1. **Extended Formulations.**

   (a) The following linear program is equivalent to the simplex embedding linear program.

$$
\begin{array}{ll}
\min \sum_{e \in E} c_e x_e & \\
\text{s.t.} & \\
\sum_{i=1}^{k} y_v^i = 1 & \forall v \in V \\
y_{t_i}^i = 1 & \forall v \in V \\
x_{uv} \geq \frac{1}{2} \sum_{i=1}^{k} d_{uv}^i & \forall u, v \in V \\
d_{uv}^i \geq y_u^i - y_v^i & \forall u, v \in V, 1 \leq i \leq k \\
d_{uv}^i \geq y_v^i - y_u^i & \forall u, v \in V, 1 \leq i \leq k \\
0 \leq x_{uv} \leq 1 & \forall u, v \in V \\
0 \leq y_v^i \leq 1 & \forall 1 \leq i \leq k, v \in V
\end{array}
$$

Figure 1: Simplex Linear Program for Multiway Cut

   (b) We introduce variables $d^i(v)$ to denote the distance of v from terminal t_i and write appropriate constraints.

$$
\begin{array}{ll}
\min \sum_{e \in E} c_e x_e & \\
\text{s.t.} & \\
d^i(t_i) = 0 & \forall 1 \leq i \leq k \\
d^i(t_j) \geq 1 & \forall 1 \leq i, j \leq k, i \neq j \\
d^i(u) \leq d^i(v) + x_{uv} & \forall (u, v) \in E \ \ \forall 1 \leq i \leq k \\
0 \leq x_e \leq 1 & \forall e \in E \\
0 \leq d^i(v) & \forall i, v
\end{array}
$$

Figure 2: Path Based Linear Program for Multiway Cut

2. **Randomized Rounding for Satisfiability**.

   (a) Consider the random assignment which sets each variable to true with probability $\frac{1}{2}$ independently. In Max-k-SAT, the probability any fixed clause is not satisfied is exactly $\frac{1}{2^k}$. Thus the expected number of satisfied clauses is $m(1 - 1/2^k)$ where m is the total number of clauses. Since $OPT \leq m$, we have the desired approximation. The same argument works for Max-SAT.

   (b) Let $C_j = \vee_{i \in P_j} x_i \bigvee \vee_{i \in N_j} \bar{x}_i$ be a clause. The probability C_j is satisfied by the random assignment where x_i set to true with probability $\frac{y_i}{2} + \frac{1}{4}$ independently, is given by

$$1 - \left(\prod_{i \in P_j} \left(1 - \frac{y_i}{2} - \frac{1}{4}\right) \prod_{i \in N_j} \left(\frac{y_i}{2} + \frac{1}{4}\right) \right)$$

We now claim that this probability is at least $\frac{3}{4} z_j$ which will give the desired approximation by linearity of expectation. In particular, we will show that it is at least $\frac{3}{4} \min \left\{1, \left(\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i)\right)\right\} \geq \frac{3}{4} z_j$. Since $OPT \leq OPT(LP) = \sum_j z_j$ we get a $\frac{3}{4}$-approximation.
To prove the inequality, by a change of variables y_i to $1 - y_i$ for each $i \in N_j$, it is enough to show

$$1 - \left(\prod_{i \in P_j \cup N_j} \left(1 - \frac{y_i}{2} - \frac{1}{4}\right) \right) \geq \frac{3}{4} \min \left\{1, \sum_{i \in P_j \cup N_j} y_i \right\} \tag{1}$$

Let $R = P_j \cup N_j$. If $\sum_{i \in R} y_i > 1$, then reducing any of the positive $y_i's$ slightly decreases the LHS but doesn't change the RHS. Thus we can assume, wlog, $\sum_{i \in R} y_i \leq 1$. Thus we need to show that

$$1 - \prod_{i \in R} \left(1 - \frac{y_i}{2} - \frac{1}{4}\right) \geq \frac{3}{4} \sum_{i \in R} y_i \tag{2}$$

$$s.t. \quad \sum_{i \in R} y_i \leq 1 \tag{3}$$

Now taking derivatives wrt y_i for some i, we see that derivative of LHS is at most $3/4$ and derivative of RHS is $3/4$. Thus we can assume wlog that $\sum_{i \in R} y_i = 1$. Applying AM-GM inequality we obtain

$$\prod_{i \in R} \left(1 - \frac{y_i}{2} - \frac{1}{4}\right) \leq \left(\frac{3}{4} - \frac{\sum_{i \in R} y_i}{2|R|}\right)^{|R|} = \left(\frac{3}{4} - \frac{1}{2|R|}\right)^{|R|} \tag{4}$$

Now, use basic calculus to verify that

$$1 - \left(\frac{3}{4} - \frac{1}{2|R|}\right)^{|R|} \geq \frac{3}{4} \tag{5}$$

for each integer value of $|R|$.

3. **Iterative Rounding.**

(a) If a variable $x_{ij} = 0$, we remove that variable and if $x_{ij} = 1$, we assign the job j to machine i, remove job j, update $T_i \leftarrow T_i - p_j$.

(b) Suppose not. Then consider the bipartite graph with jobs on one side and machines on the other and edge between a job j and machine i, if $0 < x_{ij} < 1$. Then each job vertex j must have degree at least two since $\sum_i x_{ij} = 1$. By assumption, each machine vertex has degree at least two as well (not counting the zero degree vertices). Thus the total number of edges (or variables) we have is $|M| + |J|$. But the total number of constraints is also $|M| + |J|$ but not all of them are linearly independent (Why?). Thus we obtain a contradiction.

(c) Remove the capacity constraint for the machine found in Step (b) and iterate. To be precise, the argument in (b) has to be generalized to account for machines which are present in the bipartite graph constructed with a single edge at them but no constraint.

4. **Low Dimension Embeddings.** Consider a high-dimensional space $\mathbb{R}^N$ equipped with the ℓ_1 norm. Let V be an n-point subset of $\mathbb{R}^N$. The goal of this exercise is to show that V can be isometrically embedded into $\binom{n}{2}$ dimensional ℓ_1 space i.e., there exists a map $\varphi : V \to \mathbb{R}^{\binom{n}{2}}$ such that for every $x, y \in V$,

$$\|x - y\|_1 = \|\varphi(x) - \varphi(y)\|_1,$$

where $\|x - y\|_1 = \sum_{i=1}^{N} |x_i - y_i|$ and $\|\varphi(x) - \varphi(y)\|_1 = \sum_{i=1}^{\binom{n}{2}} |\varphi_i(x) - \varphi_i(y)|$.

(a) First, we show that V can be isometrically embedded into $\mathbb{R}^{2^n}$. As we've seen in the class, every ℓ_1 metric can be represented as a sum of cut metrics with non-negative coefficients (see Lecture Notes for details). That is, there exists a sequence of non-negative numbers λ_S indexed by subsets $S \subset V$ such that for all $x, y \in V$:

$$\|x - y\|_1 = \sum_{S \subset V} \lambda_S \delta_S(x, y),$$

where $\delta_S(x, y)$ is the cut metric for set S. We define the embedding φ as follows: The image of $\varphi(x)$ is a 2^n dimensional vector indexed by subsets $S \subset V$; $\varphi_S(x) = \lambda_S$, if $x \in S$; and $\varphi_S(x) = 0$, otherwise. Observe, that $|\varphi_S(x) - \varphi_S(y)| = \lambda_S \delta_S(x, y)$. Hence,

$$\|\varphi(x) - \varphi(y)\|_1 = \sum_{S \subset V} |\varphi_S(x) - \varphi_S(y)| = \sum_{S \subset V} \lambda_S \delta_S(x, y) = \|x - y\|_1.$$

(b) We write an LP on the coefficients λ_S:

$$\sum_{S \subset V} \lambda_S \delta_S(x, y) = \|x - y\|_1 \qquad \forall x, y \in V : x \neq y.$$

Note that the LP does not have an objective function – we just want to find an extreme point feasible solution. The LP has 2^n variables and $\binom{n}{2}$ constraints. It is feasible, because the vector λ_S from part (a) satisfies all LP constraints.

(c) Since the LP has $\binom{n}{2}$ linear constraints, the number of non-zero variables λ_S in any extreme point solution is upper bounded by $\binom{n}{2}$. See e.g. Rank Lemma in "Iterative Methods in Combinatorial Optimization" by L. C. Lau, R. Ravi, M. Singh.

(d) Let λ^* be one of the extreme point solutions. We construct an embedding φ as in part (a). Then, we drop all coordinates S with $\lambda^*_S = 0$. This does not change the ℓ_1 distances between points in $\varphi(V)$, since $\varphi_S(x) = 0$ for all $x \in V$, and, hence, $|\varphi_S(x) - \varphi_S(y)| = 0$ for all $x, y \in V$.