LIMITING BEHAVIOR OF THE DERIVATIVES
OF CERTAIN TRAJECTORIES ASSOCIATED
WITH A MONOTONE HORIZONTAL LINEAR
COMPLEMENTARITY PROBLEM

R. D. C. MONTEIRO AND T. TSUCHIYA

Given a certain monotone horizontal linear complementarity problem (HLCP), we can naturally construct a family of systems of nonlinear equations parametrized by a parameter \( t \in (0, 1) \) with the property that, as \( t \) tends to 0, the corresponding system "converges" to the HLCP. Under reasonable conditions, it has been shown that each system of the family has a unique solution and that, as \( t \) tends to 0, these solutions converge to a specific solution of the HLCP. The main purpose of this paper is to study the asymptotic behavior of the derivative of the trajectory of solutions and therefore obtain information on the way the trajectory approaches the solution set of the HLCP. We show that the trajectory of solutions converges to the solution set along a unique and well-characterized direction. Moreover, if the HLCP has a solution satisfying strict complementarity then the direction forms a definite angle with any face of the feasible region which contains the limit point; otherwise, the direction is tangent to some face of the feasible region.

1. Introduction. Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space and let \( \mathbb{R}^n_+ \) denote the nonnegative orthant and the positive orthant of \( \mathbb{R}^n \), respectively. Given a vector \( x \in \mathbb{R}^n \), we denote by \( \text{diag}(x) \) the \( n \times n \) diagonal matrix having the components of \( x \) on its diagonal. Given \( n \times n \)-matrices \( P \) and \( Q \) and a vector \( q \in \mathbb{R}^n \), the horizontal linear complementarity problem (HLCP) is the problem of finding a vector \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \) such that

\[
\begin{align*}
(1a) \quad & Xy = 0, \\
(1b) \quad & Qy - Px - q = 0, \\
(1c) \quad & x \geq 0, \quad y \geq 0,
\end{align*}
\]

where \( X = \text{diag}(x) \). We note that in the presence of (1c), relation (1a) is equivalent to the condition \( x^Ty = 0 \). Observe also that when \( Q = I \), the identity matrix, (1) becomes the linear complementarity problem (LCP). In subsequent discussion, we say that \( (x, y) \) is a feasible point if it satisfies relations (1b) and (1c) and a solution only if all three conditions in (1) hold. We denote the set of all solutions of (1) by \( \mathcal{S} \).

Throughout this paper, we assume that (1) satisfies the following two assumptions.

**Assumption 1 (Monotonicity of (1)).** For every \( u \in \mathbb{R}^n \) and \( v \in \mathbb{R}^n \) such that \( Qv - Pu = 0 \), there holds \( u^Tv \geq 0 \).

**Assumption 2 (Existence of a Solution).** The set \( \mathcal{S} \) is nonempty.

We note that under Assumption 1, a necessary and sufficient condition for Assumption 2 to hold is that (1) have a feasible point. This equivalence is well known.
for monotone LCPs and holds for monotone HLCPS due to the equivalence that exists between both problems (see Sznajder and Gowda 1995 or §2 of this paper). Assumption 1 also implies that \( \mathcal{P} \) is a face of the polyhedron determined by \((1b)\) and \((1c)\). We define index sets associated with \( \mathcal{P} \) as follows:

\[
B = \{ j : x_j > 0 \text{ for some } (x, y) \in \mathcal{P} \},
\]

\[
N = \{ j : y_j > 0 \text{ for some } (x, y) \in \mathcal{P} \},
\]

\[
J = \{ j : x_j = 0 \text{ and } y_j = 0 \text{ for all } (x, y) \in \mathcal{P} \}.
\]

It is well known that the sets \( B, N \) and \( J \) form a partition of the set of indices \( \{1, \ldots, n\} \), i.e., they are pairwise disjoint and satisfy \( B \cup N \cup J = \{1, \ldots, n\} \).

In this paper, we are interested in studying the limiting behavior of certain trajectories associated with \((1)\). Define

\[
\mathcal{B}_{++} = \{ b \in \mathbb{R}^n : \exists (x, y) \in \mathbb{R}_{++}^n \text{ such that } Qy - Px - q = b \}.
\]

Given \( a \in \mathbb{R}_{++}^n \) and \( b \in \mathbb{R}^n \), consider the following system of equations parametrized by the parameter \( t \in [0, 1] \):

\[
\begin{align*}
Xy &= ta, \\
Qy - Px - q &= tb,
\end{align*}
\]

where \( X = \text{diag}(x) \). Assuming that \( Q = I, \) \( b \in \mathcal{B}_{++} \), \( P \) is positive semi-definite and \((1)\) has a solution, Kojima, Mizuno and Noma (1990) have shown that system \((4)\) has a unique solution \((x(t), y(t))\) for every \( t \in (0, 1) \) and that the limit of \((x(t), y(t))\) as \( t \) tends to 0 exists and is equal to a solution of \((1)\). Using a recent result of Sznajder and Gowda (1995) which states that any monotone HLCPS can be reduced to a monotone LCP by relabeling some pairs of complementary variables \((x_+, y_+)\) as \((y_-, x_-)\), we show in §2 that similar results can be obtained in the context of the monotone HLCPS under Assumptions 1 and 2 and the condition that \( b \in \mathcal{B}_{++} \).

Starting from a pioneering work by McLinden (1980), several papers have appeared which deal with the continuous trajectories associated with either a monotone LCP or a monotone nonlinear complementarity problem (possibly involving a multi-valued function). These include Adler and Monteiro (1991), Monteiro (1991), G"uler (1993), Kojima, Megiddo and Mizuno (1993), Kojima, Megiddo and Noma (1991), Kojima, Megiddo, Noma and Yoshise (1991), Kojima, Mizuno and Noma (1989, 1990), and Megiddo (1989). The cited papers focus mainly on the existence of the trajectories and/or their limiting behavior as \( t \) tends to zero; none of them studies the asymptotic behavior of the trajectories of derivatives. The main goal of this paper is to study the asymptotic behavior of the trajectory of derivatives \((\dot{x}(t), \dot{y}(t))\) as \( t \) tends to 0. The asymptotic behavior of the trajectory of derivatives is closely related to the superlinear and/or quadratic convergence analysis of interior point algorithms Mehrotra (1993), Tsuchiya (1991), Tsuchiya and Monteiro (1992), Ye and Anstreicher (1993), Ye, G"uler, Tapia and Zhang (1993) and is a subject worth investigating in its own right.

We observe that the results of this paper could have been derived in the context of the monotone LCP: due to the result of Sznajder and Gowda (1995) mentioned above, they would automatically carry over to the context of monotone HLCPS and/or complementarity problems determined by affine maximal monotone operators (see G"uler 1995). Our main reason to use the context of the monotone HLCPS is
due to the fact that results for this problem are easily translated to the context of linear and convex quadratic programs (see §4); these problems can be easily cast in the format of a monotone HLCP without making major changes on their data.

This paper is organized as follows. In §2 we state the result of Sznajder and Gowda (1995) and show how it can be used to reduce a monotone HLCP to a monotone LCP. Using a result of Kojima, Mizuno and Noma (1990) and this reduction scheme, we show that the trajectory $(x(t), y(t))$ converges to a solution of (1) as $t$ tends to zero.

In §3 we study the behavior of the trajectory of derivatives $(\dot{x}(t), \dot{y}(t))$ as $t$ tends to 0. The conclusions obtained depend on two cases: whether the HLCP (1) has a solution satisfying strict complementarity or not. In both cases, the trajectory of solutions of (4) converge to the set of solutions along a unique and well-characterized direction. If (1) has a solution satisfying strict complementarity then the direction forms a definite angle with any of the faces of the feasible region of (1). If (1) does not have a solution satisfying strict complementarity then the direction is tangent to some face of the feasible region of (1). The behavior in the last case is due to the fact that the components of the solution $(x(t), y(t))$ that converge to 0 have two types of behavior: some of them tend to 0 as fast as $t$ while the others tend to 0 as fast as $t^{1/2}$.

In §4, we show that the primal affine scaling continuous trajectories for a linear program and its associated dual continuous trajectories form the set of solutions of a family of systems given by (4) after a suitable change of variables is introduced. We also show that the results derived in Adler and Monteiro (1991) about the limiting behavior of the affine scaling continuous trajectories and their derivatives can be obtained as a special case of the results of this paper. Other works dealing with the study of the limiting behavior of the affine scaling continuous trajectories and the corresponding trajectory of derivatives include Güler (1994), Megiddo and Shub (1989), Monteiro (1991) and Witzgall, Boggs and Dornich (1990).

The following notation is used throughout our paper. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m \times n}$. If $J$ is a finite index set then $|J|$ denotes its cardinality, that is the number of elements of $J$. The superscript $^T$ denotes transpose. The Euclidean norm, the 1-norm and the $\infty$-norm are denoted by $\| \cdot \|_2$, $\| \cdot \|_1$ and $\| \cdot \|_\infty$, respectively. For a matrix $E$, $\text{Range}(E)$ denotes the subspace generated by the columns of $E$ and $\text{Null}(E)$ denotes the subspace orthogonal to the rows of $E$. The $i$th component of a vector $w \in \mathbb{R}^n$ is denoted by $w_i$ for every $i = 1, \ldots, n$. Given an index set $\alpha \subseteq \{1, \ldots, n\}$ and a vector $w \in \mathbb{R}^n$, we denote the subvector $[w_i]_{i \in \alpha}$ by $w_\alpha$. If $E \in \mathbb{R}^{m \times n}$, $\alpha \subseteq \{1, \ldots, m\}$ and $\beta \subseteq \{1, \ldots, n\}$ then the $|\alpha| \times |\beta|$-submatrix $[E_{ij}]_{i \in \alpha, j \in \beta}$ is denoted by $E_{\alpha\beta}$. When $\alpha = \{1, \ldots, m\}$ we denote $E_{\alpha\beta}$ simply by $E_{\beta}$. Similarly, when $\beta = \{1, \ldots, n\}$ we denote $E_{\alpha\beta}$ simply by $E_{\alpha}$. Given two vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, the product $uv$ denotes the vector whose $i$th component is $u_i v_i$ for every $i = 1, \ldots, n$. For $p \in \mathbb{R}$, $u^p$ denotes the vector whose $i$th component is given by $u_i^p$. Other notation will be introduced as the need arises.

2. Limiting behavior of the trajectories. In this section, we state the main result regarding the limiting behavior of the trajectory of solutions determined by (4) as $t$ tends to 0. This result is obtained by using the fact that problem (1) can be reduced to a monotone LCP (see Sznajder and Gowda 1995) and then using a result of Kojima, Mizuno and Noma (1990) regarding the limiting behavior of trajectories associated with a monotone LCP.

We start by recalling the result of Sznajder and Gowda. First we need to introduce some terminology and notation. A complementarity basis for (1) is any $n \times n$ nonsingular submatrix $C$ of the matrix $[Q; -P]$ such that $C_{i} = Q_{i}$ or $C_{i} = -P_{i}$, for
every \( j = 1, \ldots, n \). The following result was shown in Sznajder and Gowda (1995) and in Güler (1995).

**Theorem 2.1.** Every monotone HLCP (1) has a complementary basis.

For the sake of completeness, we include a proof of this result in the Appendix. An algorithm to compute a complementary basis of (1) is given in Tütüncü and Todd (1993). Knowing a complementary basis \( C \) for (1), we can easily reduce (1) to a monotone LCP. Indeed, consider the following problem:

\[
\tilde{x}^T \tilde{y} = 0, \\
\tilde{y} - \tilde{M} \tilde{x} = \tilde{q},
\]

(5)

\[\tilde{x} \geq 0, \quad \tilde{y} \geq 0,\]

where \( \tilde{q} \equiv C^{-1} q, \tilde{M} \equiv C^{-1} D \) and \( D \in \mathbb{R}^{n \times n} \) is defined by

\[D_j = \begin{cases} P_j, & \text{if } C_j = Q_j; \\ -Q_j, & \text{if } C_j \neq Q_j. \end{cases}\]

Consider also the following family of systems of nonlinear equations parametrized by the parameter \( t \in [0, 1] \):

\[
\tilde{X} \tilde{y} = t a, \\
\tilde{y} - \tilde{M} \tilde{x} - \tilde{q} = t b,
\]

(6)

where \( \tilde{X} \equiv \text{diag}(\tilde{x}) \) and \( \tilde{b} \equiv C^{-1} b \).

Let \( \Pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) denote the permutation map defined by \( \Pi(x, y) = (\tilde{x}, \tilde{y}) \) where \((\tilde{x}, \tilde{y})\) is given by

\[
(\tilde{x}_j, \tilde{y}_j) = \begin{cases} (x_j, y_j), & \text{if } C_j = Q_j, \\ (y_j, x_j), & \text{if } C_j \neq Q_j. \end{cases}
\]

(7)

The following proposition can be easily proved.

**Proposition 2.2.** Let \((x, y) \in \mathbb{R}^{2n}\) be given and let \((\tilde{x}, \tilde{y}) = \Pi(x, y)\). Then, \(Xy = \tilde{X} \tilde{y}\) and \(C(\tilde{y} - \tilde{M} \tilde{x} - \tilde{q} - t \tilde{b}) = Qy - Px - q - t b \) for all \( t \in \mathbb{R} \). In particular, \((x, y)\) is a solution of (1) if and only if \((\tilde{x}, \tilde{y})\) is a solution of (5); also, \((x, y)\) is a solution of (4) if and only if \((\tilde{x}, \tilde{y})\) is a solution of (6).

It has been proved in Kojima, Megiddo and Noma (1991) and Kojima, Mizuno and Noma (1990) that if (5) has a solution and the set \( \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^{2n}_+ | \tilde{y} - \tilde{M} \tilde{x} - \tilde{q} = b \} \) is nonempty then system (6) has a unique solution \((\tilde{x}(t), \tilde{y}(t))\) for every \( t \in (0, 1]\). Moreover, under the same two conditions above, Theorem 4.1 of Kojima, Mizuno and Noma (1990) guarantees that the trajectory \( t \to (\tilde{x}(t), \tilde{y}(t)) \) converges to the solution \((\tilde{x}^*, \tilde{y}^*)\) of (5) satisfying

\[
(\tilde{x}^*, \tilde{y}^*) = \text{argmax} \sum_{j \in \tilde{b}} a_j \log \tilde{x}_j + \sum_{j \in N} a_j \log \tilde{y}_j - \sum_{j \in \tilde{b}} \tilde{b}_j x_j
\]

subject to \((\tilde{x}, \tilde{y}) \in \tilde{\mathcal{J}}\),

\[
(8)
\]
where $\mathcal{S}$ denotes the set of solutions of (5) and $\bar{B}$ and $\bar{N}$ denote the index sets $B$ and $N$ with respect to (5) (cf. (2)). Using these results we can prove the following theorem with respect to the HLCP (1).

**Theorem 2.3.** Assume that $(a, b) \in \mathbb{R}_+^n \times \mathcal{B}_+$ and let $(u^0, v^0) \in \mathbb{R}^{2n}$ be an arbitrary point satisfying

$$Qv^0 - Pu^0 = b.$$  

Then the following statements hold:

(a) For each $t \in (0, 1]$, system (4) has a unique solution $(x(t), y(t))$;

(b) $\lim_{t \to 0} (x(t), y(t))$ exists and is equal to the unique solution $(\bar{x}^*, \bar{y}^*)$ of (1) satisfying

$$\begin{align*}
(x^*, y^*) &= \arg\max \sum_{j \in B} a_j \log x_j + \sum_{j \in N} a_j \log y_j - \sum_{j \in B} v_j^0 x_j - \sum_{j \in N} u_j^0 y_j \\
&\text{subject to } (x, y) \in \mathcal{S}.
\end{align*}$$

**Proof.** Using the fact that $b \in \mathcal{B}_+$, Assumption 2 and Proposition 2.2, we conclude that (5) is feasible and that the set $\{(\bar{x}, \bar{y}) \in \mathbb{R}^{2n}_+ | \bar{y} - M\bar{x} - \bar{q} = \bar{b}\}$ is nonempty. Hence, in view of the discussion preceding Theorem 2.3, we conclude that (6) has a unique solution $(\bar{x}(t), \bar{y}(t))$ for every $t \in (0, 1]$. Proposition 2.2 then implies that (4) also has a unique solution $(x(t), y(t))$ for every $t \in (0, 1]$ and hence (a) follows.

We next show (b). From the discussion preceding Theorem 2.3, we have $\lim_{t \to 0} (\bar{x}(t), \bar{y}(t)) = (\bar{x}^*, \bar{y}^*)$ where $(\bar{x}^*, \bar{y}^*)$ is given by (8). Let $(\bar{u}^0, \bar{v}^0) = \Pi(u^0, v^0)$ where $\Pi$ is the permutation map defined in (7). It is easy to see that

$$\bar{b}^0 - \bar{M}\bar{u}^0 = \bar{b}.$$  

We claim that

$$\begin{align*}
(\bar{x}^*, \bar{y}^*) &= \arg\max \sum_{j \in B} a_j \log \bar{x}_j + \sum_{j \in N} a_j \log \bar{y}_j - \sum_{j \in B} \bar{v}_j^0 \bar{x}_j - \sum_{j \in N} \bar{u}_j^0 \bar{y}_j \\
&\text{subject to } (\bar{x}, \bar{y}) \in \mathcal{S}.
\end{align*}$$

Indeed, it is enough to show that the objective functions of (8) and (12) differ by a constant term over the set $\mathcal{S}$. That is, we have to show that $\sum_{j \in B} \bar{v}_j^0 \bar{x}_j + \sum_{j \in N} \bar{u}_j^0 \bar{y}_j$ and $\sum_{j \in \bar{B}} \bar{b}_j \bar{x}_j$ differ by a constant term over the set $\mathcal{S}$, or equivalently, that

$$\begin{align*}
\bar{b}^T(\bar{x} - \bar{x}^*) &= (\bar{v}^0)^T(\bar{x} - \bar{x}^*) + (\bar{u}^0)^T(\bar{y} - \bar{y}^*) \\
\forall(\bar{x}, \bar{y}) \in \mathcal{S}, \forall(\bar{x}', \bar{y}') \in \mathcal{S}.
\end{align*}$$

First note that if $(\bar{x}, \bar{y}) \in \mathcal{S}$ and $(\bar{x}', \bar{y}') \in \mathcal{S}$ then

$$0 \leq (\bar{x} - \bar{x}')^T(\bar{M} + \bar{M}^T)(\bar{x} - \bar{x}') = 2(\bar{x} - \bar{x}')^T(\bar{y} - \bar{y}')$$

$$= 2\left[\bar{y}^T\bar{y} - \bar{y}^T\bar{x}' - \bar{x}^T\bar{y}' + (\bar{x}')^T\bar{y}'\right] = -\bar{y}^T\bar{x}' - \bar{x}^T\bar{y}'.$$
Hence, we obtain \( \tilde{M}(\tilde{x} - \tilde{x}') = -\tilde{M}^T(\tilde{x} - \tilde{x}') \). Using this relation and (11), we obtain for every \((\tilde{x}, \tilde{y}) \in \tilde{\mathcal{S}}\) and \((\tilde{x}', \tilde{y}') \in \tilde{\mathcal{S}}\) that

\[
\tilde{b}^T(\tilde{x} - \tilde{x}') = (\tilde{v}^0 - \tilde{M}\tilde{u}^0)^T(\tilde{x} - \tilde{x}') = (\tilde{v}^0)^T(\tilde{x} - \tilde{x}') - (\tilde{u}^0)^T\tilde{M}^T(\tilde{x} - \tilde{x}')
\]

\[
= (\tilde{v}^0)^T(\tilde{x} - \tilde{x}') + (\tilde{u}^0)^T(\tilde{y} - \tilde{y}'),
\]

which yields (13). Hence, the claim follows. From the way (12) is constructed, it is easy to see that \((\tilde{x}^*, \tilde{y}^*) = \Pi(x^*, y^*)\). Since \(\lim_{t \to 0} (\tilde{x}(t), \tilde{y}(t)) = (\tilde{x}^*, \tilde{y}^*)\), \((\tilde{x}(t), \tilde{y}(t)) = \Pi(x(t), y(t))\) for every \(t \in (0, 1]\) and \((\tilde{x}^*, \tilde{y}^*) = \Pi(x^*, y^*)\), we conclude that \(\lim_{t \to 0} (x(t), y(t)) = (x^*, y^*)\). □

3. Limiting behavior of the trajectories of derivatives. In this section we analyze the limiting behavior of the trajectory of derivatives \((\dot{x}(t), \dot{y}(t))\) as \(t\) goes to 0. First, we define a trajectory of estimate vectors and in Theorem 3.1 we show that this trajectory converges to the solution \((x^*, y^*)\) of (1) defined in (15). As an immediate consequence of this result, we obtain in Corollary 3.6 a characterization of the limits of the subvectors \(\dot{x}_N(t)\) and \(\dot{y}_D(t)\) as \(t\) tends to 0. We note that this corollary holds regardless of whether (1) has a solution satisfying strict complementarity or not. We then proceed to investigate the behavior of the other components of the vectors \(\dot{x}(t)\) and \(\dot{y}(t)\). Two cases are considered: (a) \(J = \emptyset\); that is, (1) has a solution satisfying strict complementarity and; (b) \(J \neq \emptyset\); that is, (1) does not have a solution satisfying strict complementarity. The main result for the first case is given in Theorem 3.8 while the main result for the second case is given in Theorem 3.16.

We start by introducing a vector of estimates which will play an important role in the analysis of this section. We define the estimate vector (or simply, estimate) \((x^E(t), y^E(t)) \in \mathbb{R}^{2n}\) as

\[
\begin{align*}
(14a) \quad x^E(t) &\equiv x(t) - \tilde{t}(t), \\
(14b) \quad y^E(t) &\equiv y(t) - \tilde{b}(t).
\end{align*}
\]

for all \(t \in (0, 1]\). Observe that \((x^E(t), y^E(t))\) satisfies \(Qy^E(t) - Px^E(t) = q, \forall t \in (0, 1]\). We can interpret the estimate \((x^E(t), y^E(t))\) as a first order approximation of the curve \(s \in (0, 1] \to (x(s), y(s))\) at \(s = t\) to obtain an estimate of \((x(0), y(0)) = \lim_{t \to 0} (x(t), y(t))\).

One of the main results of this section is stated next.

**Theorem 3.1.** \(\lim_{t \to 0} (x^E(t), y^E(t)) = (x^*, y^*)\) where \((x^*, y^*)\) is the solution of (1) which satisfies (10).

The proof of Theorem 3.1 will be given later after we state and prove several preliminary results. The following simple result provides some relations involving the estimate \((x^E(t), y^E(t))\).

**Lemma 3.2.** The following relations hold:

\[
(15) \quad (a^{-1}x(t)^2y^E(t), a^{-1}y(t)^2x^E(t)) = t^2(\dot{x}(t), \dot{y}(t)), \quad \forall t \in (0, 1],
\]

\[
(16) \quad (ay(t)^2y^E(t), ax(t)^2x^E(t)) = (\dot{x}(t), \dot{y}(t)), \quad \forall t \in (0, 1].
\]
PROOF. We know that

\[ y(t) \dot{x}(t) + x(t) \dot{y}(t) = a, \quad \forall t \in (0, 1]. \]

This relation and the fact that \( y(t) = t x(t)^{-1} \) for all \( t \in (0, 1] \) imply

\[
\begin{align*}
\dot{y}(t) &= t \left[ x(t)^{-1} a - x(t)^{-1} y(t) \dot{x}(t) \right] \\
&= y(t) - t x(t)^{-1} (t x(t)^{-1}) \dot{x}(t) \\
&= y(t) - t^2 x(t)^{-2} \dot{x}(t),
\end{align*}
\]

which implies \( y^E(t) = t^2 ax(t)^{-2} \dot{x}(t) \). Similarly, we can show that \( x^E(t) = t^2 ay(t)^{-2} \dot{y}(t) \). Hence (15) follows. Relation (16) follows immediately from (15) and the relation \( x(t)y(t) = ta, t \in (0, 1] \). □

By Theorem 2.1, we know that the matrix \([-P, Q] \in \mathbb{R}^{n \times n} \) has full rank. Let \( U \in \mathbb{R}^{n \times n} \) and \( V \in \mathbb{R}^{n \times n} \) be such that the columns of the matrix

\[
\begin{bmatrix}
U \\
V
\end{bmatrix}
\]

form a basis for the null space of the matrix \([-P, Q]\).

In the following, if \( E_{\alpha\beta} \) is a submatrix of a matrix \( E \) then the notation \( E_{\alpha\beta}^T \) denotes the matrix \( (E_{\alpha\beta})^T \). The following lemma gives some properties of the matrices \( U \) and \( V \) and is a generalization of Lemma 3.4 of Ye and Ansteicher (1993).

**Lemma 3.3.** Consider the matrices \( U \) and \( V \) as above. Then the following statements hold:

(a) \( V^T U \) is positive semi-definite;

(b) If \( \alpha \subseteq \{1, \ldots, n\} \) and \( \beta \subseteq \{1, \ldots, n\} \) are two disjoint index sets then

\[
\text{Range} \begin{bmatrix}
V^T_{\alpha} \\
U^T_{\beta}
\end{bmatrix} \subseteq \text{Range} \begin{bmatrix}
-P^T_{\alpha} \\
Q^T_{\beta}
\end{bmatrix}.
\]

**Proof.** We first prove (a). Let \( w \in \mathbb{R}^n \) be given. By the definition of \( U \) and \( V \), we know that

\[
\begin{bmatrix}
U \\
V
\end{bmatrix} w \in \text{Null}[-P, Q],
\]

or equivalently,

\[
QVw - PUw = 0.
\]

Due to monotonicity, this implies that \( w^T (V^T U)w = (Vw)^T (Uw) \geq 0 \). Since \( w \in \mathbb{R}^n \) is arbitrary, (a) follows. We next show (b). Without loss of generality, we may assume that \( \alpha \cup \beta = \{1, \ldots, n\} \) since if (18) holds for such pair \( (\alpha, \beta) \) then it would also hold for any other pair \( (\alpha', \beta') \) such that \( \alpha' \subseteq \alpha \) and \( \beta' \subseteq \beta \). By the fundamental theorem of linear algebra, (18) is equivalent to the condition that

\[
\text{Null} \begin{bmatrix}
V^T_{\alpha} \\
U^T_{\beta}
\end{bmatrix} \supseteq \text{Null}[-P_{\alpha}, Q_{\beta}].
\]
Hence, it is sufficient to show this last relation. Indeed, let

\[(22) \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \text{Null}[-P_\alpha, Q_\beta] \]

be given. By rearranging the columns of $P$ and $Q$ if necessary, we may assume that $\alpha = \{1, \ldots, l\}$ and $\beta = \{l + 1, \ldots, n\}$, where $l \in \{1, \ldots, n\}$. Using (22), it follows that

\[(23) \quad \begin{pmatrix} u \\ 0 \\ 0 \\ v \end{pmatrix} \in \text{Null}[-P, Q] = \text{Range} \left[ \begin{bmatrix} U \\ V \end{bmatrix} \right]. \]

Hence, there exists $w \in \mathbb{R}^n$ such that

\[(24a) \quad u = U_\alpha w, \]

\[(24b) \quad 0 = U_\beta w, \]

\[(24c) \quad 0 = V_\alpha w, \]

\[(24d) \quad v = V_\beta w. \]

Let $H = U_\alpha V_\alpha + V_\beta^T U_\beta$. Observe that $H + H^T = U^T V + V^T U \geq 0$, due to (a). By relations (24b) and (24c), we have $Hw = 0$. Hence, $w^T(H + H^T)w = 0$ which implies that $(H + H^T)w = 0$ since $H + H^T \geq 0$. Hence, $H^Tw = 0$. Using the definition of $H$ and relations (24a) and (24d), we obtain

\[(25) \quad V_\alpha^T u + U_\beta^T v = V_\alpha^T U_\alpha w + U_\beta^T V_\beta w = H^T w = 0, \]

that is,

\[(26) \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \text{Null} \left[ V_\alpha^T, U_\beta^T \right]. \]

We have thus shown (21) and the lemma follows. $\square$

**Lemma 3.4.** Let $(u^0, v^0)$ be any vector satisfying (9). Then, the vector $(x_B^E(t), y_N^E(t))$ is the unique optimal solution of the following convex quadratic program

\[(27) \quad \text{minimize}_{(x_B, y_N)}\ a_B^{1/2} x_B(t)^{-1} \tilde{x}_B + \frac{1}{2} a_N^{1/2} y_N(t)^{-1} \tilde{y}_N - (v_B^0)^T \tilde{x}_B - (u_N^0)^T \tilde{y}_N \]

subject to \( Q_N \tilde{y}_N - P_B \tilde{x}_B = q(t), \)

where

\[(28) \quad q(t) = q + P_J x_B^E(t) + P_N x_N^E(t) - Q_B y_B^E(t) - Q_J y_J^E(t). \]
PROOF. The vector \((x_E^E(t), y_E^E(t))\) is clearly feasible for (27). To show that \((x_E^E(t), y_E^E(t))\) is an optimal solution of (27), it is sufficient to prove that

\[
\begin{pmatrix}
a_B x_B(t)^{-2} x_B^E(t) - v_B^0 \\
a_N y_N(t)^{-2} y_N^E(t) - u_N^0
\end{pmatrix} \in \text{Range} \begin{bmatrix} -P_B^T \\ Q_N^T \end{bmatrix}.
\]

Indeed, for every \(t \in (0, 1]\) we have

\[
Q \dot{y}(t) - P \dot{x}(t) = b,
\]

which together with (9) imply that

\[
\begin{pmatrix}
\dot{x}(t) - u^0 \\
\dot{y}(t) - v^0
\end{pmatrix} \in \text{Null}[-P, Q] = \text{Range} \begin{bmatrix} U \\ V \end{bmatrix}.
\]

Using Lemma 3.2, Lemma 3.3 and relation (31), we obtain

\[
\begin{pmatrix}
a_B x_B(t)^{-2} x_B^E(t) - v_B^0 \\
a_N y_N(t)^{-2} y_N^E(t) - u_N^0
\end{pmatrix} = \begin{pmatrix}
\dot{y}_B(t) - v_B^0 \\
\dot{y}_N(t) - u_N^0
\end{pmatrix} \in \text{Range} \begin{bmatrix} V_B \\ U_N \end{bmatrix} \subseteq \text{Range} \begin{bmatrix} -P_B^T \\ Q_N^T \end{bmatrix}.
\]

Hence, (29) holds. \(\square\)

**Lemma 3.5.** The following statements hold:

(a) There exists a constant \(C > 0\) such that

\[
|\dot{x}_j(t)| \leq C \frac{x_j(t)}{t}, \quad \forall j = 1, \ldots, n,
\]

\[
|\dot{y}_j(t)| \leq C \frac{y_j(t)}{t}, \quad \forall j = 1, \ldots, n.
\]

(b) \(\lim_{t \to 0} x_{j \cup N}(t) = 0\) and \(\lim_{t \to 0} y_{j \cup N}(t) = 0\).

**Proof.** Define \(h(t) = x(t)^{1/2} y(t)^{-1/2}\) for all \(t \in (0, 1]\). We will first show that there exists a constant \(C_1 \geq 0\) such that

\[
\left\| h(t)^{-1} \dot{x}(t) \right\| \leq C_1 / \sqrt{t},
\]

\[
\left\| h(t) \dot{y}(t) \right\| \leq C_1 / \sqrt{t},
\]

for all \(t \in (0, 1]\). Indeed, using Theorem 2.3 and the fact that \(x(t)y(t) = ta\) for all \(t \in (0, 1]\), it is easy to verify that there exists a constant \(C_2 \geq 0\) such that

\[
\max\left\{ \left\| h(t) \right\|, \left\| h(t)^{-1} \right\| \right\} \leq \frac{C_2}{\sqrt{t}}, \quad \forall t \in (0, 1].
\]

We also know that

\[
y(t) \dot{x}(t) + x(t) \dot{y}(t) = a,
\]

\[
Q \dot{y}(t) - P \dot{x}(t) = b.
\]
Using (9) and (39), we obtain
\[ Q(\dot{y}(t) - v^0) - P(\dot{x}(t) - u^0) = 0, \]
which in view of monotonicity implies
\[ (\dot{y}(t) - v^0)^T(\dot{x}(t) - u^0) \geq 0, \]
or equivalently
\[ (40) \quad \dot{x}(t)^T\dot{y}(t) \geq (v^0)^T\dot{x}(t) + (u^0)^T\dot{y}(t) - (u^0)^Tv^0. \]
Multiplying (38) by \([x(t)y(t)]^{-1/2} = t^{-1/2}a^{-1/2}\), we obtain
\[ h(t)^{-1}\dot{x}(t) + h(t)\dot{y}(t) = \frac{a^{1/2}}{\sqrt{t}}. \]
Squaring both sides of this relation and using (40), we obtain
\[ \frac{||a||_1}{t} = \left\| h(t)^{-1}\dot{x}(t) + h(t)\dot{y}(t) \right\|^2 \]
\[ = \left\| h(t)^{-1}\dot{x}(t) \right\|^2 + \left\| h(t)\dot{y}(t) \right\|^2 + 2\dot{x}(t)^T\dot{y}(t) \]
\[ \geq \left\| h(t)^{-1}\dot{x}(t) \right\|^2 + \left\| h(t)\dot{y}(t) \right\|^2 + 2\left[ (v^0)^T\dot{x}(t) + (u^0)^T\dot{y}(t) - (u^0)^Tv^0 \right] \]
\[ = \left\| h(t)^{-1}\dot{x}(t) + h(t)v^0 \right\|^2 + \left\| h(t)\dot{y}(t) + h(t)^{-1}u^0 \right\|^2 \]
\[ - \left\| h(t)v^0 + h(t)^{-1}u^0 \right\|^2. \]
Using this last expression, the triangle inequality and (37), we obtain
\[ \left\| h(t)^{-1}\dot{x}(t) \right\| \leq \left( \frac{||a||_1}{t} + \left\| h(t)v^0 + h(t)^{-1}u^0 \right\|^2 \right)^{1/2} + \left\| h(t)v^0 \right\| \]
\[ \leq \frac{||a||_1^{1/2}}{\sqrt{t}} + \left\| h(t)v^0 + h(t)^{-1}u^0 \right\| + \left\| h(t)v^0 \right\| \]
\[ \leq \frac{||a||_1^{1/2}}{\sqrt{t}} + 2\left( \left\| h(t)v^0 \right\| + \left\| h(t)^{-1}u^0 \right\| \right) \]
\[ \leq \frac{||a||_1^{1/2}}{\sqrt{t}} + \frac{2C_2(||u^0|| + ||v^0||)}{\sqrt{t}} \]
\[ \leq \frac{C_1}{\sqrt{t}}, \]
where \( C_1 = ||a||_1^{1/2} + 2C_2(||u^0|| + ||v^0||) \). Hence, relation (35) holds. Similarly, we can
show that (36) holds. Now, using (35) we obtain

\[ |\dot{x}_j(t)| \leq h_j(t) \|h(t)^{-1}\dot{x}(t)\| \]

\[ \leq \left( \frac{x_j(t)^{1/2}}{y_j(t)^{1/2}} \right) \frac{C_t}{\sqrt{t}} \]

\[ \leq \left( \frac{x_j(t)}{\sqrt{a_j}} \right) \frac{C_t}{\sqrt{t}} \]

\[ \leq \frac{C_1 x_j(t)}{t^{1/2} a_j}, \quad \forall j = 1, \ldots, n. \]

(41)

which shows that (33) holds. Similarly, using (36) we can show that (34) holds. We next show (b). Since \( \lim_{t \to 0} x_j(t) = 0 \), for all \( j \in J \cup N \), relation (41) implies that \( \lim_{t \to 0} \dot{x}_j(t) = 0 \) for all \( j \in J \cup N \). Hence,

\[ \lim_{t \to 0} x_j^E(t) = \lim_{t \to 0} \{x_j(t) - \dot{x}_j(t)\} = 0, \quad \forall j \in J \cup N. \]

Similarly, using (36) we can conclude that

\[ \lim_{t \to 0} y_j^E(t) = 0, \quad \forall j \in B \cup J, \]

and hence (b) follows. \( \square \)

We are now in a position to give the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. By Lemma 3.5(b), we have

\[ \lim_{t \to 0} x_{N \cup J}^E(t) = 0 \quad \text{and} \quad \lim_{t \to 0} y_{B \cup J}^E(t) = 0. \]

Since \( x_{N \cup J}^* = 0 \) and \( y_{B \cup B}^* = 0 \), we conclude that the statement of the theorem holds with respect to the \( x_{N \cup J}^* \) component and the \( y_{J \cup B}^* \) component. Now we concentrate our efforts on showing that \( \lim_{t \to 0} x_{B}^E(t) = x_B^* \) and \( \lim_{t \to 0} y_{N}^E(t) = y_N^* \).

By Theorem 2.3, we have \( \lim_{t \to 0} x_B(t) = x_B^* > 0 \) and \( \lim_{t \to 0} y_N(t) = y_N^* > 0 \). Moreover, (28) and (42) imply that the right-hand side \( q(t) \) of (27) satisfies \( \lim_{t \to 0} q(t) = q \). Using these two facts, it is easy to verify that the optimal solution \((x_B^E(t), y_N^E(t))\) of problem (27) converges to the optimal solution of the problem

\[ \min_{x_B, y_N} \left\{ \frac{1}{2} \|a_B^{1/2}(x_B^*)^{-1}\dot{x}_B\|^2 + \frac{1}{2} \|a_N^{1/2}(y_N^*)^{-1}\dot{y}_N\|^2 - (u_B^0)^T \dot{x}_B - (u_N^0)^T \dot{y}_N \right\} \]

subject to \( Q_N \dot{y}_N - P_B \dot{x}_B = q \).

Hence, the theorem follows if we show that \((x_B^*, y_N^*)\) is the optimal solution for (43). Indeed, since \((x_B^*, y_N^*)\) is the optimal solution for (10), it satisfies the KKT condition.
for (10), that is
\[
\begin{pmatrix}
    a_B(x_B^*)^{-1} - (v_B^0) \\
    a_N(y_N^*)^{-1} - (u_N^0)
\end{pmatrix} \in \text{Range} \begin{pmatrix} -P_B^T \\ Q_N^* \end{pmatrix}, \quad Q_N y_N^* - P_B x_B^* = q.
\]

On the other hand, the KKT condition for (43) is
\[
\begin{pmatrix}
    a_B(x_B^*)^{-2} \tilde{x}_B - (v_B^0) \\
    a_N(y_N^*)^{-2} \tilde{y}_N - (u_N^0)
\end{pmatrix} \in \text{Range} \begin{pmatrix} -P_B^T \\ Q_N^* \end{pmatrix}, \quad Q_N \tilde{y}_N - P_B \tilde{x}_B = q.
\]

Comparing these formulas, we see that \((\tilde{x}_B, \tilde{y}_N) = (x_B^*, y_N^*)\) satisfies (44) and therefore, it is the unique optimal solution for (43). \(\square\)

As an immediate consequence of Theorem 3.1, we obtain the following result.

COROLLARY 3.6. There hold
\[
\lim_{t \to 0} \dot{x}_N(t) = a_N(y_N^*)^{-1},
\]
\[
\lim_{t \to 0} \dot{y}_B(t) = a_B(x_B^*)^{-1}.
\]

PROOF. By relation (16), we know that
\[
\dot{x}_N(t) = a_N y_N(t)^{-2} y_N^E(t),
\]
\[
\dot{y}_B(t) = a_B x_B(t)^{-2} x_B^E(t).
\]

The two limits (45) and (46) follow immediately from these two relations and Theorems 2.3 and 3.1. \(\square\)

The limiting behavior of \((\dot{x}(t), \dot{y}(t))\) (or, a suitable scalar multiple of this vector) determines the direction of approach of the trajectory \((x(t), y(t))\) towards the solution set \(\mathcal{S}\). The direction of approach has different properties depending on whether \(J = \emptyset\) or not. In the next two results, we characterize the direction of approach of the trajectory \((x(t), y(t))\) when \(J = \emptyset\).

LEMMA 3.7. Assume that \(J = \emptyset\). Then, the vector \((\dot{x}_B(t), \dot{y}_N(t))\) is the unique optimal solution of the following convex quadratic program:
\[
\begin{align*}
\text{minimize}_{(x_B, y_N)} & \quad \frac{1}{2} \| a_B^{1/2} x_B(t)^{-1} \dot{x}_B \|^2 + \frac{1}{2} \| a_N^{1/2} y_N(t)^{-1} \dot{y}_N \|^2 \\
\text{subject to} & \quad Q_N \dot{y}_N - P_B \dot{x}_B = p(t),
\end{align*}
\]
where
\[
p(t) = b + P_N \dot{x}_N(t) - Q_B \dot{y}_B(t).
\]

PROOF. By relations (30), (48) and the fact that \(J = \emptyset\), it follows that \((\dot{x}_B(t), \dot{y}_N(t))\) is feasible for (47). To show that \((\dot{x}_B(t), \dot{y}_N(t))\) is an optimal solution of (47), it is
sufficient to show that \((\hat{x}_B(t), \hat{y}_N(t))\) satisfies the KKT condition for (47), that is
\[
\begin{pmatrix}
    a_B x_B(t)^{-2} \hat{x}_B(t) \\
    a_N y_N(t)^{-2} \hat{y}_N(t)
\end{pmatrix} \in \text{Range} \begin{bmatrix}
    -P_B^T \\
    Q_N^T
\end{bmatrix}.
\]
Indeed, we know that
\[
Qy^E(t) - Px^E(t) = q,
\]
\[
Qy^* - Px^* = q,
\]
where \((x^*, y^*)\) is the solution of (1) defined in Theorem 2.3. Hence, we obtain
\[
\begin{pmatrix}
    x^E(t) - x^* \\
    y^E(t) - y^*
\end{pmatrix} \in \text{Null}[-P, Q] = \text{Range} \begin{bmatrix}
    U \\
    V
\end{bmatrix}.
\]
Since \(x^*_N = 0\) and \(y^*_B = 0\), (50) implies that
\[
\begin{pmatrix}
    y^E_B(t) \\
    x^E_N(t)
\end{pmatrix} \in \text{Range} \begin{bmatrix}
    V_B \\
    U_N
\end{bmatrix}.
\]
Relations (15), (51) and Lemma 3.3 then imply that
\[
\begin{pmatrix}
    a_B x_B(t)^{-2} \hat{x}_B(t) \\
    a_N y_N(t)^{-2} \hat{y}_N(t)
\end{pmatrix} = \begin{pmatrix}
    y^E_B(t)/t^2 \\
    x^E_N(t)/t^2
\end{pmatrix} \in \text{Range} \begin{bmatrix}
    V_B \\
    U_N
\end{bmatrix} \subseteq \text{Range} \begin{bmatrix}
    -P_B^T \\
    Q_N^T
\end{bmatrix}. \quad \square
\]

**THEOREM 3.8.** Assume that \(J = \emptyset\). Then, \(\lim_{t \to 0} (\hat{x}_B(t), \hat{y}_N(t))\) exists and is equal to the unique optimal solution of the following convex quadratic program:

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \left\| a_B^{1/2} (x^*_B)^{-1} \hat{x}_B \right\|^2 + \frac{1}{2} \left\| a_N^{1/2} (y^*_N)^{-1} \hat{y}_N \right\|^2 \\
\text{subject to} & \quad Q_N \hat{y}_N - P_B \hat{x}_B = b + P_N a_N (y^*_N)^{-1} - Q_B a_B (y^*_B)^{-1}.
\end{align*}
\]

**PROOF.** By Theorem 2.3, we have \(\lim_{t \to 0} x_B(t) = x^*_B > 0\) and \(\lim_{t \to 0} y_N(t) = y^*_N > 0\). Moreover, (45), (46) and (48) imply that the right-hand side \(p(t)\) of problem (47) satisfies
\[
\lim_{t \to 0} p(t) = b + P_N a_N (y^*_N)^{-1} - Q_B a_B (y^*_B)^{-1}.
\]
Using these two facts, it is easy to verify that the optimal solution \((\hat{x}_B(t), \hat{y}_N(t))\) of (47) converges to the optimal solution of (52), as \(t\) tends to 0. \(\square\)

We can now summarize the results obtained for the case in which \(J = \emptyset\), i.e., (1) has a solution satisfying strict complementarity. In this case, Corollary 3.6 and Theorem 3.8 imply that \(\lim_{t \to 0} (\hat{x}(t), \hat{y}(t))\) exists and that \(\lim_{t \to 0} \hat{x}_N(t) > 0\) and \(\lim_{t \to 0} \hat{y}_B(t) > 0\). Hence, we conclude that the trajectory \((x(t), y(t))\) converges to the solution \((x^*, y^*)\) of (1) along a direction forming a definite angle with any face of the feasible region of (1) containing \((x^*, y^*)\).

Now we turn our attention to the case where \(J \neq \emptyset\), i.e., (1) does not have a solution satisfying strict complementarity. In this case, we will show that the components of \(x(t)\) and \(y(t)\) converge to zero slower than the components of \(x_N(t)\) and \(y_B(t)\).
More precisely, we will see that the components of $x_j(t)$ and $y_j(t)$ are equal to $\Theta(\sqrt{t})$ while the components of $x_N(t)$ and $y_N(t)$ are equal to $\Theta(t)$, due to Corollary 3.6. Hence, in this case the trajectory must approach some faces faster than others and will eventually become tangent to some of the faces of the feasible region of (1).

The following notation is used in the result below. Let $u^* \in \mathbb{R}^p$ denote the vector whose $i$th component is given by $u^*_i = \max(0, u_i)$. The lemma below is an immediate consequence of Theorem 2.7 of Mangasarian and Shiau (1986).

**Lemma 3.9.** Consider the LCP (5) and assume that its solution set $\mathcal{S}$ is nonempty. Then there exists a constant $C > 0$ with the property that for any $\bar{x} \in \mathbb{R}^n$,

$$
\min \left\{ \| \bar{x} - \bar{x}^* \| : (\bar{x}^*, \bar{M}x^* + \bar{q}) \in \mathcal{S} \right\} \leq C \left[ \phi(\bar{x}) + \phi(\bar{x})^{1/2} \right],
$$

where $\phi(\bar{x}) = [\bar{x}^T(\bar{M}x + \bar{q})]^+ + \|(-\bar{x})^+\| + \|(-\bar{M}x - \bar{q})^+\|.

In the proof of the result below, we say that a quantity depending on $t > 0$ is equal to $\mathcal{O}(t^p)$ with $p \geq 0$ if the quantity is less than or equal to $\alpha t^p$ for every $t > 0$ sufficiently small and some constant $\alpha > 0$.

**Lemma 3.10.** There hold

$$
limit \sup_{t \to 0} \frac{\|x_j(t)\|}{\sqrt{t}} < \infty, \quad limit \sup_{t \to 0} \frac{\|y_j(t)\|}{\sqrt{t}} < \infty.
$$

**Proof.** Consider the solution $(\bar{x}(t), \bar{y}(t))$ of system (6). Using the fact that $(\bar{x}(t), \bar{y}(t)) > 0$ and $((\bar{x}(t), \bar{y}(t)) | t \in (0, 1]$ is bounded, we obtain

$$
\left[ \bar{x}(t)^T(\bar{M}x(t) + \bar{q}) \right]^+ = \left[ \bar{x}(t)^T(\bar{y}(t) - \bar{b}) \right]^+ = t \left[ e^T a - \bar{b}^T \bar{x}(t) \right]^+ = \mathcal{O}(t),
$$

$$
\left\| (-\bar{M}x(t) - \bar{q})^+ \right\| \leq \left\| (\bar{b} - \bar{y}(t))^+ \right\| = \mathcal{O}(t),
$$

and $\|(-\bar{x}(t))^+\| = 0$. Using these relations, we obtain that $\phi(\bar{x}(t)) = \mathcal{O}(t)$, where $\phi(\cdot)$ is defined in Lemma 3.9. Hence, it follows from Lemma 3.9 that

$$
\min \left\{ \| \bar{x}(t) - \bar{x}^* \| : (\bar{x}^*, \bar{M}x^* + \bar{q}) \in \mathcal{S} \right\} = \mathcal{O}(t^{1/2}).
$$

This implies that

$$
\text{dist}((\bar{x}(t), \bar{y}(t)), \mathcal{S})
= \min \left\{ \|(\bar{x}(t) - \bar{x}^*, \bar{y}(t) - \bar{y}^*)\| : (\bar{x}^*, \bar{y}^*) \in \mathcal{S} \right\}
= \min \left\{ \|(\bar{x}(t) - \bar{x}^*, \bar{M}(\bar{x}(t) - \bar{x}^*) + \bar{t} \bar{b})\| : (\bar{x}^*, \bar{M}x^* + \bar{q}) \in \hat{\mathcal{S}} \right\}
\leq \min \left\{ (1 + \|\bar{M}\|)\|\bar{x}(t) - \bar{x}^*\| + \|\bar{t} \bar{b}\| : (\bar{x}^*, \bar{M}x^* + \bar{q}) \in \hat{\mathcal{S}} \right\} = \mathcal{O}(t^{1/2}).
$$
Since \( \text{dist}((x(t), y(t)), \mathcal{P}) = \text{dist}((x(t), y(t)), \mathcal{P}) \), we conclude that \( \text{dist}((x(t), y(t)), \mathcal{P}) = \mathcal{O}(t^{1/2}) \). Using the fact that \( x^*_j = 0 \) and \( y^*_j = 0 \) for every \((x^*, y^*) \in \mathcal{P} \), it follows that

\[
\max\{\|x_j(t)\|, \|y_j(t)\|\} \leq \text{dist}((x(t), y(t)), \mathcal{P}) = \mathcal{O}(t^{1/2}).
\]

Hence the result follows. □

**Theorem 3.11.** There holds

\[
\lim_{t \to 0} \left( \frac{x_j(t)}{\sqrt{t}} , \frac{y_j(t)}{\sqrt{t}} \right) = (l^*_j, l^*_j),
\]

where \((l^*_j, l^*_j)\) is the unique solution of the following system:

\[
\begin{align*}
(u, v) &\in \mathbb{R}^{|J|} \times \mathbb{R}^{|J|}, \\
Q_j v - P_j u &\in \text{Range}[P_B - Q_N],
\end{align*}
\]

**Proof.** We first show that system (55) has at most one solution. Indeed, assume by contradiction that \((u^1, v^1)\) and \((u^2, v^2)\) are two distinct solutions of (55). Using the second relation in (55), we obtain

\[
Q_j (v^1 - v^2) - P_j (u^1 - u^2) \in \text{Range}[P_B - Q_N],
\]

so that

\[
Q_j (v^1 - v^2) + Q_N \eta - P_j (u^1 - u^2) - P_B \xi = 0,
\]

for some \( \xi \in \mathbb{R}^{|B|} \) and \( \eta \in \mathbb{R}^{|N|} \). Therefore, the vectors

\[
u = \begin{pmatrix} u_B \\ u_j \\ u_N \end{pmatrix} = \begin{pmatrix} \xi \\ v^1 - v^2 \\ 0 \end{pmatrix}, \\
v = \begin{pmatrix} v_B \\ v_j \\ v_N \end{pmatrix} = \begin{pmatrix} 0 \\ v^1 - v^2 \\ \eta \end{pmatrix}
\]

satisfy \( Q v - P u = 0 \). Hence, due to monotonicity, we have

\[
0 \leq u^T v = (u^1 - u^2)^T (v^1 - v^2).
\]

This relation and the fact that \((u^1, v^1)\) and \((u^2, v^2)\) are distinct imply the existence of an index \( j \in J \) such that

\[
(u_j^1 - u_j^2)(v_j^1 - v_j^2) \geq 0
\]

and either \( u_j^1 \neq u_j^2 \) or \( v_j^1 \neq v_j^2 \). Assume without loss of generality that \( u_j^1 < u_j^2 \). From the first relation in (55) we have

\[
u_j^1 v_j^1 = u_j^2 v_j^2 = a_j.
\]
This relation and the fact that \( u^1_j < u^2_j \) then imply that \( v^1_j > v^2_j \). Hence, we obtain
\[
(u^1_j - u^2_j)(v^1_j - v^2_j) < 0,
\]
contradicting (60). Thus, (55) has at most one solution. We now show that (54) holds. Recall that due to Lemma 3.10, \((x_j(t)/\sqrt{t}, y_j(t)/\sqrt{t})\) is bounded as \( t \to 0 \). Hence it is sufficient to show that every accumulation point \((\hat{a}, \hat{b})\) of \((x_j(t)/\sqrt{t}, y_j(t)/\sqrt{t})\) as \( t \) tends to 0 is a solution of (55). Indeed, \((\hat{a}, \hat{b})\) must satisfy the first relation in (55) since
\[
\frac{x_j(t)}{\sqrt{t}} \frac{y_j(t)}{\sqrt{t}} = a_j, \quad \forall t \in (0, 1].
\]
We now show that \((\hat{a}, \hat{b})\) also satisfies the second relation in (55). Indeed, we have
\[
Q \left( \frac{y(t) - \bar{y}}{\sqrt{t}} \right) - P \left( \frac{x(t) - \bar{x}}{\sqrt{t}} \right) = \sqrt{t} b
\]
and using the fact that \( \bar{x}_j = 0, \bar{x}_N = 0, \bar{y}_B = 0 \) and \( \bar{y}_j = 0 \), we obtain
\[
Q \cdot \left( \frac{y_j(t)}{\sqrt{t}} \right) + Q \cdot \left( \frac{y_B(t)}{\sqrt{t}} \right) - P \cdot \left( \frac{x_j(t)}{\sqrt{t}} \right) - P \cdot \left( \frac{x_N(t)}{\sqrt{t}} \right) = \sqrt{t} b
\]
\[
\in \text{Range}[P_B - Q_N].
\]
Hence, as \( t \) tends to 0 in (61) we obtain
\[
Q \cdot \hat{a} - P \cdot \hat{b} \in \text{Range}[P_B - Q_N],
\]
since
\[
\lim_{t \to 0} \frac{x_N(t)}{\sqrt{t}} = 0, \quad \lim_{t \to 0} \frac{y_B(t)}{\sqrt{t}} = 0.
\]
For the purpose of simplifying the notation, define
\[
d^x(t) = \sqrt{t} \hat{x}(t),
\]
\[
d^y(t) = \sqrt{t} \hat{y}(t).
\]
We have the following result.

**Lemma 3.12.** There hold
\[
\limsup_{t \to 0} \| d^x_j(t) \| < \infty,
\]
\[
\limsup_{t \to 0} \| d^y_j(t) \| < \infty.
\]

**Proof.** Follows as an immediate consequence of Lemma 3.5(a) and Lemma 3.10. \( \square \)
THEOREM 3.13. There holds
\[
\lim_{t \to 0} (d^y_J(t), d^\nu_J(t)) = \frac{1}{2} (l^y_J, l^\nu_J),
\]
where \((l^y_J, l^\nu_J)\) is defined in Theorem 3.11.

PROOF. We first show that \((u, v) = (l^y_J/2, l^\nu_J/2)\) is the unique solution of the following system of linear equations

\begin{align}
(62a) & \quad l^y_J v + l^\nu_J u = a_J, \\
(62b) & \quad Q_Jv - P_Ju \in \text{Range}[ P_B - Q_N ].
\end{align}

Indeed, in view of the definition of \((l^y_J, l^\nu_J)\), it follows that \((l^y_J/2, l^\nu_J/2)\) is feasible to (62). It remains to show that (62) has a unique solution. Assume that \((u^1, v^1)\) and \((u^2, v^2)\) are two solutions of (62). Using similar arguments as in the proof of Theorem 3.11 (see relations (56)-(59)), we obtain

\[
(63) \quad (u^1 - u^2)^T (v^1 - v^2) \geq 0.
\]

Using (62a), we obtain
\[
l^y_J (v^1 - v^2) + l^\nu_J (u^1 - u^2) = 0,
\]
so that

\[
(64) \quad (u^1 - u^2)^T (v^1 - v^2) = -\|r(u^1 - u^2)\|^2 = -\|r^{-1}(v^1 - v^2)\|^2,
\]

where \(r = [(l^y_J)^{-1} l^\nu_J]^2 > 0\). Relations (63) and (64) then imply that
\[
r(u^1 - u^2) = r^{-1}(v^1 - v^2) = 0
\]
so that \(u^1 = u^2\) and \(v^1 = v^2\). Hence, \((l^y_J/2, l^\nu_J/2)\) is the unique solution of (62). To complete the proof of the theorem, it is sufficient to show that every accumulation point \((\tilde{u}, \tilde{v})\) of \((d^y_J(t), d^\nu_J(t))\), as \(t\) tends to 0, is a solution of (62). Indeed, from the relation
\[
x(t) \dot{y}(t) + y(t) \dot{x}(t) = a,
\]
it follows that
\[
\frac{x_J(t)}{\sqrt{t}} d^y_J(t) + \frac{y_J(t)}{\sqrt{t}} d^\nu_J(t) = a_J,
\]
and this relation implies that \(l^y_J \dot{v} + l^\nu_J \dot{u} = a_J\) so that \((\tilde{u}, \tilde{v})\) satisfies (62a). Using the relation \(Q \dot{y}(t) - P \dot{x}(t) = b\) and the fact that by Corollary 3.6,
\[
\lim_{t \to 0} \sqrt{t} \dot{x}_N(t) = 0,
\]
\[
\lim_{t \to 0} \sqrt{t} \dot{y}_B(t) = 0,
\]
we can easily show that \((l^y_J/2, l^\nu_J/2)\) also satisfies (62b). The result then follows. □
Lemma 3.14. The point \((d_{B}^{x}, d_{N}^{x}) = (d_{B}^{x}(t), d_{N}^{x}(t))\) is the unique solution of the following convex quadratic program:

\[
\begin{align*}
\text{minimize}_{(d_{B}^{x}, d_{N}^{x})} & \quad \frac{1}{2} \| a_{B}^{x/2} x_{B}(t) -1 d_{B}^{x} \|^2 + \frac{1}{2} \| a_{N}^{y/2} y_{N}(t) -1 d_{N}^{x} \|^2 \\
\text{subject to} & \quad Q_{N} d_{N}^{x} - P_{B} d_{B}^{x} = w(t),
\end{align*}
\]  
(65)

where

\[
w(t) = \sqrt{t} b + P_{J} d_{J}^{x}(t) + P_{N} d_{N}^{x}(t) - Q_{B} d_{B}^{x}(t) - Q_{J} d_{J}^{y}(t).
\]

Proof. Obviously \((d_{B}^{x}(t), d_{N}^{x}(t))\) is a feasible solution of (65). To show that \((d_{B}^{x}(t), d_{N}^{x}(t))\) is the optimal solution of (65), it is enough to check

\[
\begin{pmatrix} a_{B} x_{B}(t)^{-2} x_{B}(t) \\ a_{N} y_{N}(t)^{-2} y_{N}(t) \end{pmatrix} \in \text{Range} \begin{bmatrix} -P_{B}^{T} \\ Q_{N}^{T} \end{bmatrix},
\]

since \((d^{x}(t), d^{y}(t)) = (\sqrt{t} \dot{x}(t), \sqrt{t} \dot{y}(t)).\) By repeating exactly the same argument as in Lemma 3.7, we can conclude that (66) holds. \(\square\)

Theorem 3.15. Assume that \(J \neq \emptyset.\) Then, the limit \(\lim_{t \rightarrow 0} (d_{B}^{x}(t), d_{N}^{x}(t))\) exists and is equal to the unique solution of the following convex quadratic program:

\[
\begin{align*}
\text{minimize}_{(d_{B}^{x}, d_{N}^{x})} & \quad \frac{1}{2} \| a_{B}^{x/2} (x_{B}^{*})^{-1} d_{B}^{x} \|^2 + \frac{1}{2} \| a_{N}^{y/2} (y_{N}^{*})^{-1} d_{N}^{x} \|^2 \\
\text{subject to} & \quad Q_{N} d_{N}^{x} - P_{B} d_{B}^{x} = \frac{1}{2} P_{J} l_{J}^{x} - \frac{1}{2} Q_{J} l_{J}^{y},
\end{align*}
\]  
(67)

where \((l_{J}^{x}, l_{J}^{y})\) is defined in Theorem 3.11.

Proof. Due to Theorem 2.3, we have \(\lim_{t \rightarrow 0} x_{B}(t) = x_{B}^{*} > 0\) and \(\lim_{t \rightarrow 0} y_{N}(t) = y_{N}^{*} > 0.\) Furthermore, due to Corollary 3.6 and Theorem 3.13, it easily follows that the right-hand side \(w(t)\) of (65) satisfies

\[
\lim_{t \rightarrow 0} w(t) = \frac{1}{2} P_{J} l_{J}^{x} - \frac{1}{2} Q_{J} l_{J}^{y}.
\]

Using these two facts, it is now easy to verify that the optimal solution \((d_{B}^{x}(t), d_{N}^{x}(t))\) of (65) converges to the optimal solution of (67), as \(t\) tends to 0. \(\square\)

We can combine the results obtained in Corollary 3.6, Theorem 3.13 and Theorem 3.15 into the following result.

Theorem 3.16. Assume that \(J \neq \emptyset.\) Then, the limit \((d^{x}(0), d^{y}(0)) = \lim_{t \rightarrow 0} (d^{x}(t), d^{y}(t)) = \lim_{t \rightarrow 0} (\sqrt{t} \dot{x}(t), \sqrt{t} \dot{y}(t))\) exists and,

(a) \((d_{B}^{x}(0), d_{B}^{y}(0)) = (0, 0);\)

(b) \((d_{J}^{x}(0), d_{J}^{y}(0)) = (l_{J}^{x}/2, l_{J}^{y}/2)\) where \((l_{J}^{x}, l_{J}^{y})\) is the unique solution of system (55);

(c) \((d_{B}^{x}(0), d_{N}^{x}(0))\) is the unique optimal solution of problem (67).

We can now give an interpretation of Theorem 3.16 which treats the case in which \(J \neq \emptyset,\) i.e., (1) does not have a solution satisfying strict complementarity. If \(B \cup N \neq \emptyset\) then we conclude that the trajectory \((x(t), y(t))\) converges to the solution \((x^{*}, y^{*})\) of (1) along a direction which is tangent to any face determined by one of the equalities \(x_{j} = 0\) with \(j \in N\) or \(y_{j} = 0\) with \(j \in B.\) Otherwise, if \(B \cup N = \emptyset\) then the trajectory \((x(t), y(t))\) converges to the solution \((x^{*}, y^{*}) = (0, 0)\) forming a definite
angle with any face of the feasible region of (1). Note that in the last case \((0, 0)\) is the unique solution of (1).

4. **Affine scaling continuous trajectories.** In this section we show how the approach of §§2 and 3 can be used to analyze the continuous trajectories associated with the affine scaling algorithm for linear program introduced by Dikin (1967). Our approach is to relate the affine scaling continuous trajectories associated with a linear program with the set of solutions of system (4) for some appropriate matrices \(P\) and \(Q\) and vectors \(a, b\) and \(q\). By applying the results obtained in this paper, it is then possible to duplicate all the results for the affine scaling continuous trajectories obtained in Adler and Monteiro (1991) and Monteiro (1991) (see also Witzgall, Boggs and Domich 1990).

Let us consider the following standard form linear programming problem

\[
\begin{align*}
\text{minimize}_x & \quad g^T x \\
\text{subject to} & \quad Ex = f, \quad x \geq 0,
\end{align*}
\]

(68)

and its associated dual problem

\[
\begin{align*}
\text{maximize}_{(u, y)} & \quad f^T u \\
\text{subject to} & \quad E^T u + y = g, \quad y \geq 0,
\end{align*}
\]

(69)

where \(g, x \in \mathbb{R}^n, E \in \mathbb{R}^{m \times n}, f \in \mathbb{R}^m\), and \(\text{Rank}(E) = m\). It is well known that this pair of problems is equivalent to the problem of

\[
\begin{align*}
\text{minimize}_{x, (u, y)} & \quad x^T y \\
\text{subject to} & \quad Ex = f, \\
& \quad E^T u + y = g, \\
& \quad x \geq 0, \quad y \geq 0.
\end{align*}
\]

(70)

Let \(F \in \mathbb{R}^{(n-m) \times n}\) be a matrix whose rows form a basis for the null space of \(E\). Since \(\text{Range}(E^T) = \text{Null}(F)\), the above problem is easily seen to be equivalent to the following monotone HLCP.

\[
X y = 0,
\]

(71)

\[
Q y - Px - q = 0,
\]

\[
x \geq 0, \quad y \geq 0,
\]

where

\[
P = \begin{pmatrix} -E \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ F \end{pmatrix}, \quad q = \begin{pmatrix} f \\ Fg \end{pmatrix}.
\]

We assume in the subsequent discussion that (68) has an optimal solution and an interior feasible solution, that is a feasible solution \(x > 0\). Let \(\mathcal{F}_p\) and \(\mathcal{S}_p\) denote the set of feasible solutions and optimal solutions of (68), respectively. Let \(\mathcal{F}_d\) and \(\mathcal{S}_d\) denote the set of feasible solutions and optimal solutions of (69), respectively. Also, let \(\mathcal{S}\) denote the solution set of (71). Due to the strong duality theorem for linear
programming which guarantees the existence of primal and dual optimal solutions satisfying strict complementarity, we know that the index set $J$ associated with the solution set of (71) is empty. Moreover, we have

$$
\mathcal{S} = \{(x, y) \mid Qy - Px - q = 0, x_B \geq 0, y_N \geq 0, x_N = 0, y_B = 0\}
$$

$$
\mathcal{S}_p = \{x \mid x \in \mathcal{S}_p, x_N = 0\},
$$

$$
\mathcal{S}_d = \{(u, y) \mid (u, y) \in \mathcal{S}_p, y_B = 0\}.
$$

We next describe the affine scaling continuous trajectories associated with problem (68). Given an initial interior feasible solution $x^0$ of (68), a parametrization $x : (0, 1) \to \mathbb{R}^n$ of the affine scaling continuous trajectory passing through the point $x^0$, that is satisfying $x(1) = x^0$, can be found as follows. Let $(u^0, y^0) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfying $E^T u^0 + y^0 + g$ ($y_0$ does not need to be positive) be given. Define $q^0 = y^0 - (x^0)^{-1}$ and, for each $t \in (0, 1]$, consider the following nonlinear system of equations in the variables $(x, u, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$:

$$
y - tx^{-1} - tq^0 = 0, \quad x > 0,
$$

$$
Ex - f = 0,
$$

$$
E^T u + y - g = 0.
$$

(72)

It is shown in Adler and Monteiro (1991) that (72) has a unique solution $(x(t), u(t), y(t))$ for every $t \in (0, 1]$ and that $x(1) = x^0$. The path $x : (0, 1] \to \mathbb{R}^n$ is then a parametrization of the affine scaling continuous trajectory passing through $x^0$ and $(u(t), y(t))$ is its associated dual path passing through the point $(u^0, y^0)$. We can eliminate the variable $u$ from system (72) and rewrite it as follows:

$$
x(y - tq^0) = te,
$$

(73)

$$
Qy - Px - q = 0, \quad x > 0, \quad (y - tq^0) > 0,
$$

where $e$ denotes the vector of all ones. Introducing the following change of variables

$$
\tilde{x} = x, \quad \tilde{y} = y - tq^0,
$$

(74)

system (73) becomes

$$
\tilde{x}y = te,
$$

(75)

$$
Q\tilde{y} - P\tilde{x} - q = tb.
$$

$$
\tilde{x} > 0, \quad \tilde{y} > 0,
$$

where $b = -Qq^0$. Hence, we conclude that the trajectory $(\tilde{x}(t), \tilde{y}(t))$ is of the form studied in the previous sections and can therefore be analyzed with the aid of the results obtained in §§2 and 3. This approach would yield alternative proofs for the results obtained in Adler and Monteiro (1991) regarding the limiting behavior of the affine scaling trajectory $(x(t), y(t))$ and its associated derivative trajectory $(\dot{x}(t), \dot{y}(t))$. 
Finally, we point out that the formulas (74) and (75) strongly suggest that a natural extension of the primal affine scaling continuous trajectory is obtained by considering the following system with respect to \((x, y)\):
\[
(x - tp^0)(y - tq^0) = te,
\]
\[
Qy - Px - q = 0,
\]
\[
x - tp^0 > 0, \quad y - tq^0 > 0.
\]
Here the initial point \((x(1), y(1)) = (x^0, y^0)\) is not necessarily feasible, and we can adjust \(q^0\) and \(p^0\) so that they satisfy the first and third relations when \(t = 1\), and then trace the path to \(t = 0\). Since the initial solution \((x^0, y^0)\) is not necessarily feasible, a discrete algorithm corresponding to this continuous trajectory seems to provide a natural generalization of the feasible primal affine scaling algorithm. This topic certainly deserves further attention.

**Appendix.** In this appendix we give a proof of Theorem 2.1 which was communicated to us by one of the referees.

**Proof of Theorem 2.1.** We first show that \(P + Q\) is nonsingular. To this end, we show that \((P + Q)x = 0\) implies \(x = 0\), or equivalently, that \(Qy - Px = 0\) and \(x + y = 0\) imply \(x = y = 0\). Indeed, due to Assumption 1, we have \(0 \leq 4x^Ty = \|x + y\|^2 - \|x - y\|^2 = -\|x - y\|^2\), which implies that \(x = y = 0\) and hence that \(P + Q\) is nonsingular. Now, we have
\[
0 = \det(P + Q) = \sum \{\det(C) | C \in \mathbb{R}^{n \times n}, C_j \in \{P_j, Q_j\}, \forall j = 1, \ldots, n\}.
\]
This implies that there exists at least one matrix \(C \in \mathbb{R}^{n \times n}\) such that \(\det(C) \neq 0\) and \(C_j \in \{P_j, Q_j\}\) for every \(j = 1, \ldots, n\). Clearly, multiplying by \(-1\) the columns \(C_j\) satisfying \(C_j = P_j\) gives a complementary basis. 

**Acknowledgement.** During this research, the first author was supported by the National Science Foundation (NSF) under grant DDM-9109404 and by the Office of Naval Research (ONR) under grant N00014-93-1-0234. He thanks both NSF and ONR for the financial support received during the completion of this work. This work was done while the first author was a faculty member of the Systems and Industrial Engineering Department at the University of Arizona.

The work of the second author was based on research supported by the Overseas Research Scholars of the Ministry of Education, Science and Culture of Japan. Part of this research was carried out while the second author was visiting the Center for Research on Parallel Computation and the Department of Computational and Applied Mathematics of Rice University, Houston, Texas, USA. He thanks his host Professor J. E. Dennis and the colleagues there for providing the excellent research environment during his stay at Rice University.

The second author also thanks the first author for having arranged and supported a trip with his NSF grant to visit the University of Arizona where this research was initiated. He also thanks the University of Arizona for the congenial scientific atmosphere that it provided.

**References**


R. D. C. Monteiro: School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332-0205; e-mail: monteiro@isye.gatech.edu

T. Tsuchiya: The Institute of Statistical Mathematics, 4-6-7 Mirami-Azabu, Minato-Ku, Tokyo 106, Japan; e-mail: tsuchiya@sun312.ism.ac.jp