GENERAL INTERIOR-POINT MAPS AND EXISTENCE OF WEIGHTED PATHS FOR NONLINEAR SEMIDEFINITE COMPLEMENTARITY PROBLEMS

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Extending the previous work of Monteiro and Pang (1998), this paper studies properties of fundamental maps that can be used to describe the central path of the monotone nonlinear complementarity problems over the cone of symmetric positive semidefinite matrices. Instead of focusing our attention on a specific map as was done in the approach of Monteiro and Pang (1998), this paper considers a general form of a fundamental map and introduces conditions on the map that allow us to extend the main results of Monteiro and Pang (1998) to this general map. Each fundamental map leads to a family of "weighted" continuous trajectories which include the central trajectory as a special case. Hence, for complementarity problems over the cone of symmetric positive semidefinite matrices, the notion of weighted central path depends on the fundamental map used to represent the central path.


Let \( \mathbb{R}^m \) denote the \( m \)-dimensional real Euclidean space, \( \mathcal{S}^n \) denote the space of \( n \times n \) real symmetric matrices, \( \mathcal{S}_+^n \) and \( \mathcal{S}_{++}^n \) denote the subsets of \( \mathcal{S}^n \) consisting of the positive semidefinite and positive definite matrices, respectively, and \( \mathcal{E}(\mu) \equiv \{(X, Y) \in \mathcal{S}_+^n \times \mathcal{S}_{++}^n : XY = \mu I\} \) for every \( \mu \geq 0 \). We denote the closure of a subset \( E \) of a metric space by \( \text{cl} E \). Given a continuous map \( F : \mathcal{S}_+^n \times \mathcal{S}_{++}^n \times \mathbb{R}^m \rightarrow \mathcal{S}^n \times \mathbb{R}^m \), the complementarity problem which we shall study in this paper is to find a triple \((X, Y, z) \in \mathcal{S}_+^n \times \mathcal{S}_{++}^n \times \mathbb{R}^m\) such that

\[
F(X, Y, z) = 0, \quad (X, Y) \in \mathcal{E}(0).
\]

It is known (see the cited references) that there are several equivalent equations to represent the complementarity condition \((X, Y) \in \mathcal{E}(0)\) in this problem. Associated with each of these equivalent equations, we can define an interior-point map that not only provides the foundation for developing path-following interior point methods for solving problem (1)
but also serves to generalize the notion of weighted central path from linear programming to the context of the complementarity problem (1). In the work of Monteiro and Pang (1998), the authors focus on the equivalent complementarity equation \((XY + YX)/2 = 0\) and study several properties of the associated interior-point map. The main goal of this paper is to generalize the work of Monteiro and Pang (1998) to other equivalent complementarity equations having the general form \(\Phi(X, Y) = 0\), where \(\Phi: \mathcal{D} \to \mathcal{S}^n\) is a continuous map such that:

(i) \(\mathcal{C}(0) \cup (\mathcal{S}^n_+ \times \mathcal{S}^n_+) \subseteq \mathcal{D} \subseteq \mathcal{S}^n \times \mathcal{S}^n_+\), and

(ii) \((X, Y) \in \mathcal{C}(0)\) if and only if \(\Phi(X, Y) = 0\).

In terms of \(\Phi\), problem (1) becomes equivalent to finding a triple \((X, Y, z) \in \mathcal{D} \times \mathbb{R}^m\) such that \(H(X, Y, z) = 0\), where \(H: \mathcal{D} \times \mathbb{R}^m \to \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m\) is the fundamental interior-point map (associated with \(\Phi\)) defined by

\[
H(X, Y, z) = \begin{pmatrix} \Phi(X, Y) \\ F(X, Y, z) \end{pmatrix}, \quad \text{for } (X, Y, z) \in \mathcal{D} \times \mathbb{R}^m.
\]

For the map \(\Phi(X, Y) = (XY + YX)/2\) with domain \(\mathcal{D} = \mathcal{S}^n_+ \times \mathcal{S}^n_+\) and the open convex cone \(\mathcal{V} = \mathcal{S}^n_+\), Monteiro and Pang (1998) establish under some monotonicity conditions on the map \(F\) that the system

\[
H(X, Y, z) = \begin{pmatrix} A \\ B \end{pmatrix}, \quad (X, Y, z) \in \mathcal{D} \times \mathbb{R}^m,
\]

have the following properties:

(P1) it has a solution for every \((A, B) \in \text{cl } \mathcal{V} \times \mathcal{F}_+\), where \(\mathcal{F}_+ \equiv F(\mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \mathbb{R}^m)\);

(P2) the solution, denoted \((X(A, B), Y(A, B), z(A, B))\), is unique when \((A, B) \in \mathcal{V} \times \mathcal{F}_+\);

(P3) if a sequence \(\{(A_k, B_k)\} \subseteq \mathcal{V} \times \mathcal{F}_+\) converges to a limit \((A_\infty, B_\infty) \in \mathcal{V} \times \mathcal{F}_+\), then the sequence \(\{(X(A_k, B_k), Y(A_k, B_k), z(A_k, B_k))\}\) converges to \((X(A_\infty, B_\infty), Y(A_\infty, B_\infty), z(A_\infty, B_\infty))\); and

(P4) if a sequence \(\{(A_k, B_k)\} \subseteq \mathcal{V} \times \mathcal{F}_+\) converges to a limit \((A_\infty, B_\infty) \in \text{cl } \mathcal{V} \times \mathcal{F}_+\), then the sequence \(\{(X(A_k, B_k), Y(A_k, B_k), z(A_k, B_k))\}\) is bounded, and any of its accumulation points \((\tilde{X}, \tilde{Y}, \tilde{z})\) satisfies \(H(\tilde{X}, \tilde{Y}, \tilde{z}) = (A_\infty, B_\infty)\).

Note that when \(0 \in \mathcal{F}_+\), (P1) implies the existence of a solution of (1), and (P2) implies the well-definedness of the weighted central paths passing through points in \(\mathcal{V}\), that is, paths consisting of the unique solutions of system (3) with \((A, B) = (vA^0, 0)\) for every \(v \in (0, 1]\), where \(A^0\) is a fixed point in \(\mathcal{V}\).

Under appropriate conditions on the map \(\Phi\) and the open convex cone \(\mathcal{V}\), we show in this paper that the above properties also hold with respect to the general interior-point map (2). We also illustrate how our general framework applies to the following specific central-path maps:

\[
\Phi(X, Y) = (X^{1/2} Y^{1/2} + Y^{1/2} X^{1/2})/2,
\]

\[
\Phi(X, Y) = X^{1/2} Y X^{1/2},
\]

\[
\Phi(X, Y) = L_X^T Y L_X,
\]

\[
\Phi(X, Y) = (U_Y^T L_X + L_X^T U_Y)/2,
\]

\[
\Phi(X, Y) = W^{1/2} X W^{-1/2},
\]

where \(L_X\) is the lower Cholesky factor of \(X\), \(U_Y\) is the upper Cholesky factor of \(Y\) and \(W\) is the unique symmetric positive definite matrix such that \(WXW = Y\). (The map
\( \Phi(X,Y) = (XY + YX)/2 \) can be easily verified to be a special case of our general framework using the results of Monteiro and Pang 1998. Observe that the third and fourth maps are only defined for points \((X,Y)\) in the set \(S^+ \times S^+\), and hence they illustrate the need for considering maps \(\Phi\) whose domains are not the whole set \(S^n \times S^n\). The approach of this paper is based on the theory of local homeomorphic maps. One of the preliminary steps of our analysis is to establish that \(H\) restricted to the set \(S^+ \times R^n\) is a local homeomorphism, where \(R = \Phi^{-1}(\mathcal{V})\). For this property to hold for the above cited maps, it is necessary to choose the set \(\mathcal{V}\) to be an open convex cone smaller than \(S^+ \times S^+\), which is the cone used in connection with the map \(\Phi(X,Y) = (XY + YX)/2\) in the analysis of Monteiro and Pang (1998). In fact, the sets \(\mathcal{V}\) associated with the four maps above are appropriate conical neighborhoods of the line \(\{\mu I : \mu \geq 0\}\) contained in \(S^+\).

Sturm and Zhang (1996) define a notion of weighted center for semidefinite programming based on the central path map \(\Phi(X,Y) = \Lambda(XY)\), where \(\Lambda(XY)\) denotes the eigenvalues of \(XY\) arranged in nondecreasing order. According to their definition, given a vector \(w \in S^+\) such that \(w_1 \leq \ldots \leq w_n\), a point \((X,Y,z) \in S^+ \times S^+ \times S^+\) is a \(w\)-weighted center if \(F(X,Y,z) = 0\) and \(\Lambda(XY) = w\). For linear maps \(F\) associated with semidefinite programming problems, they show the existence of a \(w\)-center for any such \(w\). However, their \(w\)-center is not unique and hence does not lead to the notion of weighted central paths as our approach does.

Finally, we observe that interior-point algorithms for solving the complementarity problem (1) which makes use of the fundamental interior-point map (2) have been proposed in Wang et al. (1996) and Monteiro and Pang (1999).

This paper is organized as follows. Section 2 contains an exposition of the main results of this paper whose proofs are given in the subsequent sections. Section 2 is divided into three subsections. Subsection 2.1 summarizes some important definitions and facts from the theory of local homeomorphic maps. Subsection 2.2 introduces the key conditions on the map \(F\) and the pair \((\Phi, \mathcal{V})\), and gives a few examples of pairs \((\Phi, \mathcal{V})\) which satisfy these conditions. In Subsection 2.3, we state the main result of this paper and some of its consequences, including its specialization to the context of the convex nonlinear semidefinite programming problem. In §3, we provide the proof of the main result. Finally, in §4, we develop some technical results which allow us to verify that the pairs \((\Phi, \mathcal{V})\) introduced in §2 satisfy our basic assumptions.

The following notation is used throughout this paper. The symbols \(\succeq\) and \(\succ\) denote, respectively, the positive semidefinite and positive definite ordering over the set of symmetric matrices; that is, for \(X,Y \in S^n\), \(X \succeq Y\) (or \(Y \preceq X\)) means \(X - Y\) is positive semidefinite, and \(X \succ Y\) (or \(Y \prec X\)) means \(X - Y\) is positive definite. The set of all \(n \times n\) matrices with real entries is denoted by \(M^n\). Let \(M^n_+\) and \(M^n_{++}\) denote the set of matrices \(X \in M^n\) such that \(X + X^T \succeq 0\) and \(X + X^T \succ 0\), respectively. Let \(S^n_+\) denote the subspace of \(M^n_+\) consisting of the skew-symmetric matrices. For \(A \in M^n\), let \(\text{tr}(A) = \sum_{i=1}^n A_{ii}\) denote the trace of \(A\). For \(P, Q \in M^n\), let \(P \cdot Q = \text{tr} P^T Q\) denote standard inner product in \(M^n\). For any matrix \(A \in M^n\), let \(\|A\| = \max\{\sqrt{\lambda} : \lambda\) is an eigenvalue of \(A^T A\}\) and \(\|A\|_F = (\text{tr}(A^T A))^{1/2}\). For a matrix \(A \in M^n\) with all real eigenvalues, we denote its smallest and largest eigenvalues by \(\lambda_{\text{min}}(A)\) and \(\lambda_{\text{max}}(A)\), respectively. Finally, define the sets

\[ R_+ \cdot I = \{\mu I : \mu \geq 0\} \quad \text{and} \quad R_{++} \cdot I = \{\mu I : \mu > 0\}. \]

2. Main assumptions and results. In this section, we state the assumptions imposed on the map \(F\) and the pair \((\Phi, \mathcal{V})\) and give a few examples of pairs \((\Phi, \mathcal{V})\) satisfying the required conditions. We also state the main result and its consequences, including its specialization to the context of the convex nonlinear semidefinite programming problem. This section is divided into three subsections. The first subsection summarizes a theory of local homeomorphic maps defined on metric spaces; the discussion is very brief. We refer
the reader to §2 and the Appendix of Monteiro and Pang (1996), Chapter 5 of Ortega and Rheinboldt (1970), and Chapter 3 of Ambrosetti and Prodi (1993) for a thorough treatment of this theory. The second subsection introduces the key conditions on the map $F$ and the pair $(\Phi, \Psi)$, and gives a few examples of pairs $(\Phi, \Psi')$ which satisfy these conditions. The third subsection states the main result and its corollaries.

2.1. Local homeomorphic maps. This subsection summarizes some important definitions and facts from the theory of local homeomorphic maps.

If $M$ and $N$ are two metric spaces, we denote the set of continuous functions from $M$ to $N$ by $C(M,N)$ and the set of homeomorphisms from $M$ onto $N$ by $\text{Hom}(M,N)$. For $G \in C(M,N)$, $D \subseteq M$, and $E \subseteq N$, we let $G(D) = \{G(u) : u \in D\}$ and $G^{-1}(E) = \{u \in M : G(u) \in E\}$. The set $G^{-1}\{v\}$ with $v \in N$ is simply denoted by $G^{-1}(v)$. Given $G \in C(M,N)$, $D \subseteq M$, and $E \subseteq N$ such that $G(D) \subseteq E$, the restricted map $G : D \to E$ defined by $G(u) \equiv G(u)$ for all $u \in D$ is denoted by $G|_{(D,E)}$; if $E = N$ then we write this $G$ simply as $G|_D$. We will also refer to $G|_{(D,E)}$ as "$G$ restricted to the pair $(D,E)$," and to $G|_D$ as "$G$ restricted to $D". The closure of a subset $E$ of a metric space will be denoted by $\text{cl}E$. Any continuous function from a closed interval of the real line $\mathbb{R}$ into a metric space will be called a path. We say that $(V_1, V_2)$ forms a partition of the set $V$ if $V_1 \subseteq V$, $V_2 \subseteq V$, $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$. A metric space $M$ is said to be connected if there exists no partition $(\mathcal{C}_1, \mathcal{C}_2)$ of $M$ for which both $\mathcal{C}_1$ and $\mathcal{C}_2$ are nonempty and open.

In the rest of this subsection, we will assume that $M$ and $N$ are two metric spaces and that $G \in C(M,N)$.

**Definition 1.** The map $G \in C(M,N)$ is said to be proper with respect to the set $E \subseteq N$ if $G^{-1}(K) \subseteq M$ is compact for every compact set $K \subseteq E$. If $G$ is proper with respect to $N$, we will simply say that $G$ is proper.

Our analysis relies upon the following two well-known results. The first one is a classical topological result whose proof can be found in Chapter 3 of Ambrosetti and Prodi (1993) and in the Appendix of Monteiro and Pang (1996).

**Proposition 1.** Assume that $G : M \to N$ is a local homeomorphism. If $N$ is connected, then $G$ is proper if and only if the number of elements of $G^{-1}(v)$ is finite and constant for $v \in N$.

The next result is derived in §2 of Monteiro and Pang (1996) and its proof is an immediate consequence of classical topological results.

**Proposition 2.** Let $M_0 \subseteq M$ and $N_0 \subseteq N$ be given sets satisfying the following conditions: $G|_{M_0}$ is a local homeomorphism and $\emptyset \neq G^{-1}(N_0) \subseteq M_0$. Assume that $G$ is proper with respect to some set $E$ such that $N_0 \subseteq E \subseteq N$. Then $G$ restricted to the pair $(G^{-1}(N_0), N_0)$ is a proper local homeomorphism. If, in addition, $N_0$ is connected, then $G(M_0) \supseteq N_0$ and $G(\text{cl}M_0) \supseteq E \cap \text{cl}N_0$.

2.2. Fundamental assumptions and examples of central-path maps. In this subsection, we state the conditions that will be imposed on the map $F$ and the pair $(\Phi, \Psi)$ during our analysis of the properties of the fundamental map (2). We also give a few examples of pairs $(\Phi, \Psi')$ which satisfy the required conditions.

We start by giving a few definitions.

**Definition 2.** A map $J(X,Y,z)$ defined on a subset $\text{dom}(J)$ of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ is said to be $(X,Y)$-equi-level-monotone on a subset $\mathcal{H} \subseteq \text{dom}(J)$ if for any $(X,Y,z) \in \mathcal{H}$ and $(X',Y',z') \in \mathcal{H}$ such that $J(X,Y,z) = J(X',Y',z')$, there holds $(X' - X) \cdot (Y' - Y) \geq 0$. When $\mathcal{H} = \text{dom}(J)$, we will simply say that $J$ is $(X,Y)$-equi-level-monotone.
In the following two definitions, we assume that $W$, $Z$ and $N$ are three normed spaces and that $\psi(w, z)$ is a function defined on a subset of $W \times Z$ with values in $N$.

**Definition 3.** The function $\psi(w, z)$ is said to be $z$-bounded on a subset $\mathcal{H} \subseteq \text{dom}(\psi)$ if for every sequence $\{(w^k, z^k)\} \subseteq \mathcal{H}$ such that $\{w^k\}$ and $\{\psi(w^k, z^k)\}$ are bounded, the sequence $\{z^k\}$ is also bounded. When $\mathcal{H} = \text{dom}(\psi)$, we will simply say that $\psi$ is $z$-bounded.

**Definition 4.** The function $\psi(w, z)$ is said to be $z$-injective on a subset $\mathcal{H} \subseteq \text{dom}(\psi)$ if the following implication holds: $(w, z) \in \mathcal{H}$, $(w', z') \in \mathcal{H}$ and $\psi(w, z) = \psi(w', z')$ implies $z = z'$. When $\mathcal{H} = \text{dom}(\psi)$, we will simply say that $\psi$ is $z$-injective.

We now state the main assumptions that will be made on the map $F$ and the pair $(\Phi, \mathcal{V})$.

**Assumption 1.** The map $F: \mathcal{L}^m_+ \times \mathcal{L}^m_+ \times \mathcal{R}^n \rightarrow \mathcal{L}^m_+ \times \mathcal{R}^n$ is continuous, $(X, Y)$-equi-level monotone, $z$-injective on $\mathcal{L}^m_+ \times \mathcal{L}^m_+ \times \mathcal{R}^n$, and $z$-bounded.

**Assumption 2.** We impose the following conditions on the pair $(\Phi, \mathcal{V})$:

(a) $\Phi: \mathcal{D} \rightarrow \mathcal{F}^n$ is a continuous map such that $\mathcal{G}(0) \cup (\mathcal{L}^m_+ \times \mathcal{L}^m_+) \subseteq \mathcal{D} \subseteq \mathcal{L}^m_+ \times \mathcal{L}^m_+$; $\mathcal{V}$ is an open connected set such that $\mathcal{V} \cap (\mathcal{R}^+_+ \cdot I) \neq \emptyset$;

(b) for any $(X, Y) \in \mathcal{D}$ and $\mu > 0$, we have $XY = \mu I$ if and only if $\Phi(X, Y) = \varphi(\mu)I$, where $\varphi: \mathcal{R}^+_+ \rightarrow \mathcal{R}^+_+$ is a continuous strictly increasing function such that $\varphi(0) = 0$;

(c) if the sequence $\{(X^k, Y^k)\} \subseteq \mathcal{D}$ is such that $\{\Phi(X^k, Y^k)\}$ is bounded, then $\{X^k \cdot Y^k\}$ is bounded;

(d) the set $\mathcal{U} \equiv \Phi^{-1}(\mathcal{V})$ is contained in $\mathcal{L}^m_+ \times \mathcal{L}^m_+$, moreover, $\Phi^{-1}(\text{cl} \mathcal{V})$ is a closed set;

(e) $\Phi$ is differentiable on $\mathcal{L}^m_+ \times \mathcal{L}^m_+$ and for every $(X, Y) \in \mathcal{U}$, the following implication holds:

$$
\left\{ \begin{array}{l}
(\Delta X, \Delta Y) \in \mathcal{L}^m_+ \times \mathcal{L}^m_+,
\Delta X \cdot \Delta Y \geq 0,
\Phi'(X, Y)(\Delta X, \Delta Y) = 0,
\end{array} \right. \implies \Delta X = \Delta Y = 0.
$$

We now make a few remarks regarding the above assumptions. First, condition (a) is stated so as to cover some relevant examples of maps $\Phi$ which are defined only in the set $\mathcal{G}(0) \cup (\mathcal{L}^m_+ \times \mathcal{L}^m_+)$ (see Examples 4 and 5 below). Second, Assumption 2(c) implies that the map $\varphi$ is onto, and hence that its inverse is defined everywhere in $\mathcal{R}^+_+$. Third, conditions (a), (b) and (d) of Assumption 2 imply that $\mathcal{U}$ is a nonempty open set. Fourth, condition (b) implies that the curve consisting of the solutions of the systems

$$
(4) \quad \Phi(X, Y) = \nu I, \quad F(X, Y, z) = 0,
$$

as $\nu > 0$ varies, is a parametrization of the central path associated with the complementarity problem (1), that is the set of solutions of the systems $XY = \mu I$ and $F(X, Y, z) = 0$ with $\mu > 0$. Fifth, condition (e) is probably the most crucial one. Under Assumption 1, this condition is equivalent to the well-definedness of the Newton direction with respect to system (4) for any point $(X, Y, z)$ in $\mathcal{U}$.

We next give some examples of pairs $(\Phi, \mathcal{V})$ which satisfy Assumption 2.

**Example 1.** Let $\mathcal{D} \equiv \mathcal{L}^m_+ \times \mathcal{L}^m_+, \Phi: \mathcal{D} \rightarrow \mathcal{F}^n$ be the map defined by $\Phi(X, Y) = (XY + YY)/2$, and $\mathcal{V} \equiv \mathcal{L}^m_+$.

**Example 2.** Let $\mathcal{D} \equiv \mathcal{L}^m_+ \times \mathcal{L}^m_+, \Phi: \mathcal{D} \rightarrow \mathcal{F}^n$ be the map defined by $\Phi(X, Y) = X^{1/2}YY^{1/2}$, and $\mathcal{V} \equiv \{Z \in \mathcal{F}^n: ||Z - \nu I||_F < \nu/(3\sqrt{2}) \text{ for some } \nu > 0\}$.

**Example 3.** Let $\mathcal{D} \equiv \mathcal{L}^m_+ \times \mathcal{L}^m_+, \Phi: \mathcal{D} \rightarrow \mathcal{F}^n$ be the map defined by $\Phi(X, Y) = (X^{1/2}Y^{1/2} + Y^{1/2}X^{1/2})/2$, and $\mathcal{V} \equiv \{Z \in \mathcal{F}^n: ||Z - \nu I||_F < \nu/(3\sqrt{2}) \text{ for some } \nu > 0\}$. 


Example 4. Let $\mathcal{D} \equiv \mathcal{C}(0) \cup (\mathcal{S}_+ \times \mathcal{S}_+)$, $\Phi : \mathcal{D} \to \mathcal{S}_+$ be the map defined by

$$
\Phi(X, Y) = \begin{cases} 
L_X^T YL_X, & \text{if } (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+,
0, & \text{if } (X, Y) \in \mathcal{C}(0),
\end{cases}
$$

where $L_X$ is the lower Cholesky factor of $X$, that is $L_X$ is the unique lower triangular matrix with positive diagonal elements satisfying $X = L_X L_X^T$. Also, let $\mathcal{V} \equiv \{Z \in \mathcal{S}_+^n : ||Z - vI||_F < v/\sqrt{2} \text{ for some } v > 0\}$.

Example 5. Let $\mathcal{D} \equiv \mathcal{C}(0) \cup (\mathcal{S}_+ \times \mathcal{S}_+)$, $\Phi : \mathcal{D} \to \mathcal{S}_+$ be the map defined by

$$
\Phi(X, Y) = \begin{cases} 
(U_Y^T L_X + L_X^T U_Y)/2, & \text{if } (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+,
0, & \text{if } (X, Y) \in \mathcal{C}(0),
\end{cases}
$$

where $L_X$ is the lower Cholesky factor of $X$ and $U_Y$ is the upper Cholesky factor of $Y$, that is $U_Y$ is the unique upper triangular matrix with positive diagonal elements satisfying $U_Y U_Y^T = Y$. Also, let $\mathcal{V} \equiv \{Z \in \mathcal{S}_+^n : ||Z - vI||_F < v/(3\sqrt{2}) \text{ for some } v > 0\}$.

Example 6. Let $\mathcal{D} \equiv \mathcal{C}(0) \cup (\mathcal{S}_+ \times \mathcal{S}_+)$, $\Phi : \mathcal{D} \to \mathcal{S}_+$ be the map defined by

$$
\Phi(X, Y) = \begin{cases} 
W^{1/2} X Y W^{-1/2}, & \text{if } (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+,
0, & \text{if } (X, Y) \in \mathcal{C}(0),
\end{cases}
$$

where $W = W(X, Y)$ is the unique symmetric positive definite matrix satisfying $W W = Y$, and $\mathcal{V} \equiv \{Z \in \mathcal{S}_+^n : ||Z - vI||_F < v/2 \text{ for some } v > 0\}$.

The map of Example 1 was extensively studied in Monteiro and Pang (1998), and is a special case of the general framework studied in this paper. The verification that these examples satisfy Assumption 2 will be given in §4. The next result shows that each map above generates a whole family of maps that satisfy Assumption 2.

Proposition 3. Let $(\Phi, \mathcal{V})$ be a pair satisfying Assumption 2. Then:

(i) for every $a > 0$, the pair $(a \Phi, a \mathcal{V})$ satisfies Assumption 2;

(ii) for every nonsingular matrix $P$, the pair $(P \Phi, \mathcal{V})$ satisfies Assumption 2, where $P \Phi : \mathcal{D} \to \mathcal{S}_+$ is the map defined as $P \Phi(X, Y) = \Phi(P X P^T, P^{-T} Y P^{-1})$ with domain $\mathcal{D} \equiv \{(X, Y) : (P X P^T, P^{-T} Y P^{-1}) \in \mathcal{D}\}$.

Proof. The proof is just a simple verification. □

In all six examples above, the domain $\mathcal{D}_P$ of the map $\Phi_P$ remains invariant, that is $\mathcal{D}_P = \mathcal{D}$ for every $P$. For a pair $(\mathcal{V}, \Phi)$ and a nonsingular matrix $P$, let $\mathcal{U}_P \equiv \Phi_P^{-1}(\mathcal{V})$. In general, $\mathcal{U}_P$ may differ from $\mathcal{U}$, but for the maps of Examples 2 and 4, we have $\mathcal{U}_P = \mathcal{U}$ for every nonsingular matrix $P$. Moreover, the maps $\Phi_P$ corresponding to Examples 1–6 become

$$
\Phi_P(X, Y) = \begin{cases} 
[ P X Y P^{-1} + (P X Y P^{-1})^T ]/2, & \text{if } (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+,
(P X P^T)^{1/2} (P^{-T} Y P^{-1}) (P X P^T)^{1/2}, & \text{if } (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+,
[(P X P^T)^{1/2} (P^{-T} Y P^{-1})^{1/2} + (P^{-T} Y P^{-1})^{1/2} (P X P^T)^{1/2}] / 2, & \text{if } (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+,
L_X^T (P^{-T} Y P^{-1}) L_X, & \text{if } (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+,
(U_Y^T L_X + L_X^T U_Y)/2, & \text{if } (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+,
(P^{-T} W P^{-1})^{1/2} (P X Y P^{-1}) (P^{-T} W P^{-1})^{-1/2}, & \text{if } (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+,
\end{cases}
$$
respectively, where \( L_J \) is the lower Cholesky factor of \( \tilde{X} = P X P^T \) and \( U_J \) is the upper Cholesky factor of \( \tilde{Y} = P^{-1} TP^{-1} \). In the last equation, we used the fact that \( \tilde{W} \equiv P^{-1} WP^{-1} \) is the unique symmetric matrix such that \( \tilde{W} X \tilde{W} = \tilde{Y} \).

2.3. The main result and its consequences. In this subsection we state the main result of this paper. We also state some of its consequences, including its specialization to the context of the convex nonlinear semidefinite programming problem. The main result extends Theorem 2 of Monteiro and Pang (1998) to fundamental maps (2) associated with pairs \((\Phi, \nu)\) satisfying Assumption 2. It essentially states that the fundamental map (2) has some nice homeomorphic properties.

**Theorem 1.** Assume that \( F: \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \to \mathcal{S}_+^n \times \mathbb{R}^m \) satisfies Assumption 1 and the pair \((\Phi, \nu)\) satisfies Assumption 2. Then the following statements hold for the map \( H \):

(a) \( H \) is proper with respect to \( \text{cl} \nu \times \mathcal{S}_+^n \);

(b) \( H \) maps \( \nu \times \mathbb{R}^m \) homeomorphically onto \( \nu \times \mathcal{S}_+^n \), where \( \nu \equiv \Phi^{-1}(\nu) \);

(c) \( H(\nu \times \mathbb{R}^m) \supseteq H(\text{cl} \nu \times \mathbb{R}^m) \supseteq \text{cl} \nu \times \mathcal{S}_+^n \).

Theorem 1 establishes the claimed properties (P1)-(P4) of the map \( H \) stated in the Introduction. Indeed, (P1) follows from conclusion (c); (P2) and (P3) follow from (b); and (P4) follows from (a). In what follows, we give two important consequences of the above theorem, assuming that \( 0 \in \mathcal{S}_+^n \). The first one, Corollary 1, has to do with the central path for the semidefinite complementarity problem (1); the second one, Corollary 2, is a solution existence result for the same problem.

**Corollary 1.** Suppose that \( F: \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \to \mathcal{S}_+^n \times \mathbb{R}^m \) satisfies Assumption 1 and the pair \((\Phi, \nu)\) satisfies Assumption 2. Assume that \((0,0) \in \text{cl} \nu \times \mathcal{S}_+^n \) and let \( P: [0,1] \to \mathcal{S}_+^n \) and \( Q: [0,1] \to \mathcal{S}_+^n \) be paths such that \( P(0) = 0 \), \( Q(0) = 0 \) and \( P(t) \in \nu \) for every \( t \in (0,1] \). Then there exist (unique) paths \( X: (0,1] \to \mathcal{S}_+^n \), \( Y: (0,1] \to \mathcal{S}_+^n \) such that \( (X(t),Y(t)) \in \nu \), and \( z: (0,1] \to \mathbb{R}^m \) such that

\[
\Phi(X(t), Y(t)) = P(t), \quad F(X(t), Y(t), z(t)) = Q(t), \quad \text{for all } t \in (0,1].
\]

Moreover, every accumulation point of \((X(t),Y(t),z(t))\) as \( t \) tends to 0 is a solution of the complementarity problem (1).

**Corollary 2.** Assume that \( F: \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \to \mathcal{S}_+^n \times \mathbb{R}^m \) satisfies Assumption 1 and the pair \((\Phi, \nu)\) satisfies Assumption 2. If there exists \((X^0, Y^0, z^0) \in \nu \times \mathbb{R}^m \) such that \( F(X^0, Y^0, z^0) = 0 \), then, for every \( A \in \text{cl} \nu \), the system

\[
\Phi(X, Y) = A, \quad F(X, Y, z) = 0, \quad (X, Y) \in \text{cl} \nu
\]

has a solution, which is unique when \( A \in \nu \).

We now discuss the specialization of Theorem 1 to the context of the convex nonlinear semidefinite program. The complementarity problem (1) arises as the set of first-order necessary optimality conditions for the following nonlinear semidefinite program (see for example Shapiro 1997):

\[
\begin{align*}
\text{minimize} & \quad \theta(x) \\
\text{subject to} & \quad G(x) \in -\mathcal{S}_+^n, \\
& \quad h(x) = 0,
\end{align*}
\]

where \( \theta: \mathbb{R}^m \to \mathbb{R} \), \( G: \mathbb{R}^m \to \mathcal{S}_+^n \) and \( h: \mathbb{R}^m \to \mathbb{R}^p \) are given smooth maps. Indeed, it is well known (see for example Shapiro 1997) that, under a suitable constraint qualification,
if $x^*$ is a local optimal solution of the semidefinite program, then there must exist $\eta^* \in \mathbb{R}^p$ and $U^* \in \mathcal{S}_+^n$ such that

$$\nabla_x L(x^*, U^*, \eta^*) = 0, \quad U^* G(x^*) = 0,$$

where $L: \mathbb{R}^m \times \mathcal{S}_+^n \times \mathbb{R}^p \to \mathbb{R}$ is the Lagrangian function defined by

$$L(x, U, \eta) = \theta(x) + U \cdot G(x) - \eta^T h(x), \quad \text{for } (x, U, \eta) \in \mathbb{R}^m \times \mathcal{S}_+^n \times \mathbb{R}^p.$$

Letting

$$F(U, V, x, \eta) = \begin{pmatrix} V + G(x) \\ \nabla_x L(x, U, \eta) \\ h(x) \end{pmatrix}, \quad \text{for } (U, V, x, \eta) \in \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \times \mathbb{R}^p,$$

we see that the first-order necessary optimality conditions for problem (5) are equivalent to the complementarity problem (1) with $X = U$, $Y = V$ and $Z = (x, \eta)$.

The map $G: \mathbb{R}^m \to \mathcal{S}_+^n$ is said to be positive semidefinite convex (psd-convex) if

$$G(tx + (1-t)y) \leq tG(x) + (1-t)G(y), \quad \forall x, y \in \mathbb{R}^m, \forall t \in (0, 1).$$

The following result follows as an immediate consequence of Theorem 1 and the results of §5 of Monteiro and Pang (1996b).

**Theorem 2.** Suppose that $(\Phi, \Psi)$ is a pair satisfying Assumption 1, the function $	heta: \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable and convex, $G: \mathbb{R}^m \to \mathcal{S}_+^n$ is continuously differentiable and psd-convex, $h: \mathbb{R}^m \to \mathbb{R}^p$ is an affine function such that the (constant) gradient matrix $\nabla h(x)$ has full column rank and the feasible set $\mathcal{F} = \{x \in \mathbb{R}^m : G(x) \leq 0, h(x) = 0\}$ is bounded. If any one of the following conditions holds:

(a) $\theta$ is strictly convex; 
(b) for every $U \in \mathcal{S}_+^n$, the map $x \mapsto U \cdot G(x) \in \mathbb{R}$ is strictly convex; 
(c) each $G_{ij}$ is an analytic function,

then the following statements hold for the maps $F$ and $H$ given by (8) and (2), respectively:

(i) $H$ is proper with respect to $\text{cl} \mathcal{F} \times F(\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \times \mathbb{R}^p)$;
(ii) $H$ maps $\mathcal{U} \times \mathbb{R}^m \times \mathbb{R}^p$ homeomorphically onto $\mathcal{F} \times F(\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \times \mathbb{R}^p)$;
(iii) $H(\mathcal{U} \times \mathbb{R}^m \times \mathbb{R}^p) \supseteq H(\text{cl} \mathcal{U} \times \mathbb{R}^m \times \mathbb{R}^p) \supseteq \text{cl} \mathcal{F} \times F(\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \times \mathbb{R}^p)$;
(iv) the set $F(\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \times \mathbb{R}^p)$ is convex.

**Proof.** The assumptions of the theorem together with Proposition 4, Lemmas 7, 8 and 9 of Monteiro and Pang (1996b) imply that the map $F$ given by (8) is $(U, V)$-equi- level-monotone on $\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \times \mathbb{R}^p$, $(x, \eta)$-injective on $\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \times \mathbb{R}^p$ and $(x, \eta)$-bounded on $\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \times \mathbb{R}^p$. Hence, the conclusions (i), (ii) and (iii) of the theorem follow directly from Theorem 1. It was shown in Theorem 4(iii) of Monteiro and Pang (1996b) that the set $F(\mathcal{U} \times \mathbb{R}^m \times \mathbb{R}^p)$ is convex, where $\mathcal{U} = \{(X, Y) \in \mathcal{S}_+^n \times \mathcal{S}_+^n : XY + YX \in X + \mathcal{S}_+^n\}$. Since $\mathcal{U}(\mu) \subseteq \mu \mathcal{U} \subseteq \mathcal{S}_+^n \times \mathcal{S}_+^n$ for all $\mu > 0$, it follows from Lemma 1(b) that $F(\mu \mathcal{U} \times \mathbb{R}^m \times \mathbb{R}^p) = F(\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \times \mathbb{R}^p)$. Hence, (iv) holds. □

3. **Proof of the main results.** The goal of this section is to give the proof of Theorem 1. In the process of doing this, we will first establish Theorem 1 for the case in which the map $F$ is affine. Apart from the fact that an affine map $F$ considerably simplifies the analysis, we obtain through this case an important technical result (Lemma 5) which plays an important role in establishing Theorem 1 for the case of a nonlinear map $F$ satisfying Assumption 1.
We first establish two important technical lemmas that hold for nonlinear maps $F$ satisfying Assumption 1.

**Lemma 1.** Let $F : \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \to \mathcal{S}_+^n \times \mathbb{R}^m$ be a map satisfying Assumption 1. Then, the following statements hold:

(a) $F$ restricted to $\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m$ is an open map; in particular, $\mathcal{F}_+^+$ is an open set;

(b) $\mathcal{F}_+^+ = F(\mathcal{C}(\mu) \times \mathbb{R}^m)$ for every $\mu > 0$.

**Proof.** We first establish (a). Let $\tilde{F} : \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \to \mathcal{S}_+^n \times \mathbb{R}^m$ be defined by $\tilde{F}(X, Y, z) = F((X + X^T)/2, Y, z)$, for every $(X, Y, z) \in \mathcal{M}_+^n \times \mathcal{M}_+^n \times \mathbb{R}^m$. Using the fact that $F$ satisfies Assumption 1, it is easy to see that $\tilde{F}$ is $(X, Y)$-equilevel-monotone, $z$-injective on $\mathcal{M}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m$ and $z$-bounded. Then, it follows from Lemma 11 of Monteiro and Pang (1998) that the map $\tilde{H} : \mathcal{M}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \to \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m$ defined by

$$\tilde{H}(X, Y, z) = \left( \begin{array}{c} XY \\ \tilde{F}(X, Y, z) \end{array} \right)$$

maps $\mathcal{M}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m$ homeomorphically onto $\tilde{H}(\mathcal{M}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m)$. By the domain invariance theorem, this implies that $\tilde{H}$, and hence $\tilde{F}$, restricted to $\mathcal{M}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m$ is an open map. This conclusion together with a simple argument shows that $F$ restricted to $\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m$ is also an open map.

We now show (b). Clearly, $F(\mathcal{C}(\mu) \times \mathbb{R}^m) \subseteq \mathcal{F}_+^+$ for every $\mu > 0$. To prove the other inclusion, let $B$ be an arbitrary element of $\mathcal{F}_+^+$. By Corollary 3 of Monteiro and Pang (1998), the system $F(X, Y, z) = B$ and $XY = \mu I$ has a (unique) solution in $\mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m$ for every $\mu > 0$, we conclude that $B \in F(\mathcal{C}(\mu) \times \mathbb{R}^m)$ for every $\mu > 0$. □

A consequence of Lemma 1(b) is as follows. Assumption 2(a) implies the existence of $\mu_0 > 0$ such that $\mu_0 I \in \mathcal{V}$. This together with (b) and (d) of Assumption 2 imply that $\mathcal{C}(\mu_0) \subseteq \mathcal{C}(\mathcal{C}(\mu_0)) \subseteq \mathcal{C}(\mathcal{F}_+^+ \times \mathcal{F}_+^+)$. Hence, in view of Lemma 1(b), it follows that $F(\mathcal{C}(\mathbb{R}^m)) = \mathcal{F}_+^+$.

**Lemma 2.** Let $F : \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m \to \mathcal{S}_+^n \times \mathbb{R}^m$ be a map satisfying Assumption 1, $(\Phi, \mathcal{V})$ be a pair satisfying Assumption 2, and $H : \mathcal{D} \times \mathbb{R}^m \to \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m$ be the map defined in (2). Then, $H$ is proper with respect to $\text{cl} \mathcal{V} \times \mathcal{F}_+^+$.

**Proof.** Let $K$ be a compact subset of $\text{cl} \mathcal{V} \times \mathcal{F}_+^+$. We will show that $H^{-1}(K)$ is compact from which the result follows. We first show that $H^{-1}(K)$ is closed. Since $K$ is closed and $H$ is continuous, $H^{-1}(K)$ is closed with respect to $\text{dom}(H) = \mathcal{D} \times \mathbb{R}^m$. Hence, the closeness of $H^{-1}(K)$ follows if we show that $H^{-1}(K)$ is contained in a closed subset of $\text{dom}(H)$. Indeed, the definition of $H$ implies that $H^{-1}(K)$ is contained in $\Phi^{-1}(\text{cl} \mathcal{V}) \times \mathbb{R}^m$, which by Assumption 2(d), is a closed set. We next show that $H^{-1}(K)$ is bounded. Indeed, suppose for contradiction that there exists a sequence $\{(X^k, Y^k, z^k)\} \subseteq H^{-1}(K)$ such that $\lim_{k \to \infty} \| (X^k, Y^k, z^k) \| = \infty$. Since $K$ is compact and $\{H(X^k, Y^k, z^k)\} \subseteq K$, we may assume without loss of generality that there exists $F^\infty \in \mathcal{F}_+^+$ such that

$$F^\infty = \lim_{k \to \infty} F(X^k, Y^k, z^k).$$

Clearly, we have $F^\infty = F(X^\infty, Y^\infty, z^\infty)$ for some $(X^\infty, Y^\infty, z^\infty) \in \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m$. Since $X^\infty > 0$ and $Y^\infty > 0$, there exists $\eta > 0$ such that the set

$$\mathcal{N}_\infty \equiv \{(X, Y, z) \in \mathcal{S}_+^n \times \mathcal{S}_+^n \times \mathbb{R}^m : \eta^{-1} I \succ X \succ \eta I, \ \eta^{-1} I \succ Y \succ \eta I\}$$

is contained in $\text{cl} \mathcal{V} \times \mathcal{F}_+^+$.
contains \((X^\infty, Y^\infty, z^\infty)\). We clearly have that \(\mathcal{N}^\infty\) is open, and using Lemma 1(a), it follows that \(F(\mathcal{N}^\infty)\) is an open set. Thus, by (10) and the fact that \(F^\infty \in F(\mathcal{N}^\infty)\), we conclude that for all \(k\) sufficiently large, say \(k \geq k_0\), we have \(F(X^k, Y^k, z^k) \in F(\mathcal{N}^\infty)\), and hence that \(F(X^k, Y^k, z^k) = (\tilde{X}^k, \tilde{Y}^k, \tilde{z}^k)\) where \((\tilde{X}^k, \tilde{Y}^k, \tilde{z}^k) \in \mathcal{N}^\infty\). Since \(F\) is \((X, Y)\)-equilevel-monotone, we have \((X^k - \tilde{X}^k, Y^k - \tilde{Y}^k, z^k - \tilde{z}^k) \geq 0\) for all \(k \geq k_0\). This inequality together with the fact that \((\tilde{X}^k, \tilde{Y}^k, \tilde{z}^k) \in \mathcal{N}^\infty\) imply that

\[X^k \cdot Y^k + \eta k^{-2} \geq X^k \cdot \tilde{Y}^k + \tilde{X}^k \cdot \tilde{Y}^k \geq X^k \cdot \tilde{Y}^k + Y^k \cdot \tilde{X}^k \geq \eta (X^k \cdot \tilde{Y}^k + Y^k \cdot \tilde{X}^k),\]

for every \(k \geq k_0\). Using the fact that \(\{H(X^k, Y^k, z^k)\} \subseteq K\) and \(K\) is bounded, we conclude that \(\{X^k, Y^k\}\) is bounded. Using Assumption 2(c), we have that \(\{X^k, Y^k\}\) is bounded. This fact together with the above inequality implies that the sequences \(\{X^k\}\) and \(\{Y^k\}\) are bounded. Since \(\lim_{k \to \infty} ||(X^k, Y^k, z^k)|| = \infty\), we must have \(\lim_{k \to \infty} ||z^k|| = \infty\). From the fact that \(F\) is \(z\)-bounded, we conclude that \(\lim_{k \to \infty} ||F(X^k, Y^k, z^k)|| = \infty\), thereby contradicting (10).

We now turn our attention to the case of an affine map \(F\) and derive a version of Theorem 1 in this context (see Theorem 3 below). We begin with a result that contains some elementary properties of affine maps.

**Lemma 3.** Assume that \(G : \mathcal{S}^n \times \mathcal{S}^m \times \mathfrak{R}^m \to \mathcal{S}^n \times \mathfrak{R}^m\) is an affine map and let \(G^0 : \mathcal{S}^n \times \mathcal{S}^m \times \mathfrak{R}^m \to \mathcal{S}^n \times \mathfrak{R}^m\) denote its linear part. Then the following statements hold:

(a) \(G\) is \((X, Y)\)-equilevel-monotone if and only if

\[G^0(\Delta X, \Delta Y, \Delta z) = 0 \implies \Delta X \cdot \Delta Y \geq 0;\]

(b) \(G\) is \(z\)-injective if and only if

\[|\Delta z| \in \mathfrak{R}^m, G^0(0, 0, \Delta z) = 0 \implies \Delta z = 0;\]

(c) \(G\) is \(z\)-injective if and only if \(G\) is \(z\)-bounded.

**Proof.** See Lemma 3 in Monteiro and Pang (1998). \(\square\)

The next lemma is an important step towards establishing the main result for the case in which \(F\) is affine.

**Lemma 4.** Let \(F : \mathcal{U} \times \mathcal{U} \times \mathfrak{R}^m \to \mathcal{U} \times \mathfrak{R}^m\) be an affine map which is \((X, Y)\)-equilevel-monotone and \(z\)-injective, and let \((\Phi, \nu)\) be a pair satisfying Assumption 2. Then the map \(H\) defined in (2) restricted to \(\mathcal{U} \times \mathfrak{R}^m\) is a local homeomorphism.

**Proof.** Since \(\mathcal{U} \times \mathfrak{R}^m\) is an open set, it is sufficient to show that the derivative map \(H'(X, Y, z) : \mathcal{U} \times \mathcal{U} \times \mathfrak{R}^m \to \mathcal{U} \times \mathfrak{R}^m\) is an isomorphism for every \((X, Y, z)\) \(\in \mathcal{U} \times \mathfrak{R}^m\). For this purpose, fix any \((X, Y, z)\) \(\in \mathcal{U} \times \mathfrak{R}^m\). Since \(H'(X, Y, z)\) is linear and is a map between identical spaces, it is enough to show that

\[\begin{aligned}
(\Delta X, \Delta Y, \Delta z) \in \mathcal{U} \times \mathcal{U} \times \mathfrak{R}^m \quad & \implies (\Delta X, \Delta Y, \Delta z) = 0.
\end{aligned}\]

Indeed, assume that the left-hand side of the above implication holds. By the definition \(H\), we have

\[F^0(\Delta X, \Delta Y, \Delta z) = 0,\]

\[\Phi'(X, Y)(\Delta X, \Delta Y) = 0.\]
By (14) and Lemma 3(a), we have $\Delta X \cdot \Delta Y \geq 0$. This together with (15) and Assumption 2(e) imply that $\Delta X = \Delta Y = 0$. Hence, by (14) and Lemma 3(b) we conclude that $\Delta z = 0$. We have thus shown that the implication (13) holds.

We are now ready to establish a version of the main result for the affine case.

**Theorem 3.** Assume that $F: \mathcal{P} \times \mathcal{P} \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ is an affine map which is $(X,Y)$-equilevel-monotone and $z$-injective, and $(\Phi, \psi)$ is a pair satisfying Assumption 2. Then, the following statements hold:

- (a) $H$ maps $\mathcal{U} \times \mathcal{P}$ homeomorphically onto $\mathcal{V} \times \mathcal{P}^+$;
- (b) $H(\mathcal{D} \times \mathcal{P}) \supseteq H(\mathcal{C} \times \mathcal{P}) \supseteq \mathcal{C} \psi \times \mathcal{P}^+$.

**Proof.** Let $M \equiv \mathcal{D} \times \mathcal{P}$, $N \equiv \mathcal{P} \times \mathcal{P} \times \mathcal{P}$, $M_0 \equiv \mathcal{U} \times \mathcal{P}$, $N_0 \equiv \mathcal{V} \times \mathcal{P}^+$ and $E \equiv \mathcal{C} \psi \times \mathcal{P}^+$. We will show first that these sets and the map $G \equiv H$ satisfy the assumptions of Proposition 2. Indeed, first observe that $H|_{M_0}$ is a local homeomorphism due to Lemma 4. The assumption that $F$ is $(X,Y)$-equilevel-monotone and $z$-injective together with Lemma 3(c) imply that $F$ is $z$-bounded. Hence, it follows from Lemma 2 that $H$ is proper with respect to $E$. By the definition of $M_0$, $N_0$ and $H$, we have $M_0 = H^{-1}(N_0)$. To show that $H^{-1}(N_0) \neq \emptyset$, note that by Assumption 2(a), there exists $\mu_0 > 0$ such that $\mu_0 \in \mathcal{V}$. Letting $v_0 \equiv (\phi^{-1}(\mu_0))^{1/2}$, we have by Assumption 2(b) that $\Phi(v_0 I, v_0 J) = \mu_0 I \in \mathcal{V}$. Hence, it follows that $(v_0 I, v_0 J) \in H^{-1}(N_0)$ for every $v \in \mathcal{P}$. By Assumption 2(a) and the fact that a continuous map carries connected sets onto connected sets, we conclude that $N_0 \equiv \mathcal{V} \times \mathcal{P}^+$ is a connected set. By Proposition 2, we conclude that $H$ restricted to the pair $(H^{-1}(N_0), N_0) = (M_0, N_0)$ is a proper local homeomorphism, $H(M_0) \supseteq N_0$ and $H(\mathcal{C} \times \mathcal{P}) \supseteq E \cap \mathcal{C} \psi \times \mathcal{P}^+$. Clearly, the last inclusion implies the second inclusion in (b).

We now show (a). We have already shown that $H$ maps $M_0$ onto $N_0$ and that $\tilde{H} = H|_{M_0}$ is a proper local homeomorphism. It remains to show that $\tilde{H}$ is one-to-one. Since $N_0$ is connected, by Proposition 1 it remains to show that for some $(A, B) \in N_0 = \mathcal{V} \times \mathcal{P}^+$, the inverse image $\tilde{H}^{-1}((A, B))$ has at most one element. Indeed, let $A = I$ and take $B \in \mathcal{P}^+$ arbitrarily. Assume that $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathcal{V} \times \mathcal{P}^+ = M_0$ is such that $\tilde{H}(\tilde{X}, \tilde{Y}, \tilde{Z}) = H(\tilde{X}, \tilde{Y}, \tilde{Z}) = (I, B)$. Then, by the definition of $H$, we have $F(\tilde{X}, \tilde{Y}, \tilde{Z}) = B$ and $\Phi(\tilde{X}, \tilde{Y}) = I$, which by Assumption 2(b) implies that $\tilde{X} \tilde{Y} = \phi^{-1}(1) I$. Since by Corollary 3 of Monteiro and Pang (1998), the system $F(X, Y, z) = B$ and $XY = vI$ has a unique solution in $\mathcal{P}^+ \times \mathcal{P}^+ \times \mathcal{P}^+$, for every $v > 0$ and $B \in F(\mathcal{P}^+ \times \mathcal{P}^+ \times \mathcal{P}^+)$, we conclude that $(\tilde{X}, \tilde{Y}, \tilde{Z})$ is the only solution of $\tilde{H}(X, Y, z) = H(X, Y, z) = (I, B)$ in $M_0$.

We use the above result to prove the following very important technical lemma.

**Lemma 5.** Let $(\Phi, \psi)$ be a pair satisfying Assumption 2. Let $(X_0, Y_0)$ and $(X_1, Y_1)$ be elements of $\mathcal{U}$ such that

$$\begin{align*}
(X_1 - X_0) \cdot (Y_1 - Y_0) & \geq 0 \quad \text{and} \quad \Phi(X_0, Y_0) = \Phi(X_1, Y_1).
\end{align*}$$

Then, $(X_0, Y_0) = (X_1, Y_1)$.

**Proof.** Assume for contradiction that $(X_0, Y_0) \neq (X_1, Y_1)$. It has been shown in the proof of Lemma 5 of Monteiro and Pang (1998) that there exists an affine map $F: \mathcal{P} \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ which is $(X, Y)$-equilevel-monotone and satisfies $F(X_0, Y_0) = F(X_1, Y_1)$. Observe that $F$ satisfies Assumption 1 with $m = 0$ (no variable $z$ is present). By Theorem 3(a), it follows that the associated map $H$ (with $m = 0$) restricted to $\mathcal{U}$ is one-to-one. Moreover, (16) and the relation $F(X_0, Y_0) = F(X_1, Y_1)$ imply that $H(X_0, Y_0) = H(X_1, Y_1)$. The
last two conclusions together with the fact that \((X_0, Y_0), (X_1, Y_1) \in \mathcal{U}\) then imply that 
\((X_0, Y_0) = (X_1, Y_1)\). □

We are now ready to give the proof of Theorem 1.

**Proof of Theorem 1.** The proof is close to the one given for Theorem 3. It consists of showing that the sets 
\(M = \mathcal{S}^n \times \mathcal{S}_{++}^n\) and 
\(N = \mathcal{S}_{++}^n \times \mathcal{S}^n\), together with the map \(G = H\), satisfy the assumptions of Proposition 2. But instead of using Lemma 4 to show that \(H|_{\mathcal{M}_0}\) is a local homeomorphism, we use Lemma 5 to prove that \(H|_{\mathcal{M}_0}\) maps \(M_0\) homeomorphically onto \(H(M_0)\). Since \(H|_{\mathcal{M}_0}\) is a continuous map from an open subset of the vector space \(\mathcal{S}^n \times \mathcal{S}_{++}^n \times \mathcal{S}^n\) into the same space, by the domain invariance theorem it suffices to show that \(H|_{\mathcal{M}_0}\) is one-to-one. For this purpose, assume that \(H(\vec{X}, \vec{Y}, \vec{Z}) = H(\vec{X}', \vec{Y}', \vec{Z}')\) for some \((\vec{X}, \vec{Y}, \vec{Z}) \in \mathcal{U} \times \mathcal{R}^m\) and \((\vec{X}', \vec{Y}', \vec{Z}') \in \mathcal{U} \times \mathcal{R}^m\). Then, by the definition of \(H\), we have \(F(\vec{X}, \vec{Y}, \vec{Z}) = F(\vec{X}', \vec{Y}', \vec{Z}')\) and \(\Phi(\vec{X}, \vec{Y}) = \Phi(\vec{X}', \vec{Y}')\). Since \(F\) is \((X, Y)\)-equilevel-monotone, we conclude that 
\((\vec{X} - \vec{X}') \cdot (\vec{Y} - \vec{Y}') \geq 0\). Hence, by Lemma 5, we have 
\((\vec{X}, \vec{Y}) = (\vec{X}', \vec{Y}')\). This implies that 
\(F(\vec{X}, \vec{Y}, \vec{Z}) = F(\vec{X}', \vec{Y}', \vec{Z}')\) and by the \(z\)-injectiveness of \(F\) it follows that \(\vec{Z} = \vec{Z}'\). We have thus proved that \(H|_{\mathcal{M}_0}\) maps \(M_0\) homeomorphically onto \(H(M_0)\). Since \(H|_{\mathcal{M}_0}\) is clearly a local homeomorphism and \(F\) is \((X, Y)\)-equilevel-monotone and \(z\)-bounded by assumption, it follows from Lemma 2 that (a) holds. The proofs of (b) and (c) use the same arguments as in the proof of Theorem 3. □

4. Verification of Assumption 2 for several central-path maps. In this section we verify that the maps of Examples 1 to 6 satisfy Assumption 2.

**Verification for Example 1.** Let \(D = \mathcal{S}^n \times \mathcal{S}^n_+\), \(\Phi: \mathcal{D} \to \mathcal{S}^n\) be the map defined by \(\Phi(X, Y) = (XY + YX)/2\), and \(\mathcal{V} = \mathcal{S}^n_+\). Conditions (a), (b) and (c) of Assumption 2 are straightforward, and clearly we have that \(\mathcal{U}\) is contained in \(\mathcal{S}^n_+ \times \mathcal{S}^n_+\). Since \(\Phi\) is continuous and its domain is closed, the set \(\Phi^{-1}(\mathcal{V})\) is closed, and hence Assumption 2(d) holds. Assumption 2(e) follows from Theorem 3.1(iii) in Shida, Shindoh and Kojima (1998).

**Verification for Example 2.** Let \(D = \mathcal{S}^n_+ \times \mathcal{S}^n_+, \Phi: \mathcal{D} \to \mathcal{S}^n\) be the map defined by 
\(\Phi(X, Y) = X^{1/2} Y X^{1/2}\), and \(\mathcal{V} = \{Z \in \mathcal{S}^n_+: \|Z - VL\| < v/\sqrt{2} \text{ for some } v > 0\}\). Conditions (a), (b) and (c) of Assumption 2 are straightforward. As in Example 1, we easily see that \(\Phi^{-1}(\mathcal{V})\) is closed. Hence, Assumption 2(d) holds. Finally, it follows from the proof of Lemma 2.3 of Monteiro and Tsuchiya (1999) (with \(H = 0\)) that the implication of Assumption 2(e) holds.

**Verification for Example 3.** Let \(D = \mathcal{S}^n_+ \times \mathcal{S}^n_+, \Phi: \mathcal{D} \to \mathcal{S}^n\) be the map defined by 
\(\Phi(X, Y) = (X^{1/2} Y^{1/2} + Y^{1/2} X^{1/2})/2\), and \(\mathcal{V} = \{Z \in \mathcal{S}^n_+: \|Z - VL\| < v/(3\sqrt{2}) \text{ for some } v > 0\}\). Conditions (a), (b) and (c) of Assumption 2 are straightforward. As in Example 1, we easily see that \(\Phi^{-1}(\mathcal{V})\) is closed. Hence, Assumption 2(d) holds. Finally, Assumption 2(e) follows from Proposition 4 below with \(H = 0\).

**Verification for Example 4.** Let \(D = \mathcal{C}(0) \cup (\mathcal{S}^n_+ \times \mathcal{S}^n_+), \Phi: \mathcal{D} \to \mathcal{S}^n\) be the map defined by 
\[
\Phi(X, Y) = \begin{cases} 
L_X^T Y L_X, & \text{if } (X, Y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+; \\
0, & \text{if } (X, Y) \in \mathcal{C}(0),
\end{cases}
\]
where \(L_X\) is the lower Cholesky factor of \(X\). Also, let \(\mathcal{V} = \{Z \in \mathcal{S}^n_+: \|Z - VL\| < v/\sqrt{2} \text{ for some } v > 0\}\). Observe that the continuity of the map \(\Phi\) required in Assumption 2(a) follows from Lemma 6. The remaining requirements in (a) and conditions (b) and (c) of Assumption 2 are straightforward. We easily see that 
\[
\Phi^{-1}(\mathcal{V}) = \mathcal{C}(0) \cup \{(X, Y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+: \|L_X^T Y L_X - VL\| \leq v/\sqrt{2} \text{ for some } v > 0\},
\]
which, in view of Lemma 6, is a closed set. Hence, Assumption 2(d) holds. Finally, Assumption 2(e) follows from Lemma 2.4 of Monteiro and Zanjácomo (1997) with $H = 0$.

**Verification for Example 5.** Let $\mathcal{D} \equiv \mathcal{C}(0) \cup (\mathcal{S}_{++}^n \times \mathcal{S}_{++}^n)$, $\Phi : \mathcal{D} \to \mathcal{S}^n$ be the map defined by

$$
\Phi(X, Y) = \begin{cases} 
(U_f^T L_X + L_f^T U_Y)/2, & \text{if } (X, Y) \in \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n, \\
0, & \text{if } (X, Y) \in \mathcal{C}(0),
\end{cases}
$$

where $L_X$ and $U_Y$ are the lower Cholesky factor of $X$ and the upper Cholesky factor of $Y$, respectively. Also, let $\mathcal{C} \equiv \{ Z \in \mathcal{S}_{++}^n : \| Z - vI \|_F < \sqrt{3}/2 \text{ for some } v > 0 \}$. Observe that the continuity of the map $\Phi$ required in Assumption 2(a) follows from Lemma 6. The remaining requirements in (a) and conditions (b) and (c) of Assumption 2 are straightforward. We easily see that

$$
\Phi^{-1}(\text{cl } \mathcal{C}) = \mathcal{C}(0) \cup \left\{ (X, Y) \in \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n : \frac{\| U_f^T L_X + L_f^T U_Y - vI \|_F}{2} \leq \frac{v}{3\sqrt{2}} \text{ for some } v > 0 \right\},
$$

which, in view of Lemma 6, is a closed set. Hence, Assumption 2(d) holds. Finally, Assumption 2(e) follows from Proposition 5 below with $H = 0$.

**Verification for Example 6.** Let $\mathcal{D} \equiv \mathcal{C}(0) \cup (\mathcal{S}_{++}^n \times \mathcal{S}_{++}^n)$, $\Phi : \mathcal{D} \to \mathcal{S}^n$ be the map defined by

$$
\Phi(X, Y) = \begin{cases} 
W^{1/2} X Y W^{-1/2}, & \text{if } (X, Y) \in \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n, \\
0, & \text{if } (X, Y) \in \mathcal{C}(0),
\end{cases}
$$

where $W = W(X, Y)$ is the unique positive definite matrix such that $W X W = Y$, and $\mathcal{C} \equiv \{ Z \in \mathcal{S}_{++}^n : \| Z - vI \| < \sqrt{2} \text{ for some } v > 0 \}$. Observe that the continuity of the map $\Phi$ required in Assumption 2(a) follows from Lemma 6. The remaining requirements in (a) and conditions (b) and (c) of Assumption 2 are straightforward. We easily see that

$$
\Phi^{-1}(\text{cl } \mathcal{C}) = \mathcal{C}(0) \cup \left\{ (X, Y) \in \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n : \| W^{1/2} X Y W^{-1/2} - vI \| \leq \frac{v}{2} \text{ for some } v > 0 \right\},
$$

which, in view of Lemma 6, is a closed set. Hence, Assumption 2(d) holds. Assumption 2(e) follows from Proposition 6 below.

Note that all the sets $\mathcal{C}$ in the above examples are convex cones containing the line $\{ vI : v > 0 \}$. We now present the technical results needed for the verification of the Examples 1 to 6 described above.

**Lemma 6.** Let $\| \cdot \|$ be an arbitrary norm in $\mathcal{S}^n$ such that $\| H \| \geq \| H \|$ for all $H \in \mathcal{S}^n$. Let $\Phi : \mathcal{C}(0) \cup (\mathcal{S}_{++}^n \times \mathcal{S}_{++}^n) \to \mathcal{S}^n$ denote the map of Example 4, 5 or 6 described above. Then, $\Phi$ is a continuous map and, for any $v \in (0, 1)$, the set defined by

$$
\mathcal{C}(0) \cup \{(X, Y) \in \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n : \| \Phi(X, Y) - vI \| \leq \gamma v \text{ for some } v > 0 \}
$$

is a closed set.
PROOF. First note that for \((X, Y) \in \mathcal{S}_+^n \times \mathcal{S}_+^n\), we have

\begin{align}
(18) \quad \|U_+^T L X\|^2 = \|L_+^T Y L X\| = \lambda_{\max}(L_+^T Y L X) = \lambda_{\max}(XY) \in \left[\frac{X \cdot Y}{n}, X \cdot Y\right],
\end{align}

and

\begin{align}
(19) \quad \|W^{1/2} X Y W^{-1/2}\| = \lambda_{\max}(W^{1/2} X Y W^{-1/2}) = \lambda_{\max}(XY) \in \left[\frac{X \cdot Y}{n}, X \cdot Y\right],
\end{align}

where the first equality of (19) follows from the fact that \(W^{1/2} X Y W^{-1/2}\) is a symmetric matrix. To establish the continuity of the three maps, it is sufficient to show that if \(\{(X^*, Y^*)\} \subset \mathcal{S}_+^n \times \mathcal{S}_+^n\) converges to a point \((\tilde{X}, \tilde{Y}) \in \mathcal{G}(0)\) then the sequences \(\{(W^k)^{1/2} X^k Y^k (W^k)^{-1/2}\}, \{U_{+k}^T L X_k\}\) and \(\{L_{+k}^T Y_k L X_k\}\) converge to 0. Indeed, \((\tilde{X}, \tilde{Y}) \in \mathcal{G}(0)\) implies that \(\{X^k, Y^k\}\) converges to 0, which, in view of (18) and (19), implies that these three sequences converge to 0.

We now establish the closedness of the set (17), which we denote by \(\mathcal{N}_{\tilde{X}}\). It is sufficient to show that if \(\{(X^k, Y^k)\} \subset \mathcal{S}_+^n \times \mathcal{S}_+^n\) converges to \((\tilde{X}, \tilde{Y})\), then \((\tilde{X}, \tilde{Y}) \in \mathcal{N}_{\tilde{X}}\). Clearly, \((\tilde{X}, \tilde{Y}) \in \mathcal{S}_+^n \times \mathcal{S}_+^n\). Let \(E_k \equiv \Phi(X^k, Y^k)\) for all \(k\). Since \((X^k, Y^k) \in \mathcal{N}_{\tilde{X}} \setminus \mathcal{G}(0)\), there exists \(v_k > 0\) such that

\begin{align}
(20) \quad \|E_k - v_k I\| \leq \gamma v_k,
\end{align}

and hence

\begin{align}
(21) \quad (1 - \gamma) v_k \leq \lambda_{\min}(E^k) \leq \lambda_{\max}(E^k) \leq (1 + \gamma) v_k,
\end{align}

due to the fact that \(\|\cdot\| \geq \|\cdot\|\). Using (18), we easily see that \(\{E_k\}\) is bounded. In view of (21), this implies that \(\{v_k\}\) is bounded. Without loss of generality, we may assume that \(\{v_k\}\) converges to some \(\bar{v} \geq 0\). We now consider the two possible cases: \(\bar{v} = 0\) and \(\bar{v} > 0\). If \(\bar{v} = 0\) then (21) implies that \(\lim_{k \to \infty} \|E^k\| = 0\), or equivalently \(\lim_{k \to \infty} \Phi(X^k, Y^k) = 0\).

For the map of Example 4, this means that \(\lim_{k \to \infty} L_{+k}^T Y_k L X_k = 0\), and for the one in Example 5, that \(\lim_{k \to \infty} U_{+k}^T L Y_k + L_{+k}^T U X_k = 0\). Using the fact that \(U_{+k}^T L Y_k\) is lower triangular, the last limit implies that \(\lim_{k \to \infty} U_{+k}^T L Y_k = 0\). Hence, for both maps \(\Phi\), it follows from (18) that \(\{X^k, Y^k\}\) converges to 0, and hence that \((\tilde{X}, \tilde{Y}) \in \mathcal{G}(0) \subseteq \mathcal{N}_{\tilde{X}}\). Assume now that \(\bar{v} > 0\). We will show that \((\tilde{X}, \tilde{Y}) \in \mathcal{S}_+^n \times \mathcal{S}_+^n\), from which the conclusion that \((\tilde{X}, \tilde{Y}) \in \mathcal{N}_{\tilde{X}} \setminus \mathcal{G}(0)\) follows by letting \(k\) tend to infinity in (20) and using the fact that \(\bar{v} > 0\) and \(\Phi\) is continuous on \(\mathcal{S}_+^n \times \mathcal{S}_+^n\). Indeed, assume for contradiction that \((\tilde{X}, \tilde{Y}) \notin \mathcal{S}_+^n \times \mathcal{S}_+^n\). Without loss of generality, we may assume that \(\tilde{X}\) is singular. Consider first the map \(\Phi\) of Example 4. We have

\begin{align}
\lambda_{\min}(E^k) = \lambda_{\min}(L_{+k}^T Y_k L X_k) = \lambda_{\min}(X^k Y^k),
\end{align}

and since \(\lim_{k \to \infty} \lambda_{\min}(X^k Y^k) = \lambda_{\min}(\tilde{X} \tilde{Y}) = 0\), we conclude that \(\lim_{k \to \infty} \lambda_{\min}(E^k) = 0\). By (21) this implies that \(\bar{v} = \lim_{k \to \infty} v_k = 0\), contradicting the assumption that \(\bar{v} > 0\).

Consider now the map \(\Phi\) of Example 5. We have

\begin{align}
\lambda_{\min}(E^k) = \frac{1}{2} \lambda_{\min}(U_{+k}^T L X_k + L_{+k}^T U Y_k) \leq \lambda_{\min}(U_{+k}^T L X_k),
\end{align}

where the inequality follows from the fact that the real part of the spectrum of a real matrix is contained between the largest and the smallest eigenvalues of its Hermitian part (see p. 187 of Horn and Johnson 1991, for example) and the fact that \(U_{+k}^T L X_k\) being lower triangular has only real eigenvalues. Note that the sequences \(\{U_{+k}\}\) and \(\{L_{X_k}\}\)
are bounded, since \( \|U_Y\| = \|Y^k\|^{1/2} \) and \( \|L_X\| = \|X^k\|^{1/2} \). Let \( \bar{U} \) and \( \bar{L} \) be accumulation points of \( \{U_Y\} \) and \( \{L_X\} \), respectively. Clearly, \( \bar{L}^T = \bar{X} \), and since \( \bar{X} \) is singular, it follows that \( \bar{L} \) is also singular. Letting \( k \) tend to infinity in (22), we conclude that
\[
0 = \lambda_{\text{min}}(\bar{U}^T\bar{L}) = \lim_{k \to 0} \lambda_{\text{min}}(U_Y^T L_X) \geq \lim_{k \to 0} \lambda_{\text{min}}(E^k),
\]
which, as above, yields the desired contradiction.

Finally, consider the map \( \Phi \) of Example 6. We have
\[
\lambda_{\text{min}}(E^k) = \lambda_{\text{min}}((W^k)^{1/2}X^kY^k(W^k)^{-1/2}) = \lambda_{\text{min}}(X^kY^k),
\]
and since \( \lim_{k \to 0} \lambda_{\text{min}}(X^kY^k) = \lambda_{\text{min}}(\bar{X}\bar{Y}) = 0 \), we conclude that \( \lim_{k \to 0} \lambda_{\text{min}}(E^k) = 0 \). By (21) this implies that \( \bar{v} = \lim_{k \to 0} v_k = 0 \), contradicting the assumption that \( \bar{v} > 0 \). □

The next three lemmas are used in the proof of Proposition 4.

**Lemma 7.** Let \( G \in \mathcal{M}^n \) be such that \( \text{tr} \, (G^2) \geq 0 \). Then
\[
\|G\|_F \leq \sqrt{2}\|(G + G^T)/2\|_F.
\]

**Proof.** Defining \( H \equiv (G + G^T)/2 \in \mathcal{S}^n \) and \( A \equiv (G - G^T)/2 \in \mathcal{S}^n \), and using the identities \( H \cdot A = \text{tr} \, (AH) = \text{tr} \, (HA) = 0 \), \( \|H\|_F^2 = \text{tr} \, (H^2) \) and \( \|A\|_F^2 = -\text{tr} \, (A^2) \), we easily see that
\[
\text{tr} \, (G^2) = \|H\|_F^2 - \|A\|_F^2 \quad \text{and} \quad \|G\|_F^2 = \|H\|_F^2 + \|A\|_F^2.
\]
The result now follows by adding the two identities and using the assumption that \( \text{tr} \, (G^2) \geq 0 \). □

**Lemma 8.** Let \( E \in \mathcal{M}^n \) be a matrix whose eigenvalues are all real and let \( \gamma \in (0, 1/\sqrt{2}) \) be given. Then, the following implication holds:
\[
\left\| \frac{E + E^T}{2} - I \right\|_F \leq \gamma \implies \|E - I\|_F \leq \sqrt{2}\gamma \quad \text{and} \quad \|E^{-1}\| \leq \frac{1}{1 - \sqrt{2}\gamma},
\]

**Proof.** Assume that the left-hand side of the implication holds. Using the fact that \( E \), and hence \( E - I \), has only real eigenvalues, it is easy to see that \( \text{tr}((E - I)^2) \geq 0 \). Using Lemma 7 with \( G = E - I \), it follows that
\[
\|E - I\|_F \leq \sqrt{2}\|(E + E^T)/2 - I\|_F \leq \sqrt{2}\gamma,
\]
and hence the first inequality of the right-hand side of the implication holds. The second inequality follows from Lemma 2.3.3 of Golub and Van Loan (1989) and the fact that \( \|E - I\| \leq \|E - I\|_F \leq \sqrt{2}\gamma < 1 \). □

**Lemma 9.** The following statements hold:
(a) for every \( A \in \mathcal{S}^n_+ \) and \( H \in \mathcal{S}^n \), the equation \( AU + UA = H \) has a unique solution \( U \in \mathcal{S}^n_+ \); moreover, this solution satisfies \( \|AU\|_F \leq \|H\|_F/\sqrt{2} \);
(b) if \( \theta : \mathcal{S}^n_+ \to \mathcal{S}^n_+ \) denotes the square root function \( \theta(X) = X^{1/2} \), then \( \theta \) is an analytic function and \( \theta'(X)H = U \), where \( U \in \mathcal{S}^n_+ \) is the unique solution of \( X^{1/2}U + UX^{1/2} = \theta(X) \) is the derivative of \( \theta \) at \( X \) and \( \theta'(X)H \) is the linear map \( \theta'(X) \) evaluated at \( H \).

**Proof.** See Lemmas 2.2 and 2.3 of Monteiro and Tsuchiya (1999). □

**Proposition 4.** Suppose that \( 0 < \gamma < 1/(3\sqrt{2}) \), \( H \in \mathcal{S}^n \) and \( \Phi : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \to \mathcal{S}^n \) is the map of Example 3. If \( (X, Y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+ \) is a point such that \( \|(X^{1/2}Y^{1/2} + Y^{1/2}X^{1/2})/2 - X\|_F \leq \sqrt{2}\gamma \)
\[ \|X\|_F \leq \gamma v \text{ for some } v > 0, \text{ then} \]
\[
\begin{aligned}
(\Delta X, \Delta Y) \in \mathcal{G}^n \times \mathcal{G}^n \\
(23) \quad \Phi'(X, Y)(\Delta X, \Delta Y) = H \\
\Delta X \cdot \Delta Y \geq 0
\end{aligned}
\]
where \( \Delta X \equiv X^{-1/2} \Delta XY^{1/2} \) and \( \Delta Y \equiv X^{1/2} \Delta YY^{-1/2} \).

**Proof.** Using Lemma 9(a), it is easy to see that the equation \( \Phi'(X, Y)(\Delta X, \Delta Y) = H \) is equivalent to
\[
(24) \\
V X^{1/2} + X^{1/2}V + U Y^{1/2} + Y^{1/2}U = 2H,
\]
where \( U \) and \( V \) are the unique solutions of the Lyapunov equations
\[
(25) \\
UX^{1/2} + X^{1/2}U = \Delta X, \quad VY^{1/2} + Y^{1/2}V = \Delta Y.
\]
Multiplying the first (resp., second) equation on the left and on the right by \( X^{-1/2} \) (resp., \( Y^{-1/2} \)) and using Lemma 9(a), we obtain
\[
(26) \\
\|X^{-1/2}U\|_F \leq \frac{\|X^{-1/2}\Delta X X^{-1/2}\|_F}{\sqrt{2}}, \quad \|Y^{-1/2}V\|_F \leq \frac{\|Y^{-1/2}\Delta YY^{-1/2}\|_F}{\sqrt{2}}.
\]
Using the definition of \( \Delta X \) and \( \Delta Y \), it follows from (24) and (25) that
\[
\Delta X + \Delta Y = X^{-1/2} \Delta XY^{1/2} + X^{1/2} \Delta YY^{-1/2} \\
= X^{1/2} Y^{1/2} V Y^{-1/2} + X^{1/2} V + X^{-1/2} U X^{1/2} Y^{1/2} + U Y^{1/2} \\
= 2H + X^{1/2} Y^{1/2} V Y^{-1/2} - V X^{-1/2} + X^{-1/2} U X^{1/2} Y^{1/2} - Y^{1/2} U \\
= 2H + (X^{1/2} Y^{1/2} - v I) V Y^{-1/2} + V Y^{-1/2} (v I - Y^{1/2} X^{1/2}) \\
+ X^{-1/2} U (X^{1/2} Y^{1/2} - v I) + (v I - Y^{1/2} X^{1/2}) X^{-1/2} U.
\]
Taking the Frobenius norm of both sides of the last equation and using (26), Lemma 8 with \( E \equiv (X^{1/2} Y^{1/2})/v \), the triangular inequality for norms and the fact that \( \Delta X \cdot \Delta Y = X^{-1/2} \Delta X Y^{-1/2} \geq 0 \), we obtain
\[
(\|\Delta X\|_F^2 + \|\Delta Y\|_F^2)^{1/2} = (\|X^{-1/2} \Delta XY^{1/2}\|_F^2 + \|X^{1/2} \Delta YY^{-1/2}\|_F^2)^{1/2} \\
\leq 2\|H\|_F + 2\|X^{-1/2} Y^{-1/2} - v I\|_F (\|Y^{-1/2} V\|_F + \|X^{-1/2} U\|_F) \\
\leq 2\|H\|_F + 2\sqrt{2} \gamma v (\|Y^{-1/2} V\|_F + \|X^{-1/2} U\|_F) \\
\leq 2\|H\|_F + 2\gamma v (\|X^{-1/2} \Delta XY^{1/2}\|_F + \|Y^{-1/2} \Delta YY^{-1/2}\|_F) \\
\leq 2\|H\|_F + 2\gamma v (\|\Delta X\|_F + \|\Delta Y\|_F) \\
\leq 2\|H\|_F + \frac{2\sqrt{2} \gamma}{1 - \sqrt{2} \gamma} (\|\Delta X\|_F^2 + \|\Delta Y\|_F^2)^{1/2},
\]
from which (23) follows. □

The next result has a proof similar to the one of Proposition 4, and hence we omit it. Instead of Lemma 9, its proof relies on Lemma 2.2 of Monteiro and Zanjácomo (1997).
Proposition 5. Suppose that $0 < \gamma < 1/(3\sqrt{2})$, $H \in S^n$ and $\Phi : C(0) \cup (S_+^n \times S_+^n) \to S^n$ is the map in Example 5. If $(X, Y) \in S_+^n \times S_+^n$ is a point such that $\|(U_T^y L_X + L_T^y U_Y)/2 - vI\|_F \leq \gamma v$ for some $v > 0$, then

\[
\begin{align*}
(\Delta X, \Delta Y) &\in S^n \\
(27) \quad \Phi'(X, Y)(\Delta X, \Delta Y) &= H \\
\Delta X \cdot \Delta Y &\geq 0
\end{align*}
\]

where $\Delta X = U_T^y \Delta X L_X^{-T}$ and $\Delta Y = U_T^y \Delta Y L_X$.

Proposition 6. Let $\Phi : C(0) \cup (S_+^n \times S_+^n) \to S^n$ denote the map of Example 6 and let $(X, Y) \in S_+^n \times S_+^n$ be a point such that $\|\Phi(X, Y) - vI\| \leq \gamma v$ for some $v > 0$ and $\gamma \in (0, 1/2)$. Then,

\[
\begin{align*}
(\Delta X, \Delta Y) &\in S^n \\
(28) \quad \Phi'(X, Y)(\Delta X, \Delta Y) &= 0 \\
\Delta X \cdot \Delta Y &\geq 0
\end{align*}
\]

Proof. Using the identity $WXW = Y$, we easily see that $\Phi(X, Y) = (W^{1/2}XW^{1/2})^2$. Letting $V = V(X, Y) \equiv W^{1/2}XW^{1/2}$, we have that $\Phi(X, Y) = V^2$. Hence, the directional derivative of $\Phi(X, Y)$ along $(\Delta X, \Delta Y)$ is

$\Phi'(X, Y)(\Delta X, \Delta Y) = VV' + V'V,$

where $V' \in S^n$ denote the directional derivative of the function $V$ at the point $(X, Y)$ along the direction $(\Delta X, \Delta Y)$. Since $V \in S_+^n$, the condition $\Phi'(X, Y)(\Delta X, \Delta Y) = 0$ is equivalent to $V' = 0$. Now, let $T \in S^n$ denote the directional derivative of the function $W$ at the point $(X, Y)$ along the direction $(\Delta X, \Delta Y)$. Then, $T$ satisfies the following equation:

\[
(29) \quad TXW + WXT + W\Delta X W = \Delta Y,
\]

obtained from the identity $WXW = Y$. Using Lemma 9(b) and the definition of $V$, it is easy to see that the condition $V' = V'(X, Y)(\Delta X, \Delta Y) = 0$ is equivalent to

\[
(30) \quad V' = UXW^{1/2} + W^{1/2}XU + W^{1/2}\Delta X W^{1/2} = 0,
\]

where $U$ is the unique solution of the Lyapunov equation

\[
(31) \quad UW^{1/2} + W^{1/2}U = T.
\]

Letting $\Delta X \equiv W^{1/2}\Delta X W^{1/2}$, $\Delta Y \equiv W^{-1/2}\Delta Y W^{-1/2}$, $\bar{U} \equiv W^{-1/2}U$ and $\bar{T} \equiv W^{-1/2}TW^{-1/2}$, the identities (29), (30) and (31) become

\[
\begin{align*}
(32) \quad \bar{U}V + V\bar{U} + \Delta X &= \Delta Y, \\
(33) \quad \bar{U}^TV + V\bar{U} + \Delta X &= 0, \\
(34) \quad \bar{U}^T + \bar{U} &= \bar{T}.
\end{align*}
\]

Subtracting (33) from (32) and using (34), we obtain that $\Delta Y = \bar{U}V + V\bar{U}^T$. This identity together with (33) and the fact that $V^2 = \Phi(X, Y)$, $tr(\bar{U}^2) = \|W^{-1/4}UW^{-1/4}\|_F^2 \geq 0$ and $\Delta X \cdot \Delta Y = \Delta X \cdot \Delta Y \geq 0$ imply that

\[
0 \geq \frac{1}{2} tr [(\bar{U}V + V\bar{U}^T)(\bar{U}^TV + V\bar{U})] = \|V^{1/2}\bar{U}V^{1/2}\|_F^2 + tr(V^2\bar{U}^2)
\]
\[ \geq \| V^{1/2} \tilde{U} V^{1/2} \|_F^2 + \nu \text{tr} (\tilde{U}^2) + \text{tr} [(V^2 - \nu I) \tilde{U}^2] \]
\[ \geq \| V^{1/2} \tilde{U} V^{1/2} \|_F^2 - \| V^2 - \nu I \|_F \| \tilde{U} \|_F \geq \lambda_{\min}(V^2) \| \tilde{U} \|_F^2 - \| V^2 - \nu I \|_F \| \tilde{U} \|_F^2 \]
\[ \geq (1 - \gamma) \nu \| \tilde{U} \|_F^2 - \gamma \nu \| \tilde{U} \|_F^2 = (1 - 2\gamma) \nu \| \tilde{U} \|_F^2. \]

Since \( \gamma \in [0, 1/2] \), the last relation implies that \( \tilde{U} = 0 \). Hence, by (32)–(34), we conclude that \( \Delta X = \Delta Y = 0 \), or equivalently \( \Delta X = \Delta Y = 0 \). \( \square \)

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