Approximation Algorithms CS599 **Homework 3**
Date: December 5, 2014 **Due:** December 12, 2014

1. **Discrepancy.** Consider a set system $S_1, \ldots, S_n$ where each $S_i \subseteq [n]$. The discrepancy of the set system is defined to be
$$\min_{\chi:[n]\to\{+1,-1\}} \max_i |\sum_{j\in S_i} \chi(j)|.$$
   (a) Show that a random coloring will have discrepancy $O(\sqrt{n\log n})$ with high probability and show there is a set system where the the discrepancy of a random coloring can be $\Omega(\sqrt{n\log n})$ with high probability.
   (b) In the class we claimed that there is a way to get a coloring with discrepancy $O(\sqrt{n})$. Construct a set system with n sets over $[n]$ in which every coloring has discrepancy at least $\Omega(\sqrt{n})$. (Hint: think random.)

2. **Embeddings into ℓ_2.** Give a polynomial-time algorithm that given a metric space (X, d) finds an embedding $\varphi : X \to \mathbb{R}^N$ with the minimum possible distortion D: That is, for all $x, y \in X$,
$$d(x, y) \leq \|\varphi(x) - \varphi(y)\|_2 \leq D \times d(x, y).$$
   (N is a sufficiently large number e.g. $N = |X|$).

3. **An integrality gap for MAX CUT.** Denote by C_k be the cycle graph on k vertices.
   (a) Find the maximum cut in C_k for every $k > 1$.
   (b) Show that the SDP value for MAX CUT in C_k is at least
$$k \times \frac{1 - \cos\pi(1 - 1/k)}{2}.$$
   (c) What is the worst integrality gap for graphs C_k?

   Note: The integrality gap for MAX CUT equals $\alpha_{GW} \approx 0.878$. However, the worst integrality gap for C_k is greater than α_{GW}.

4. **(Densest Subgraph)** Given an undirected graph $G = (V, E)$, the density of a subset S is defined to be
$$\rho(S) = \frac{|E(S, S)|}{|S|}; \quad (1)$$
   i.e., the number of edges with both endpoints in S, divided by the number of vertices in S. The goal of this problem is to find the subset S which maximizes $\rho(S)$, the max being taken over all subsets of V.

(a) Consider the following LP relaxation for the problem: have a variable x_{ij} for each edge $\{i, j\}$, and variable y_i for each vertex $i \in V$. The goal is to maximize $\sum_{ij} x_{ij}$ subject to the constraints:

$$
\begin{aligned}
x_{ij} &\leq \min\{y_i, y_j\} \qquad\qquad \forall \{i, j\} \in E \\
\textstyle\sum_{i \in V} y_i &\leq 1.
\end{aligned}
$$

Is this a <u>linear</u> program? If not, write an equivalent LP for the problem. Show that this is a valid relaxation for the Densest Subgraph problem.

(b) Give the best (upper and lower) bounds you can on the integrality gap of this LP relaxation.