**Approximation Algorithms CS599** **Homework 2**
**Date:** November 18 , 2014 **Due:** December 5, 2014

1. **The Little Grothendieck Inequality.** We prove the little Grothendieck inequality: For every positive semidefinite matrix $A$ the following inequality holds:

$$\max_{x_i \in \{\pm 1\}} \sum_{i,j} a_{ij} x_i x_j \geq \frac{2}{\pi} \max_{\|v_i\|_2 = 1} \sum_{i,j} a_{ij} \langle v_i, v_j \rangle.$$

Consider the set of unit vectors $v_i$ that maximizes the right hand side. Pick a random Gaussian vector $g$, and let $x_i = \operatorname{sgn}(\langle v_i, g\rangle)$. We want to show that

$$\mathbf{E}\Big[\sum_{i,j} a_{ij} x_i x_j\Big] \geq \frac{2}{\pi} \sum_{i,j} a_{ij} \langle v_i, v_j \rangle. \tag{1}$$

(a) Prove that

$$\mathbf{E}\Big[\sum_{i,j} a_{ij} \langle g, v_i \rangle \cdot \langle g, v_j \rangle\Big] = \sum_{i,j} a_{ij} \langle v_i, v_j \rangle.$$

(b) Compute $\mathbf{E}[|g_1|]$, where $g_1$ is a (one dimensional) Gaussian random variable with mean 0 and standard deviation 1 i.e. $g_1 \sim \mathcal{N}(0, 1)$.

(c) Show that for all unit vectors $u$ and $v$:

$$\mathbf{E}\big[\operatorname{sgn}(\langle u, g\rangle) \cdot \langle v, g \rangle\big] = \sqrt{\frac{2}{\pi}}\, \langle u, v \rangle.$$

(Note that the left hand side is not symmetric with respect to $u$ and $v$.)

(d) Prove that for every $\lambda$,

$$\mathbf{E}\Big[\sum_{i,j} a_{ij} (\langle v_i, g\rangle - \lambda \operatorname{sgn}(\langle v_i, g \rangle)) \cdot (\langle v_j, g\rangle - \lambda \operatorname{sgn}(\langle v_j, g \rangle))\Big] \geq 0.$$

(e) Using parts (a), (c), and (d) prove (1).

2. **Isometric Embeddings of $\ell_2$.** We show that every $n$-point subset of $\ell_2$ isometrically embeds into $\ell_1$. To this end, we define an $L_1$ norm on functions from $\mathbb{R}^n$ to $\mathbb{R}$. We let

$$\|f\|_{L_1} = \mathbf{E}[|f(g)|],$$

where $g$ is an $n$-dimensional Gaussian vector. Note that the expectation on the right hand side may be undefined. The distance between two functions $f_1$ and $f_2$ equals

$$\|f_1 - f_2\|_{L_1} = \mathbf{E}[|f_1(g) - f_2(g)|].$$

(a) Show that the $L_1$ norm on functions satisfies the triangle inequality. That is,

$$\|f_1 - f_3\|_{L_1} \le \|f_1 - f_2\|_{L_1} + \|f_2 - f_3\|_{L_1}$$

for all functions $f_1$, $f_2$ and $f_3$ assuming that the norms above are defined.

(b) Let $V \subset \mathbb{R}^n$ be an $n$-point subset of $n$-dimensional Euclidean space. Construct a map $v \mapsto f_v$ from $V$ to the space of functions such that for all $u, v \in V$:

$$\|f_u - f_v\|_{L_1} = \|u - v\|_2.$$

**Hint:** Compute $\|f_u - f_v\|_{L_1}$ for $f_v(x) = \langle x, v\rangle$ (here $x \in \mathbb{R}^n$).

(c) Now we embed functions $f_v$, constructed in the previous part, into $\ell_1$. For every ordering of vertices $\pi$ (i.e., a one-to-one map from $\{1, \dots, n\}$ to $V$) define a set $\mathcal{E}_\pi$ as follows:

$$\mathcal{E}_\pi = \{x : f_{\pi(i)}(x) \le f_{\pi(i+1)}(x) \text{ for all } i\}.$$

Define a map $\varphi$ from the set of functions $f_v$ to an $n!$ dimensional space as follows:

$$\varphi_\pi(f_v) = \mathbf{E}[f_v(g) \cdot \mathbf{1}(g \in \mathcal{E}_\pi)],$$

here $\varphi_\pi(f_v)$ is the $\pi$ coordinate of $\varphi(f_v)$; $\mathbf{1}(g \in \mathcal{E}_\pi)$ is the indicator of the event $\{g \in \mathcal{E}_\pi\}$. Prove that

$$\|\varphi(f_u) - \varphi(f_v)\|_1 = \|f_u - f_v\|_{L_1}.$$

(d) Use all parts above to show than every $n$-point subset of Euclidean space embeds isometrically into $\ell_1$.

3. (From the book of Williamson and Shmoys.) In the maximum $k$-cut problem, we are given an undirected graph $G = (V, E)$, and non-negative weights $w_{ij} > 0$ for all $(i, j) \in E$. The goal is to partition the vertex set $V$ into $k$ parts $V_1, \dots, V_k$ so as to maximize the weight of all edges whose endpoints are in different parts. Give a $(k-1)/k$-approximation algorithm for the MAX $k$-CUT problem.

4. **Initialization of Interior Point Methods.** Given an original linear program as follows. Here $A$ is an integral $m \times n$ matrix which has full row rank and $b$ and $c$ are integral vectors

$$\begin{aligned} &\min c^T x \\ &\text{s.t.} \\ &Ax = b \\ &x \ge 0 \end{aligned}$$

Figure 1: Original Linear Program

of length $m$ and $n$. Let

$$L = \log(\text{ largest absolute value of the determinant of a square submatrix of } A+1) + \\ + \log(1+\max_j |c_j|) + \log(1+\max_i |b_i|) + 3.$$

Let $\alpha = 2^{4L}$ and $\lambda = 2^{2L}$. Let $K_b = \alpha\lambda(n+1) - \lambda c^T e$ where $e$ is all ones vector and $K_c = \alpha\lambda$. Now consider the augmented linear program.

$$\begin{aligned}
&\min c^T x + K_c x_{n+2} \\
&\text{s.t.} \\
&Ax + (b - \lambda Ae)x_{n+2} = b \\
&(\alpha e - c)^T x + \alpha x_{n+1} = K_b \\
&x \geq 0
\end{aligned}$$

Figure 2: Modified Linear Program

(a) Construct the dual for the the modified linear program where we have the dual variables $y_i$ for $i = 1, \ldots, m$ for the original $m$ constraints and $y_{m+1}$ for the new constraint added in the following form where $\hat{b}, \hat{A}, \hat{c}$ need to be defined. Moreover, let $s \in \mathbb{R}^{n+2}$ denote the slack variables.

$$\begin{aligned}
&\max \hat{b}^T y \\
&\text{s.t.} \\
&\hat{A}y + s = \hat{c} \\
&s \geq 0
\end{aligned}$$

Figure 3: Modified Linear Program

(b) Let $x^* = (\lambda, \ldots, \lambda, 1)^T \in \mathbb{R}^{n+2}$, $y^* = (0, \ldots, 0, -1)^T \in \mathbb{R}^{m+1}$ and let slack variables $s^* = (\alpha, \ldots, \alpha, \alpha\lambda) \in \mathbb{R}^{n+2}$. Show $x^*$ is feasible for modified primal LP and $y^*$ is feasible for its dual and $s^*$ are the corresponding slack variables for the dual constraints, i.e., $\hat{A}y^* + s^* = \hat{c}$. Also verify that $x_i^* s_i^* = \alpha\lambda$ for each $i$ and therefore they are on the central path with $\mu = \alpha\lambda$.

(c) Since modified primal and modified dual have feasible solutions, they will have optimal solutions. Let $\bar{x}, \bar{y}, \bar{s}$ denote the optimal primal and dual solutions. Show the following.

  i. If $\bar{x}_{n+2} = 0$ and $\bar{s}_{n+1} = 0$, then $x' = (\bar{x}_1, \ldots, \bar{x}_n)$ is optimal for original primal and $y' = (\bar{y_1}, \ldots, \bar{y_m})$ is dual optimal with slack variables $s' = (\bar{s}_1, \ldots, \bar{s}_n)$.

  ii. If $\bar{x}_{n+2} \neq 0$ and $\bar{s}_{n+1} = 0$, then original primal is infeasible.

  iii. If $\bar{s}_{n+1} \neq 0$ and $\bar{x}_{n+2} = 0$, then original primal is unbounded.