langle \tilde{a}, x \rangle = 0\}$ satisfies the following:

$$\forall x \in \{-R, ..., R\}^n : x \in H \implies x \in \tilde{H}.$$

   Solution: Applying Dirichlet's theorem, we get $p_1, \ldots, p_n, q$ integers and $0 \leq q \leq (2nR)^n$ such that

$$|\frac{p_i}{q} - a_i| \leq \frac{1}{2nRq}.$$

   Let $\tilde{a}_i = \frac{p}{q}$. A simple calculations shows that for any $x \in \{-R, \ldots, R\}^n$, such that $x \in H$, we have

$$|\tilde{a}^T x| \leq \frac{1}{2q}.$$

   Since the LHS is $\frac{1}{q}$-integral, it must be 0 as required.

3. Can you solve the following problem in polynomial time. Given matrices $A \in \mathbb{R}^{m \times n_1}, B \in \mathbb{R}^{m \times n_2}$ and vector $b \in \mathbb{R}^m$, all rational, such that

$$Ax + By = b$$

for some $x \in \mathbb{Z}^{n_1}$ and $y \in \mathbb{R}^{n_2}$.

Solution: Using Gaussian elimination (for variables in y), we can transform to an equivalent system of following form

$$\begin{bmatrix} B' & A' \\ 0 & A'' \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b' \\ b'' \end{bmatrix}$$

where B' is upper-triangular with non-zero diagonal entries (Why?) of appropriate dimension (the rank of B). Now it enough to find a solution to $A''x = b'', x \in \mathbb{Z}^{n_1}$ if one exists. Then for every such x, we can obtain a $y \in \mathbb{R}^{n_2}$ such that $B'y = b' - A'x$ since the B' is upper-triangular with non-zero diagonal.

4. State and prove the Farkas' Lemma for the following version $Ax \leq z, x \geq 0$ where x and z are variables and A is a matrix.

   Solution: This is always feasible. Do check via Farkas lemma.

5. Prove the Farkas' lemma for following general constraints. For compatible matrices A, B, C and vectors u, v, w either there exists a solution vector x for

$$Ax = u, Bx \geq v, Cx \leq w,$$

   or there exists row vectors a, b, c such that

$$aA + bB + cC = 0, b \leq 0, c \geq 0, au + bv + cw < 0.$$

   Solution: Apply any variant to tranform to this form.