On the Capacitated Lot-Sizing and Continuous 0–1 Knapsack Polyhedra

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Abstract

We consider the single item capacitated lot-sizing problem, a well-known production planning model that often arises in practical applications, and derive new classes of valid inequalities for it. We also derive new easily computable valid inequalities for the continuous 0–1 knapsack problem. The continuous 0–1 knapsack problem has been introduced recently and has been shown to provide a useful relaxation of mixed 0–1 integer programs. Finally, we show that by studying an appropriate continuous 0–1 knapsack relaxation of the capacitated lot-sizing problem, it is possible to derive strengthenings of most of the known classes of inequalities for the capacitated lot-sizing problem.

Keywords: Capacitated lot-sizing, continuous 0-1 knapsack problem

December 1998

\footnote{This research was supported by NSF Grant No. DDM-9700285 and by Philips Electronics North America}
1 Introduction

The single item capacitated lot-sizing problem (CLSP) is a production planning model that often arises in practical applications, either in its pure form or as a substructure of a more complicated model, and that has been studied extensively. The standard formulation is

$$\begin{align*}
\min & \quad \sum_{j=1}^{T} p_j x_j + \sum_{j=1}^{T} h_j s_j + \sum_{j=1}^{T} q_j y_j \\
\text{subject to} & \quad x_j + s_{j-1} - s_j = d_j, \; j = 1, ..., T \\
& \quad x_j \leq c_j y_j, \; j = 1, ..., T \\
& \quad s_0 = 0; x_j, s_j \geq 0, \; j = 1, ..., T \\
& \quad y_j \in \{0, 1\}, \; j = 1, ..., T.
\end{align*}$$

The number of time periods in the problem is $T$. The demand and production capacity in period $j$ are given by $d_j$ and $c_j$, respectively; we assume that $d_j > 0$, $c_j > 0$, and $d_j \leq c_j$, $j = 1, ..., T$. The parameter $p_j$ is the unit cost of production during period $j$, $h_j$ is the cost of holding stock or inventory at the end of period $j$, and $q_j$ is the fixed set-up cost that must be paid before any production can occur in period $j$. The nonnegative variable $x_j$ is continuous and represents the amount produced in period $j$. The variable $s_j$, also continuous, represents inventory or stock held over at the end of period $j$, and its nonnegativity ensures that backorders are not permitted. The binary variable $y_j$ indicates whether or not we produce in period $j$. Throughout this paper, it is convenient to use the identity $s_0 = 0$.

Constraints (2) are inventory balance constraints that ensure that demand is met. Constraints (3) enforce the production capacity restriction in each time period, and also ensure that a set-up cost is paid for each period in which production occurs. By letting $d_{jl} = \sum_{k=j}^{l} d_k$, $j \leq l$, we can tighten (3) to

$$x_j \leq \min\{c_j, d_{jT}\} y_j, \; j = 1, ..., T.$$  

We refer to valid inequalities for the set defined by (2), (4)—(6) as valid inequalities for CLSP, and we refer to the convex hull of the set defined by (2), (4)—(6) as $\text{conv}(\text{CLSP})$. This paper studies the polyhedral structure of $\text{conv}(\text{CLSP})$.

Much research has been done on the polyhedral structure of $\text{conv}(\text{CLSP})$. Most of this work has focused on one of two special cases: the uncapacitated lot-sizing problem (ULSP), in which $c_j \geq d_{jT}$, $j = 1, ..., T$ (Barany, Van Roy, and Wolsey [1984]); and the constant capacity lot-sizing problem (CCLP), in which $c_j = c > 0$, $j = 1, ..., T$.
Although CLSP in general is NP-hard, both of these special cases are polynomially solvable. Polynomial-time algorithms were given by Wagner and Whitin [1958] for ULS and by Florian and Klein [1971] for CCLP.

In Section 2, we derive four classes of inequalities for conv(CLSP). The first was introduced by Pochet [1988] and is based on the concept of a cover. We believe that the other three are new. One of these is also based on a cover, and is closely related to Pochet’s class, while the other two classes are based on the concept of a reverse cover. The only other class of valid inequalities for CLSP, of which we are aware, recently appeared in Marchand [1998].

Marchand and Wolsey [1997] studied the polyhedral structure of the convex hull of solutions of the continuous 0-1 knapsack problem (CKP). A continuous 0-1 knapsack set has the form

\[ Y = \{(y, s) \in B^n \times R_+^n : \sum_{j=1}^{n} a_j y_j \leq b + s \}, \]  

where \( a_j \in Z_+ \), \( j = 1, ..., n \), and \( b \in Z_+ \). Marchand and Wolsey derive two classes of easily computable, facet-defining inequalities for CKP through sequential lifting. Covers and reverse covers also play a fundamental role in the derivation of these inequalities. Furthermore, Marchand and Wolsey show that these inequalities can often be used effectively to solve mixed 0-1 integer programs by constructing continuous 0-1 knapsack relaxations.

In Section 3, we continue the investigation of the polyhedral structure of CKP and derive two new classes of easily computable inequalities for CKP.

Then, in Section 4, we examine the relationship between the valid inequalities for CKP and the inequalities presented in Section 2 for CLSP. In particular, we will show that by considering an appropriate CKP relaxation of CLSP, it is possible to derive strengthenings of each of the classes of inequalities presented for CLSP. This clearly demonstrates the value of studying the polyhedral structure of simple mixed 0-1 sets that can arise as relaxations of more complicated problems.

Finally, in Section 5, we will relate our results to what is known about CLSP, ULS, and CCLP.

2 Valid Inequalities for CLSP

Let \([k, ..., l] \subseteq [1, ..., T]\) be a set of consecutive indices (hereafter called an interval). All the families of inequalities in this section have the general form

\[ \sum_{j \in U} \beta_j y_j + \sum_{j \in V} x_j + s_{l-1} - y_j \geq \beta_0, \]  

where \( \beta_j \) and \( x_j \) are variables, \( s_{l-1} \) is a slack variable, and \( y_j \) is a binary variable.
where \( U \cup V = [k, ..., l], U \cap V = \emptyset, \beta_j \geq 0, j = k, ..., l, \) and \( \beta_0 \geq 0. \) When we set \( V = \emptyset, \) we get a subset of each of the families, and these subsets have the simpler form

\[
\sum_{j \in [k, ..., l]} \beta_j y_j + s_{k-1} \geq \beta_0. \tag{9}
\]

For ease of notation and in order to facilitate understanding and comparison, for each of the classes of inequalities in this section we will present only the subset that has this simpler form (9). We will present the entire sets of each class, which have the general form (8), in Section 4.

It is clear that \( \min\{c_j, d_{jli}\} \) is an upper bound on production in period \( j \) that can satisfy demand in periods \( j, ..., l, j \leq l, \) since we do not allow backorders. Therefore, we will let \( \bar{c}_{jl} = \min\{c_j, d_{jli}\}, j \leq l. \)

### 2.1 Cover Inequalities

Given an interval \([k, ..., l] \subset [1, ..., T], \) we say that \( S \subset [k, ..., l] \) is cover of \([k, ..., l] \) if

\[
\mu = \sum_{j \in S} \bar{c}_{jl} - d_{kl} \geq 0.
\]

We then let \( L = [k, ..., l] \setminus S. \)

#### 2.1.1 Family 1—Pochet’s Class

A subset of the class of inequalities introduced by Pochet can be expressed as follows.

**Theorem 1** (Pochet [1988]) Let \( S \) be a cover of an interval \([k, ..., l] \subset [1, ..., T]. \) The inequality

\[
\sum_{j \in S} (\bar{c}_{jl} - \mu)^+ y_j + \sum_{j \in L} \min\{\bar{c}_{jl}, (\bar{c}_{jl}, \bar{c}_{max}) - \mu\}^+ y_j + s_{k-1} \geq \sum_{j \in S} (\bar{c}_{jl} - \mu)^+,
\]

where \( \bar{c}_{max} = \max_{j \in S} \{\bar{c}_{jl}\} \) and \( a^+ = \max\{a, 0\}, \) is valid for CLSP.

When \( \mu < \bar{c}_{max} \leq \min_{j \in L}\{\bar{c}_{jl}\}, \) that is, when the periods are ordered in nondecreasing values of \( \bar{c}_{jl} \) and then placed into \( S \) until production capacity in \( S \) covers demand in \([k, .., l], \) (10) simplifies to

\[
\sum_{j \in S} (\bar{c}_{jl} - \mu)^+ y_j + \sum_{j \in L} (\bar{c}_{jl} - \mu) y_j + s_{k-1} \geq \sum_{j \in S} (\bar{c}_{jl} - \mu)^+.
\]
2.1.2 Family 2

We have identified another class of inequalities that can be derived from a cover of 
[k, ..., l].

**Theorem 2** Let S be a cover of an interval [k, ..., l] ⊂ [1, ..., T]. The inequality

\[
\sum_{j \in S} (\min \{\bar{c}_{jl}, \bar{c}_{\min}\} - \mu)^+ y_j + \sum_{j \in L} \min \{\bar{e}_{jl}, (\max \{\bar{e}_{jl}, \bar{c}_{\min}\} - \mu)\} y_j + s_{k-1} \\
\geq \sum_{j \in S} (\min \{\bar{c}_{jl}, \bar{c}_{\min}\} - \mu)^+,
\]

where

\[
\bar{c}_{\min} = \max \{\min_{j \in S, \bar{e}_{jl} \geq \mu} \{\bar{e}_{jl}\}, \min_{j \in L} \{\bar{e}_{jl}\}\},
\]

is valid for CLSP.

(When minimizing over an empty set, we let the minimum value be +∞. Therefore
\(\bar{c}_{\min} = +\infty\) if \(\{j \in S : \bar{e}_{jl} \geq \mu\} = \emptyset\), and (12) becomes

\[
\sum_{j \in L} \bar{e}_{jl} y_j + s_{k-1} \geq 0,
\]

which is trivially valid.)

The proof of Theorem 2 is similar to the proof given by Pochet for Theorem 1.

When \(\mu < \max_{j \in S} \{\bar{e}_{jl}\} \leq \min_{j \in L} \{\bar{e}_{jl}\}\), that is, when the periods are ordered in non-
decreasing values of \(\bar{e}_{jl}\) and then placed in \(S\) until \(S\) covers \([k, ..., l]\), (12) also simplifies
to (11).

2.2 Reverse Cover Inequalities

Given an interval [k, ..., l] ⊂ [1, ..., T], we choose a set \(S \subset [k, ..., l]\) such that

\[
\lambda = d_{kl} - \sum_{j \in S} \bar{e}_{jl} > 0.
\]

We refer to \(S\) as a reverse cover for \([k, ..., l]\). Again, we let \(L = [k, ..., l] \setminus S\).
2.2.1 Family 1

Theorem 3 Let $S$ be a reverse cover of an interval $[k, ..., l] \subset [1, ..., T]$. The inequality

$$\sum_{j \in S}(\lambda - (\bar{e}_{\min} - \bar{e}_{ji})^+)y_j + \sum_{j \in L} \min\{\bar{e}_{ji}, \lambda + (\bar{e}_{ji} - \bar{e}_{\min})^+\}y_j
+ s_{k-1} \geq \lambda + \sum_{j \in S}(\lambda - (\bar{e}_{\min} - \bar{e}_{ji})^+),$$

where

$$\bar{e}_{\min} = \max\{\min\{\bar{e}_{ji}, \lambda \}, \min\{\bar{e}_{ji} \}, \lambda \},$$

is valid for CLSP.

Proof: If $\{j \in L : \bar{e}_{ji} \geq \lambda\} = \emptyset$, then (14) reduces to a trivially valid inequality, so in the remainder of the proof we assume $\{j \in L : \bar{e}_{ji} \geq \lambda\} \neq \emptyset$.

We consider a feasible solution $(x^*, y^*, s^*)$ to CLSP, and show that it satisfies (14). Let

$$S^* = \{j \in S : y_j = 1\} \cup \{j \in S : \bar{e}_{ji} \leq \bar{e}_{\min} - \lambda\}$$

and

$$L^* = \{j \in L : y_j = 1\}.$$

Partition $L^*$ as follows:

$$L^*_1 = \{j \in L^* : \bar{e}_{ji} > \bar{e}_{\min}\}$$
$$L^*_2 = \{j \in L^* : \bar{e}_{\min} \geq \bar{e}_{ji} \geq \lambda\}$$
$$L^*_3 = \{j \in L^* : \lambda > \bar{e}_{ji}\}$$

(It is possible that one or more of these sets is empty.) To show that $(x^*, y^*, s^*)$ satisfies (14), we need to show

$$\sum_{j \in L^*_1}(\bar{e}_{ji} - \bar{e}_{\min} + \lambda) + \sum_{j \in L^*_2} \lambda + \sum_{j \in L^*_3} \bar{e}_{ji} + s_{k-1} \geq 
\lambda + \sum_{j \in S \setminus S^*}(\lambda - (\bar{e}_{\min} - \bar{e}_{ji})^+).$$

(15)

The proof proceeds by cases.

Case 1: $|L^*_1 \cup L^*_2| \leq |S \setminus S^*|$. Consider

$$\sum_{j \in L^*_1}(\bar{e}_{ji} - \bar{e}_{\min}) + \sum_{j \in L^*_3} \bar{e}_{ji} + s_{k-1} \geq \lambda + \sum_{j \in S \setminus S^*}(\lambda - (\bar{e}_{\min} - \bar{e}_{ji})^+) - |L^*_1 \cup L^*_2|\lambda$$

(16)
The inequality (15) is obtained from (16) by adding \([L_1^* \cup L_2^*]\) to both sides, so it suffices to prove (16). Now, since \(d_{kl} - \sum_{j \in S^*} \bar{c}_{jl} = \lambda + \sum_{j \in S\setminus S^*} \bar{c}_{jl}\), since \(\bar{c}_{jl}\) is an upper bound on any production in period \(j\) that can satisfy demand in \([k, ..., l]\), and since \((x^*, y^*, s^*)\) is feasible,

\[
\sum_{j \in L_1^*} \bar{c}_{jl} + s^*_{k-1} \geq d_{kl} - \sum_{j \in S^*} \bar{c}_{jl} - \sum_{j \in L_1^* \cup L_2^*} \bar{c}_{jl} \\
\geq (d_{kl} - \sum_{j \in S^*} \bar{c}_{jl}) - |L_1^* \cup L_2^*| \bar{c}_{min} - \sum_{j \in L_1^*} (\bar{c}_{jl} - \bar{c}_{min}) \\
= (\lambda + \sum_{j \in S\setminus S^*} \bar{c}_{jl}) - |L_1^* \cup L_2^*| \bar{c}_{min} - \sum_{j \in L_1^*} (\bar{c}_{jl} - \bar{c}_{min}). \tag{17}
\]

This implies

\[
\sum_{j \in L_1^*} (\bar{c}_{jl} - \bar{c}_{min}) + \sum_{j \in L_2^*} \bar{c}_{jl} + s^*_{k-1} \geq \lambda + \sum_{j \in S\setminus S^*} \bar{c}_{jl} - |L_1^* \cup L_2^*| \bar{c}_{min}. \tag{18}
\]

By definition \(\bar{c}_{min} \geq \lambda\), so

\[
\lambda + \sum_{j \in S\setminus S^*} \bar{c}_{jl} - |L_1^* \cup L_2^*| \bar{c}_{min} \geq \lambda + (|S \setminus S^*| - |L_1^* \cup L_2^*|) \bar{c}_{min}
\]

\[
= \lambda + \frac{|S \setminus S^*| - |L_1^* \cup L_2^*| \lambda}{\sum_{j \in S\setminus S^*} (\bar{c}_{min} - \bar{c}_{jl})^+}
\]

\[
= \lambda + \sum_{j \in S\setminus S^*} (\lambda - (\bar{c}_{min} - \bar{c}_{jl})^+) - |L_1^* \cup L_2^*| \lambda, \tag{19}
\]

and (16) holds.

**Case 2:** \(|L_1^* \cup L_2^*| > |S \setminus S^*|\). In this case
\[
\sum_{j \in L_s} (\bar{c}_{jl} - \bar{c}_{\min} + \lambda) + \sum_{j \in L_s} \lambda + \sum_{j \in L_s} \bar{c}_{jl} + s_{k-1}^* \geq (|S \setminus S^*| + 1)\lambda \\
\geq \lambda + \sum_{j \in S \setminus S^*} (\lambda - (\bar{c}_{\min} - \bar{c}_{jl})^+) ,
\]

and (15) is satisfied. \(\Box\)

When \(\min_{j \in S}\{\bar{c}_{jl}\} \geq \max_{j \in L}\{\bar{c}_{jl}\} \geq \lambda\) (that is, the elements of \([k, ..., l]\) are ordered in nonincreasing order and then put into \(S\) until the reverse cover is maximal), (14) simplifies to

\[
\sum_{j \in S} \lambda y_j + \sum_{j \in L} \min\{\bar{c}_{jl}, \lambda\} y_j + s_{k-1} \geq (|S| + 1)\lambda.
\]

(20)

2.2.2 Family 2

**Theorem 4** Let \(S\) be a reverse cover of an interval \([k, ..., l] \subset [1, ..., T]\). The inequality

\[
\sum_{j \in S} \min\{\bar{c}_{jl}, (\lambda - (\bar{c}_{\max} - \bar{c}_{jl})^+)\} y_j + \sum_{j \in L} \min\{\bar{c}_{jl}, \lambda\} y_j + s_{k-1} \geq \\
\lambda + \sum_{j \in S} \min\{\bar{c}_{jl}, (\lambda - (\bar{c}_{\max} - \bar{c}_{jl})^+)\} ,
\]

(21)

where \(\bar{c}_{\max} = \max_{j \in L}\{\bar{c}_{jl}\}\), is valid for CLSP.

The proof of Theorem 4 is similar to the one given for Theorem 3.

Note that when \(\lambda \leq \bar{c}_{\max} \leq \min_{j \in S}\{\bar{c}_{jl}\}\), which, as above, occurs when the elements of \([k, ..., l]\) are ordered in nonincreasing order and then put into \(S\) until the reverse cover is maximal, (21) also simplifies to (20).

3 Continuous Knapsack Inequalities

Covers and reverse covers also form the basis for classes of inequalities for CKP. Let \(N = \{1, ..., n\}\). We define the continuous 0–1 knapsack set by
where $a_j \in \mathbb{Z}_+, j = 1, \ldots, n$, and $b \in \mathbb{Z}_+$. We say that the set $Y$ defines an instance of CKP.

An $(i, C, T)$ cover pair for $Y$ is an index $i$ and sets $C$ and $T$ such that

- $C \cap T = i, C \cup T = N$;
- $\lambda = \sum_{j \in C} a_j - b > 0$;
- $a_i > \lambda$.

Note that these conditions imply

$$
\mu = a_i - \lambda = \sum_{j \in T} a_j - (\sum_{j \in N} a_j - b) > 0.
$$

One way to derive classes of inequalities for $Y$ is through sequential lifting. We first take an $(i, C, T)$ cover pair and fix all $y_j, j \in C \setminus i$ to 1 and all $y_j, j \in T \setminus i$ to 0, thus obtaining the two-dimensional polyhedron $Y_0 = \{(y, s) \in B^1 \times R^1_+ : a_i y_i \leq b + s\}$. The unique non-trivial facet of \text{conv}(Y_0)$ is given by

$$
(a_i - \mu) y_i \leq s,
$$

or equivalently, $\lambda y_i \leq s$. We then lift the fixed variables back into (22). Different lifting orders lead to different inequalities, which are facets for \text{conv}(Y)$ if the lifting is maximal. For two general lifting sequences, Marchand and Wolsey [1997] identified the lifting function and showed that it is superadditive, and thus lifting is sequence independent and can be done with negligible computation (see Wolsey [1977] and Gu, Nemhauser, and Savelsbergh [1995] for a more elaborate discussion of sequence independent lifting). These two lifting sequences give rise to two classes of inequalities, namely \textit{continuous cover inequalities} and \textit{continuous reverse cover inequalities}.

We have identified two other general lifting sequences which yield easily computable valid inequalities for $Y$, and we have identified conditions under which these inequalities induce facets of \text{conv}(Y)$.
3.1 Continuous Cover and Continuous Reverse Cover Inequalities

**Theorem 5** *(Marchand and Wolsey [1997])*  Given an \((i, C, T)\) cover pair for \(Y\), order the elements of \(C\) such that \(a_1 \geq \ldots \geq a_{r_C}\), where \(r_C\) is the number of elements of \(C\) with \(a_j > \lambda\). Let \(A_0 = 0\), \(A_j = \sum_{p=1}^j a_{[p]}\), \(j = 1, \ldots, r_C\), and define

\[
\phi_C(u) = \begin{cases} 
(j - 1)\lambda, & \text{if } A_{j-1} \leq u \leq A_j - \lambda, \ j = 1, \ldots, r_C, \\
(j - 1)\lambda + [u - (A_j - \lambda)], & \text{if } A_j - \lambda \leq u \leq A_j, \ j = 1, \ldots, r_C - 1, \\
(r_C - 1)\lambda + [u - (A_{r_C} - \lambda)], & \text{if } A_{r_C} - \lambda \leq u. 
\end{cases}
\]

The inequality

\[
\sum_{j \in C} \min\{\lambda, a_j\} y_j + \sum_{T \backslash i} \phi_C(a_j) y_j \leq \sum_{j \in C \backslash i} \min\{\lambda, a_j\} + s
\]

defines a facet of \(\text{conv}(Y)\).

Inequalities of the form (24) are called **continuous cover inequalities**, and are obtained from (22) by first lifting all the variables in \(C \backslash i\), then all those in \(T \backslash i\). Because the lifting function for each of the two sets is superadditive, the lifting order within the two sets is immaterial.

**Theorem 6** *(Marchand and Wolsey [1997])*  Given an \((i, C, T)\) cover pair for \(Y\), order the elements of \(T\) such that \(a_1 \geq \ldots \geq a_{r_T}\), where \(r_T\) is the number of elements of \(T\) with \(a_j > \mu\). Let \(A_0 = 0\), \(A_j = \sum_{p=1}^j a_{[p]}\), \(j = 1, \ldots, r_T\), and define

\[
\psi_T(u) = \begin{cases} 
 u - j\mu, & \text{if } A_j \leq u \leq A_{j+1} - \mu, \ j = 0, \ldots, r_T - 1, \\
 A_j - j\mu, & \text{if } A_j - \mu \leq u \leq A_j, \ j = 1, \ldots, r_T - 1, \\
 A_{r_T} - r_T\mu, & \text{if } A_{r_T} - \mu \leq u.
\end{cases}
\]

The inequality

\[
\sum_{j \in T} (a_j - \mu)^+ y_j + \sum_{j \in C \backslash i} \psi_T(a_j) y_j \leq \sum_{j \in C \backslash i} \psi_T(a_j) + s
\]

defines a facet of \(\text{conv}(Y)\).

Inequalities of the form (26) are called **continuous reverse cover inequalities**, and are derived from (22) by lifting all variables in \(T \backslash i\) first, then all variables in \(C \backslash i\). As before, the coefficients do not depend on the sequence chosen within \(T \backslash i\) and \(C \backslash i\).
3.2 New Lifting Orders for Continuous Knapsack Inequalities

In this section we examine two new lifting orders that yield easily computable inequalities for \( Y \). In addition, we give conditions under which these inequalities define facets of \( \text{conv}(Y) \).

As with continuous cover and continuous reverse cover inequalities, in deriving both of these classes we first fix all \( y_j, j \in C \setminus i \) to 1, and all \( y_j, j \in T \setminus i \) to 0, to obtain (22). From now on, we will let \( X = X \cup \epsilon \), where \( X \) is a set and \( \epsilon \not\in X \).

3.2.1 First New Lifting Sequence

Given an \((i, C, T)\) cover pair, let \( v = \max\{a_i, \min_j \in C \setminus \{a_j\} \} \). Also, let \( T' = \{ j \in T \setminus i : a_j \leq v \} \), \( T'' = T \setminus T' \), \( C' = \{ j \in C \setminus i : a_j < v \} \), and \( C'' = C \setminus C' \). The first sequence is as follows:

1. Lift all \( j \in T' \).
2. Lift all \( j \in C'' \).
3. Lift all \( j \in T'' \).
4. Lift all \( j \in C' \).

Also, let

\[
\hat{C} = \begin{cases} 
C'', & \text{if } a_i \leq \min_{j \in C \setminus \{a_j\}} \\
C', & \text{otherwise.}
\end{cases}
\]

Thus we add \( i \) to \( C'' \) to form the set \( \hat{C} \) if and only if \( C' \neq \emptyset \). The reason for doing so will become apparent in the proof of the following theorem.

**Theorem 7** Given an \((i, C, T)\) cover pair, order the elements of \( \hat{C} \) such that \( a_{[1]} \leq \ldots \leq a_{[|\hat{C}|]} \). Let \( B_0 = 0, B_j = \sum_{p=1}^{j} a_{[p]}, j = 1, \ldots, |\hat{C}|, \) and define

\[
f_{\hat{C}}(u) = \begin{cases} 
B_j - j\mu, & \text{if } B_j \leq u \leq B_j + \mu, j = 0, \ldots, |\hat{C}|, \\
B_j - j\mu + u - (B_j + \mu), & \text{if } B_j + \mu \leq u \leq B_{j+1}, j = 0, \ldots, |\hat{C}| - 1, \\
|\hat{C}|\mu + u - (|\hat{C}|\mu + \mu), & \text{if } B_{|\hat{C}|} + \mu \leq u.
\end{cases}
\]

The inequality

\[
\sum_{j \in C} f_{\hat{C}}(a_j) y_j + \sum_{j \in C'} \min\{a_j, \lambda\} y_j + \sum_{j \in C'} (a_j - \mu) y_j \\
\leq \sum_{j \in C'} \min\{a_j, \lambda\} + \sum_{j \in C'} (a_j - \mu) + s
\]

is valid for \( Y \).
Proof: It can be checked that fixing and lifting variables as specified in steps 1 and 2 above yields the continuous reverse cover inequality

\[
\sum_{j \in T'} (a_j - \mu) y_j + (a_i - \mu) y_i + \sum_{j \in C''} \psi_{T_i'}(a_j) y_j \leq \sum_{j \in C''} \psi_{T_i'}(a_j) + s,
\]

(29)

for \( Y_{T^n,C'} = \{(y,s) \in Y : y_j = 0, j \in T''; y_j = 1, j \in C'\} \). Here \((i, C', T_i')\) is an \(i\)-cover pair for \( Y_{T^n,C'} \) with the same capacity deficiency \( \mu \) that \((i, C, T)\) has for \( Y \). (I.e., \( \mu = b - \sum_{j \in C'} a_j \) obviously implies \( \mu = (b - \sum_{j \in C'} a_j) - \sum_{j \in C''} a_j \).) Thus the lifting function \( \psi_{T_i'}(\cdot) \) for \( y_j, j \in C'' \) is given by (25), where the elements \( A_j \) are obtained by ordering and summing the coefficients of \( T'_i \) with \( a_j > \mu \). Since \( \psi_{T_i'}(u) \leq (u - \mu), u \geq a_i \), and since \( a_j \geq v \geq a_i, j \in C' \),

\[
\sum_{j \in T'} (a_j - \mu) y_j + (a_i - \mu) y_i + \sum_{j \in C''} (a_j - \mu) y_j \leq \sum_{j \in C''} (a_j - \mu) + s
\]

(30)
is valid for \( Y_{T^n,C'} \). To lift \( y_j, j \in T'' \), we define

\[
F(u) = \max\left\{ \sum_{j \in T'} (a_j - \mu) y_j + (a_i - \mu) y_i + \sum_{j \in C''} (a_j - \mu) y_j - s : \sum_{j \in T'' \cup C''} a_j y_j \leq u + s \right\}.
\]

(31)
The lifting function is then defined as

\[
\hat{f}_{T' \cup C''}(u) = F(b - \sum_{j \in C'} a_j) - F((b - \sum_{j \in C'} a_j) - u).
\]

Let \( j \in C'' \) be ordered such that \( a_{[1]} \geq ... \geq a_{[|C''|]} \), and let \( A_0 = 0, A_j = \sum_{p=1}^j a_{[p]}, j = 1, ..., |C'\|. Then

\[
F(u) = \begin{cases} 
  u, & \text{if } u \leq 0, \\
  A_j - j \mu, & \text{if } A_j \leq u \leq A_j + \mu, j = 0, ..., |C'\|, \\
  A_j - j \mu + (u - (A_j + \mu)), & \text{if } A_j + \mu \leq u \leq A_{j+1}, j = 0, ..., |C'| - 1.
\end{cases}
\]

(32)

This follows from the fact that it is possible to solve (31) greedily, by taking \( j \in C'' \) in nondecreasing values of \( a_j \). Given (32), it is easy to verify algebraically that
where the elements $\hat{B}_j$ are obtained by ordering $j \in C''$ such that $a_{[1]} \leq \ldots \leq a_{[|C''|]}$, and letting $\hat{B}_0 = 0$, $\hat{B}_j = \sum_{p=1}^j a_{[p]}$, $j = 1, \ldots, |C''|$. Thus from (27) $f_{\hat{C}}(\cdot) \leq \hat{f}_{T'_{\hat{U}C''}}(\cdot)$.

Moreover, both $\hat{f}_{T'_{\hat{U}C''}}(\cdot)$ and $f_{\hat{C}}(\cdot)$ are superadditive on $[0, \infty)$, and therefore

$$
\sum_{j \in T'} (a_j - \mu)^+ y_j + \sum_{j \in T''} f_{\hat{C}}(a_j) y_j + (a - \mu) y; + \sum_{j \in C''} (a_j - \mu) y_j \
\leq \sum_{j \in C''} (a_j - \mu) + s
$$

is valid for $Y_{C'} = \{(y, s) \in Y : y_j = 1, j \in C'\}$.

Finally, we lift $y_j$, $j \in C'$ into (33). Assuming that $C' \neq \emptyset$ (thus $a_i = v$ and $\hat{C} = C''$), we proceed by complementing binary variables and lifting $\tilde{y}_j$, $j \in C'$ into

$$
- \sum_{j \in T'} (a_j - \mu)^+ \tilde{y}_j - \sum_{j \in T''} f_{\hat{C}}(a_j) \tilde{y}_j - \sum_{j \in C''} (a_j - \mu) \tilde{y}_j \
\leq - \sum_{j \in T'} (a_j - \mu)^+ - \sum_{j \in T''} f_{\hat{C}}(a_j) + s.
$$

(34)

To compute $\sigma_{T_{\hat{U}C''}}(\cdot)$, the function to lift $\tilde{y}_j$, $j \in C'$ into (34), we first define

$$
G(u) = \max \{- \sum_{j \in T'} (a_j - \mu)^+ \tilde{y}_j - \sum_{j \in T''} f_{\hat{C}}(a_j) \tilde{y}_j - \sum_{j \in C''} (a_j - \mu) \tilde{y}_j - s : \\
- \sum_{j \in TUC''} a_j \tilde{y}_j \leq u + s\}.
$$

Since $0 < a_j < a_i$, $j \in C'$, we need to compute

$$
\sigma_{T_{\hat{U}C''}}(u) = G(b - \sum_{j \in N} a_j) - G(b - \sum_{j \in N} a_j) - u), u \in (-a_i, 0).
$$

For this we use the following lemma, whose proof is given in the appendix.

Lemma 8

$$
G(u) = - \sum_{j \in T'} (a_j - \mu)^+ - \sum_{j \in T''} f_{\hat{C}}(a_j) - ((b - \sum_{j \in N} a_j - u)^+, \\
b - \sum_{j \in N} a_j \leq u \leq b - \sum_{j \in N \backslash i} a_j.
$$

(35)
From Lemma 8 it follows that $\sigma_{T \cup C^u}(u) = -\min(-u, \lambda), -a_i < u < 0$, and therefore $-\sigma_{T \cup C^u}(u) = \min(u, \lambda), 0 < u < a_i$. Thus $\sigma_{T \cup C^u}(\cdot)$ is superadditive on $(-a_i, 0)$. (These facts would not necessarily hold if we replaced $f_C(\cdot)$ in (33) with $f_{T_i \cup C^u}(\cdot)$.) Noting that $f_C(a_j) = (a_j - \mu)^+$, $j \in T_i'$, we conclude that (28) is a valid inequality for $Y$. □

The proof of Theorem 7 also yields

**Corollary 9** The inequality (28) defines a facet of $\text{conv}(Y)$ if and only if (i) $\psi_{T_i}(a_j) = (a_j - \mu), \forall j \in C''$; and (ii) $\hat{f}_{T_i \cup C^u}(a_j) = f_C(a_j), \forall j \in T''$.

**Corollary 10** If $C' = \emptyset$, (28) defines a facet of $\text{conv}(Y)$ if and only if $\psi_{T_i}(a_j) = (a_j - \mu), \forall j \in C''$.

### 3.2.2 Second New Lifting Sequence

In this section, given an $(i, C, T)$ cover pair, we let $v = \max\{a_i; \min_{j \in T \setminus \{i\}} a_j\}$. Let $C' = \{j \in C \mid a_j \leq v\}$, $C'' = C \setminus C'$, $T' = \{j \in T \setminus i : a_j < v\}$, and $T'' = T \setminus T'$. The second sequence is as follows:

1. Lift all $j \in C'$.
2. Lift all $j \in T''$.
3. Lift all $j \in C''$.
4. Lift all $j \in T'$.

Also, let

$$\hat{T} = \begin{cases} 
T'', & \text{if } a_i \leq \min_{j \notin T \setminus i} a_j \\
T''_i, & \text{otherwise}
\end{cases}$$

As before, $\hat{T}$ is formed from $T''$ by adding $i$ if and only if $T' \neq \emptyset$. Since the proofs of the theorem and corollaries in this section parallel those presented in the last section, they are omitted here.

**Theorem 11** Given an $(i, C, T)$ cover pair, order the elements of $\hat{T}$ such that $a_{[1]} \leq \ldots \leq a_{[|\hat{T}|]}$. Let $B_j = 0$, $B_j = \sum_{p=1}^{j} a_{[p]}, j = 1, \ldots, |\hat{T}|$, and define
The inequality

\[
\sum_{j \in T} (a_j - \mu)^+ y_j + \sum_{j \in T'} \lambda y_j + \sum_{j \in C} g_T(a_j) y_j \leq \sum_{j \in C \setminus i} g_T(a_j) + s
\]

is valid for \( Y \).

Observe that \(-g_T(\cdot)\) is superadditive on \((-\infty, 0]\). For the next corollary, we define \( \phi_{C_i}(\cdot) \) as equal to \( \phi_C(\cdot) \) in (23), where the elements \( A_j \) are obtained by ordering and summing \( j \in C_i \) with \( a_j > \lambda \). Also, we define

\[
\hat{g}_{T' \cup C_i}(u) = \begin{cases} 
  j\lambda + (u - \hat{B}_j), & \text{if } \hat{B}_j \leq u \leq \hat{B}_j + \lambda, \ j = 0, \ldots, |T'|, \\
  (j + 1)\lambda, & \text{if } \hat{B}_j + \lambda \leq u \leq \hat{B}_{j+1}, \ j = 0, \ldots, |T'| - 1, \\
  ([T'| + 1)\lambda, & \text{if } \hat{B}_{|T'|} + \lambda \leq u,
\end{cases}
\]

where we first order \( a_j, j \in T' \) such that \( a_{[1]} \leq \ldots \leq a_{|T'|} \), and then let \( \hat{B}_0 = 0 \), \( \hat{B}_j = \sum_{p=1}^j a_{[p]}, j = 1, \ldots, |T'| \). The function \( \hat{g}_{T' \cup C_i}(\cdot) \) is also superadditive, and \( g_T(\cdot) \geq \hat{g}_{T' \cup C_i}(\cdot) \).

**Corollary 12** The inequality (37) defines a facet of \( \text{conv}(Y) \) if and only if (i) \( \phi_{C_i}(a_j) = \lambda, \forall j \in T' \); and (ii) \( \hat{g}_{T' \cup C_i}(a_j) = g_T(a_j), \forall j \in C_i \).

**Corollary 13** If \( T' = \emptyset \), the inequality (37) defines a facet of \( \text{conv}(Y) \) if and only if \( \phi_{C_i}(a_j) = \lambda, \forall j \in T'' \).

# 4 Deriving CLSP Inequalities from CKP Inequalities

In this section we use results for CKP to strengthen each of the four classes of inequalities presented in Section 2. We first give a simple valid inequality for CLSP which generates an instance of CKP.
Proposition 14 For any interval \([k, ..., l] \subset [1, ..., T]\), the inequality

\[
\sum_{j=k}^{l} \bar{c}_{jl} \bar{y}_j \leq \left( \sum_{j=k}^{l} \bar{c}_{jl} - d_{kl} \right) + s_{k-1},
\]  

(38)

where \(\bar{y}_j = (1 - y_j), j = 1, ..., n\), is valid for CLSP.

Proof: Because backorders are not permitted, \(\bar{c}_{jl}\) is an upper bound on production in \(j\) that can be used to satisfy demand in \([k, ..., l], k \leq j\). Thus, for any \([k, ..., l] \subset [1, ..., T]\),

\[
s_{k-1} + \sum_{j=k}^{l} \bar{c}_{jl} y_j \geq d_{kl}
\]  

(39)

is valid for CLSP. By complementing the \(y\) variables, i.e., setting \(\bar{y}_j = (1 - y_j), j \in [k, ..., l]\), we obtain (38). □

We now define

\[
\bar{Y}_{kl} = \{(\bar{y}_k, ..., \bar{y}_l, s_{k-1}) \in \mathbb{R}^{l-k+1} \times \mathbb{R}_+^l : \sum_{j=k}^{l} \bar{c}_{jl} \bar{y}_j \leq \left( \sum_{j=k}^{l} \bar{c}_{jl} - d_{kl} \right) + s_{k-1}\},
\]  

(40)

which is an instance of CKP, and thus valid inequalities for \(\bar{Y}_{kl}\) are also valid for CLSP, after recomplementation. We will explore the relationship between the inequalities given in Section 2 for CLSP and valid inequalities for \(\bar{Y}_{kl}\).

4.1 CLSP Cover Inequalities

4.1.1 Family 1

We first restate (26) for \(\bar{Y}_{kl}\) as

\[
\sum_{j \in T} (\bar{c}_{jl} - \mu)^+ \bar{y}_j + \sum_{j \in C \setminus i} \psi_T(\bar{c}_{jl}) \bar{y}_j \leq \sum_{j \in C \setminus i} \psi_T(\bar{c}_{jl}) + s_{k-1}.
\]  

(41)

Proposition 15 Given any CLSP cover inequality of Family 1 of the form (40), there is a continuous reverse cover inequality for \(\bar{Y}_{kl}\) of the form (41) which implies it.
Proof: If $\mu = 0$, both (10) and (41) reduce to (39), which is trivially valid. If \( \{ j \in S : \bar{c}_{ij} > \mu \} = \emptyset \), then both (10) and (41) reduce to $s_{k-1} \geq 0$, which is also trivially valid.

If neither of these cases applies, take any $i \in S$ with $\bar{c}_{ij} > \mu$ and take $T = S$ and $C = L \cup i$. Since $S$ is a cover of $[k, \ldots, l]$ with excess capacity $\mu$, $(i, C, T)$ is a cover pair for $\bar{c}_{il}$ with capacity deficiency $\mu$. Thus we can redefine Family 1 of the cover inequalities to have the form (41) and obtain

$$\sum_{j \in S} (\bar{c}_{ij} - \mu)^+ y_j + \psi_T(\bar{c}_{ij}) y_j + s_{k-1} \geq \sum_{j \in S} (\bar{c}_{ij} - \mu)^+,$$

which implies (10) because

$$\psi_T(\bar{c}_{ij}) \leq \min\{\bar{c}_{ij}, (\max\{\bar{c}_{ij}, \bar{c}_{max}\} - \mu)^+\}.$$ (To see this, order $j \in T$ such that $\bar{c}_{[1]i} \geq \ldots \geq \bar{c}_{[T]i}$. Thus $\bar{c}_{max} = \bar{c}_{[1]i}$. Then from (25), $\psi_T(\bar{c}_{ij}) = \min\{\bar{c}_{ij}, (\max\{\bar{c}_{ij}, \bar{c}_{max}\} - \mu)^+\}$ if $\bar{c}_{ij} \leq \bar{c}_{[1]i} + \bar{c}_{[2]i} - \mu$, and $\psi_T(\bar{c}_{ij}) < \min\{\bar{c}_{ij}, (\max\{\bar{c}_{ij}, \bar{c}_{max}\} - \mu)^+\}$ if $\bar{c}_{ij} > \bar{c}_{[1]i} + \bar{c}_{[2]i} - \mu$. \)

We can now redefine Family 1 of the cover inequalities to have the form (42).

**Example 1**

In this example we show a facet of $\text{conv}(\text{CLSP})$ that is induced by a Family 1 cover inequality. Consider the instance with capacity and demand vectors

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_j$</td>
<td>19</td>
<td>9</td>
<td>19</td>
<td>9</td>
<td>10</td>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>$d_j$</td>
<td>7</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

Observe that $\bar{c}_{j7} = c_j, j = 1, \ldots, 7$. Take $k = 2$ and $l = 7$, so that $[k, \ldots, l] = [2, \ldots, 7]$. The total demand for this interval is $d_{27} = 31$. Take the cover $S$ to be $\{2, 4, 5, 7\}$ and let $L = \{3, 6\}$. This gives $\mu = 9 + 9 + 10 + 7 - 31 = 4$. In order to define the function $\psi_T(\cdot)$ for this instance, take $T = S$. Then $A_0 = 0, A_1 = 10, A_2 = 19$. Therefore

$$\psi_T(u) = \begin{cases} 
  u, & \text{if } 0 \leq u \leq 6 \\
  6, & \text{if } 6 \leq u \leq 10 \\
  u - 4, & \text{if } 10 \leq u \leq 15 \\
  11, & \text{if } 15 \leq u \leq 19.
\end{cases}$$

Although $\psi(\cdot)$ is defined on the interval $[0, +\infty)$, it is sufficient for this instance to consider $\psi(\cdot)$ only on the interval $[0, 19]$, since $\bar{c}_{j7} \leq 19, j \in L$. Now (42) yields

$$s_1 + 5y_2 + 11y_3 + 5y_4 + 6y_5 + 6y_6 + 3y_7 \geq 19,$$
a facet of \( \text{conv}(\text{CLSP}) \) for this instance. \( \square \)

We can extend (42) to obtain the full class of Family 1 cover inequalities:

\[
\sum_{j \in S'} (\bar{c}_{j \ell} - \mu)^+ y_j + \sum_{j \in S''} x_j + \sum_{j \in L'} \psi_T(\bar{c}_{j \ell}) y_j + \sum_{j \in L''} x_j + s_{k-1} \geq \sum_{j \in S}(\bar{c}_{j \ell} - \mu)^+,
\]

where \( L' \cup L'' = L \) is any partition of \( L \), and \( S' \cup S'' = S \) is any partition of \( S \). Here, to compute \( \psi_T(\cdot) \), we take \( T = S' \), and thus order and sum the elements of \( S' \) with \( \bar{c}_{j \ell} > \mu \), rather than all of \( S = T \), as in (42).

The complete set of inequalities given by Pochet [1988] is

\[
\sum_{j \in S} (\bar{c}_{j \ell} - \mu)^+ y_j + \sum_{j \in L} \min\{\bar{c}_{j \ell}, (\max\{\bar{c}_{j \ell}, \bar{c}_{\ell \max}\} - \mu)^+\} y_j + \sum_{j \in L''} x_j + s_{k-1} \geq \sum_{j \in S}(\bar{c}_{j \ell} - \mu)^+,
\]

where \( L' \cup L'' = L \) is any partition of \( L \). The extended form (44) either includes or implies all inequalities of the form (45), i.e., Pochet’s class.

### 4.1.2 Family 2

**Proposition 16** Let \( S \) be a cover for demand in \([k, \ldots, l]\) such that \( \{j \in S : \bar{c}_{ij} \geq \mu\} \neq \emptyset \), and let \( L = [k, \ldots, l] \setminus S \). Choose any \( i \in S \) such that \( \bar{c}_{ij} \geq \mu \). The inequality

\[
\sum_{j \in S} f_{\hat{C}}(\bar{c}_{j \ell}) y_j + \sum_{j \in L} \min\{\bar{c}_{j \ell}, (\max\{\bar{c}_{j \ell}, v_i\} - \mu)^+\} y_j + s_{k-1} \geq \sum_{j \in S} f_{\hat{C}}(\bar{c}_{j \ell}),
\]

where

\[
v_i = \max\{\bar{c}_{ij}, \min_{j \in L}\{\bar{c}_{j \ell}\}\},
\]

and \( f_{\hat{C}}(\cdot) \) is defined as in (27) with

\[
\hat{C} = \begin{cases} L, & \text{if } v_i \leq \min_{j \in L}\{\bar{c}_{j \ell}\} \\ \{j \in L : \bar{c}_{j \ell} \geq v_i\} \cup i, & \text{otherwise} \end{cases}
\]

is valid for \( \text{CLSP} \).
Proof: If \( v_i > \mu > 0 \), then for the given interval \([k, ..., l]\), define \( Y_{kl} \) as in (40). Take \( T = S \) and \( C = L \cup i \), and define \( C', C'', T' \), and \( T'' \) for \( Y_{kl} \) as they are defined in Theorem 7. Observe that because of our choice of \( i \), \( \lambda = (v_i - \mu) \) if \( C' = \{ j \in L : \bar{e}_{ji} < \bar{e}_{\min} \} \neq \emptyset \). Thus, applying Theorem 7 to \( \bar{Y}_{kl} \) yields

\[
\sum_{j \in T} f_C(\bar{e}_{ji}) \bar{y}_j + \sum_{j \in C'} \min\{\bar{e}_{ji}, v_i - \mu\} \bar{y}_j + \sum_{j \in C''}(\bar{e}_{ji} - \mu) \bar{y}_j \\
\leq \sum_{j \in C'} \min\{\bar{e}_{ji}, v_i - \mu\} + \sum_{j \in C''}(\bar{e}_{ji} - \mu) + s_{k-1}.
\]

(47)

Recompose variables, and the desired result follows. If \( v_i = \mu \), the argument is similar; if \( \mu = 0 \), (46) is trivially valid. \( \square \)

Corollary 17 Given any cover inequality of Family 2 of the form (12) with \( \bar{e}_{\min} \neq +\infty \), there is a valid inequality for \( \bar{Y}_{kl} \) of the form (48) which implies it.

Proof: Choose \( i = \arg\min_{j \in S : \bar{e}_{ji} \geq \mu} \{\bar{e}_{ji}\} \). Apply Proposition 16 with this choice of \( i \), note that \( v_i = \bar{e}_{\min} \), where \( \bar{e}_{\min} \) is defined as in (12), and we obtain

\[
\sum_{j \in S} f_C(\bar{e}_{ji}) \bar{y}_j + \sum_{j \in L} \min\{\bar{e}_{ji}, (\max\{\bar{e}_{ji}, \bar{e}_{\min}\} - \mu)\} \bar{y}_j + s_{k-1} \geq \sum_{j \in S} f_C(\bar{e}_{ji}).
\]

(48)

Since \( f_C(u) \geq (\min\{u, \bar{e}_{\min}\} - \mu)^+, u \geq 0 \), for this choice of \( i \), (46) implies (12). \( \square \)

While Corollary 17 allows us to strengthen this class of inequalities for CLSP, Proposition 16 also allows us to expand it, since we are no longer restricted to choosing \( i = \arg\min_{j \in S : \bar{e}_{ji} \geq \mu} \{\bar{e}_{ji}\} \). There are facets in this family which can only be induced by choosing an \( i \in S \) such that \( i > \arg\min_{j \in S : \bar{e}_{ji} \geq \mu} \{\bar{e}_{ji}\} \). We can now redefine Family 2 of the cover inequalities to have the form (46).

Example 2

In this example we show a facet that is induced by a Family 2 cover inequality. Consider the instance with capacity and demand vectors

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_j )</td>
<td>15</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>12</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td>( d_j )</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>11</td>
<td>9</td>
<td>12</td>
<td>9</td>
</tr>
</tbody>
</table>

In this instance \( \bar{e}_{j7} = c_j, j = 1, ..., 7 \). Take \( k = 2 \) and \( l = 7 \), which gives a total demand \( d_{27} = 49 \). Take \( S = \{2, 4, 5, 7\} \), \( L = \{3, 6\} \), and choose \( i = 2 \); then \( \mu = 2 \),
\(v_2 = \max\{\overline{e}_{2i}, \min_{j \in L} \{e_{ji}\}\} = \max\{10, 12\} = 12\). In order to compute \(f_C(\cdot)\), take \(C = L \cup i\). Since \(\overline{e}_{ji} \geq v_2 = 12, j \in L\), \(\hat{C} = \hat{L} = \{3, 6\}\). So \(B_0 = 0, B_1 = 12,\) and \(B_2 = 27\), and

\[
f_C(u) = \begin{cases} 
0, & \text{if } 0 \leq u \leq 2 \\
\overline{e}_{ii} - 2, & \text{if } 2 \leq u \leq 12 \\
10, & \text{if } 12 \leq u \leq 14 \\
\overline{e}_{ii} - 4, & \text{if } 14 \leq u \leq 27.
\end{cases}
\]

Since \(\overline{e}_{ji} < 27, j \in S\), it is not necessary for this instance to consider the rest of the domain of \(f_C(\cdot)\). So (46) yields

\[
s_1 + 8y_2 + 13y_3 + 16y_4 + 10y_5 + 10y_6 + 7y_7 \geq 41,
\]

a facet for this instance. \(\square\)

The full class of Family 2 cover inequalities can be obtained by extending (46) to

\[
\sum_{j \in S'} f_\hat{C}(\overline{e}_{ji}) y_j + \sum_{j \in S''} x_j + \sum_{j \in L'} \min\{\overline{e}_{ji}, \left(\max\{\overline{e}_{ji}, v_i\} - \mu\right)\} y_j + \\
\sum_{j \in L''} x_j + \sum_{j \in S} f_\hat{C}(\overline{e}_{ji}),
\]

where \(S' \cup S'' = S\) is any partition of \(S\), and \(L' \cup L'' = L\) is any partition of \(L\). To compute \(f_\hat{C}(\cdot)\) in this inequality, we let

\[
\hat{C} = \begin{cases} 
L', & \text{if } v_i \leq \min_{j \in L'} \{\overline{e}_{ji}\} \\
\{j \in L' : \overline{e}_{ji} \geq v_i\} \cup i, & \text{otherwise.}
\end{cases}
\]

### 4.2 CLSP Reverse Cover Inequalities

#### 4.2.1 Family 1

Since the proofs of the results in this section are similar to those given for the results in Section 4.1.2, they are omitted here.

**Proposition 18** Let \(S\) be a reverse cover of demand in \([k, ..., l]\), and let \(L = [k, ..., l] \setminus S\), such that \(\{j \in L : \overline{e}_{ji} \geq \lambda\} \neq \emptyset\). Choose any \(i \in L\) with \(\overline{e}_{ii} \geq \lambda\). The inequality

\[
\sum_{j \in S} (\lambda - (v_i - \overline{e}_{ji})^+) y_j + \sum_{j \in L} g_f(\overline{e}_{ji}) y_j + s_{k-1} \geq \sum_{j \in S} (\lambda - (v_i - \overline{e}_{ji})^+) \lambda,
\]

where

\[
v_i = \max\{\overline{e}_{ii}, \min_{j \in S} \{\overline{e}_{ji}\}\},
\]

20
and $g_\hat{T}(\cdot)$ is defined as in (27) with

$$
\hat{T} = \begin{cases} 
S, & \text{if } v_i \leq \min_{j \in S} \{\bar{c}_{ji}\} \\
\{j \in S : \bar{c}_{ji} \geq v_i\} \cup i, & \text{otherwise},
\end{cases}
$$

is valid for CLSP.

**Corollary 19** Given any reverse cover inequality of Family 1 of the form (14) with $\bar{c}_{\text{min}} \neq +\infty$, there is a valid inequality for $\hat{Y}_{k1}$ of the form (51) which implies it.

We can now redefine Family 1 of the reverse cover inequalities to have the form (51). Note that, as in Section 4.1.2, this redefinition not only strengthens but also expands this family. As with Family 2 of the cover inequalities (46), there are facets in this family which can only be induced by choosing an $i \in L$ such that $i > \arg\min_{j \in L} c_{ji} \geq \lambda \{\bar{c}_{ji}\}$.

**Example 3**

In this example we show a facet that is induced by a Family 1 reverse cover inequality. Consider the instance with capacity and demand vectors

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_j$</td>
<td>15</td>
<td>9</td>
<td>18</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td>$d_j$</td>
<td>3</td>
<td>7</td>
<td>5</td>
<td>6</td>
<td>9</td>
</tr>
</tbody>
</table>

In this instance $\bar{c}_{j5} = c_j$, $j = 1, ..., 5$. Take $k = 1$ and $l = 5$; $d_{15} = 30$. Choose $S = \{1, 4\}$, $L = \{2, 3, 5\}$, and $i = 2$; thus $\lambda = 3$ and $v_i = 12$. In order to compute $g_\hat{T}(\cdot)$, take $T = S \cup i$. Since $\bar{c}_{ji} \geq v_i, i \in S, \hat{T} = S$. Therefore $B_0 = 0, B_1 = 12, B_2 = 27$, and

$$
g_\hat{T}(u) = \begin{cases} 
3, & \text{if } 0 \leq u \leq 3 \\
3 - u, & \text{if } 3 \leq u \leq 12 \\
6, & \text{if } 12 \leq u \leq 15 \\
6, & \text{if } 15 \leq u \leq 27.
\end{cases}
$$

For this instance, we do not need to consider the rest of the domain of $g_\hat{T}(\cdot)$. So (51) yields

$$
3y_1 + 3y_2 + 6y_3 + 3y_4 + 3y_5 \geq 9,
$$

a facet for this instance. □

The full class of Family 1 reverse cover inequalities can be obtained by extending (51) to
\[
\sum_{j \in S}(\lambda - (v_i - \bar{c}_{ji})^+)y_j + \sum_{j \in S''}x_j + \sum_{j \in L'}g_T(\bar{c}_{ji})y_j + \\
\sum_{j \in L''}x_j + s_{k-1} \geq \sum_{j \in S}(\lambda - (v_i - \bar{c}_{ji})^+) + \lambda,
\]

where \( S' \cup S'' = S \) is any partition of \( S \) and \( L' \cup L'' = L \) is any partition of \( L \). To compute \( g_T(\cdot) \), we let

\[
\hat{T} = \begin{cases} 
S', & \text{if } v_i \leq \min_{j \in S'}\{\bar{c}_{ji}\} \\
\{j \in S' : \bar{c}_{ji} \geq v_i\} \cup i, & \text{otherwise.}
\end{cases}
\]

### 4.2.2 Family 2

We first restate (24) for \( \bar{Y}_{kl} \) as

\[
\sum_{j \in C} \min\{\lambda, \bar{c}_{ji}\}y_j + \sum_{T \neq i} \phi_C(\bar{c}_{ji})y_j \leq \sum_{j \in C \setminus i} \min\{\lambda, \bar{c}_{ji}\} + s_{k-1}.
\]

#### Proposition 20

Given any reverse cover inequality of Family 2 of the form (21), there is a continuous cover inequality for \( \bar{Y}_{kl} \) of the form (54) which implies it.

**Proof:** If \( \{j \in L : \bar{c}_{ji} > \lambda\} = \emptyset \), then (21) and (54) are both trivially valid. Otherwise, choose \( i \) to be any \( i \in L \) with \( \bar{c}_{il} > \lambda \), and take \( C = L \). Then (54) gives

\[
\sum_{j \in S} \phi_C(\bar{c}_{ji})y_j + \sum_{j \in L} \min\{\bar{c}_{ji}, \lambda\}y_j + s_{k-1} \geq \lambda + \sum_{j \in S} \phi_C(\bar{c}_{ji}).
\]

By definition

\[
\phi_C(\bar{c}_{ji}) \geq \min\{\bar{c}_{ji}, (\lambda - (\bar{c}_{\max} - \bar{c}_{ji})^+)\},
\]

and the result follows. \( \square \)

We can now redefine Family 2 of the reverse cover inequalities to have the form (55).

#### Example 4

In this example we show a facet that is induced by a Family 2 reverse cover inequality. Consider the instance with capacity and demand vectors

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
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<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_j )</td>
<td>19</td>
<td>9</td>
<td>12</td>
<td>19</td>
<td>9</td>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>( d_j )</td>
<td>10</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>7</td>
</tr>
</tbody>
</table>
In this instance \( \bar{e}_{j7} = e_j, j = 1, \ldots, 7. \) Choose \( k = 2 \) and \( l = 7; d_{27} = 35. \) Choose \( S = \{3, 4\}, \) then \( \lambda = 4, C = \{2, 5, 6, 7\}, \) and (55) yields

\[
s_1 + 4y_2 + 4y_3 + 8y_4 + 4y_5 + 4y_6 + 4y_7 \geq 16, \tag{56}
\]
a facet of \( \text{conv}(\text{CLSP}) \) for this instance. \( \Box \)

The full class of Family 2 reverse cover inequalities can be obtained by extending (55) to

\[
\sum_{j \in S'} \phi_C(\bar{e}_{jl}) y_j + \sum_{j \in S''} x_j + \sum_{j \in L'} \min\{\bar{e}_{jl}, \lambda\} y_j + \sum_{j \in L''} x_j + s_{k-1} \geq \lambda + \sum_{j \in S} \phi_C(\bar{e}_{jl}), \tag{57}
\]
where \( L' \cup L'' = L \) is any partition of \( L, \) and \( S' \cup S'' = S \) is any partition of \( S. \) Here we compute \( \phi_C(\cdot) \) by taking \( C = \{j \in L' : \bar{e}_{jl} > \lambda\}. \)

5 Strength of the Inequalities and Special Cases

In this section we discuss the relationship between the four families of valid inequalities we have presented for CLSP and what is known about the polyhedral structure of CLSP, ULSP, and CCLP.

5.1 General Case

One consequence of our being able to tighten all four classes of inequalities presented in Section 2 is that we can now state

**Proposition 21** Each of the four classes of inequalities for CLSP presented in this paper, in their lifted forms (42), (46), (51), and (55), induce non-trivial facets of CLSP that are not induced by any of the other three families.

**Proof:** Each of the four examples in Section 4 shows a facet of CLSP which can be induced only by the family which that example illustrates. \( \Box \)

Marchand [1998] very recently derived a class of valid inequalities for CLSP by first relaxing CLSP to an instance of the **dynamic knapsack set**, a special structure defined by more than one continuous knapsack constraint, and obtaining valid inequalities for this set. This class has a large intersection both with Family 1 of the CLSP cover inequalities and with Family 2 of the reverse cover inequalities. However, it does not dominate any of
the classes presented here. Indeed, none of the facets in the examples in Section 4 can be induced by inequalities of Marchand's class. While inequalities of this class often define high-dimensional faces of $\text{conv}(\text{CLSP})$, they have the disadvantage that computation of the coefficients is often nontrivial.

5.2 Uncapacitated Case

For ULSP, $c_j \geq d_{jT}, j = 1, \ldots, T$. A well known class of valid inequalities for ULSP is the family of $(l, S)$ inequalities. These can be stated as follows:

$$\sum_{j \in S^*} d_{jl} y_j + \sum_{j \in [1, q]} x_j \geq d_{1l}$$  \hspace{1cm} (58)

where $S^*$ is any subset of an interval $[1, \ldots, l]$ (i.e., $k = 1$). Barany, Van Roy, and Wolsey [1984] proved that these inequalities, along with trivial facets, describe $\text{conv}(\text{ULSP})$.

**Proposition 22** Each of the four families of inequalities for CLSP that we have presented, in their extended forms (i.e., (44), (50), (53), and (57)) contain the set of $(l, S)$ inequalities.

**Proof:** For each set of cover inequalities, take $S = \{1\}$. Then $\mu = 0$, and we set $S^* = S' \cup I'$. Observe that $\overline{c}_{jl} = d_{jl}, j = 1, \ldots, l$, and the result follows. For each reverse cover class take $S = \emptyset$, thus causing $\lambda = d_{1l}$. Then take $S^* = I'$ and observe again that $\overline{c}_{jl} = d_{jl}, j = 1, \ldots, l$, and the result follows. \hfill \Box

5.3 Constant Capacity

For CCLP, $c_j = c > 0, j = 1, \ldots, T$. One family of valid inequalities derived specifically for CCLP is the class of $(Q, R, D)$ inequalities (Leung, Magnanti, and Vachani [1989]). These can be stated as follows:

$$\sum_{j \in Q} x_j + \sum_{j \in R} \min\{r, d_{ji}\} y_j + s_{k-1} \geq r \lceil d_{kl}/c \rceil$$  \hspace{1cm} (59)

where $Q \cup R = [k, \ldots, l], Q \cap R = \emptyset$, and $r = d_{kl} - \lceil d_{kl}/c \rceil - 1)c$. Leung, et al., give conditions under which these inequalities define facets of $\text{conv}(\text{CCLP})$.

**Proposition 23** Family 2 of the CLSP cover inequalities (50) and the two classes of CLSP reverse cover inequalities (53) and (57) each imply the set of $(Q, R, D)$ inequalities.
Proof: For the class of Family 2 cover inequalities, given a \((Q, R, D)\) inequality, let \(i' = k + \lceil d_{kl} / c \rceil - 1\). Let \(S' = \{ j \in R : j < i' \}, S'' = \{ j \in Q : j < i' \}, L' = \{ j \in R : j > i' \}, L'' = \{ j \in Q : j > i' \}\), and choose \(i = i'\). Thus \(S = S' \cup S''\) is a cover of \([k, ..., l]\) that consists of the first \(\lceil d_{kl} / c \rceil\) periods of \(k, ..., l\). So \(v_i = \bar{r}_{ii}\). If \(d_{ii} \geq c\), then \(v_i = c\), \(c - \mu = v_i - \mu = r \leq c \leq d_{ji}, \forall j \in S\), and (50) becomes

\[
\sum_{j \in S'} (v_i - \mu) y_j + \sum_{j \in S''} x_j + \sum_{j \in L'} \min\{d_{ji}, v_i - \mu\} y_j + \sum_{j \in L''} x_j + s_{kl} \geq |S|(v_i - \mu).
\]

Since \(Q = S'' \cup L''\) and \(R = S' \cup L'\), and since \(r [d_{kl} / c] = |S|(v_i - \mu)\), (59) and (60) are equivalent. If \(d_{ii} < c \leq d_{ii} + \mu\), the argument is similar. If \(d_{ii} + \mu < c\), the argument is again similar, except that in this case, since \(v_i = \bar{r}_{ii} = d_{ii}, f_C(c) > v_i - \mu\), and we get an inequality that strictly implies (59).

For Family 1 of the reverse cover inequalities, given a \((Q, R, D)\) inequality, again let \(i' = k + \lceil d_{kl} / c \rceil - 1\). Let \(S' = \{ j \in R : j < i' \}, S'' = \{ j \in Q : j < i' \}, L' = \{ j \in R : j \geq i' \}, L'' = \{ j \in Q : j \geq i' \}\), and choose \(i = i'\). Thus \(S = S' \cup S''\) is a reverse cover of \([k, ..., l]\) that consists of the first \(\lceil d_{kl} / c \rceil - 1\) periods of \([k, ..., l]\). Moreover, \(\lambda = r \leq v_i \leq c \leq d_{ji}, \forall j \in S\). So (53) becomes

\[
\sum_{j \in S'} \lambda y_j + \sum_{j \in S''} x_j + \sum_{j \in L'} \min\{d_{ji}, \lambda\} y_j + \sum_{j \in L''} x_j + s_{kl} \geq (|S| + 1)\lambda.
\]

Since \(Q = S'' \cup L''\) and \(R = S' \cup L'\), and since \(r [d_{kl} / c] = (|S| + 1)\lambda\), the result follows.

The argument for Family 2 of the reverse cover inequalities parallels that given for Family 1. □

The strongest set of inequalities known for CCLP is the \((k, l, S, I)\) family (Pochet and Wolsey [1993] and [1994]), in the sense that all known, non-trivial facets of CCLP are \((k, l, S, I)\) inequalities. There are instances of CCLP in which \((k, l, S, I)\) inequalities induce facets which cannot be induced by any of the families we present.

6 Acknowledgments

We would like to thank Laurence Wolsey for his valuable comments relating to this research.
References


Appendix

Proof of Lemma 8: Recall that Lemma 8 states that

\[ G(u) = - \sum_{j \in T'} (a_j - \mu)^+ - \sum_{j \in T''} f_C(a_j) - \left( \left( b - \sum_{j \in N \setminus i} a_j - \mu \right)^+ \right), \]

where \( G(\cdot) \) is defined by

\[ G(u) = \max \left\{ - \sum_{j \in T'_i} (a_j - \mu)^+ \bar{y}_j - \sum_{j \in T''} f_C(a_j) \bar{y}_j - \sum_{j \in C''} (a_j - \mu) \bar{y}_j - s : \right. \]
\[ \left. - \sum_{j \in T \cup C''} a_j \bar{y}_j \leq u + s \right\}. \quad (62) \]

To prove this, it suffices to show that the following three equations hold:

\[ G(b - \sum_{j \in N} a_j) = - \sum_{j \in T'_i} (a_j - \mu)^+ - \sum_{j \in T''} f_C(a_j), \quad (63) \]
\[ G(b - \sum_{j \in N \setminus i} a_j - \mu) = - \sum_{j \in T'_i} (a_j - \mu)^+ - \sum_{j \in T''} f_C(a_j), \quad (64) \]
\[ G(b - \sum_{j \in N \setminus i} a_j) = - \sum_{j \in T'} (a_j - \mu)^+ - \sum_{j \in T''} f_C(a_j). \quad (65) \]

Because of the presence of the continuous variable \( s \), decreasing \( u \) by 1 never decreases the optimal value of the maximization problem (62) by more than 1. Therefore, if (63)–(65) are true, then the function \( G(u) \) has slope 1 on the interval \([b - \sum_{j \in N} a_j, b - \sum_{j \in N \setminus i} a_j - \mu]\), which has length \( \lambda \), and is constant on the interval \([b - \sum_{j \in N \setminus i} a_j - \mu, b - \sum_{j \in N \setminus i} a_j]\), which has length \( \mu \). Thus the desired result will follow. We will prove only (65), since (63) and (64) can be proven in the same way. So consider the maximization problem

\[ G(b - \sum_{j \in N \setminus i} a_j) = \max \left\{ - \sum_{j \in T'_i} (a_j - \mu)^+ \tilde{y}_j - \sum_{j \in T''} f_C(a_j) \tilde{y}_j - \sum_{j \in C''} (a_j - \mu) \tilde{y}_j - s : \right. \]
\[ \left. - \sum_{j \in T \cup C''} a_j \tilde{y}_j \leq b - \sum_{j \in N \setminus i} a_j + s \right\}. \quad (66) \]

A feasible solution \((\tilde{y}', s')\) to (66) with objective function value given by (65) is
\[
\begin{align*}
\bar{y}_j' &= 1, \text{ if } j \in T \setminus i; \\
\bar{y}_j' &= 0, \text{ otherwise}; \\
\bar{s}' &= 0.
\end{align*}
\]

To prove (65), we need to show that \((\bar{y}', \bar{s}')\) is also an optimal solution to (66). We will prove this by contradiction. So assume \((\bar{y}', \bar{s}')\) is not optimal, and consider an optimal solution \((\bar{y}^*, s^*)\) to (66). Let \(T^* = \{j \in T \setminus i : \bar{y}_j^* = 0\}, C_i^* = \{j \in C_i^p : \bar{y}_j^* = 1\}\). Recall that \(f_C(a_j) = (a_j - \mu)^+, j \in T'\). Then for \((\bar{y}', \bar{s}')\) not to be optimal,

\[
\sum_{j \in T^*} f_C(a_j) > \sum_{j \in C_i^*} (a_j - \mu) + s^* \tag{67}
\]

must hold. However, for \((\bar{y}^*, s^*)\) to be feasible,

\[
\sum_{j \in T^*} a_j \leq \sum_{j \in C_i^*} a_j + s^* + \mu \tag{68}
\]

must be true. However, (67) and (68) cannot both be true. To see this, let \(K = \sum_{j \in C_i^*} a_j + s^*\). Now

\[
\min_{(y,s) \in B^{|C^p_i|} \times R^+} \left\{ \sum_{j \in C_i^p} (a_j - \mu)y_j + s : \sum_{j \in C_i^p} a_jy_j + s = K \right\} = f_C(B_{\bar{r}}) + K - B_{\bar{r}}, \tag{69}
\]

where \(B_0, \ldots, B_{|C^p_i|}\), are defined as in Theorem 7 and \(\bar{r}\) is the largest integer such that \(K \geq B_{\bar{r}}\). (Intuitively, (69) says that the sum \(\sum_{j \in C_i^*} (a_j - \mu) + s^*\) is smallest, for a fixed value of \(\sum_{j \in C_i^*} a_j + s^*\), if the elements of \(C_i^*\) are chosen from the smallest elements of \(C_i^p\), with \(s^*\) then being chosen so that \(s^* = K - \sum_{j \in C_i^*} a_j\). Thus \(\sum_{j \in C_i^*} (a_j - \mu) + s^* \geq f_C(B_{\bar{r}}) + K - B_{\bar{r}}\). Now \(\sum_{j \in T^*} a_j \leq K + \mu\) must be true. Therefore, by the definition of \(f_C(\cdot)\), and by the fact that \(f_C(\cdot)\) is superadditive,

\[
\sum_{j \in T^*} f_C(a_j) \leq f_C(\sum_{j \in T^*} a_j) \\
\leq f_C(K + \mu) \\
dleq f_C(B_{\bar{r}} + K - B_{\bar{r}} + \mu) \\
\leq f_C(B_{\bar{r}} + \mu) + K - B_{\bar{r}} \\
= f_C(B_{\bar{r}}) + K - B_{\bar{r}} \\
\leq \sum_{j \in C_i^*} (a_j - \mu) + s^*, \tag{70}
\]

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and (67) is violated. Thus, (67) and (68) cannot both be true, \((\tilde{g}', s')\) is optimal, and (65) is true. □

Note that the proof of this lemma, and particularly of (65), depends strongly on using \(f(\cdot)\) instead of \(\tilde{f}(\cdot)\) in the definitions of \(G(\cdot)\).