PROPERTIES OF AN INTERIOR-POINT MAPPING FOR MIXED COMPLEMENTARITY PROBLEMS

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Using a unified theory of local homeomorphic maps, we establish some basic properties of a fundamental mapping associated with the family of path-following interior-point methods for solving a mixed nonlinear complementarity problem.

1. Introduction. The existence of interior points and interior paths is an issue of central importance in the family of path-following interior-point algorithms for solving linear programs and complementarity problems. In the context of the monotone nonlinear complementarity and related problems, this issue has been treated rather extensively in several papers (e.g., see Güler 1993, 1995; Kojima et al. 1991, McLinden 1980). In this paper, we shall consider the same issue for a mixed nonlinear complementarity problem.

Let \( H: R_+^{2n} \times R^n \rightarrow R^{2n+m} \) be a given continuous mapping. The problem we shall study in this paper is to find a vector \( (x, y, z) \in R_+^{2n} \times R^n \) such that

\[
H(x, y, z) = 0, \quad x \perp y,
\]

where the notation \( \perp \) means "is perpendicular to." The standard nonlinear complementarity problem is clearly a special case of this problem in which \( m = 0 \) and \( H(x, y) = y - f(x) \) for some mapping \( f: R_+^{2n} \rightarrow R^n \). The horizontal linear complementarity problem, which has been considered in several recent papers dealing with the convergence of interior-point methods (see Bonnans and Gonzaga 1993, Güler 1995, Monteiro and Tsuchiya 1992, Tütüncü and Todd 1993, Zhang 1994), is another special case of (1) in which \( m = 0 \) and \( H(x, y) = Ax + By + a \) for some vector \( a \in R^n \) and matrices \( A, B \in R^{n \times n} \). The Karush-Kuhn-Tucker (KKT) system of a variational inequality (see Pang 1994) is an important example of a mixed complementarity problem of type (1) in which the dimension \( m \) of the free variable \( z \) is positive. This system is defined by a function \( H \) of the form: for \( (u, v, \lambda, \eta) \in R^N \times R_+^{2p} \times R^q \),

\[
H(u, v, \lambda, \eta) \equiv \begin{pmatrix}
F(u) + \sum_{i=1}^{p} \lambda_i \nabla g_i(u) + \sum_{j=1}^{q} \eta_j \nabla h_j(u) \\
v + g(u) \\
h(u)
\end{pmatrix},
\]

where \( u \in R^N \) is the primary variable of the variational inequality, \( v \) and \( \lambda \) are, respectively, the slack variable and multiplier of the constraint \( g(u) \leq 0 \), (thus, both \( v \) and \( \lambda \) are nonnegative, and they are complementary), \( \eta \) is the multiplier of the

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constraint \( h(u) = 0 \), and \( F \) is a mapping from \( R^N \) into itself. In the notation of (1), \( v \) and \( \lambda \) play the role of \( x \) and \( y \) respectively, and \( (u, \eta) \) plays the role of \( z \).

The goal of this paper is to establish some basic properties of the mapping \( G : R_+^{2n} \times R^m \to R_+^n \times R^{n+m} \) defined by

\[
G(x, y, z) = \begin{pmatrix} x \circ y \\ H(x, y, z) \end{pmatrix}, \quad \text{for} \ (x, y, z) \in R_+^{2n} \times R^m,
\]

where the notation \( \circ \) denotes the Hadamard product of two vectors; i.e. \( x \circ y \) is the vector whose \( i \)th component is equal to \( x_i y_i \) for all \( i = 1, \ldots, n \). Our motivation for studying this mapping \( G \) stems from the family of path-following interior-point methods for solving complementarity problems. In the cited references, the fundamental role of \( G \) has been examined in the contexts of the standard nonlinear complementarity problem and the horizontal linear complementarity problem. Briefly, the role of \( G \) in a path-following interior-point method for solving problem (1) is as follows. A typical method of this kind relies on the existence of a certain continuous "interior" trajectory which joins a given vector \( (x^0, y^0, s^0) \in R_+^{2n} \times R^m \) to a desired solution of (1); the numerical tracing of this trajectory is the focus of various computational schemes. Theoretically, since

\[
G(R_+^{2n} \times R^m) \subseteq R_+^n \times H(R_+^{2n} \times R^m),
\]

a natural question to ask is when equality will hold in this inclusion. Indeed, we will show that \( G \) maps \( R_+^{2n} \times R^m \) onto \( R_+^n \times H(R_+^{2n} \times R^m) \) homeomorphically under several special cases of the mapping \( H \). In one such case, we impose a monotonicity condition on the mapping \( H \) which generalizes the notion of monotonicity for the standard nonlinear complementarity problem or the horizontal linear complementarity problem. We will also show that the inclusion

\[
G(R_+^{2n} \times R^m) \supseteq R_+^n \times H(R_+^{2n} \times R^m)
\]

holds, and that in some cases, we actually have \( H(R_+^{2n} \times R^m) = R^{n+m} \). Clearly, the inclusion (4) implies that a solution of problem (1) exists whenever \( 0 \in H(R_+^{2n} \times R^m) \). In the context of the standard nonlinear complementarity problem or the horizontal linear complementarity problem, this last condition is equivalent to the existence of an interior feasible solution. We will also give conditions that guarantee the convexity of the set \( H(R_+^{2n} \times R^m) \) in the monotone case; in fact, the answer to this issue leads us to a second and stronger notion of monotonicity for the mapping \( H \).

The tools used in the analysis of the above issues are presented in §2 in a very general context in which a local homeomorphism \( F \) between two metric spaces is considered; conditions are then stated which guarantee that \( F \) is a (global) homeomorphism. Such conditions have been systematically studied in the literature (see for example Ambrosetti and Prodi 1993, Browder 1954, Lima 1977, Ortega and Rheinboldt 1970) and are based on the method of analytic continuation.

It is interesting to note that system (1) can be considered as a complementarity problem of the type considered in Güler (1993) and McLinden (1980). By defining the set-valued map \( \mathcal{A} : R_+^n \to R_+^n \) where for each \( x \in R_+^n \),

\[
\mathcal{A}(x) = \{ y \in R_+^n | H(x, y, z) = 0 \ \text{for some} \ z \in R^m \},
\]

problem (1) clearly becomes that of finding \( (x, y) \in R_+^{2n} \cap \text{Gr}(\mathcal{A}) \) such that \( x \perp y \), where \( \text{Gr}(\mathcal{A}) \) denotes the graph of \( \mathcal{A} \). The monotonicity condition on the mapping
$H$, which we shall introduce in §6, will imply that the resulting operator $\mathcal{A}$ is monotone, and more interestingly, that $\mathcal{A}$ is maximal with respect to $R^2_{++}$; see §6 for the definition of this concept and the precise result.

This point of view of problem (1) raises the question of whether the theory developed in Güler (1993) and McLinden (1980) is applicable to yield some of the results in this paper. In order for this to be the case, we need to demonstrate that the operator $\mathcal{A}$ is maximal monotone with respect to the entire space $R^2_{++}$, because this is the basic requirement of the theory in the references. It turns out that this demonstration is not a trivial matter; indeed, we are able to establish only a kind of restricted maximality of $\mathcal{A}$ (with respect to $R^2_{++}$). Although the theory in Güler (1993) and McLinden (1980) could probably be extended to operators which are maximal in a restricted sense, we should stress that the proof of the restricted maximality of the operator $\mathcal{A}$ is elaborate and is a consequence of the results developed in §2. Hence, the approach based on the consideration of the operator $\mathcal{A}$, although potentially feasible, would invariably involve the analysis in this paper. Moreover, our approach is direct and makes full use of the special structure of $\mathcal{A}$, namely as being defined by the mapping $H$.

2. A general theory. In this section, we state some basic results that form the framework upon which the entire paper is based. The main result (see Theorem 1) gives equivalent conditions for a local homeomorphism between two metric spaces to be a homeomorphism. Its main consequences are subsequently stated as several corollaries. The developments in this section are based on the theory of analytic continuation methods described for example in Chapter 5 of Ortega and Rheinboldt (1970) and Chapter 3 of Ambrosetti and Prodi (1993).

If $M$ and $N$ are two metric spaces, we denote the set of continuous functions from $M$ to $N$ by $C(M, N)$ and the set of homeomorphisms from $M$ onto $N$ by $\text{Hom}(M, N)$. For $F \in C(M, N)$, $D \subseteq M$, and $E \subseteq N$, we let $F(D) \equiv \{F(u)|u \in D\}$ and $F^{-1}(E) \equiv \{u \in M|F(u) \in E\}$.

The set $F^{-1}(\{v\})$ with $v \in N$ is simply denoted by $F^{-1}(v)$ and the number of elements in $F^{-1}(v)$ is denoted by $[v]$; in general, $[v]$ is either 0, a positive integer, or infinite. Given $F \in C(M, N)$, $D \subseteq M$, and $E \subseteq N$ such that $F(D) \subseteq E$, the restricted mapping $G: D \rightarrow E$ defined by $G(u) = F(u)$ for all $u \in D$ is denoted by $F|_{(D, E)}$, if $E = N$ then we write this $G$ simply as $F|_{D}$. We will also refer to $F|_{(D, E)}$ as “$F$ restricted to the pair $(D, E)$.”’’ and to $F|_{D}$ as “$F$ restricted to $D$.” The closure of a subset $E$ of a metric space will be denoted by $\text{cl} E$. Any continuous function from a closed interval of the real line $R$ into a metric space will be called a path. We say that $(V_1, V_2)$ forms a partition of the set $V$ if $V_1 \subseteq V$, $V_2 \subseteq V$, $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$. A metric space $M$ is said to be connected if there exists no partition $(\mathcal{E}_1, \mathcal{E}_2)$ for which both $\mathcal{E}_1$ and $\mathcal{E}_2$ are nonempty and open. A metric space $M$ is said to be path-connected if for any two points $u_0, u_1 \in M$, there exists a path $p: [0, 1] \rightarrow M$ such that $p(0) = u_0$ and $p(1) = u_1$. It is well known that any path-connected metric space is connected; the converse however does not always hold. A metric space $M$ is said to be simply-connected if it is path-connected and for any path $p: [0, 1] \rightarrow M$ with $p(0) = p(1) = u$, there exists a continuous mapping $\alpha: [0, 1] \times [0, 1] \rightarrow M$ such that $\alpha(s, 0) = p(s)$ and $\alpha(s, 1) = u$ for all $s \in [0, 1]$ and $\alpha(0, t) = \alpha(1, t) = u$ for all $t \in [0, 1]$. If $x$ and $y$ are two points of a vector space, we denote the line segment joining them by $[x, y] \equiv \{(1 - t)x + ty|t \in [0, 1]\}$. We say that a path $p: [a, b] \rightarrow V$ in a vector space $V$ is affine if it is of the form $p(t) = x + tw$ for some $x, w \in V$.

In the remainder of this section, we will assume that $M$ and $N$ are two metric spaces and that $F \in C(M, N)$.

DEFINITION 1. Given a path $q: [a, b] \rightarrow N$ and a point $(t_0, u_0) \in [a, b] \times M$ such that $F(u_0) = q(t_0)$, we say that $F$ is invertible along $q$ through $(t_0, u_0)$ if there exists...
a unique path \( p: [a, b] \to M \) such that \( p(t_0) = u_0 \) and \( F(p(t)) = q(t) \) for all \( t \in [a, b] \). In this case, we say that \( p \) inverts \( F \) along \( q \) through the point \((t_0, u_0)\). We say that \( F \) is invertible along \( q \) if \( F \) is invertible along \( q \) through any \((t_0, u_0) \in [a, b] \times M \) such that \( F(t_0) = q(t_0) \).

A referee commented that the terminology "path inverting" is unusual and that the usual terminology is "path lifting." The former terminology is used in the book Ambrosetti and Prodi (1993) which we follow.

**Definition 2.** The mapping \( F \in C(M, N) \) is said to be proper with respect to the set \( E \subseteq N \) if \( F^{-1}(K) \subseteq M \) is compact for every compact set \( K \subseteq E \). If \( F \) is proper with respect to \( N \), we will simply say that \( F \) is proper.

We next state the main result of this section. Its proof can be found in the references Ambrosetti and Prodi (1993), Browder (1954), Lima (1977), Ortega and Rheinboldt (1970); Chapter 4 of the Lima monograph (1977) contains the proof of most of its statements but, unfortunately, it is written in Portuguese and is not accessible to most of the readers. For the sake of completeness, we give a direct proof of Theorem 1 in the Appendix using whenever possible the results that we found in the other references.

**Theorem 1.** Assume that \( F: M \to N \) is a local homeomorphism. If \( N \) is connected then the following two statements are equivalent:

(a) \([v] \) is constant (and finite) for every \( v \in N \); in particular, \( F \) is onto;

(b) \( F \) is proper.

If \( N \) is path-connected then statements (a) and (b) are also equivalent to:

(c) \( F \) is invertible along any path \( q: [a, b] \to N \) and \([v_0]\) is finite for some \( v_0 \in N \).

Moreover, if \( M \) is path-connected and \( N \) is simply-connected, then the above three conditions are also equivalent to:

(d) \( F \in \text{Hom}(M, N) \).

Finally, if \( M \) is path-connected and \( N \) is a convex subset of a normed vector space, then we can add the following two statements to our list of equivalences:

(e) \( F \) is proper along line segments, that is \( F^{-1}([y_{0}, y_{1}]) \) is compact for any \( y_{0}, y_{1} \in N \);

(f) \( F \) is invertible along any affine path \( q: [a, b] \to N \) and \([v_0]\) is finite for some \( v_0 \in N \).

We next give some consequences of the Theorem 1 that are useful in our presentation.

**Corollary 1.** Let \( M_{0} \subseteq M \) and \( N_{0} \subseteq N \) be given sets satisfying the following conditions: \( F|_{M_0} \) is a local homeomorphism, \( F(M_{0}) \cap N_{0} \neq \emptyset \) and \( F(M \setminus M_{0}) \cap N_{0} = \emptyset \). Assume that \( F \) is proper with respect to some set \( E \) such that \( N_{0} \subseteq E \subseteq N \). Then \( F \) restricted to the pair \((M_{0} \cap F^{-1}(N_{0}), N_{0})\) is a proper local homeomorphism. If, in addition, \( N_{0} \) is connected, then \( F(M_{0}) \supseteq N_{0} \) and \( F(\text{cl} \ M_{0}) \supseteq E \cap \text{cl} \ N_{0} \).

**Proof.** Let \( \tilde{G}: M_{0} \cap F^{-1}(N_{0}) \to N_{0} \) denote the map \( F \) restricted to the pair \((M_{0} \cap F^{-1}(N_{0}), N_{0})\). Observe that the domain of \( \tilde{G} \) is nonempty since \( F(M_{0}) \cap N_{0} \neq \emptyset \) by assumption. Using the fact that \( F|_{M_0} \) is a local homeomorphism, one can easily verify that \( \tilde{G} \) is also a local homeomorphism. We will now verify that \( \tilde{G} \) is proper. Indeed, let \( K \subseteq N_{0} \) be a compact set. Using the fact that \( F(M \setminus M_{0}) \cap N_{0} = \emptyset \), one can easily verify that \( \tilde{G}^{-1}(K) = F^{-1}(K) \). By the properness assumption of \( F \) with respect to \( E \), it follows that \( F^{-1}(K) = \tilde{G}^{-1}(K) \) is compact. Hence, \( \tilde{G} \) is proper. Assume now that \( N_{0} \) is connected. By Theorem 1, it follows that \( \tilde{G} \) is onto and so, \( F(M_{0}) \supseteq N_{0} \). To prove the inclusion \( E \subseteq \text{cl} \ F(\text{cl} \ M_{0}) \), let \( \tilde{y} \in E \subseteq \text{cl} \ N_{0} \) be given. Then, \( \tilde{y} = \lim_{k \to \infty} y^{k} \) for some sequence \( \{y^{k}\} \subseteq N_{0} \). Since \( F(M_{0}) \supseteq N_{0} \), there exists a sequence \( \{x^{k}\} \subseteq M_{0} \) such that \( F(x^{k}) = y^{k} \) for all \( k \). Now the set \( K = \{y^{k}|k = 0, 1, \ldots\} \cup \{\tilde{y}\} \) is a compact subset of \( E \) since \( \tilde{y} \in E \) and \( \{y^{k}\} \subseteq N_{0} \subseteq E \). Hence, by
the properness assumption, $F^{-1}(K)$ is compact. Since $(x^k) \subseteq F^{-1}(K)$, there exists a subsequence $(x^k)_k \in K$ converging to some $\bar{x} \in M$. Clearly, we have $F(\bar{x}) = \bar{y}$ and $\bar{x} \in \text{cl } M_0$. □

**Corollary 2.** Assume that $E \subseteq N$ is connected and $E \cap F(M) \neq \emptyset$. Assume also that $F$ is a local homeomorphism which is proper with respect to $E$. Then, $F(M) \supseteq E$.

**Proof.** This result follows immediately from Corollary 1 by letting $M_0 = M$ and $N_0 = E$. □

**Corollary 3.** Let $M$ be a path-connected metric space. Assume that $F: M \to \mathbb{R}^n$ is a local homeomorphism and that $F^{-1}([y_0, y_1])$ is compact for any pair of points $y_0, y_1 \in F(M)$. Then, $F|_{(M, F(M))} \in \text{Hom}(M, F(M))$ and $F(M)$ is convex.

**Proof.** Let $y_0, y_1$ be a pair of points in $F(M)$. It follows by the compactness assumption of $F^{-1}([y_0, y_1])$ that $F$ is proper with respect to $[y_0, y_1]$. Hence, it follows from Corollary 2 that $F(M) \supseteq [y_0, y_1]$. We have thus proved that $F(M)$ is convex. The conclusion that $F|_{(M, F(M))} \in \text{Hom}(M, F(M))$ now follows immediately from Theorem 1. □

3. **The differentiable case.** Starting in this section, we shall apply the results of the last section to the interior-point mapping $G$ defined in (3). This section considers the case of a differentiable mapping $H$ satisfying a $P_\alpha$-property and some properness condition. The next section considers the case of a mapping $H$ associated with a mixed nonlinear complementarity problem (NCP) with a uniform $P$-function. Section 5 deals with a “monotone” mapping $H$. In both §§4 and 5, $H$ is not assumed to be differentiable.

One important distinction between the main results in this and the next section (Theorems 2 and 4) and the one in §5 (Theorem 5) lies in the choice of the space $N$ in Theorem 1 on which these results are all based. In the former results, $N \equiv R_{1,+}^{n+m}$; and in the latter result, $N \equiv R_{++}^{n+m} \times H(R_{++}^{2n} \times \mathbb{R}^m)$. In what follows, we first state a general result that concerns the case where $N \equiv R_{++}^{n+m} \times R^{n+m}$. In the remainder of this paper, we shall use the following notation:

$$H_+ \equiv H(R_{++}^{n+m} \times \mathbb{R}^m) \quad \text{and} \quad H_{++} \equiv H(R_{++}^{2n} \times \mathbb{R}^m).$$

**Lemma 1.** Assume that the mapping $G: R_+^{2n} \times \mathbb{R}^m \to R_+^{n+m}$ defined in (3) is continuous and proper. Assume also that $G$ satisfies either one of the following conditions:

(A) $G$ is injective on $R_+^{2n} \times \mathbb{R}^m$; or

(B) for every $(x, y, z) \in R_+^{2n} \times \mathbb{R}^m$, the Jacobian matrix $\nabla G(x, y, z)$ exists and is nonsingular; moreover, $\nabla G$ is continuous on $R_+^{2n} \times \mathbb{R}^m$.

Then the following statements hold:

(a) $G$ maps $R_+^{2n} \times \mathbb{R}^m$ homeomorphically onto $R_+^{n+m}$;

(b) $H_{++} = H_+ = R_{++}^{n+m}$;

(c) $G(R_+^{2n} \times \mathbb{R}^m) = R_+^{n+m}$; in particular, problem (1) has a solution.

If, instead of (A), the stronger assumption holds:

(A') $G$ is injective on $R_+^{2n} \times \mathbb{R}^m$, then,

(d) $G$ maps $R_+^{2n} \times \mathbb{R}^m$ homeomorphically onto $R_+^{n+m}$.

**Proof.** Let $\tilde{G}$ denote the mapping $G$ restricted to the pair $(M, N) \equiv (R_+^{2n} \times \mathbb{R}^m, R_+^{n+m} \times R^{n+m})$. Either one of conditions (A) or (B) implies that the mapping $\tilde{G}$ is a local homeomorphism. Indeed, if (A) holds, then this conclusion follows from the domain invariance theorem; if (B) holds, then the same conclusion
follows from the inverse function theorem. Using the fact that \( G \) is proper and that \( \overline{G}^{-1}(S) = \overline{G}^{-1}(S) \) for every subset \( S \subseteq R^m_+ \times R^{n+m}_+ \), we conclude that \( \overline{G} \) is proper. Thus, Theorem 1 implies that \( \overline{G} \) is a homeomorphism, and hence (a) follows. The inclusions
\[
G(R^{2n}_+ \times R^m) \subseteq R^n_+ \times H_+ \subseteq R^n_+ \times H_+ \subseteq R^n_+ \times R^{n+m}
\]
and the equality
\[
G(R^{2n}_+ \times R^m) = R^n_+ \times R^{n+m}
\]
impied by (a) obviously give statement (b). To show (c), observe that since \( G \) is proper, it follows that \( G(R^{2n}_+ \times R^m) \) is a closed set. Hence, we have
\[
G(R^{2n}_+ \times R^m) = \text{cl} \ G(R^{2n}_+ \times R^m) \supseteq \text{cl} \ G(R^{2n}_+ \times R^m) = \text{cl}(R^n_+ \times R^{n+m}) = R^n_+ \times R^{n+m}.
\]
Since the reverse inclusion \( G(R^{2n}_+ \times R^m) \subseteq R^n_+ \times R^{n+m} \) is obvious, (c) follows.

Assume now that \( G \) is injective on \( R^{2n}_+ \times R^m \). By (c), it follows that \( G \) is a continuous bijection from \( R^{2n}_+ \times R^m \) onto \( R^n_+ \times R^{n+m} \). Since \( G \) is proper, it maps closed sets onto closed sets, by Lemma 12(a) in Appendix A. Hence \( G^{-1} \) is continuous and (d) follows.

In what follows, we shall give a sufficient condition, in terms of \( H \), for \( G \) to satisfy condition (B) of Lemma 1. We first explain some notation and terminology. If \( H \) has a Fréchet derivative at a point \( (x, y, z) \in R^{2n}_+ \times R^m \), its Jacobian matrix
\[
\nabla H(x, y, z) = \begin{bmatrix} \nabla_x H(x, y, z) & \nabla_y H(x, y, z) & \nabla_z H(x, y, z) \end{bmatrix}
\]
is an \((n + m) \times (2n + m)\) matrix, where \( \nabla_x H(x, y, z), \nabla_y H(x, y, z), \) and \( \nabla_z H(x, y, z) \) denote the partial Jacobian submatrices with respect to \( x, y, \) and \( z \) which are of order \((n + m) \times n, (n + m) \times n, \) and \((n + m) \times m, \) respectively. If a matrix \( A \in R^{(n+m)\times m} \) is of full (column) rank, any nonsingular submatrix \( B \) of order \( m \) is called a basis of \( A \). Given a matrix \( Q \) in partitioned form
\[
Q = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
\]
where the (principal) submatrix \( A \) is nonsingular, the matrix \( D - CA^{-1}B \), which need not be square, is called the Schur complement of \( A \) in \( Q \) (see Cottle 1974). In general, if \( A \) is an arbitrary nonsingular submatrix of a matrix \( Q \), we can permute the columns and rows of \( Q \) so that \( A \) becomes a leading principal submatrix of \( Q \); in such a permuted matrix \( Q' \), the Schur complement of \( A \) in \( Q' \) is thus well defined. With an abuse of language, we also say that the latter Schur complement is the Schur complement of \( A \) in \( Q \). If \( A \) and \( B \) are two square matrices of the same order, we say that the pair \((A, B)\) has the \( P_0 \)-property (or, \((A, B)\) is a \( P_0 \)-pair) if for every nonzero pair of vectors \((u, v)\) satisfying \( Au + Bu = 0 \), there exists an index \( i \) with \( |u_i| + |v_i| > 0 \) such that \( u_i, v_i \geq 0 \). The class of \( P_0 \)-pairs of matrices has recently been studied in Sznajder and Gowda (1995) in connection with various generalizations of the linear complementarity problem. Consider a matrix \( Q \) in partitioned form
\[
Q = [A \ B \ C],
\]
where \( A \) and \( B \) are of order \((n + m) \times n\) and \( C \) is of order \((n + m) \times m\). Suppose the matrix \( C \) is of full (column) rank. Then there is an \( m \times m \) nonsingular submatrix \( C_1 \) of \( C \). By permuting the rows of \( Q \) if necessary, we may assume without loss of generality that \( Q \) is of the form:

\[
Q = \begin{bmatrix}
A_1 & B_1 & C_1 \\
A_2 & B_2 & C_2
\end{bmatrix},
\]

where \( A_1 \) and \( B_1 \) are of order \( m \times n \), \( A_2 \) and \( B_2 \) are square matrices of order \( n \), and \( C_2 \) is of order \( n \times m \). The Schur complement of \( C_1 \) in \( Q \) comprises of the pair of matrices:

\[
A_2' = A_2 - C_2 C_1^{-1} A_1 \quad \text{and} \quad B_2' = B_2 - C_2 C_1^{-1} B_1,
\]

which are square matrices of order \( n \). We shall say that \( Q \) (or the triple \((A, B, C)\)) has the mixed \( P_0 \)-property if \( C \) has full rank and there is a basis \( C_1 \) of \( C \) such that the pair \((A_2', B_2')\) has the \( P_0 \)-property. If \( C \) is of full rank, this mixed \( P_0 \)-property holds for \( Q \) if and only if the implication below holds:

\[
Au + Bv + Cw = 0 \quad (u, v) \neq 0 \quad \Rightarrow \quad u_i, v_i \geq 0 \quad \text{for some} \quad i \quad \text{such that} \quad |u_i| + |v_i| > 0.
\]

**Lemma 2.** Let \( H: R_+^{2n} \times R^m \rightarrow R_+^{n+m} \) be continuously differentiable at every point \((x, y, z) \in R_+^{2n} \times R^m\) and assume that the Jacobian matrix

\[
\nabla H(x, y, z) = \begin{bmatrix}
\nabla_x H(x, y, z) & \nabla_y H(x, y, z) & \nabla_z H(x, y, z)
\end{bmatrix}
\]

has the mixed \( P_0 \)-property. Then \( G \) satisfies condition (B) of Lemma 1; in particular, \( G \) is a local homeomorphism at each point in \( R_+^{2n} \times R^m \).

**Proof.** It suffices to show that for a vector \((x, y, z) \in R_+^{2n} \times R^m\), the mixed \( P_0 \)-property of the Jacobian matrix \( \nabla H(x, y, z) \) implies the nonsingularity of the Jacobian matrix \( \nabla G(x, y, z) \). Clearly, by letting \( X \) and \( Y \) be the diagonal matrix with diagonal entries consisting of the components of \( x \) and \( y \) respectively, we have

\[
\nabla G(x, y, z) = \begin{bmatrix}
Y & X & 0 \\
\nabla_x H(x, y, z) & \nabla_y H(x, y, z) & \nabla_z H(x, y, z)
\end{bmatrix}.
\]

Suppose that this matrix is singular. Then there exists a nonzero vector \((u, v, w) \in R_+^{2n+m}\) such that

\[
Yu + Xv = 0,
\]

(6)

\[
\nabla_x H(x, y, z) u + \nabla_y H(x, y, z) v + \nabla_z H(x, y, z) w = 0.
\]

(7)

If \((u, v) = 0\), then the full rank assumption of \( \nabla_x H(x, y, z) \) implies \( w = 0 \). Hence, \((u, v) \neq 0\). By the mixed \( P_0 \)-property of \( \nabla H(x, y, z) \), there is an index \( i \) with \(|u_i| + |v_i| > 0 \) such that \( u_i, v_i \geq 0 \). But this easily contradicts (6). \( \square \)

We next discuss a sufficient condition, also in terms of \( H \), for \( G \) to be a proper mapping. Specifically, we say that \( H \) is uniformly norm coercive if for every sequence \( \{(x^k, y^k, z^k) \in R_+^{2n} \times R^m\} \) and for every subset \( \alpha \) of \( \{1, \ldots, n\} \) for which \( \{(x_{\alpha}^k, y_{\alpha}^k)\} \) is
bounded and \( \lim_{k \to \infty} \| (x^k, y^k, z^k) \| = \infty \), where \( \bar{\alpha} \) is the complement of \( \alpha \) in \( \{1, \ldots, n\} \), we have
\[
\lim_{k \to \infty} \| H(x^k, y^k, z^k) \| = \infty.
\]

The above definition can be equivalently stated as: for every sequence \( \{(x^k, y^k, z^k)\} \subset R_+^{2n} \times R^n \) and for every subset \( \alpha \) of \( \{1, \ldots, n\} \) for which \( \{H(x^k, y^k, z^k)\} \) and \( \{(x^k_\alpha, y^k_\alpha)\} \) are bounded, we have \( \{(x^k_\alpha, y^k_\alpha, z^k)\} \) bounded. Subsequently, we shall give a sufficient condition for this norm coercivity condition to hold. But now, we show that norm coercivity of \( H \) is sufficient for \( G \) to be proper with respect to \( R^n_+ \times R^{n+m} \).

**Lemma 3.** Suppose that \( H: R_+^{2n} \times R^n \to R^{n+m} \) is continuous and uniformly norm coercive. Then for every compact subset \( S \subset R^n_+ \times R^{n+m} \), the inverse image
\[
G^{-1}(S) \equiv \{(x, y, z) \in R_+^{2n} \times R^n | G(x, y, z) \in S\}
\]
is a compact subset of \( R_+^{2n} \times R^n \).

**Proof.** Since \( G \) is continuous, \( S \) is closed and the domain of \( G \) is closed, it follows that \( G^{-1}(S) \) is a closed set. It remains to show that \( G^{-1}(S) \) is a bounded set. Indeed, assume for contradiction that there exists a sequence \( \{(x^k, y^k, z^k)\} \subset G^{-1}(S) \) such that
\[
\lim_{k \to \infty} \| (x^k, y^k, z^k) \| = \infty.
\]
Observe that since \( S \) is bounded, we have the \( \{x^k \circ y^k\} \) and \( \{H(x^k, y^k, z^k)\} \) are bounded. It is easy to verify the existence of a subset \( \alpha \subseteq \{1, \ldots, n\} \) (possibly \( \alpha = \emptyset \) or \( \alpha = \{1, \ldots, n\} \)) and an infinite index set \( \mathcal{I} \) such that \( \{(x^k_\alpha)\} \) is bounded and \( \lim_{k \to \infty} x^k_i = \infty \) for all \( i \in \bar{\alpha} \). Since \( \{(x^k_\circ y^k)\} \) is bounded, this implies that \( \lim_{k \to \infty} y^k_\alpha = 0 \). Hence, the sequence \( \{(x^k_\circ y^k)\} \) is bounded and, since \( \{H(x^k, y^k, z^k)\} \) is bounded, it follows by the uniform norm coercivity of \( H \) that \( \{(x^k_\circ y^k, z^k)\} \) is also bounded. Hence \( \{(x^k, y^k, z^k)\} \) is bounded and this contradicts (8). Thus, the set \( G^{-1}(S) \) is bounded and the result follows. \( \square \)

As an immediate consequence of Lemma 1 and the above discussion, we obtain the following result about the mapping \( G \).

**Theorem 2.** Let \( H: R_+^{2n} \times R^n \to R^{n+m} \) be continuous and uniformly norm coercive. Suppose that for every \( (x, y, z) \in R_+^{2n} \times R^n \) the Jacobian matrix \( \nabla H(x, y, z) \) exists, is continuous and has the mixed \( P_0 \) property. Then the following statements hold:

(a) \( G \) maps \( R_+^{2n} \times R^n \) homeomorphically onto \( R_+^n \times R^{n+m} \);
(b) \( H_+ = H_\circ = R^{n+m} \);
(c) \( G(R_+^{2n} \times R^n) = R_+^{n} \times R^{n+m} \); in particular, problem (1) has a solution.

**Proof.** By Lemma 3, it follows that \( G \) is proper. Moreover, by Lemma 2, we know that \( G \) satisfies condition (B) of Lemma 1. Hence, statements (a), (b), and (c) hold. \( \square \)

As a consequence of the above result, we illustrate a homotopy approach to find a solution of the system
\[
H(x, y, z) = 0, \quad x \circ y = a, \quad (x, y, z) \in R_+^{2n} \times R^n,
\]
where \( a \in R^n_+ \). Note that when \( a = 0 \), the above system is equivalent to problem (1). Let \( (x^0, y^0, z^0) \in R_+^{2n} \times R^n \) be given and let \( \alpha: [0, 1] \to R^n_+ \) and \( \beta: [0, 1] \to R^{n+m} \)
be paths satisfying
\[ \alpha(t) \in R^+_n, \quad \forall t \in (0, 1), \]
\[ \alpha(t) = x^0 \circ y^0 > 0, \]
\[ \beta(1) = H(x^0, y^0, z^0), \]
\[ \beta(0) = 0, \quad \alpha(0) = a. \]

It follows from Theorem 2(a) that, for each \( t \in (0, 1) \), there exists a unique triple \((x(t), y(t), z(t)) \in R^+_n \times R^n\) such that
\[ x(t) \circ y(t) = \alpha(t), \]
\[ H(x(t), y(t), z(t)) = \beta(t). \]

Moreover, \((x(t), y(t), z(t))\) is continuous in \( t \) and \((x(1), y(1), z(1)) = (x^0, y^0, z^0); \)
hence, the path has one endpoint at the given initial point \((x^0, y^0, z^0)\). By Lemma 3, it follows that \([(x(t), y(t), z(t)) | t \in (0, 1)]\) is bounded, and every accumulation point of the path \( t \to (x(t), y(t), z(t)) \), as \( t \to 0 \), is a solution of system (9). Hence, if we follow the path \( t \to (x(t), y(t), z(t)) \) starting from the point \((x^0, y^0, z^0)\), we will eventually approach a solution of (9). A trivial way to construct the paths \( \alpha(t) \) and \( \beta(t) \) is as follows:
\[ \alpha(t) = (1 - t)a + t^0 \circ y, \quad \forall t \in [0, 1]; \]
\[ \beta(t) = tH(x^0, y^0, z^0), \quad \forall t \in [0, 1]. \]

We end this section by giving some sufficient conditions for the assumptions, hence the conclusions, of Theorem 2 to hold. One such condition concerns the case where \( H \) is affine:

\[ H(x, y, z) = Ax + By + Cz + q \quad \text{for } (x, y, z) \in R^+_n \times R^n, \]

for some matrices \( A \) and \( B \) of order \((n + m) \times n\), matrix \( C \) of order \((n + m) \times m\), and vector \( q \in R^{n+m}. \) A representative of the pair \((A, B)\) is an \((n + m) \times n\) matrix whose \( i \)th column is equal to the \( i \)th column of either \( A \) or \( B \) (Sznajder and Gowda 1995). Another condition concerns a mixed nonlinear complementarity problem in which \( H \) is given by

\[ H(x, y, z) = \begin{pmatrix} f(x, z) - y \\ g(x, z) \end{pmatrix} \quad \text{for } (x, y, z) \in R^+_n \times R^n \]

where \( f: R^n \times R^m \to R^n \) and \( g: R^n \times R^m \to R^m \) are two continuously differentiable functions. Associated with these functions \( f \) and \( g \), define \( F: R^+_n \times R^m \to R^{n+m} \) by

\[ F(x, z) = \begin{pmatrix} f(x, z) \\ g(x, z) \end{pmatrix} \quad \text{for } (x, z) \in R^+_n \times R^m. \]

For a given function \( F: C \subseteq R^N \to R^N \), we shall say that \( F \) is inverse Lipschitz at a
vector \( \bar{c} \in C \) with modulus \( \gamma > 0 \) if for all vectors \( c \in C \),

\[
\|F(c) - F(\bar{c})\| \geq \gamma \|c - \bar{c}\|.
\]

We note that if \( F \) is a global homeomorphism on \( C \) with a Lipschitzian inverse, then \( F \) must be inverse Lipschitz at every vector \( \bar{c} \in C \).

Let \( F: \prod_{i=1}^{N} C_i \subseteq \mathbb{R}^N \rightarrow \mathbb{R}^N \) be given. For each subset \( \alpha \subseteq \{1, \ldots, N\} \) with complement \( \bar{\alpha} \) and each vector \( a \in \prod_{i \in \bar{\alpha}} C_i \), define \( f: \prod_{i \in \bar{\alpha}} C_i \rightarrow \mathbb{R}^k \), where \( k \equiv |\bar{\alpha}| \), by

\[
f(x_{\bar{\alpha}}) = F_{\bar{\alpha}}(a_{\bar{\alpha}}, x_{\bar{\alpha}}) \quad \text{for} \quad x_{\bar{\alpha}} \in \prod_{i \in \bar{\alpha}} C_i.
\]

This \( f \) is a principal subfunction of \( F \) obtained by fixing the \( x_{\alpha} \) components at the constant \( a_{\alpha} \) and removing the corresponding \( \alpha \) components from the function \( F \).

Consider the function \( F \) defined by (12). Let \( \mathcal{F} \) be the collection of principal subfunctions of \( F \) obtained by fixing the \( x_{\alpha} \) components at some nonnegative vector \( a_{\alpha} \) and removing the corresponding \( f_{\alpha} \) functions, for all index subsets \( \alpha \) of \( \{1, \ldots, n\} \) and nonnegative vectors \( a_{\alpha} \).

**THEOREM 3.** Theorem 2 holds under either one of the following two assumptions:

(a) \( H \) is given by (10) where \([A \ B \ C] \) has the mixed \( P_0 \)-property, and for every representative matrix \( E \) of \((A, B)\), the homogeneous system

\[
Eu + Cv = 0, \quad u \geq 0
\]

has \((u, v) = 0\) as the unique solution;

(b) \( H \) is given by (11), \( f \) and \( g \) are continuously differentiable on \( \mathbb{R}^n_{++} \times \mathbb{R}^m \) and satisfy the conditions: (i) the function \( F \) defined by (12) is Lipschitz continuous on \( \mathbb{R}^n_{++} \times \mathbb{R}^m \), (ii) for all \((x, y, z) \in \mathbb{R}^{2n}_{++} \times \mathbb{R}^m \), the matrix

\[
\begin{bmatrix}
\nabla_x f(x, z) \\
\nabla_z g(x, z)
\end{bmatrix}
\]

has full rank, and

\[
\begin{align*}
\nabla_x g(x, z)u + \nabla_z g(x, z)w &= 0 \\
(u, \nabla_x f(x, z)u + \nabla_z f(x, z)w) &\neq 0
\end{align*}
\]

\[
\Rightarrow \quad u_i(\nabla_x f(x, z)u + \nabla_z f(x, z)w)_i \geq 0 \quad \text{for some} \quad i
\]

\[
\text{with} \quad |u_i| + |(\nabla_x f(x, z)u + \nabla_z f(x, z)w)_i| > 0,
\]

and (iii) each function in the family \( \mathcal{F} \) is inverse Lipschitz at some vector in its domain.

**PROOF.** For part (a), it suffices to note that \( H \) is uniformly norm coercive. The proof of this property is rather easy by a standard normalization followed by a limiting argument. For part (b), we need to verify that \( \nabla H(x, y, z) \) has the mixed \( P_0 \)-property and that \( H \) has the uniform norm coercivity property. The \( P_0 \)-property is easily verified using the expression:

\[
\nabla H(x, y, z) = \begin{bmatrix}
\nabla_x f(x, z) & -I & \nabla_z f(x, z) \\
\nabla_z g(x, z) & 0 & \nabla_z g(x, z)
\end{bmatrix}.
\]
To verify the uniform norm coercivity condition, let \((x^k, y^k, z^k) \in R_+^{2n} \times R^m\) be a sequence and \(\alpha \subseteq \{1, \ldots, n\}\), with complement \(\bar{\alpha}\), be an index set for which \((x^k_{\bar{\alpha}}, y^k_{\bar{\alpha}})\) is bounded and \((H(x^k, y^k, z^k))\) is bounded. We will show that \((x^k_{\bar{\alpha}}, y^k_{\bar{\alpha}}, z^k)\) is bounded. Indeed, without loss of generality, we may assume that \((x^k_{\bar{\alpha}})\) converges to the vector \(a_{\alpha}\). Consider the principal subfunction in the family \(\tilde{F}\) defined by the vector \(a_{\alpha}\) and the index set \(\alpha\); this subfunction, which we denote \(\tilde{F}\), is obtained by fixing the \(x_{\bar{\alpha}}\) components at \(a_{\alpha}\) and removing the \(f_{\alpha}\) functions. We have

\[
\begin{pmatrix}
    f_{\bar{\alpha}}(x^k, z^k) - y^k_{\bar{\alpha}} \\
    g(x^k, z^k)
\end{pmatrix}
= \begin{pmatrix}
    f_{\bar{\alpha}}(x^k, z^k) - f_{\bar{\alpha}}(a_{\alpha}, x^k_{\bar{\alpha}}, z^k) - y^k_{\bar{\alpha}} \\
    g(x^k, z^k) - g(a_{\alpha}, x^k_{\bar{\alpha}}, z^k)
\end{pmatrix} + \tilde{F}(x^k_{\bar{\alpha}}, z^k).
\]

Note that the first term in the right-hand side of this equation is bounded, by the Lipschitzian assumption of \(F\). Also, the left-hand side is bounded due to the fact that \((H(x^k, y^k, z^k))\) is bounded. Hence \((\tilde{F}(x^k_{\bar{\alpha}}, z^k))\) is bounded, and by the inverse Lip- schitzian assumption of \(\tilde{F}\), this implies that \((x^k_{\bar{\alpha}}, z^k)\) is also bounded. Hence \((x^k, z^k)\) is bounded and so is \((F(x^k, z^k))\), by continuity of \(F\). Since both \((H(x^k, y^k, z^k))\) and \((F(x^k, z^k))\) are bounded, we obtain the conclusion that \((y^k)\) is bounded. We have thus shown that \((x^k, y^k, z^k)\) is bounded.

The condition about the representative matrices \(E\) of \((A, B)\) in condition (a) of Theorem 3 is a generalization of the class of \(R_0\)-matrices in linear complementarity theory (Cottle et al. 1992). Indeed, when \(C\) is vacuous, \(A = I\), and \(B = -M\), this condition is equivalent to the \(R_0\)-property of \(M\). As we recall, one characterizing condition of an \(R_0\)-matrix \(M\) is that for all vectors \(q\), the linear complementarity problem:

\[
x \geq 0, \quad q + Mx \geq 0, \quad x^T(q + Mx) = 0,
\]

has a bounded (possibly empty) solution set. The following result generalizes this characterization to the assumption in Theorem 3(a).

**Proposition 1.** The system (13) has \((u, v) = (0, 0)\) as the unique solution for all representative matrices \(E\) of \((A, B)\) if and only if for all vectors \(q \in R^n\), the problem

\[
(14) \quad Ax + By + Cz + q = 0, \quad (x, y) \geq 0, \quad x \perp y
\]

has a bounded (possibly empty) solution set.

**Proof.** "Necessity." Suppose that for some vector \(q\), there is an unbounded sequence \((x^k, y^k, z^k)\) of solutions to (14). By a standard normalization followed by a limiting argument, it is easy to exhibit a representative matrix \(E\) of \((A, B)\) and a nonzero solution to (13). This is a contradiction.

"Sufficiency." Since every solution \((u, v)\) of (13) for some representative matrix \(E\) of \((A, B)\) trivially extends to a solution of (14) with \(q = 0\), the assumption that the latter (homogeneous) problem has a bounded solution set (which must be a singleton because of homogeneity) implies that such a pair \((u, v)\) must equal to zero. □

We note that condition (ii) in Theorem 3(b) holds if \(\nabla_z g(x, z)\) is nonsingular and the Schur complement

\[
\nabla_x f(x, z) - \nabla_z f(x, z) \nabla_z g(x, z)^{-1} \nabla_z q(x, z)
\]

is a \(P_0\)-matrix. We also note that the Lipschitz continuity assumption of \(F\) is somewhat strong; but we can replace it by a strengthened inverse Lipschitz assump-
tion on the principal subfunctions of \( F \). We shall omit the discussion of the latter strengthened condition as it would involve some cumbersome notation.

4. A mixed NCP with a uniform \( P \)-function. In this section, we consider the case in which the function \( H \) is given by (11), where the function \( F \) defined by (12) is assumed to be a uniform \( P \)-function; i.e. there exists a scalar \( \gamma > 0 \) such that for all pairs \( w = (x, z) \) and \( w' = (x', z') \) in \( R^n_+ \times R^m \),

\[
\max_{1 \leq i \leq n + m} (w_i - w'_i) (F_i(w) - F_i(w')) \geq \gamma \| w - w' \|^2.
\]

Throughout this section, we do not assume that \( F \) is differentiable or Lipschitz continuous.

**Lemma 4.** Suppose that \( F \) in (12) is a continuous uniform \( P \)-function on \( R^n_+ \times R^m \). Then the function \( G: R^{2n}_+ \times R^m \to R^n_+ \times R^{n+m} \) defined in (3) is injective and proper.

**Proof.** Suppose \( G(x, y, z) = G(x', y', z') \) for some \( (x, y, z), (x', y', z') \in R^{2n}_+ \times R^m \). Then we have

\[
y - y' = f(x, z) - f(x', z') \quad \text{and} \quad 0 = g(x, z) - g(x', z'),
\]

which implies, by Lemma 1 of Kojima et al. (1989) and the fact that \( x \circ y = x' \circ y' \),

\[
(x_i - x'_i)(f_i(x, z) - f_i(x', z')) \leq |x_i y_i - x'_i y'_i| = 0.
\]

Since we also have

\[
(z_i - z'_i)(g_i(x, z) - g_i(x', z')) = 0,
\]

the uniform \( P \)-function of \( F \) implies \((x, z) = (x', z')\) which in turn yields \( y = y' \). This establishes that \( G \) is injective.

According to Lemma 3, the properness of \( G \) will follow if we can verify that \( H \) is uniformly norm coercive on \( R^{2n}_+ \times R^m \). We proceed as in the proof of Theorem 3. Let \((x^k, y^k, z^k) \in \{x^k, y^k, z^k\} \subset R^{2n}_+ \times R^m \) be a sequence and \( \alpha \subseteq \{1, \ldots, n\} \), with complement \( \bar{\alpha} \), be an index set for which \( \{(x^\alpha, y^\alpha)\} \) and \( \{(H(x^k, y^k, z^k)\) are bounded. We will show that \( \{(x^\alpha, y^\alpha)\} \) is bounded. Using the uniform \( P \)-property of \( F \) and the boundedness of \( \{x^k\} \), we prove that there exists a scalar \( \beta \geq 0 \) such that for all \( k \),

\[
\| \begin{pmatrix} f_{\alpha}(x^k, z^k) \\ g(x^k, z^k) \end{pmatrix} \| \geq \gamma \| (x^\alpha, z^k) \| - \beta.
\]

Indeed, defining \( u^k \) by \( u^k_{\alpha} = x^k_{\alpha}, u^k_{\bar{\alpha}} = 0 \), we obtain

\[
\max \left\{ \max_{i \in \alpha} x^k_i (f_i(x^k, z^k) - f_i(u^k, 0)), \max_{1 \leq j \leq m} z^k_j (g_j(x^k, z^k) - g_j(u^k, 0)) \right\}
\]

\[
\geq \gamma \| (x^\alpha, z^k) \|^2;
\]

thus by letting

\[
\beta = \sup_k \| (f(u^k, 0), g(u^k, 0)) \|,
\]
and by the Cauchy-Schwartz inequality, (15) follows. Since both \( \{H(x^k, y^k, z^k)\} \) and \( \{y^k_a\} \) are bounded, and

\[
f_a(x^k, z^k) = H_a(x^k, y^k, z^k) + y^k_a,
\]

it follows that the left-hand side of (15) is bounded, and hence that the sequence \( \{(x^k, z^k)\} \) is bounded. Thus, \( \{(x^k, z^k)\} \) is bounded, and by continuity it follows that \( \{F(x^k, z^k)\} \) is bounded. Hence, \( \{y^k\} \) is also bounded and the result follows. □

With the above lemma, we may invoke Lemma 1 to obtain the following strengthening of Theorem 3 for a uniform \( P \)-function without assuming differentiability. This result generalizes the theory in Kojima et al. (1989) for the standard NCP.

**Theorem 4.** Suppose that \( F \) in (12) is a continuous uniform \( P \)-function on \( R^n_+ \times R^m \). Then with \( H \) given by (11), the mapping \( G \) maps \( R^n_+ \times R^m \) homeomorphically onto \( R^n \times R^{n+m} \).

**Proof.** Lemma 4 implies that \( G \) satisfies condition (A)' in Lemma 1. Consequently, the desired conclusion follows from part (d) of the latter lemma. □

We conclude this section by mentioning that in Kojima et al. (1991), a result similar to Theorem 3 was established for the standard NCP under a \( P_0 \)-property and a certain properness assumption (Condition 1.5 in the reference). This result also follows as a special case of Lemma 1.

**5. The monotone case.** In §3, we have imposed a mixed \( P_0 \)-property on the Jacobian matrix of a general mapping \( H \) and relied on the uniform norm coercivity property in order to apply Theorem 1 to the corresponding mapping \( G \). In this section, we shall impose a certain monotonicity assumption on \( H \) and establish some results similar to Theorem 2. Throughout this section, \( H \) is not assumed to be differentiable. The following definition plays a central role.

**Definition 3.** Let \( X \subseteq R^n \) and \( Y \subseteq R^m \) be given. Let \( U \subseteq X \times Y \) and a mapping \( J: X \times Y \times R^m \rightarrow R^{n+m} \) be given. Then,

(a) \( J \) is said to satisfy the \((x, y)\)-monotonicity on \( U \) if for any \((x, y, z)\) and \((x', y', z')\) in \( U \times R^m \) such that \( J(x, y, z) = J(x', y', z') \), there holds \((x - x')^T(y - y') \geq 0\);

(b) \( J \) is said to satisfy the \(z\)-injectiveness on \( U \) if for each \((x, y) \in U \), the function \( J(x, y, \cdot) \) is injective on \( R^m \);

(c) \( J \) is said to satisfy the \(z\)-boundedness on \( U \) if for any sequence \( \{(x^k, y^k, z^k)\} \subseteq U \times R^m \) with \( \{(x^k, y^k)\} \) bounded and \( \{z^k\} \) such that \( \lim_{k \to \infty} \|z^k\| = \infty \), there holds:

\[
\lim_{k \to \infty} \|J(x^k, y^k, z^k)\| = \infty.
\]

The above definition of monotonicity may seem a little unusual at first glance. As we will see subsequently, this definition is actually quite broad and includes several known monotonicity concepts in the literature; see Proposition 3 and Lemma 7. Moreover, the monotonicity property (a) provides a sufficient condition for a certain "natural" set-valued map associated with problem (1) to be a restricted maximal monotone operator; see Theorem 7 (and also Theorem 8).

The following simple lemma is stated without proof.

**Lemma 5.** Assume that \((x, y)\) and \((\hat{x}, \hat{y})\) are two points in \( R^n \times R^n \) satisfying \((x - \hat{x})^T(y - \hat{y}) \geq 0\). Then, the following two implications hold:

(a) if \( x + y = \hat{x} + \hat{y} \) then \((x, y) = (\hat{x}, \hat{y})\);

(b) if \((x, y) > 0\), \((\hat{x}, \hat{y}) > 0\) and \( x \circ y = \hat{x} \circ \hat{y} \) then \((x, y) = (\hat{x}, \hat{y})\).
The next lemma asserts two important properties of the mapping $G$ defined in (3) under the assumptions of $H$ described in Definition 3.

**LEMMA 6.** Suppose that the mapping $H: R_+^{2n} \times R^n \to R^{n+m}$ is continuous and satisfies the $(x, y)$-monotonicity on $R_+^{2n}$, the $z$-injectiveness on $R_+^{2n}$, and the $z$-boundedness on $R_+^{2n}$. Then $G: R_+^{2n} \times R^n \to R_+^n \times R^{n+m}$ is injective on $R_+^{2n} \times R^n$ and proper with respect to $R_+^n \times H_+$. 

**PROOF.** We first show that $G$ is injective on $R_+^{2n} \times R^n$. Indeed, assume that $(x, y, z)$ and $(x', y', z')$ are two points in $R_+^{2n} \times R^n$ such that $G(x, y, z) = G(x', y', z')$. Then, we have $x \odot y = x' \odot y'$ and $H(x, y, z) = H(x', y', z')$. The monotonicity of $H$ then implies $(x - y)^T(x' - y') \geq 0$. By Lemma 5, we then obtain that $x = x'$ and $y = y'$. The $z$-injectiveness of $H$ on $R_+^{2n}$ then implies $z = z'$. We have thus shown that $(x, y, z) = (x', y', z')$, which shows that $G$ is injective on $R_+^{2n} \times R^n$. 

We next show that $G$ is proper with respect to $R_+^n \times H_+$. Indeed, let $S$ be a compact subset of $R_+^n \times H_+$. By continuity of $G$, it follows that $G^{-1}(S)$ is a closed subset. We next show that $G^{-1}(S)$ is a bounded subset from which it follows that $G^{-1}(S)$ is a compact subset, and hence that $G$ is proper with respect to $R_+^{2n} \times H_+$. Indeed, suppose for contradiction that there exists a sequence $\{(x_k, y_k, z_k)\} \subset G^{-1}(S)$ for which

$$
\lim_{k \to \infty} \|(x_k, y_k, z_k)\| = \infty.
$$

Since $S$ is compact, we may assume without loss of generality that there exists a vector $H_+^e \in H_+$ such that

$$
H_+^e = \lim_{k \to \infty} H(x_k, y_k, z_k).
$$

By the definition of $H_+$, there exists $(x_\infty, y_\infty, z_\infty) \in R_+^{2n} \times R^n$ such that $H(x_\infty, y_\infty, z_\infty) = H_+^e$. Let

$$
\epsilon = \frac{1}{2} \min \{\min(x_i^e, y_i^e) : 1 \leq i \leq n\},
$$

and let $B \subset R_+^{2n} \times R^n$ be the open ball with center at $(x_\infty, y_\infty)$ and radius $\epsilon$. Clearly, $B \subset R_+^{2n} \times R^n$. Moreover, since $G$ is injective, it follows from the domain invariance theorem that $G(B \times R_+^n)$, and hence $H(B \times R_+^n)$, is an open set. Thus, for all $k$ sufficiently large, $H(x_k, y_k, z_k) \in H(B \times R_+^n)$. Hence there exists $(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k) \in B \times R_+^n$ such that $H(x_k, y_k, z_k) = H(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)$. By the $(x, y)$-monotonicity of $H$, we deduce

$$
(x_k - \tilde{x}_k)^T(y_k - \tilde{y}_k) \geq 0,
$$

which yields

$$
(x_k)^T\tilde{y}_k + (y_k)^T\tilde{x}_k \leq (\tilde{x}_k)^T\tilde{y}_k + (x_k)^Ty_k.
$$

Since $(x_k \odot y_k, H(x_k, y_k, z_k)) \in S$, it follows that $\{(x_k)^Ty_k, H(x_k, y_k, z_k)\}$ is bounded. Moreover, we must have

$$
\min(\tilde{x}_i^k, \tilde{y}_i^k) \geq \epsilon, \quad (\tilde{x}_k)^T\tilde{y}_k \leq (x_\infty)^Ty_\infty + n \epsilon,
$$

for all $k$ sufficiently large and all $i = 1, \ldots, n$. Consequently, by (16), it follows that \$((x_k, y_k))$ is bounded. Thus, by the $z$-boundedness condition and the fact that
\( (H(x^k, y^k, z^k)) \) is bounded, it follows that \((z^k)\) must also be bounded. But this is a contradiction. Consequently, \(G^{-1}(S)\) is bounded. \(\Box\)

Unlike the norm coercivity case, the set \(H_{++}\) is not necessarily convex under the monotonicity assumption stated above. This is illustrated by the following example.

**Example.** Let \(n = 2\) and \(m = 0\) and let \(H\) be given by

\[
H(x, y) = (x_1 + x_2, x_1 + x_2 + x_3^2) \quad \text{for} \quad (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2.
\]

It is trivial to verify that \(H(x, y) = H(x', y')\) implies \(x = x'\). Thus \(H\) is monotone in the sense defined above. Nevertheless, the set

\[
H(R^4_{++}) = \{(r + s, r + s + s^3)| (r, s) > 0\}
\]

is easily seen to be nonconvex.

In spite of the possible nonconvexity of the set \(H_{++}\), we can still establish the solvability of problem (1) under a "strict feasibility" condition. We note that the two conclusions in the next result are not as strong as the conclusions in Theorem 2. The main reason for this weakening is due to the restricted properness of \(G\) under the assumptions made in this section; cf. Lemma 6. Indeed, the proof of the following result makes use of Theorem 1, and not Lemma 1.

**Theorem 5.** Suppose that the mapping \(H: R^{2n}_{++} \times R^m \rightarrow R^{n+m}_{++}\) is continuous and satisfies the \((x, y)\)-monotonicity on \(R^{2n}_{++}\), the \(z\)-injectiveness on \(R^{2n}_{++}\), and the \(z\)-boundedness on \(R^{2n}_{++}\). Then the following statements hold:

(a) \(G\) maps \(R^{2n}_{++} \times R^m\) onto \(R^{n+m}_{++} \times H_{++}\) homeomorphically;

(b) there holds

\[
R^{n}_{++} \times H_{++} \subseteq G(R^{2n}_{++} \times R^m);
\]

as a consequence, if \(0 \in H_{++}\) then system (9) has a solution for every vector \(a \in R^n\); in particular, so does problem (1).

**Proof.** By Lemma 6 and the domain invariance theorem, it follows that \(G\) maps \(R^{2n}_{++} \times R^m\) homeomorphically onto \(G(R^{2n}_{++} \times R^m) \subset R^{n+m}_{++} \times H_{++}\). In particular, it follows that \(G\) restricted to the pair \((R^{2n}_{++} \times R^m, R^{n+m}_{++} \times H_{++})\) is a local homeomorphism. Moreover, this restriction is also proper due to Lemma 6. Hence, by Theorem 1, we deduce that \(G\) restricted to the pair \((R^{2n}_{++} \times R^m, R^{n+m}_{++} \times H_{++})\) is an onto map. Thus, \(G(R^{2n}_{++} \times R^m) = R^{n+m}_{++} \times H_{++}\), and statement (a) follows.

Statement (b) follows from Corollary 1 by letting \(M = R^{2n}_{++} \times R^m, M_0 = R^{2n}_{++} \times R^m, N = R^n_{++} \times H_{++}, N_0 = R^n_{++} \times H_{++}, E = R^n_{++} \times H_{++}\), and the mapping \(F\) to be the restriction of \(G\) to the pair \((R^{2n}_{++} \times R^m, R^n_{++} \times H_{++})\). \(\Box\)

Part of the reason why the set \(H_{++}\) is not convex under the \((x, y)\)-monotonicity stated above is that this condition is rather weak; more specifically, it does not say anything about two triples \((x, y, z)\) and \((x', y', z')\) when \(H(x, y, z) \neq H(x', y', z')\). Instead, suppose that

\[
\text{(a') there exist continuous functions } h: R^{2n}_{++} \times R^m \rightarrow R^{n+m} \text{ and } c: R^{(n+m)} \rightarrow R \text{ with } c(r, r) = 0 \text{ for all } r \in R^{n+m} \text{ such that }
\]

\[
\begin{align*}
H(x, y, z) &= r \\
H(x', y', z') &= r' \\
(x, y, z), (x', y', z') &\in R^{2n}_{++} \times R^m
\end{align*}
\]

\[
\Rightarrow (x - x')^T(y - y') \\
&\geq (r - r')^T(h(x, y, z) - h(x', y', z')) + c(r, r').
\]
The next result shows that the set $H_{++}$ is indeed convex under this stronger notion of monotonicity and the same conditions on the $z$-component. Incidentally, the paper by Kojima et al. (1993) treats this convexity issue for the special case of the standard nonlinear complementarity problem.

PROPOSITION 2. Suppose that the mapping $H: R^{2n}_{++} \times R^m \rightarrow R^{n+m}$ is continuous and satisfies the monotonicity assumption (a'), the $z$-injectiveness on $R^{2n}_{++}$, and the $z$-boundedness on $R^{2n}_z$. Then the set $H_{++}$ is convex.

PROOF. It suffices to show that the set $G(R^{2n}_{++} \times R^m)$ is convex. Indeed, let $M = R^{2n}_{++} \times R^m$ and $F = G|_{R^{2n}_{++} \times R^m}$. We next show that $M$ and $F$ satisfy the hypotheses of Corollary 3. Observe that the convexity of $F(R^{2n}_{++} \times R^m) = G(R^{2n}_{++} \times R^m)$ follows from the conclusion of Corollary 3. The set $M$ is obviously path-connected, and the map $F$ is a local homeomorphism due to Theorem 5(a). It suffices to show that for any two triples $(x^i, y^i, z^i) \in M$, $i = 1, 2$, the set $F^{-1}(E)$ is compact, where $E$ is the line segment $[F(x^1, y^1, z^1), F(x^2, y^2, z^2)]$. Let $(x, y, z)$ be an arbitrary triple in the set $G^{-1}(E) \subseteq F^{-1}(E)$. By the definition of $G$, there exists $\tau \in [0, 1]$ such that

$$x \circ y = \tau (x^1 \circ y^1) + (1 - \tau) (x^2 \circ y^2),$$

$$H(x, y, z) = \tau H(x^1, y^1, z^1) + (1 - \tau) H(x^2, y^2, z^2).$$

Using the first relation and the fact that $(x^1, y^1), (x^2, y^2) \in R^{2n}_{++}$, we easily see that $(x, y) \in R^{2n}_{++}$. This implies that $(x, y, z) \in F^{-1}(E)$, and hence that $G^{-1}(E) = F^{-1}(E)$. Since $E$ and the domain of $G$ are closed sets and $G$ is continuous, it follows that $G^{-1}(E) = F^{-1}(E)$ is closed. It remains to show that $F^{-1}(E)$ is bounded, or equivalently that the arbitrary point $(x, y, z)$ as above is bounded. Write $b^i = H(x^i, y^i, z^i)$ for $i = 1, 2$, and $b(\tau) = \tau b^1 + (1 - \tau) b^2$. By the monotonicity assumption (a'), we have

$$(x^1 - x)^T (y^1 - y) \geq (1 - \tau) (b^1 - b^2)^T (h(x^1, y^1, z^1) - h(x, y, z)) + c(b^1, b(\tau)), $$

which implies

$$x^Ty^1 + y^Tx^1 \leq C_1 + (1 - \tau) (b^1 - b^2)^T h(x, y, z) \quad \text{where}$$

$$C_1 = (x^1)^T y^1 + \max \left( (x^1)^T y^1, (x^2)^T y^2 \right)$$

$$+ \max_{\tau \in [0, 1]} \left| c(b^1, b(\tau)) \right| + \left| (b^1 - b^2)^T h(x^1, y^1, z^1) \right|.$$

Similarly, we can deduce

$$x^Ty^2 + y^Tx^2 \leq C_2 + \tau (b^2 - b^1)^T h(x, y, z) \quad \text{where}$$

$$C_2 = (x^2)^T y^2 + \max \left( (x^1)^T y^1, (x^2)^T y^2 \right)$$

$$+ \max_{\tau \in [0, 1]} \left| c(b^2, b(\tau)) \right| + \left| (b^1 - b^2)^T h(x^1, y^2, z^2) \right|.$$

Multiplying (18) by $\tau$ and (19) by $1 - \tau$ and adding, we deduce

$$\left( \tau y^1 + (1 - \tau) y^2 \right)^T x + \left( \tau x^1 + (1 - \tau) x^2 \right)^T y \leq \max(C_1, C_2).$$

Thus $(x, y)$ is bounded. By the $z$-boundedness assumption, so is the vector $z$. □
In what follows, we show that the monotonicity assumption (a)' holds in two special cases of the mapping $H$; these correspond to the mixed monotone nonlinear complementarity problem and the mixed monotone horizontal linear complementarity problem. Subsequently, we shall discuss in detail the special mapping $H$ in (2) that corresponds to the KKT system of a monotone VI defined on the solution set of a system of finitely many differentiable convex inequalities; this last discussion covers the case of a convex program.

**Proposition 3.** The monotonicity assumption (a)' holds if $H$ is of either one of the two forms:

(a) $$H(x, y, z) = \begin{pmatrix} f(x, z) - y \\ g(x, z) \end{pmatrix}$$

where $f$: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g$: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy the standard monotonicity assumption: for any two pairs $(x, z), (x', z') \in \mathbb{R}^n \times \mathbb{R}^m$,

$$(x - x')^T (f(x, z) - f(x', z')) + (z - z')^T (g(x, z) - g(x', z')) \geq 0;$$

(b) $$H(x, y, z) = \begin{pmatrix} A_1 x + B_1 y + C_1 z + q_1 \\ A_2 x + B_2 y + C_2 z + q_2 \end{pmatrix}$$

where $C_1$ is a nonsingular matrix of order $m$ and the pair

$A_2' = A_2 - C_2 C_1^{-1} A_1$ and $B_2' = B_2 - C_2 C_1^{-1} B_1$,

is (column) monotone; i.e.

$$A_2' u + B_2' v = 0 \Rightarrow u^T v \geq 0.$$

**Proof.** To prove that the mapping $H$ given by (a) satisfies the monotonicity property (a)', define $h$: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n+m}$ and $c$: $\mathbb{R}^{2(n+m)} \to \mathbb{R}$ by

$$h(x, y, z) = -(x, z) \quad \text{and} \quad c \equiv 0.$$

Suppose $H(x^i, y^i, z^i) = (r^i, s^i)$ for $i = 1, 2$. We have

$$\begin{align*}
0 & \leq (x^1 - x^2)^T (f(x^1, z^1) - f(x^2, z^2)) + (z^1 - z^2)^T (g(x^1, z^1) - g(x^2, z^2)) \\
& = (x^1 - x^2)^T (y^1 - y^2 + r^1 - r^2) + (z^1 - z^2)^T (s^1 - s^2),
\end{align*}$$

which implies

$$(x^1 - x^2)^T (y^1 - y^2) \geq (H(x^1, y^1, z^1) - H(x^2, y^2, z^2))^T (h(x^1, y^1, z^1) - h(x^2, y^2, z^2)), $$

as desired.

Next, we prove that the mapping $H$ given by (b) satisfies the monotonicity property (a)' . By a property of a column monotone pair of square matrices (Sznajder and Gowda 1995), we may assume without loss of generality that $A_2$ is nonsingular and
that \(-(A_2')^{-1}B_2'\) is positive semidefinite. As before, let \(H(x', y', z') = (r^i, s^i)\) for \(i = 1, 2\). By a simple manipulation, we can easily deduce that
\[
x^1 - x^2 = -(A_2')^{-1}B_2'(y^1 - y^2) + (A_2')^{-1}[(s^1 - s^2) - C_2(C_1)^{-1}(r^1 - r^2)].
\]

Hence by defining
\[
h(x, y, z) \equiv \begin{pmatrix} (A_2')^{-T}y \\ -[(A_2')^{-1}C_2(C_1)^{-1}]^{-T}y \end{pmatrix},
\]
and \(c = 0\), we can easily verify the desired monotonicity property. \(\square\)

We end this section by discussing the application of Theorem 5 to the mapping \(H\) given by (2). We wish to specify some conditions on the functions \(F, g\) and \(h\) in order for the resulting function \(H\) to satisfy the assumptions of this theorem, and thus for the conclusions of the theorem to hold. The following lemma asserts the monotonicity of \(H\). We recall the correspondence of variables: \((u, \lambda) \leftrightarrow (x, y)\) and \((u, \eta) \leftrightarrow z\) in the notation of (2) versus that of \(H(x, y, z)\) in Theorem 5; we also recall the respective dimensions of the functions \(F, g,\) and \(h\); specifically, \(F: R^N \to R^N, g: R^N \to R^p,\) and \(h: R^N \to R^q.\)

**Lemma 7.** Suppose that the function \(F: R^N \to R^N\) is continuous and monotone, each \(g_i: R^N \to R\) is continuously differentiable and convex, and each \(h_j: R^N \to R\) is affine. Then \(H\) defined by (2) is \((\nu, \lambda)\)-monotone on \(R_+^{2p}\).

**Proof.** Suppose \(H(u, \nu, \lambda, \eta) = H(u', \nu', \lambda', \eta')\) for some \((u, \nu, \lambda, \eta)\) and \((u', \nu', \lambda', \eta')\) with \((u, \nu)\) and \((u', \nu')\) nonnegative. By the linearity of \(h\), we have for all \(j\),
\[
\nabla h_j(u)^T(u - u') = \nabla h_j(u')^T(u - u') = 0.
\]

We also have \(g(u) - g(u') = \nu' - \nu\) which implies
\[
(\lambda' - \lambda)_i(\nu' - \nu)_i = (\lambda' - \lambda)_i(g(u) - g(u'))_i.
\]

Since
\[
F(u) + \sum_{i = 1}^{p} \lambda_i \nabla g_i(u) + \sum_{j = 1}^{q} \eta_j \nabla h_j(u) = F(u') + \sum_{i = 1}^{p} \lambda'_i \nabla g_i(u') + \sum_{j = 1}^{q} \eta'_j \nabla h_j(u'),
\]
we deduce, by (20) and a simple algebraic manipulation,
\[ 0 = (u - u')^T (F(u) - F(u')) + \sum_{i=1}^{p} (\lambda_i - \lambda_i')(g_i(u) - g_i(u')) + \sum_{i=1}^{p} \left[ \lambda_i'(g_i(u) - g_i(u') - \nabla g_i(u')^T (u - u')) \right] \\
+ \lambda_i'(g_i(u') - g_i(u) - \nabla g_i(u)^T (u' - u)) \right] \\
= (u - u')^T (F(u) - F(u')) - \sum_{i=1}^{p} (\lambda_i - \lambda_i')(v_i - v_i') + \sum_{i=1}^{p} \left[ \lambda_i'(g_i(u) - g_i(u') - \nabla g_i(u')^T (u - u')) \right] \\
+ \lambda_i'(g_i(u') - g_i(u) - \nabla g_i(u)^T (u' - u)) \right] \\
\geq - \sum_{i=1}^{p} (\lambda_i - \lambda_i')(v_i - v_i'), \]

where the last inequality follows from the monotonicity of \( F \), the convexity of each \( g_i \), the nonnegativity of \( \lambda_i \) and \( \lambda_i' \), and (21). This establishes the desired monotonicity of \( H \). \( \square \)

**Remark.** By slightly refining the above proof, we can show that the stronger monotonicity assumption (a)' also holds for the same function \( H \) under the assumptions of Lemma 7.

We next establish the \((u, \eta)\)-injectiveness of \( H \) under the strict monotonicity assumption of \( F \) (or the strict convexity of some \( g_i \)) and a linear independence assumption on the matrix \( \nabla h(u) \), which is a constant for all vectors \( u \).

**Lemma 8.** Suppose that the function \( F: \mathbb{R}^N \to \mathbb{R}^N \) is continuous and monotone, each \( g_i: \mathbb{R}^N \to \mathbb{R} \) is continuously differentiable and convex, and each \( h_j: \mathbb{R}^N \to \mathbb{R} \) is affine. Assume that either \( F \) is strictly monotone or one of the functions \( g_i \) is strictly convex. Assume also that the vectors \( \{ \nabla h_j(u): j = 1, \ldots, q \} \) are linearly independent. Then \( H \) is \((u, \eta)\)-injective on \( \mathbb{R}^{2p}_{++} \).

**Proof.** Suppose \( H(u, v, \lambda, \eta) = H(u', v, \lambda, \eta') \) for some \((v, \lambda) > 0\). By using either the strict monotonicity of \( F \) or the strict convexity of some \( g_i \), the fact that \( \lambda < 0 \) and expression (23), we deduce that \( u = u' \). By (22) and the linear independence assumption of the Jacobian matrix \( \nabla h(u) \), it follows that \( \eta = \eta' \). \( \square \)

The next lemma establishes the \((u, \eta)\)-boundedness of \( H \) under the same linear independence assumption on \( \nabla h(u) \) and the assumption that the set

\[ X = \{ u \in \mathbb{R}^N | g(u) \leq 0, h(u) = 0 \} \]

is bounded and nonempty.

**Lemma 9.** Suppose that the function \( F: \mathbb{R}^N \to \mathbb{R}^N \) is continuous, each \( g_i: \mathbb{R}^N \to \mathbb{R} \) is continuously differentiable and convex, and each \( h_j: \mathbb{R}^N \to \mathbb{R} \) is affine. Assume also that the vectors \( \{ \nabla h_j(u): j = 1, \ldots, q \} \) are linearly independent and the set \( X \) defined in (24) is bounded and nonempty. Then \( H \) is \((u, \eta)\)-bounded on \( \mathbb{R}^{2p}_{+} \).
Proof. By the convexity of each $g_i$ and the affine property of each $h_j$, it can be shown that for all vectors $c \in R^p$ and nonnegative scalars $d$, the set

$$
X_{c,d} = \{u \in R^N | g(u) \leq c, \|h(u)\| \leq d \}
$$

is bounded. Indeed, consider the convex function:

$$
f(u) = \max \left\{ |h(u)|, \max_{1 \leq i \leq p} g_i(u) \right\}, \quad u \in R^N,
$$

whose level set

$$
\{ u \in R^N | f(u) \leq 0 \} = X
$$

is bounded and nonempty by assumption. Thus, by a well-known property of convex functions (Corollary 8.7.1 of Rockafellar 1970), it follows that every level set of $f(u)$ is bounded. Since $X_{c,d} \subseteq \{ u \in R^N | f(u) \leq \beta \}$, where $\beta = \max(d, \max_{i=1}^p c_i)$, we conclude that $X_{c,d}$ is bounded.

Let $\{(u^k, v^k, \lambda^k, \eta^k)\}$ be a sequence such that each $(v^k, \lambda^k)$ is nonnegative and the sequences $(\{u^k\})$ and $(\{H(u^k, v^k, \lambda^k, \eta^k)\})$ are bounded. We need to show that $\{(u^k, \eta^k)\}$ is bounded. Write $r^k = v^k + g(u^k)$ and $s^k = h(u^k)$. Since $(r^k)$ and $(s^k)$ are both bounded, let $c \in R^p$ and $d \in R$ be such that $c \geq r^k$ and $d \geq \|s^k\|$ for all $k$. Then $\{u^k\} \subseteq X_{c,d}$. By what has been proved above, it follows that $\{u^k\}$ is bounded. Hence by the boundedness of the sequence $\{L(u^k, v^k, \lambda^k, \eta^k)\}$, where

$$
L(u, v, \lambda, \eta) = F(u) + \sum_{i=1}^p \lambda_i \nabla g_i(u) + \sum_{j=1}^q \eta_j \nabla h_j(u),
$$

and the linear independence of the vectors $\{(\nabla h_j(u))\}$, it follows that $\{\eta^k\}$ is bounded.

Finally, we examine the assumption $0 \in H_{++}$ (required in part (b) of Theorem 5) for the function $H$ under consideration. In essence, the following result shows that this assumption is satisfied under the assumption of Lemma 9 and a Slater condition on the set $X$ in (24).

Lemma 10. Suppose that the function $F : R^N \to R^N$ is continuous, each $g_i : R^N \to R$ is continuously differentiable and convex, and each $h_j : R^N \to R$ is affine. Assume also that the set $X$ defined in (24) is bounded and there is a vector $\tilde{u} \in X$ such that $g(\tilde{u}) < 0$. Then there exists $(u^*, \nu^*, \lambda^*, \eta^*)$ with $(\nu^*, \lambda^*) > 0$ satisfying $H(u^*, \nu^*, \lambda^*, \eta^*) = 0$.

Proof. Let $c \equiv \frac{1}{2} g(\tilde{u})$ and write $\tilde{X} \equiv X_{c,0}$. Then by the first part of the proof of Lemma 9 (which incidentally does not require the linear independence assumption of the gradients of $h$), it follows that $\tilde{X}$ is compact. Define the function

$$
\tilde{F}(u) = F(u) - \sum_{i=1}^p g_i(u)^{-1} \nabla g_i(u), \quad u \in \tilde{X}.
$$

Noting that $g(u) < 0$ for all $u \in \tilde{X}$, we see that this function $\tilde{F}$ is well defined and continuous on $\tilde{X}$. Moreover, since $\tilde{u} \in \tilde{X}$, by a fundamental existence result for variational inequalities (see Pang 1994), it follows that the variational inequality defined by the pair $\langle \tilde{F}, \tilde{X} \rangle$ has a solution, say $u^*$. Moreover, since $\tilde{u}$ satisfies the Slater condition for the set $\tilde{X}$, there exist multipliers $(\lambda, \eta^*)$ with $\lambda \geq 0$ such that
\[ \tilde{L}(u^*, v^*, \tilde{\lambda}, \eta^*) = 0, \text{ where } v^* = -g(u^*) > 0 \text{ and} \]
\[ \tilde{L}(u, v, \lambda, \eta) = \tilde{F}(u) + \sum_{i=1}^{p} \lambda_i \nabla g_i(u) + \sum_{j=1}^{q} \eta_j \nabla h_j(u). \]

By defining \( \lambda_i^* = \tilde{\lambda}_i - 1/g_i(u^*) \) for each \( i \), it is easily seen that \( (u^*, v^*, \lambda^*, \eta^*) \) is the desired vector. \( \Box \)

Combining the above lemmas, we obtain the following result for the function \( H \) given by (2) that arises from a variational inequality defined by a monotone mapping \( F \). This corollary follows easily from Theorem 5 and requires no further proof.

**Corollary 4.** Suppose that the function \( F: R^N \to R^N \) is continuous and monotone, each \( g_i: R^N \to R \) is continuously differentiable and convex, and each \( h_j: R^N \to R \) is affine. Assume that either \( F \) is strictly monotone or one of the functions \( g_i \) is strictly convex. Assume also that the vectors \( \{\nabla h_j(u); j = 1, \ldots, q\} \) are linearly independent, that there exists a vector \( \tilde{u} \in R^N \) satisfying \( g(\tilde{u}) < 0, h(\tilde{u}) = 0 \), and that the set \( X \) defined in (24) is bounded. Then the following statements hold for the function \( H \) defined in (2):

(a) \( G \) maps \( R^N \times R^{2p} \times R^q \) homeomorphically onto \( R^p_+ \times H_{++} \);

(b) \( R^p_+ \times H_{++} \subseteq G(R^N \times R^{2p} \times R^q) \);

(c) for every vector \( a \in R^p_+ \), the system

\[ H(u, v, \lambda, \eta) = 0, \quad v \circ \lambda = a \]

has a solution \( (u, v, \lambda, \eta) \in R^N \times R^{2p} \times R^q \); and

(d) the set \( H_{++} \) is convex.

6. **Maximal monotonicity.** In this section we return to a general function \( H: R^{2n}_+ \times m \to R^{n+m} \); we are interested in establishing conditions under which the operator \( \mathcal{A}: R^m_+ \to R^m_+ \) defined in (5) is maximal monotone. For this purpose, we introduce some definition and notation.

If \( X \) and \( Y \) are two metric spaces, we shall denote a set-valued map \( M \) from \( X \) into subsets of \( Y \) by \( M: X \to Y \); the graph of \( M \) is the set

\[ \text{Gr}(M) = \{(x, y) \in X \times Y | y \in M(x) \}. \]

A subset \( \Gamma \subseteq R^{2n} \) is said to be a monoton e set if for every \( (x, y) \in \Gamma \) and \((x', y') \in \Gamma \), there holds \( (x - x')^T(y - y') \geq 0 \). A set-valued map \( M: X \to Y \), with \( X \) and \( Y \) being subsets of \( R^n \), is called a monotone operator if \( \text{Gr}(M) \) is a monotone set. The monotone operator \( M: X \to Y \) is said to be maximal with respect to a subset \( W \subseteq X \times Y \) if there exists no monotone set \( \Gamma \subseteq W \) that properly contains \( W \cap \text{Gr}(M) \). If \( W = X \times Y \), we will simply say that the monotone operator \( M: X \to Y \) is maximal.

The following result gives sufficient conditions for the maximality of a monotone operator. The first condition is the "easy" implication of the "if and only if" characterization for the maximality of a monotone operator \( M: R^n \to R^n \) due to Minty (1962).

**Theorem 6.** Assume that \( M: X \to Y \) is a monotone operator, where \( X \) and \( Y \) are subsets of \( R^n \). The following two statements hold:

(a) if \( X = Y = R^n \) and, for every \( a \in R^n \), the system

\[ x + y = a, \quad y \in M(x) \]

has a solution, then \( M \) is maximal.
(b) If \( R^+_+ \times R^+_+ \subseteq X \times Y \) and, for every \( a \in R^+_+ \), the system

\[
x \circ y = a, \quad y \in M(x)
\]

has a solution \((x, y) \in R^{2n}_+\), then \( M \) is maximal with respect to \( R^{2n}_+ \).

**Proof.** We only prove (b) since the proof of (a) is similar. Let \( \Gamma \subseteq R^{2n}_+ \) be a monotone set containing \( R^{2n}_+ \cap \text{Gr}(M) \). We have to show that \( \Gamma = R^{2n}_+ \cap \text{Gr}(M) \).

Indeed, let \((x^0, y^0) \in \Gamma \) be given. By assumption, system (26) with \( a = x^0 \circ y^0 > 0 \) has a solution \((\tilde{x}, \tilde{y}) \in R^{2n}_+\), that is

\[
\tilde{x} \circ \tilde{y} = x^0 \circ y^0,
\]

\[
(\tilde{x}, \tilde{y}) \in \text{Gr}(M).
\]

Since both \((x^0, y^0)\) and \((\tilde{x}, \tilde{y})\) are in \( \Gamma \) and the set \( \Gamma \) is monotone, we conclude that

\[
(x^0 - \tilde{x})^T(y^0 - \tilde{y}) \geq 0.
\]

Relations (27) and (29) together with Lemma 5(b) then imply that \((x^0, y^0) = (\tilde{x}, \tilde{y})\) and this, in view of (28), shows that \((x^0, y^0) \in \text{Gr}(M)\). Hence, \( \Gamma = R^{2n}_+ \cap \text{Gr}(M) \) and (b) follows. \( \square \)

We can now state two results which guarantee the maximality of certain operators associated with a mapping \( H(x, y, z) \).

**Theorem 7.** Suppose that the mapping \( H: R^{2n}_+ \times R^m \to R^{n+m} \) is continuous and satisfies the \((x, y)\)-monotonicity on \( R^{2n}_+ \), the \( z \)-injectiveness on \( R^{2n}_+ \), and the \( z \)-boundedness on \( R^{2n}_+ \). Then, for any \( b \in H^{++} \), the set-valued map \( \mathcal{A}_b: R^n \mapsto R^n \) defined by

\[
\mathcal{A}_b(x) = \{ y \in R^n | H(x, y, z) = b \text{ for some } z \in R^m \}
\]

is monotone and maximal with respect to \( R^{2n}_+ \).

**Proof.** The operator \( \mathcal{A}_b \) is trivially monotone. It remains to show the maximality of \( \mathcal{A}_b \) with respect to \( R^{2n}_+ \). In view of Theorem 6(b), it is sufficient to show that for every \( a \in R^{2n}_+ \), the system \( x \circ y = a \) and \( y \in \mathcal{A}_b(x) \) has a solution \((x, y) \in R^{2n}_+\), or equivalently, that the system \( x \circ y = a \) and \( H(x, y, z) = b \) has a solution \((x, y, z) \in R^{2n}_+ \times R^m \). That last system has a solution in \( R^{2n}_+ \times R^m \) follows immediately from Theorem 5(a). \( \square \)

**Theorem 8.** Suppose that the mapping \( H: R^{2n}_+ \times R^m \to R^{n+m} \) is continuous and satisfies the \((x, y)\)-monotonicity on \( R^{2n}_+ \), the \( z \)-injectiveness on \( R^{2n}_+ \), and the \( z \)-boundedness on \( R^{2n}_+ \). Then, for any \( b \in H(R^{2n+m}) \), the multivalued operator \( \mathcal{B}_b: R^n \mapsto R^n \) defined by

\[
\mathcal{B}_b(x) = \{ y \in R^n | H(x, y, z) = b \text{ for some } z \in R^m \}
\]

is maximal monotone.

**Proof.** The proof of this result is similar to the one of Theorem 7 except that now we use Theorem 6(a) and the lemma stated below. \( \square \)

**Lemma 11.** Suppose that the mapping \( H: R^{2n}_+ \times R^m \to R^{n+m} \) is continuous and satisfies the \((x, y)\)-monotonicity on \( R^{2n}_+ \), the \( z \)-injectiveness on \( R^{2n}_+ \), and the \( z \)-boundedness
on $R^{2n}$. Then, the mapping $F: R^{2n+m} \to R^n \times R^{n+m}$ defined by

\[ F(x, y, z) = \begin{pmatrix} x + y \\ H(x, y, z) \end{pmatrix} \quad \text{for } (x, y, z) \in R^{2n} \times R^m, \]

maps $R^{2n+m}$ homeomorphically onto $R^n \times H(R^{2n+m})$.

**Proof.** Using the $(x, y)$-monotonicity and the $z$-injectiveness of $H$, one can easily show that $F$ is one-to-one on $R^{2n+m}$. By the domain invariance theorem, it follows that $F(R^{2n+m})$, and hence $H(R^{2n+m})$, is an open set and that $F$ maps $R^{2n+m}$ onto $F(R^{2n+m})$ homeomorphically. In particular, it follows that $F$ is a local homeomorphism. The result now follows from Corollary 2 once we show that $F$ is proper with respect to $R^n \times H(R^{2n+m})$. The proof of this fact is very similar to the proof of Lemma 6. Indeed, let $S$ be a compact subset of $R^n \times H(R^{2n+m})$. By continuity of $F$, it follows that $F^{-1}(S)$ is a closed subset. We next show that $F^{-1}(S)$ is a bounded subset from which it follows that $F^{-1}(S)$ is a compact subset, and hence that $F$ is proper with respect to $R^n \times H(R^{2n+m})$. Indeed, suppose for contradiction that there exists a sequence $((x_k, y_k, z_k)) \subset F^{-1}(S)$ for which

\[ \lim_{k \to \infty} \|(x_k, y_k, z_k)\| = \infty. \]

Since $S$ is compact, we may assume without loss of generality that there exists a vector $H^\infty \in H(R^{2n+m})$ such that

\[ H^\infty = \lim_{k \to \infty} H(x_k, y_k, z_k). \]

Let $(x^\infty, y^\infty, z^\infty) \in R^{2n} \times R^m$ be such that $H(x^\infty, y^\infty, z^\infty) = H^\infty$ and let $B^\infty$ be the open ball with center at $(x^\infty, y^\infty)$ and radius 1. By the observations made above, we know that $H(B^\infty \times R^m)$ is an open set which contains $H^\infty = (H(x^\infty, y^\infty, z^\infty)$. Thus, for all $k$ sufficiently large, $H(x_k, y_k, z_k) \in H(B^\infty \times R^m)$. Hence there exists $(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k) \in B^\infty \times R^m$ such that $H(x_k, y_k, z_k) = H(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)$. By the $(x, y)$-monotonicity of $H$, we deduce that

\[ 0 \leq (x_k - \tilde{x}_k)^T (y_k - \tilde{y}_k) \]

\[ = \frac{1}{2} \|(x_k + y_k) - (\tilde{x}_k + \tilde{y}_k)\|^2 - \frac{1}{2} \|(x_k - y_k) - (\tilde{x}_k - \tilde{y}_k)\|^2, \]

which in turn implies that

\[ \|(x_k - y_k)\| \leq \|(x_k + y_k) - (\tilde{x}_k + \tilde{y}_k)\| + \|\tilde{x}_k - \tilde{y}_k\|. \]

Since the sequences $\{\tilde{x}_k\}$, $\{\tilde{y}_k\}$ and $\{x_k + y_k\}$ are bounded, it follows from (31) that the sequence $\{x_k - y_k\}$ is bounded. The fact that both sequences $\{x_k + y_k\}$ and $\{x_k - y_k\}$ are bounded obviously imply that $\{x_k\}$ and $\{y_k\}$ are bounded. In view of the $z$-boundedness of $H$ and the fact that $(H(x_k, y_k, z^k))$ is bounded, it also follows that $\{z_k\}$ is bounded. But this is a contradiction and the result follows.

**Appendix A.** In this appendix we give the proof of Theorem 1. We first state three lemmas and give references where the reader can find their proofs.
LEMMA 12. Assume that $F$ is a local homeomorphism which is proper. Then, the following statements hold:

(a) $F$ maps closed subsets of $M$ onto closed subsets of $N$; in particular, $F(M)$ is a closed subset of $N$;

(b) $[v]$ is finite for every $v \in N$;

(c) $[v]$ is locally constant, that is, for every $v \in N$, there exists an open set $V \ni v$ such that $[w] = [v]$ for every $w \in V$.

PROOF. See Theorem 1.2 of Chapter 3 of Ambrosetti and Prodi (1993). \hfill \Box

LEMMA 13. Assume that $F$ is a local homeomorphism and let $q: [a, b] \to N$ and $(t_0, u_0) \in [a, b] \times M$ satisfy $F(u_0) = q(t_0)$. Then the following statements hold:

(a) there exists at most one path $p: [a, b] \to M$ such that $p(t_0) = u_0$ and $F(p(t)) = q(t)$ for all $t \in [a, b]$;

(b) a sufficient condition for a path $p: [a, b] \to M$ in (a) to exist is that the set $F^{-1}(q([a, b])) \subseteq M$ be compact.

PROOF. See the proof of Lemma 1.11 in Chapter 3 of Ambrosetti and Prodi (1993). \hfill \Box

LEMMA 14. Assume that $F$ is a local homeomorphism and let continuous mappings $\beta: [a, b] \times [c, d] \to N$ and $\theta: [a, b] \to M$ be given such that $\beta(s, 0) = F(\theta(s))$ for all $s \in [a, b]$. Assume also that $F$ is invertible along the path $\beta_0: [c, d] \to N$ for any $s \in [a, b]$ where $\beta_s$ is defined by $\beta_s(t) = \beta(s, t)$ for all $t \in [c, d]$. Then there exists a unique continuous mapping $\alpha: [a, b] \times [c, d] \to M$ such that $\alpha(s, 0) = \theta(s)$ for all $s \in [a, b]$ and $F(\alpha(s, t)) = \beta(s, t)$ for all $(s, t) \in [a, b] \times [c, d]$. Moreover, if $\beta(s, 1) = \beta(0, t) = \beta(1, t) = v$ for all $(s, t) \in [a, b] \times [c, d]$, then $\theta(0) = \theta(1)$.

PROOF. See the proof of Theorem 5.3.4 in Ortega and Rheinboldt (1970). \hfill \Box

Using the above three lemmas, we are now in a position to give the proof of Theorem 1.

PROOF OF THEOREM 1. We first show the equivalence between (a) and (b).

(a) $\Rightarrow$ (b): Let $K$ be a compact subset of $N$. To show that $F^{-1}(K)$ is compact, we will show that every sequence $\{x^k\}$ in $F^{-1}(K)$ has an accumulation point in $F^{-1}(K)$. Indeed, let $y^k = F(x^k)$ for all $k$. Since $(y^k) \subseteq K$ and $K$ is compact, there is a subsequence $(y^k)_{k \in \mathcal{K}}$ converging to some $\bar{y} \in K$. Let $F^{-1}(\bar{y}) = \{\bar{x}_1, \ldots, \bar{x}_m\}$. Since $F$ is a local homeomorphism, we can find disjoint open sets $\bar{x}_i \in U_i, \bar{K} \subseteq M, i = 1, \ldots, m$ and an open set $V \subseteq N$ containing $\bar{y}$ such that $F(U_i) = V$ and $F|_{U_i \cap V} \in \text{Hom}(U_i, V)$ for all $i = 1, \ldots, m$. Since $(y^k)_{k \in \mathcal{K}}$ converges to $\bar{y}$, we may assume that $y^k \in V$ for all $k \in \mathcal{K}$. Each $U_i$ contains a unique point $u^k_i$ such that $F(u^k_i) = y^k$. Observe that the points $u^k_i, i = 1, \ldots, m$, are distinct since the sets $U_i, i = 1, \ldots, m$, are disjoint. By assumption, we have $[y^k] = [\bar{y}] = m$, and so we must have $F^{-1}(y^k) = \{u^k_i | i = 1, \ldots, m\}$. Since $x^k \in F^{-1}(y^k)$, we conclude that $x^k = u^k_i$ for some $i = i(k) \in \{1, \ldots, m\}$. Thus, there exists an index $i \in \{1, \ldots, m\}$ and an infinite index set $\mathcal{K} \subseteq \mathcal{K}$ such that $\{x^k\}_{k \in \mathcal{K}} \subseteq U_i$. Since $x^k = (F|_{U_i \cap V})^{-1}(y^k)$ for all $k \in \mathcal{K}$ and $(y^k)_{k \in \mathcal{K}}$ converges to $\bar{y}$, it follows that $(x^k)_{k \in \mathcal{K}}$ converges to $\bar{x}_i \in F^{-1}(K)$. We have thus shown that $F^{-1}(K)$ is compact.

(b) $\Rightarrow$ (a): Fix some $v_0 \in N$ and let $\sigma_1 = \{v \in N | [v] = [v_0]\}$ and $\sigma_2 = \{v \in N | [v] \neq [v_0]\}$. Clearly, $(\sigma_1, \sigma_2)$ is partition of $N$ and, by Lemma 12(c), both sets $\sigma_1$ and $\sigma_2$ are open. Since $N$ is connected, one of these sets must be empty. The only possibility is to have $\sigma_2 = \emptyset$. Hence, (a) holds.

We next show the equivalence of (a) and (b) with statement (c).

(b) $\Rightarrow$ (c): Let $q: [a, b] \to N$ be given. Since $F$ is proper and $q([a, b])$ is compact, it follows that $F^{-1}(q([a, b])) \subseteq M$ is compact. Hence, Lemma 13 implies that $F$ is
invertible along $q$. By the local homeomorphism and properness of $F$, it follows easily that $[v]$ is a finite positive integer for all $v \in F(M) \subseteq N$.

(c) $\Rightarrow$ (a): We will first show that $[v_0] \geq 1$. Indeed, choose a point $\bar{v} \in F(M)$ and let $\bar{u} \in M$ be such that $F(\bar{u}) = \bar{v}$. Since $N$ is path-connected, there exists a path $\bar{q}: [0, 1] \to N$ such that $\bar{q}(0) = \bar{v}$ and $\bar{q}(1) = v_0$. By assumption, $F$ is invertible along $\bar{q}$ and hence, there exists a path $\bar{p}$ such that $\bar{p}(0) = \bar{u}$ and $F(\bar{p}(t)) = \bar{q}(t)$ for all $t \in [0, 1]$. In particular, we have $F(\bar{p}(1)) = \bar{q}(1) = v_0$ which shows that $[v_0] \geq 1$. We will next show that $[v_0] = [v]$ for all $v \in N$. Indeed, let $v \in N$ be given. Since $N$ is path-connected, there exists a path $q: [0, 1] \to N$ such that $q(0) = v_0$ and $q(1) = v$. Using this path, we will construct a bijection $\phi: F^{-1}(v_0) \to F^{-1}(v)$, which therefore shows that $[v_0] = [v]$. Indeed, given $u_0 \in F^{-1}(v_0)$, let $p: [0, 1] \to M$ be the unique path such that $p(0) = u_0$ and $F(p(t)) = q(t)$ for all $t \in [0, 1]$. Define $\phi(u_0) = p(1) \in F^{-1}(v)$. Clearly, $\phi$ is well defined. To show that $\phi$ is one-to-one, assume that $\phi(u_0) = \phi(u_1) = u$ for some $u_0, u_1 \in F^{-1}(v_0)$. By definition of $\phi$, we have $\phi(u_i) = p_i(1)$ for some path $p_i: [0, 1] \to M$ such that $p_i(0) = u_i$ and $F(p_i(t)) = q(t)$ for all $t \in [0, 1]$ ($i = 0, 1$). Hence, $p_0(1) = p_1(1) = u$. Therefore, both paths $p_0$ and $p_1$ invert $F$ along $q$ through the point $(1, u)$, and so $p_0 = p_1$. In particular, we have $u_0 = p_0(0) = p_1(0) = u_1$. We now show that $\phi$ is onto. Indeed, let $u \in F^{-1}(v)$ be given. By assumption, there exists a path $p: [0, 1] \to M$ which inverts $F$ along $q$ through the point $(1, u)$. Hence, $F(p(0)) = q(0) = v_0$ and so $p(0) \in F^{-1}(v_0)$. Moreover, by the definition of $\phi$, we have $\phi(p(0)) = p(1) = u$. Hence, $\phi$ is onto.

We now show the equivalence of (d) with the first three conditions. The implication (d) $\Rightarrow$ (a) is straightforward. Conversely, (a) implies that $F$ is onto. We will show that (c) implies that $F$ is one-to-one. Let $u_0, u_1 \in M$ be such that $F(u_0) = F(u_1) = u$. Since $M$ is path-connected, there exists a path $\theta: [0, 1] \to M$ such that $\theta(0) = u_0$ and $\theta(1) = u_1$. Since $N$ is simply connected, there exists a continuous mapping $\beta: [0, 1] \times [0, 1] \to N$ such that $\beta(s, 0) = F(\theta(s))$ and $\beta(s, 1) = u$ for all $s \in [0, 1]$ and $\beta(0, t) = \beta(1, t) = v$ for all $t \in [0, 1]$. Since the mappings $\beta$ and $\theta$ satisfy the hypothesis of Lemma 14, it follows that $\theta(0) = \theta(1)$, that is $u_0 = u_1$. Hence $F$ is a local homeomorphism which is both one-to-one and onto, so $F \in \text{Hom}(M, N)$.

We next show the equivalence of (e) and (f) with the other conditions. The implication (e) $\Rightarrow$ (f) is straightforward. The implication (e) $\Rightarrow$ (f) is proved exactly like the implication (b) $\Rightarrow$ (c) except that now we use the fact that $q(\alpha, b)$ is a segment in order to guarantee the compactness of the set $F^{-1}(q(\alpha, b))$. The proof of the implication (f) $\Rightarrow$ (d) is exactly like the one given for the implication [(a), (b), and (c)] $\Rightarrow$ (d) except that now we can construct the mapping $\beta: [0, 1] \times [0, 1] \to N$ explicitly as $\beta(s, t) = (1 - t)F(\theta(s)) + tv$ for all $(s, t) \in [0, 1] \times [0, 1]$. Observe that since the path $\beta_t$ is affine, the mapping $F$ is invertible along $\beta_t$ for any $t \in [0, 1]$.

\[ \square \]

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