ISYE 7661: Linear Inequalities
Homework 2 (Due 10/5)

You are free to discuss the homework with one other person but write your solutions on your own.

1. . Let $\Lambda = \Lambda(B)$ with $B \in \mathbb{R}^{n\times n}$ be a lattice. Show that for any $\epsilon > 0$ there is a radius $R := R(\epsilon, n, B)$ so that

$$(1-\epsilon)\cdot \frac{vol(B(0,R))}{det(\Lambda)} \leq |B(0,R)\cap\Lambda| \leq (1+\epsilon)\cdot\frac{vol(B(0,R))}{det(\Lambda)}.$$

   Here $B(0,R)$ is the ball of radius R around the origin.

2. This is an application of Dirichlet's Theorem: Let $a \in]0,1]^n$ be a real vector and consider the hyperplane $H := \{x \in \mathbb{R}^n : \langle a, x\rangle = 0\}$. Then there is a rational vector $\tilde{a} \in \frac{\mathbb{Z}^n}{q}$ with $q \leq (2nR)^n$ so that $\tilde{H} := \{x \in \mathbb{R}^n : \langle \tilde{a}, x\rangle = 0\}$ satisfies the following:

$$\forall x \in \{-R, ..., R\}^n : x \in H \implies x \in \tilde{H}.$$

3. Can you solve the following problem in polynomial time. Given matrices $A \in \mathbb{R}^{m\times n_1}, B \in \mathbb{R}^{m\times n_2}$ and vector $b \in \mathbb{R}^m$, all rational, such that

$$Ax + By = b$$

   for some $x \in \mathbb{Z}^{n_1}$ and $y \in \mathbb{R}^{n_2}$.

4. State and prove the Farkas' Lemma for the following version $Ax \leq z, x \geq 0$ where x and z are variables and A is a matrix.

5. Prove the Farkas' lemma for following general constraints. For compatible matrices A, B, C and vectors u, v, w either there exists a solution vector x for

$$Ax = u, Bx \geq v, Cx \leq w,$$

   or there exists row vectors a, b, c such that

$$aA + bB + cC = 0, b \leq 0, c \geq 0, au + bv + cw < 0.$$