INEXACT CUTS IN SDDP APPLIED TO MULTISTAGE
STOCHASTIC NONDIFFERENTIABLE PROBLEMS
VINCENT GUIGUES∗, RENATO MONTEIRO†, AND BENAR SVAITER‡

Abstract. In [13], an Inexact variant of Stochastic Dual Dynamic Programming (SDDP) called
ISDDP was introduced which uses approximate (instead of exact with SDDP) primal dual solutions
of the problems solved in the forward and backward passes of the method. That variant of SDDP
was studied in [13] for linear and for differentiable nonlinear Multistage Stochastic Programs (MSPs).
In this paper, we extend ISDDP to nondifferentiable MSPs. We first provide formulas for inexact
cuts for value functions of convex nondifferentiable optimization problems. We then combine these
cuts with SDDP to describe ISDDP for nondifferentiable MSPs and analyze the convergence of the
method. More precisely, for a problem with 7 stages, we show that for errors bounded from above
by , the limit superior and limit inferior of sequences of upper and lower bounds on the optimal
value of the problem are at most at distance 3 to the optimal value and that for asymptotically
vanishing errors ISDDP converges to an optimal policy.

Key words. Stochastic optimization, SDDP, Inexact cuts for value functions, Inexact SDDP.

AMS subject classifications. 90C15, 90C90, 90C30

1. Introduction. Multistage stochastic programs (MSPs) offer a framework to
model many real-life applications but are challenging to solve, see [26] for a thorough
review on MSPs.

A possible approach to approximately solve such problems is to restrict the poli-
cies to be decision rules belonging to specific classes of parametric functions, see for
instance [19] and references therein. In this situation, most studies have focused on
classes of problems and of decision rules allowing for a reformulation of the problem
(either tight or with controlled accuracy) as a tractable optimization problem, i.e., a
well structured convex optimization problem. This strategy has also been used in the
context of Robust Optimization where uncertain parameters are assumed to belong
to convex, nonempty, compact sets (see [2] for a thorough presentation of Robust
Optimization) for instance in [3].

Another approach to solve MSPs formulated using Dynamic Programming equa-
tions is to approximate the recourse functions. Two important classes of such methods
are Approximate Dynamic Programming [24] and Stochastic Dual Dynamic Program-
ming (SDDP) [21] which is a sampling-based extension of the Nested Decomposition
method [5], closely related to Stochastic Decomposition [16].

Several variants of SDDP have been proposed such as CUPPS [7], ReSa [17], the
Abridged Nested Decomposition [6], MIDAS [22] for monotonic Bellman functions, or
risk-averse variants [15], [25], [11], [18]. For convergence analysis of the method and
variants see [23], [9], [12], [1]. We also refer to [8] which explains how to take advantage
of the stationarity of the underlying stochastic processes to solve MSPs with SDDP
and to [13], [20] for variants which can accelerate the convergence of SDDP. In partic-
ular, in [13], an Inexact variant of SDDP called ISDDP was introduced which allows

∗School of Applied Mathematics, Fundação Getulio Vargas, 190 Praia de Botafogo, Rio de Janeiro,
Brazil, (vincent.guigues@fgv.br). Research of this author was partially supported by CNPq grants
† School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA
30332-0205, USA, (renato.monteiro@isye.gatech). Research of this author was partly supported by
CNPq grant 401371/2014-0.
‡ IMPA, Estrada Dona Castorina, 110, Jardim Botânico, Rio de Janeiro, Brazil. (benar@impa.br).
Research of this author was partly supported by CNPq grant 401371/2014-0.
us to solve approximately the optimization subproblems of the forward and backward passes of SDDP and to increase the accuracy of the solutions of these subproblems along the iterations of the method. ISDDP can be seen as an extension to multistage and both linear and nonlinear problems of [28] where inexact cuts were combined with Benders Decomposition [4] to solve two-stage stochastic linear programs. An inexact Stochastic Dynamic Cutting Plane (another variant of SDDP solving approximately the subproblems along the iterations of the method) was also introduced in [14] to solve MSPs. For all these inexact variants, convergence can be shown for vanishing noises and numerical experiments in [28], [13] have shown that convergence can be achieved quicker with these inexact variants.

An important tool in the development of these inexact variants is the computation of inexact cuts for value functions of optimization problems, i.e., affine lower bounding functions for the value function on the basis of approximate primal-dual solutions. This task can be easily achieved for value functions of linear programs, see for instance Proposition 2.1 in [13]. For nonlinear differentiable problems, the derivation of inexact cuts is given in Propositions 2.2 and 2.3 in [13] and Proposition 3.8 in [10]. However, this task is more delicate for nondifferentiable optimization problems.

In this paper, we extend these results developing tools to compute inexact cuts for value functions of nondifferentiable optimization problems. The problem can be stated as follows. Let $Q : X \to \mathbb{R}$ be the value function given by

$$Q(x) = \left\{ \begin{array}{l}
\min_{y \in \mathbb{R}^m} f(y, x) \\
y \in Y, Ay + Bx = b, g_i(y, x) \leq 0, i = 1, \ldots, p,
\end{array} \right. \quad (1.1)$$

where $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ and where

$$(H0) \ X \text{ and } Y \text{ are convex, closed, and nonempty sets and } f, g_i : Y \times X \to [-\infty, +\infty] \text{ are proper, lower semicontinuous, convex, and possibly nondifferentiable.}$$

Due to (H0) value function $Q$ is convex and if $\bar{x} \in \text{ri}(\text{dom}(Q))$ then $Q$ is subdifferentiable at $\bar{x}$ and there exists a cut (a lower bounding affine function) for $Q$ at $\bar{x}$ which coincides with $Q$ at $\bar{x}$. More generally, under some assumptions, the characterization of the subdifferential of $Q$ at $\bar{x} \in X$ was given in [12, Lemma 2.1] and formulas for affine lower bounding functions for $Q$ were derived in [10, Proposition 3.2] on the basis of optimal primal-dual solutions to (1.1). When only approximate primal-dual solutions are available, we can only compute inexact cuts which are still lower bounding functions for the value function but which do not coincide with this function at the point $\bar{x}$ used to compute the cut. Formulas for computing inexact cuts on the basis of approximate primal-dual solutions to (1.1) were derived in [13, 10] when functions $f, g_i$ are differentiable.

In this paper, we extend in Sections 2, 3 this analysis considering possibly nondifferentiable functions $f, g_i$. More precisely, Section 3 provides formulas for inexact cuts when the objective $f$ has a Fenchel-type representation while Section 2 derives inexact cuts using a reformulation of the problem that adds some variables and constraints. In the case when $f$ and $g_i$ are differentiable, we compare in Section 4 the formulas for inexact cuts from [13] and the formulas from Section 2. In Section 5, we describe ISDDP for possibly nondifferentiable MSPs combining the framework of SDDP with the inexact cuts derived in Sections 2 and 3. In this section, we also study the convergence of ISDDP. A useful tool for the convergence analysis of SDDP and ISDDP is Lemma 5.2 in [9] for vanishing errors and Lemma 4.1 in [13] for bounded errors. We provide different proofs of these lemmas with slightly different assump-
tions (see the corresponding Lemmas 5.1 and 5.2) and derive a stronger conclusion. More precisely, one of our assumptions is stronger (the continuity of $f$ [which is satisfied when the lemmas are applied to study the convergence of ISDDP]) and two are weaker. These proofs also shed some new light on the convergence of SDDP and ISDDP, showing the almost sure uniform convergence of the approximate Bellman functions to a continuous function which coincides with the true Bellman functions at all accumulation points of the sequences of trial points. Interestingly, as for ISDDP applied to linear programs studied in [13], we show that for a problem with $T$ stages and errors bounded from above by $\varepsilon$, the limit superior and limit inferior of sequences of upper and lower bounds on the optimal value of the problem are at most at distance $3\varepsilon T$ to the optimal value. Finally, similarly to ISDDP for nonlinear differentiable programs developed in [13], we show the convergence of ISDDP to an optimal policy for vanishing noises.

2. Inexact cuts for value functions of convex optimization problems.

In the sequel, the usual scalar product in $\mathbb{R}^n$ is denoted by $\langle x, y \rangle = x^\top y$ for $x, y \in \mathbb{R}^n$. The corresponding norm is $\|x\| = \|x\|_2 = \sqrt{\langle x, x \rangle}$.

The objective of this section is to compute inexact cuts with controlled accuracy $\varepsilon$ for value functions $Q$ of form (1.1) on the basis of approximate primal-dual solutions to (1.1) solved for a given $x = \bar{x}$. We will call these cuts $\varepsilon$-inexact cuts at $\bar{x}$:

**Definition 2.1 (\(\varepsilon\)-inexact cut.).** Let $Q : X \rightarrow \mathbb{R}$ be a convex function with $X$ convex, $X \subset \text{ri}(\text{dom}(Q))$, and let $\varepsilon \geq 0$. We say that $C : X \rightarrow \mathbb{R}$ is an $\varepsilon$-inexact cut for $Q$ at $\bar{x} \in X$ if $C$ is an affine function satisfying $Q(x) \geq C(x)$ for all $x \in X$ and $Q(\bar{x}) - C(\bar{x}) \leq \varepsilon$.

**Remark 2.1.** A $0$-inexact cut for $Q$ at $\bar{x}$, i.e., an $\varepsilon$-inexact cut at $\bar{x}$ with $\varepsilon = 0$ will be called an exact cut for $Q$ at $\bar{x}$.

2.1. **Affine functions of the argument in the constraints.** We start computing inexact cuts for particular value functions $Q$ where the argument of this function only appears in the constraints through affine functions of this argument. The study of this case will help us discuss the general case of a value function of form (1.1) considered in the next Section 2.2.

More precisely, we consider value functions $Q$ of form:

\[
Q(x) = \begin{cases}
\min_{y \in \mathbb{R}^m} f(y) \\
g(y) \leq Cx, \\
Ay + Bx = b, \\
y \in Y,
\end{cases}
\]

along with the corresponding dual problem given by

\[
\begin{aligned}
\max_{\lambda, \mu} \theta_x(\lambda, \mu) \\
\mu \geq 0, \lambda,
\end{aligned}
\]

where dual function $\theta_x(\lambda, \mu)$ is given by

\[
\theta_x(\lambda, \mu) = \min \{ L_x(y, \lambda, \mu) : y \in Y \}
\]

for the Lagrangian

\[
L_x(y, \lambda, \mu) = f(y) + \langle \lambda, Ay + Bx - b \rangle + \langle \mu, g(y) - Cx \rangle.
\]
Proposition 2.2 provides a formula for computing inexact cuts for value function \( Q \) given by (2.2):

**Proposition 2.2.** Assume that \( f : \mathbb{R}^m \to ]-\infty, +\infty[ \) and component functions \( g_i : \mathbb{R}^m \to ]-\infty, +\infty[ \), \( i = 1, \ldots, p \), of \( g \) are proper, convex, and lower semicontinuous. Assume that \( \hat{y} \) is an \( \varepsilon_P \)-optimal feasible solution of problem (2.2) for \( x = \bar{x} \) and that \( (\hat{\lambda}, \hat{\mu}) \) is an \( \varepsilon_D \)-optimal feasible solution of the corresponding dual problem (2.3) for \( x = \bar{x} \). Assume that \( f \) is finite on \( \{ y \in Y : Ay + B\bar{x} = b, g(y) \leq C\bar{x} \} \) and that Slater constraint qualification holds for (2.2) written for \( x = \bar{x} \), i.e., there is \( y_{\bar{x}} \in ri(Y) \), such that \( Ay_{\bar{x}} + B\bar{x} = b, g(y_{\bar{x}}) < C\bar{x} \). Then

\[
C(x) = f(\hat{y}) - (\varepsilon_P + \varepsilon_D) + \langle B^T \hat{\lambda} - C^T \hat{\mu}, x - \bar{x} \rangle
\]

is an \( (\varepsilon_P + \varepsilon_D) \)-inexact cut for \( Q \) at \( \bar{x} \).

**Proof.** By definition of \( \hat{y} \), we get

\[
(2.5) \quad f(\hat{y}) \leq Q(\bar{x}) + \varepsilon_P.
\]

The assumptions of the Convex Duality theorem are satisfied for problem (2.2) and its dual (2.3), both written for \( x = \bar{x} \). Therefore the optimal value of dual problem (2.3) written for \( x = \bar{x} \) is the optimal value \( Q(\bar{x}) \) of the corresponding primal problem. Using the definition of \( \hat{\lambda}, \hat{\mu} \), it follows that

\[
(2.6) \quad \theta_{\bar{x}}(\hat{\lambda}, \hat{\mu}) \geq Q(\bar{x}) - \varepsilon_D.
\]

Next,

\[
\begin{align*}
Q(x) & \geq \theta_{\bar{x}}(\hat{\lambda}, \hat{\mu}) \text{ by weak duality and feasibility of } \hat{\mu}, \hat{\lambda}, \\
& = \min\{ L_{\bar{x}}(y, \hat{\lambda}, \hat{\mu}) : y \in Y \}, \\
& = \langle \hat{\lambda}, B(x - \bar{x}) \rangle + \langle \hat{\mu}, -C(x - \bar{x}) \rangle + \min\{ L_{\bar{x}}(y, \hat{\lambda}, \hat{\mu}) : y \in Y \}, \\
& = \langle B^T \hat{\lambda} - C^T \hat{\mu}, x - \bar{x} \rangle + \theta_{\bar{x}}(\hat{\lambda}, \hat{\mu}), \\
& \overset{(2.6)}{=} \langle B^T \hat{\lambda} - C^T \hat{\mu}, x - \bar{x} \rangle + Q(\bar{x}) - \varepsilon_D, \\
& \overset{(2.5)}{\geq} C(x) := \langle B^T \hat{\lambda} - C^T \hat{\mu}, x - \bar{x} \rangle + f(\hat{y}) - \varepsilon_P - \varepsilon_D.
\end{align*}
\]

Moreover, since \( f(\hat{y}) \geq Q(\bar{x}) \), we get

\[
Q(\bar{x}) - C(\bar{x}) = \varepsilon_P + \varepsilon_D + Q(\bar{x}) - f(\hat{y}) \leq \varepsilon_P + \varepsilon_D,
\]

and we have shown that \( C \) is an \( (\varepsilon_P + \varepsilon_D) \)-inexact cut for \( Q \) at \( \bar{x} \). \( \square \)

**Remark 2.2.** The proof of Proposition 2.2 also shows that if \( \theta_{\bar{x}}(\hat{\lambda}, \hat{\mu}) \) can be computed exactly (i.e., if optimization problem (2.4) written for \( x = \bar{x}, \lambda = \hat{\lambda}, \mu = \hat{\mu} \) is solved to optimality) then \( C(x) = \theta_{\bar{x}}(\hat{\lambda}, \hat{\mu}) + \langle B^T \hat{\lambda} - C^T \hat{\mu}, x - \bar{x} \rangle \) is an \( \varepsilon_D \)-inexact cut for \( Q \) at \( \bar{x} \).

**2.2. General value functions.** We now consider general value functions of form

\[
(2.7) \quad Q(x) = \begin{cases} 
\min_{y \in \mathbb{R}^m} f(y, x) \\
g(y, x) \leq 0, \\
Ay + Bx = b, \\
y \in Y.
\end{cases}
\]
Analyzing the proof of Proposition 2.2 dedicated to the special case of value functions of form (2.2), we observe that the linearity in $x$ of Lagrangian function $L$ was crucial to derive our formula for inexact cuts. The Lagrangian obtained dualizing coupling constraints in problem (2.7) does not satisfy this property anymore. However, we can reformulate equivalently the problem in such a way that the Lagrangian of the reformulated problem satisfies this property. This reformulation is obtained adding variable $z \in \mathbb{R}^n$ together with the constraint $z = x$. We obtain the equivalent representation of problem (2.7) under the form

$$Q(x) = \begin{cases} \min_{y \in \mathbb{R}^m, z \in \mathbb{R}^n} f(y, z) \\
g(y, z) \leq 0, \\
Ay + Bz = b, \\
y \in Y, \\
z = x. \end{cases}$$

(2.8)

Denoting by $S$ the set

$$S = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n : g(y, z) \leq 0, Ay + Bz = b, y \in Y\},$$

(2.9)

and dualizing the coupling constraint $z = x$ in problem (2.8), we obtain the dual problem given by

$$\begin{cases} \max_{\lambda} \theta_x(\lambda) \\
\lambda \in \mathbb{R}^n, \end{cases}$$

(2.10)

where dual function $\theta_x(\lambda)$ is given by

$$\theta_x(\lambda) = \min \{L_x(y, z, \lambda) : (y, z) \in S\}$$

(2.11)

now for the Lagrangian

$$L_x(y, z, \lambda) = f(y, z) + \langle \lambda, x - z \rangle,$$

which, as in the special case considered in the previous section, is a linear function of $x$. Therefore, for every $x, \bar{x} \in X$, for every $(y, z) \in S$, and $\lambda$, we have

$$L_x(y, z, \lambda) = \langle \lambda, x - \bar{x} \rangle + L_{\bar{x}}(y, z, \lambda)$$

and the optimal value $\theta_x(\lambda)$ of problem (2.11) is the sum of $\langle \lambda, x - \bar{x} \rangle$ and of $\theta_{\bar{x}}(\lambda)$. Observing that from Weak Duality $\theta_x(\lambda)$ is a lower bound on $Q(x)$, this sum is an affine function of $x$ which is a lower bounding function for $Q$. It can be bounded from below in terms of a computable affine function (which therefore is an inexact cut for $Q$ at $\bar{x}$) using an approximate primal-dual solution if problem (2.7) and its dual (2.10) written for $x = \bar{x}$ satisfy the Slater assumption.

The details of these computations are given in the proof of Proposition 2.3 below which provides formulas for inexact cuts for value function (2.7). The proof of the proposition is given for completeness but, due to our previous observations, it is similar to the proof of Proposition 2.2.

**Proposition 2.3.** Let Assumption (H0) hold. Assume that $\hat{y}$ is an $\varepsilon_P$-optimal feasible solution of problem (2.7) for $x = \bar{x}$ and that $\hat{\lambda}$ is an $\varepsilon_D$-optimal feasible solution of dual problem (2.10) written for $x = \bar{x}$. Assume that $f(\cdot, \bar{x})$ is finite on
\{ y \in Y : Ay + b \tilde{x} = b, g(y, \tilde{x}) \leq 0 \} and that the following Slater constraint qualification holds for (2.7) written for \( x = \tilde{x} \):

\begin{equation}
\exists y_z \text{ such that } (y_z, \tilde{x}) \in ri(S)
\end{equation}

where \( S \) is given by (2.9). Then

\[ C(x) = f(\hat{y}, \tilde{x}) - (\varepsilon_P + \varepsilon_D) + \langle \hat{\lambda}, x - \tilde{x} \rangle \]

is an \((\varepsilon_P + \varepsilon_D)\)-inexact cut for \( Q \) at \( \tilde{x} \).

Proof. By definition of \( \hat{y} \), we get

\[ f(\hat{y}, \tilde{x}) \leq Q(\tilde{x}) + \varepsilon_P. \]

The assumptions of the Convex Duality theorem for dual problem (2.10) and primal problem (2.7) written for \( x = \tilde{x} \) are satisfied and therefore the optimal value of dual problem (2.10) written for \( x = \tilde{x} \) is the optimal value \( Q(\tilde{x}) \) of the corresponding primal problem. Therefore, using the definition of \( \hat{\lambda} \), we get

\[ \theta_{\tilde{x}}(\hat{\lambda}) \geq Q(\tilde{x}) - \varepsilon_D. \]

Next,

\[ Q(x) \geq \theta_{\tilde{x}}(\hat{\lambda}) \text{ by weak duality and feasibility of } \hat{\lambda}, \]

\[ = \min \{ L_x(y, z, \lambda) : (y, z) \in S \}, \]

\[ = \langle \lambda, x - \tilde{x} \rangle + \min \{ L_x(y, z, \lambda) : (y, z) \in S \}, \]

\[ = \langle \lambda, x - \tilde{x} \rangle + \theta_{\tilde{x}}(\hat{\lambda}), \]

\[ \geq \langle \lambda, x - \tilde{x} \rangle + Q(\tilde{x}) - \varepsilon_D, \]

\[ \geq C(x) := \langle \lambda, x - \tilde{x} \rangle + f(\hat{y}, \tilde{x}) - \varepsilon_P - \varepsilon_D. \]

Moreover, since \( f(\hat{y}, \tilde{x}) \geq Q(\tilde{x}) \), we get

\[ Q(\tilde{x}) - C(\tilde{x}) = \varepsilon_P + \varepsilon_D + Q(\tilde{x}) - f(\hat{y}, \tilde{x}) \leq \varepsilon_P + \varepsilon_D, \]

which achieves the proof. ☐

As before, observe that if \( \theta_{\tilde{x}}(\hat{\lambda}) \) is available, i.e., if optimization problem (2.11) written for \( x = \tilde{x} \) and \( \lambda = \hat{\lambda} \) is solved to optimality then \( \langle \lambda, x - \tilde{x} \rangle + \theta_{\tilde{x}}(\hat{\lambda}) \) is an \( \varepsilon_D \)-inexact cut for \( Q \) at \( \tilde{x} \).

It is also worth mentioning that if we have access to an optimal primal-dual solution to (2.7) then we can obtain an exact cut for \( Q \) at \( \tilde{x} \) directly solving (2.7) and its dual, without adding constraint \( z = x \). More precisely, a characterization of the subdifferentiable of \( Q \) and formulas for exact cuts for \( Q \) given by (2.7) can be found in Lemma 2.1 in [12] and Proposition 3.2 in [10].

3. Inexact cuts for value functions with Fenchel-type representation of the objective. We now consider value functions \( Q \) of form (1.1) where the objective \( f \) admits a Fenchel-type representation: if \( p = (y, x) \), function \( f \) is given by

\begin{equation}
f(p) = p^T a + \max_{w \in W} [p^T C_0 w - \phi_0(w)]
\end{equation}

for some known convex, proper, lower semicontinuous function \( \phi_0 \), some known convex, compact, nonempty set \( W \), vector \( a \), and matrix \( C_0 \). In this situation, we will derive inexact cuts for \( Q \) without additional variables \( z \in \mathbb{R}^n \) and constraints \( z = x \) introduced in the previous section.

"Well structured" convex functions have Fenchel-type representations.
We start considering value functions of form
\[ Q(x) \]
with \( Y \) compact, convex, and nonempty. Let \( \theta = [a_2; a_1] \) and let us write matrix \( C_0 = [A_0; B_0] \) where \( A_0 \) contains the first \( m \) rows and \( B_0 \) the last \( n \) rows of \( C_0 \).

Representation (3.13) can then be written
\[ Q(x) = \min_{y \in \mathbb{R}^m} f(y, x) \]
and problem (3.14) becomes the saddle point problem
\[ Q(x) = \min_{y \in Y} \max_{w \in \mathbb{W}} y^T a_1 + y^T a_2 + \max_{w \in \mathbb{W}} y^T A_0 w + x^T B_0 w - \phi_0(w) \]
and problem (3.14) becomes the saddle point problem
\[ Q(x) = \min_{y \in \mathbb{R}^m} \max_{w \in \mathbb{R}^n} f(y, x) \].

Since \( Y \) and \( \mathbb{W} \) are compact, convex and nonempty, this saddle point problem can be equivalently written as the convex problem
\[ Q(x) = x^T a_1 + \max_{w \in \mathbb{W}} \theta_x(w) \]
where concave function \( \theta_x \) is given by
\[ \theta_x(w) = \min_{y \in \mathbb{Y}} L_x(y, w) \]
where
\[ L_x(y, w) = y^T (a_2 + A_0 w) + x^T B_0 w - \phi_0(w). \]

Once again, the linearity in \( x \) of this new Lagrangian function \( L_x(y, w) \) will allow us to derive inexact cuts. However, contrary to the previous section, this linearity was achieved using a Fenchel-type representation of \( f \). The following proposition provides inexact cuts for \( Q \) given by (3.14) with \( f \) of the form (3.15).  

**Proposition 3.2.** Consider problem (3.14) with \( f \) having a Fenchel-type representation of form (3.15). Assume that \( Y \) and \( \mathbb{W} \) are compact, convex, and nonempty. Let \( \hat{w} \in \mathbb{W} \) be an \( \varepsilon \)-optimal solution of problem (3.17) written with \( x = \bar{x} \) and let \( \hat{y} \in Y \) be a \( \tau \)-optimal solution of problem (3.18) written with \( x = \bar{x}, \hat{w} = \hat{w} \). Then the affine function
\[ C(x) := x^T (a_1 + B_0 \hat{w}) + \hat{y}^T (a_2 + A_0 \hat{w}) - \phi_0(\hat{w}) - \tau \]
is a \((\varepsilon + \tau)\)-inexact cut for \( Q \) at \( \bar{x} \).

**Proof.** Let \( (\hat{y}, \hat{w}) \) be an optimal solution of saddle point problem (3.16) with \( x = \bar{x} \). By definition of \( \hat{w} \) and \( \hat{y} \), we have
\[ \theta_{\bar{x}}(\hat{w}) - \varepsilon \leq \theta_{\bar{x}}(\hat{w}) \] and \( \theta_{\bar{x}}(\hat{w}) + \tau \geq L_{\bar{x}}(\hat{y}, \hat{w}) \geq \theta_{\bar{x}}(\hat{w}). \]
By linearity of $L(y,w)$ we get for every $y \in Y, w \in W$, that
\[(3.22)\quad L_x(y,w) = L\bar{x}(y,w) + (x - \bar{x})^T B_0 w.\]

Next, using representation (3.17) of $Q$ and the fact that $\hat{w} \in W$ we have
\[Q(x) \geq x^T a_1 + \theta_x(\hat{w}) = x^T a_1 + \min_y L_x(y,\hat{w}) \geq x^T a_1 + (x - \bar{x})^T B_0 \hat{w} + \theta_x(\hat{w}) \geq x^T a_1 + (x - \bar{x})^T B_0 \hat{w} + \theta_x(\hat{w}) - \tau \geq C(x). \]

Moreover,
\[0 \leq Q(\bar{x}) - C(\bar{x}) = \tau + \theta_x(\hat{w}) - L\bar{x}(\hat{y},\hat{w}) \leq \tau + \theta_x(\hat{w}) - \theta_x(\hat{w}) \leq \tau + \varepsilon, \]
which achieves the proof of the proposition.

Now consider value function $Q$ given by
\[(3.23)\quad Q(x) = \left\{ \begin{array}{l} \min_y f(y,x) \\ y \in Y, Ay + Bx = b. \end{array} \right\}
\]
with $Y$ convex, nonempty, and compact. If $f$ has a Fenchel-type representation of form (3.15) with $W$ convex, nonempty, and compact, value function (3.23) can be written
\[(3.24)\quad Q(x) = x^T a_1 + \left\{ \max_w \theta_x(w) \right\}_{w \in W}
\]
where
\[(3.25)\quad \theta_x(w) = \left\{ \begin{array}{l} \min_y y^T (a_2 + A_0 w) + x^T B_0 w - \phi_0(w) \\ y \in Y, Ay + Bx = b. \end{array} \right\}
\]
For problem (3.25) define the Lagrangian
\[(3.26)\quad L_{x,w}(y,\lambda) = y^T (a_2 + A_0 w) + x^T B_0 w - \phi_0(w) + \lambda^T (Ay + Bx - b) = L_x(y,w) + \lambda^T (Ay + Bx - b),
\]
where $L_x(y,w)$ is given by (3.19). Let us fix $\bar{x} \in \mathbb{R}^n$ and assume that there is $y_0 \in \text{ri}(Y)$ such that $A\bar{y}_0 + B\bar{x} = b$. Then by the Convex Duality theorem, we can express $\theta_x(w)$ as the optimal value of the dual of (3.25):
\[(3.27)\quad \theta_x(w) = \max_{\lambda} h_{\bar{x},w}(\lambda) \]
for the dual function
\[(3.28)\quad h_{\bar{x},w}(\lambda) = \left\{ \begin{array}{l} \min_y L_{\bar{x},w}(y,\lambda) \\ y \in Y. \end{array} \right\}
\]
Proposition 3.3. Consider problem (3.23) with \( f \) having a Fenchel-type representation of form (3.15). Assume that sets \( Y \) and \( W \) are nonempty, convex, and compact. Let us fix \( \bar{x} \in \mathbb{R}^n \) and assume that there is \( y_0 \in \text{ri}(Y) \) such that \( Ay_0 + B\bar{x} = b \). Let \((\hat{y},\hat{w})\) be an optimal solution of saddle point problem (3.24) with \( x = \bar{x} \) and let \( \hat{w} \in W \) be an \( \varepsilon \)-optimal solution of problem (3.24) written with \( x = \bar{x} \):

\[
\theta_{\bar{x}}(\hat{w}) \geq \theta_{\bar{x}}(\hat{w}) - \varepsilon,
\]

and let \( \hat{\lambda} \in Y \) be a \( \delta \)-optimal solution of problem

\[
\theta_{\bar{x}}(\hat{w}) = \max_{\lambda} h_{\bar{x},\hat{w}}(\lambda)
\]
i.e.,

\[
h_{\bar{x},\hat{w}}(\hat{\lambda}) \geq \theta_{\bar{x}}(\hat{w}) - \delta.
\]

Let \( \hat{y} \) be a \( \tau \)-optimal feasible solution of

\[
\theta_{\bar{x}}(\hat{w}) = \left\{ \begin{array}{l}
\min_y y^T(a + A\hat{w}) + \bar{x}^T B_0 \hat{w} - \phi_0(\hat{w}) \\
y \in Y, Ay + B\bar{x} = b,
\end{array} \right.
\]
i.e.,

\[
\hat{y} \in Y, A\hat{y} + B\bar{x} = b, L_{\bar{x}}(\hat{y}, \hat{w}) \leq \theta_{\bar{x}}(\hat{w}) + \tau.
\]

Then the affine function

\[
C(x) = x^T(a + B_0 \hat{w} + B^T \hat{\lambda}) + \hat{y}^T(a + A_0 \hat{w}) - \bar{x}^T B^T \hat{\lambda} - \phi_0(\hat{w}) - \tau - \delta
\]
is a \((\varepsilon + \tau + \delta)\)-inexact cut for \( Q \) at \( \bar{x} \).

Proof. By linearity of \( L_{x,w}(y,\lambda) \) we get for every \( y \in Y, w \in W \), that

\[
L_{x,w}(y,\lambda) = L_{\bar{x},\hat{w}}(y,\lambda) + (x - \bar{x})^T(B_0 \hat{w} + B^T \hat{\lambda}).
\]

Next, using representation (3.24) of \( Q \) and the fact that \( \hat{w} \in W \) we have

\[
Q(x) \geq x^T a_1 + \theta_{\bar{x}}(\hat{w}) \\
\geq x^T a_1 + h_{\bar{x},\hat{w}}(\hat{\lambda}),
\]

\[(3.28)\]

\[
x^T a_1 + \left\{ \begin{array}{l}
\min_y L_{x,w}(y,\lambda) \\
y \in Y,
\end{array} \right.
\]

\[(3.33)\]

\[
x^T a_1 + (x - \bar{x})^T(B_0 \hat{w} + B^T \hat{\lambda}) + \left\{ \begin{array}{l}
\min_y L_{\bar{x},\hat{w}}(y,\hat{\lambda}) \\
y \in Y,
\end{array} \right.
\]

\[(3.28)\]

\[
x^T a_1 + (x - \bar{x})^T(B_0 \hat{w} + B^T \hat{\lambda}) + h_{\bar{x},\hat{w}}(\hat{\lambda})
\]

\[(3.30)\]

\[
x^T a_1 + (x - \bar{x})^T(B_0 \hat{w} + B^T \hat{\lambda}) + \theta_{\bar{x}}(\hat{w}) - \delta
\]

\[(3.31)\]

\[
x^T a_1 + (x - \bar{x})^T(B_0 \hat{w} + B^T \hat{\lambda}) + L_{\bar{x}}(\hat{y}, \hat{w}) - \tau - \delta
\]

\[(3.32)\]

\[
C(x).
\]

Moreover, if \( \hat{w} \) is an optimal solution of (3.24) written for \( x = \bar{x} \), i.e., \( Q(\bar{x}) = \bar{x}^T a_1 + \theta_{\bar{x}}(\hat{w}) \) we obtain

\[
0 \leq Q(\bar{x}) - C(\bar{x}) = \tau + \delta + \theta_{\bar{x}}(\hat{w}) - L_{\bar{x}}(\hat{y}, \hat{w}) \leq \tau + \delta + \theta_{\bar{x}}(\hat{w}) - \theta_{\bar{x}}(\hat{w}) \leq \tau + \delta + \varepsilon,
\]

which achieves the proof of the proposition. \( \Box \)
4. Particular case of differentiable problems and comparison with the inexact cuts from [13]. The following proposition, taken from [13], provides an inexact cut for \( Q \) given by (2.7) when functions \( f, g_i \) are differentiable.

**Proposition 4.1.** Consider value function \( Q \) given by (2.7). Let Assumption (H0) hold, take \( \bar{x} \in X \), and assume that

\[
\text{(4.34) there exists } y_{z_{\bar{x}}} \in \text{ri}(Y) \text{ such that } Ay_{z_{\bar{x}}} + B\bar{x} = b \text{ with } g(y_{z_{\bar{x}}}, \bar{x}) < 0.
\]

Assume that \( f \) and \( g \) are differentiable on \( Y \times X \). Let \( \varepsilon \geq 0 \), let \( \hat{y} \) be an \( \varepsilon \)-optimal feasible primal solution for problem (2.7) written for \( x = \bar{x} \) and let \( (\lambda, \mu) \) be an \( \varepsilon \)-optimal feasible solution of the corresponding dual problem given by

\[
\max_{\mu \geq 0, \lambda} \theta_z(\lambda, \mu)
\]

where the dual function \( \theta_z(\lambda, \mu) \) is given by

\[
\theta_z(\lambda, \mu) = \min_{y \in Y} L_z(y, \lambda, \mu)
\]

for the Lagrangian

\[
L_z(y, \lambda, \mu) = f(y, x) + \langle \lambda, Bx + Ay - b \rangle + \langle \mu, g(y, x) \rangle.
\]

Assume that \( \ell(\cdot, \bar{x}) \) is finite on

\[
\text{(4.36) } S(\bar{x}) = \{ y \in Y : Ay + B\bar{x} = b, g(y, \bar{x}) \leq 0 \}
\]

and that \( \eta(\varepsilon) = \ell(\hat{y}, \bar{x}, \hat{\lambda}, \hat{\mu}) \) is finite where

\[
\ell(\hat{y}, \bar{x}, \hat{\lambda}, \hat{\mu}) = \max\{ \langle \nabla_y L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), \hat{y} - y \rangle : y \in Y \}.
\]

Then the affine function

\[
\text{(4.37) } C(x) := L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}) - \eta(\varepsilon) + \langle \nabla_x L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle
\]

is an \( (\varepsilon + \ell(\hat{y}, \bar{x}, \hat{\lambda}, \hat{\mu})) \)-inexact cut for \( Q \) at \( \bar{x} \).

We want to compare the inexact cuts given by Propositions 2.3 and 4.1 obtained taking \( \varepsilon_D = \varepsilon_P = \varepsilon \) in Proposition 2.3. For the cut given by Proposition 4.1 to be valid, we assume that the assumptions of this proposition are satisfied. In particular, (4.34) holds. Let us show that if in addition \( Y \times X \subset \text{dom}(g_i) \) for all \( i = 1, \ldots, p \), this implies that (2.12) holds which will imply that the assumptions of Proposition 2.3 are also satisfied and the inexact cut given by that proposition is valid. Indeed, write set \( S \) given by (2.9) as \( S = S_1 \cap S_2 \cap (Y \times \mathbb{R}^n) \) where \( S_1 = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n : g(y, z) \leq 0 \} \) and \( S_2 = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n : Ay + Bz = b \} \). We have that \( \text{ri}(S_2) = S_2 \) and

\[
\text{(4.38) } \text{ri}(\{g_i \leq 0\}) = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n : (y, z) \in \text{ri}(\text{dom}(g_i)), g_i(y, z) < 0, i = 1, \ldots, p \}.
\]

Since \( Y \times \{\bar{x}\} \subset \text{dom}(g_i), i = 1, \ldots, p \), we have \( \text{ri}(Y) \times \{\bar{x}\} \subset \text{ri}(\text{dom}(g_i)), i = 1, \ldots, p \), implying that set \( \bigcap_{i=1}^p \text{ri}(\{g_i \leq 0\}) \) is nonempty since it contains the nonempty set \( \text{ri}(Y) \times \{\bar{x}\} \) (this set contains \( (y_{\bar{x}}, \bar{x}) \)). Therefore \( \text{ri}(S_1) = \bigcap_{i=1}^p \text{ri}(\{g_i \leq 0\}) = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n : (y, z) \in \text{ri}(\text{dom}(g_i)), g_i(y, z) < 0, i = 1, \ldots, p \} \). It follows that convex sets \( S_1, S_2 \), and \( Y \times \mathbb{R}^n \) are convex and satisfy \( \text{ri}(S_1) \cap \text{ri}(S_2) \cap (\text{ri}(Y) \times \mathbb{R}^n) \neq \emptyset \) (they contain the point \( (y_{\bar{x}}, \bar{x}) \)) which implies that \( \text{ri}(S) = \text{ri}(S_1) \cap \text{ri}(S_2) \cap (\text{ri}(Y) \times \mathbb{R}^n) \) and
recalling the representations of \( \text{ri}(S_1) \) and \( \text{ri}(S_2) \), we see that \((\hat{y}, \hat{x})\) which satisfies (4.34) also belongs to \( \text{ri}(S) \), i.e., Slater condition (2.12) holds. Therefore, Proposition 2.3 provides a valid \( 2\varepsilon \)-inexact cut for \( \mathcal{Q} \).

Let us use the notation \( C_1(x) = \theta_1 + \langle \beta_1, x - \bar{x} \rangle \) and \( C_2(x) = \theta_2 + \langle \beta_2, x - \bar{x} \rangle \) for respectively the inexact cuts given by Propositions 2.3 and 4.1. In Proposition 4.2 below, we derive upper and lower bounds on \( \theta \) respectively the inexact cuts given by Propositions 2.3 and 4.1. In Proposition 4.2 below, we derive upper and lower bounds on \( \theta_1 - \theta_2 = C_1(\bar{x}) - C_2(\bar{x}) \) (observe that in the exact case, i.e., when \( \varepsilon = 0 \), clearly \( \theta_1 = \theta_2 \) and \( \beta_1 = \beta_2 \)). This will be done using characterizations of \( \varepsilon \)-optimal feasible primal-dual solutions to obtain bounds for the terms \( \langle \hat{\mu}, g(\hat{y}, \bar{x}) \rangle \) and \( \max_{y \in Y} (\nabla_y L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), \hat{y} - y) \) (which are clearly null if \( \hat{y} \) and \((\hat{\lambda}, \hat{\mu})\) are optimal primal-dual solutions). In particular, we will show that \( \langle \hat{\mu}, g(\hat{y}, \bar{x}) \rangle \) is between \(-2\varepsilon\) and \(0\). To derive these bounds, we will assume that

(A0) the gradient of objective function \( f(\cdot, \bar{x}) \) (resp. of constraint function \( g_i(\cdot, \bar{x}) \)) is \( L_0 \) (resp. \( L_i \))-co-coercive with \( L_i > 0, i = 0, \ldots, p \).

Recall that \( F : \text{Dom}(F) \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^m \) is \( L \)-co-coercive on \( \Omega \subseteq \text{Dom}(F) \) if

\[
L\langle y - x, F(y) - F(x) \rangle \geq \|F(y) - F(x)\|^2, \quad \forall x, y \in \Omega.
\]

**Proposition 4.2.** Let the assumptions of Proposition 4.1 hold and assume that \( Y \times X \subseteq \text{dom}(g_i) \) for all \( i = 1, \ldots, p \). Take \( \bar{x} \in X \) and let \( L_{\bar{x}} \) be any lower bound on \( Q(\bar{x}) \). Let \( C_1(x) = \theta_1 + \langle \beta_1, x - \bar{x} \rangle \) and \( C_2(x) = \theta_2 + \langle \beta_2, x - \bar{x} \rangle \) be respectively the inexact cuts given by Propositions 2.3 and 4.1 taking \( \varepsilon_D = \varepsilon_P = \varepsilon \). Assume that \( f \) and \( g_i, i = 1, \ldots, p \), satisfy (A0), that \( Y \) is compact, and set

\[
U_{\bar{x}} = \frac{f(y_{\bar{x}}, \bar{x}) - L_{\bar{x}} + \varepsilon}{\min_{i = 1, \ldots, p} g_i(y_{\bar{x}}, \bar{x})}, \quad L = L_0 + \max_{i = 1, \ldots, p} L_i.
\]

Then we have

\[-2\varepsilon \leq C_1(\bar{x}) - C_2(\bar{x}) \leq 2\varepsilon + 2D_Y \sqrt{L\varepsilon},\]

where \( D_Y \) is the diameter of \( Y \).

**Proof.** Recall that

\[
C_1(\bar{x}) = f(\hat{y}, \bar{x}) - 2\varepsilon,
\]

\[
C_2(\bar{x}) = f(\hat{y}, \bar{x}) + \langle \hat{\mu}, g(\hat{y}, \bar{x}) \rangle - \max_{y \in Y} (\nabla_y L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), \hat{y} - y),
\]

and that \((\hat{\mu}, \hat{\lambda}, \hat{\mu})\) satisfy

(4.39) \( \hat{y} \in S(\bar{x}), \hat{\mu} \geq 0, f(\hat{y}, \bar{x}) \leq Q(\bar{x}) + \varepsilon, \theta_2(\hat{\lambda}, \hat{\mu}) \geq Q(\bar{x}) - \varepsilon, \)

where \( S(x) \) is defined in (4.36) and \( \theta_2 \) is the dual function given by (4.35).

By the subgradient inequality, if \( L_{\bar{x}} \) is the Lagrangian given in Proposition 4.1, we get

(4.40) \[
\theta_2(\hat{\lambda}, \hat{\mu}) = \min_{y \in Y} L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}) \geq L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}) + \min_{y \in Y} (\nabla_y L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), y - \hat{y}) = f(\hat{y}, \bar{x}) + \langle \hat{\mu}, g(\hat{y}, \bar{x}) \rangle + \min_{y \in Y} (\nabla_y L_{\bar{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), y - \hat{y}) = C_2(\bar{x}).
\]
Therefore,
\[ C_1(\bar{x}) = f(\hat{y}, \bar{x}) - 2\varepsilon \]
(4.41)
\[ \geq \theta_\varepsilon(\hat{\lambda}, \hat{\mu}) - 2\varepsilon \text{ by weak duality}, \]
(4.40)
\[ \geq C_2(\bar{x}) - 2\varepsilon. \]

We next provide an upper bound for \( C_1(\bar{x}) - C_2(\bar{x}) \). Indeed, (4.39) implies that
\[ f(\hat{y}, \bar{x}) \leq \theta_\varepsilon(\hat{\lambda}, \hat{\mu}) + 2\varepsilon = \min_{y \in Y} \{ L_{\hat{x}}(y, \hat{\lambda}, \hat{\mu}) : y \in Y \} + 2\varepsilon \]
and hence that
\[ L_{\hat{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}) = f(\hat{y}, \bar{x}) + \langle \hat{\mu}, g(\hat{y}, \bar{x}) \rangle \leq \min_{y \in Y} \{ L_{\hat{x}}(y, \hat{\lambda}, \hat{\mu}) : y \in Y \} + \langle \hat{\mu}, g(\hat{y}, \bar{x}) \rangle + 2\varepsilon \]
where the first equality is due to \( \hat{y} \in S(\bar{x}) \). The last inequality in turn is equivalent to \( \hat{\varepsilon} := 2\varepsilon + \langle \hat{\mu}, g(\hat{y}, \bar{x}) \rangle \) satisfying
\[ 2\varepsilon \geq \hat{\varepsilon} = \hat{\varepsilon} \geq 0, \quad 0 \in \partial_{\hat{\varepsilon}} \left( L_{\hat{x}}(\cdot, \hat{\lambda}, \hat{\mu}) + \delta_Y(\cdot) \right)(\hat{y}) \]
where \( \delta_Y(\cdot) \) is the indicator function of set \( Y \) given by
\[ \delta_Y(y) = \begin{cases} 0 & \text{if } y \in Y, \\ +\infty & \text{otherwise.} \end{cases} \]

It is easy to check that \( \|\hat{\mu}\| \leq U_{\hat{x}} \) (see for instance the proof of Proposition 2.3 in [13]) which easily implies that \( L_{\hat{x}}(\cdot, \hat{\lambda}, \hat{\mu}) \) is \( L_{\hat{x}} \)-coercive (for the interested reader, we provide in Lemma 7.1 in the appendix the proof that a sum of \( L_{\hat{x}} \)-coercive mappings \( f_i \) is \((\sum_{i=1}^{n} L_{\hat{x}})\)-co-coercive). Combining this observation with (4.42) and Lemma 3.2 in [27], we obtain that there exists \( v \) satisfying:
\[ v \in \nabla_y L_{\hat{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}) + \partial_{\hat{\varepsilon}} \delta_Y(\hat{y}), \quad \|v\| \leq \sqrt{2L_{\hat{x}}} \leq 2\sqrt{L_{\hat{x}}}. \]
(4.43)

It is well known that set \( \partial_{\hat{\varepsilon}} \delta_Y(\hat{y}) \) is the \( \hat{\varepsilon} \)-normal set to \( Y \) at \( \hat{y} \) given by
\[ \partial_{\hat{\varepsilon}} \delta_Y(\hat{y}) = \{ z \in \mathbb{R}^m : \langle z, y - \hat{y} \rangle \leq \hat{\varepsilon} \ \forall y \in Y \} \]
and therefore \( v \) which satisfies (4.43) also satisfies
\[ (\nabla_y L_{\hat{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}) - v, \hat{y} - y) \leq \hat{\varepsilon}, \ \forall y \in Y \Leftrightarrow \max_{y \in Y} (\nabla_y L_{\hat{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}) - v, \hat{y} - y) \leq \hat{\varepsilon}. \]
(4.44)

We then obtain the following upper bound for \( C_1(\bar{x}) - C_2(\bar{x}) \):
\[ C_2(\bar{x}) = f(\hat{y}, \bar{x}) + \langle \hat{\mu}, g(\hat{y}, \bar{x}) \rangle - \max_{y \in Y} (\nabla_y L_{\hat{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), \hat{y} - y) \]
(4.42)
\[ \geq C_1(\bar{x}) + 2\varepsilon + \langle \hat{\mu}, g(\hat{y}, \bar{x}) \rangle - \max_{y \in Y} (\nabla_y L_{\hat{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), \hat{y} - y) \]
\[ \geq C_1(\bar{x}) - \max_{y \in Y} (\nabla_y L_{\hat{x}}(\hat{y}, \hat{\lambda}, \hat{\mu}), \hat{y} - y) \]
(4.44)
\[ \geq C_1(\bar{x}) - \hat{\varepsilon} - \|v\| D_Y \geq C_1(\bar{x}) - 2\varepsilon - 2D_Y \sqrt{L_{\hat{x}}}, \]
which achieves the proof of the proposition. \( \Box \)
The upper and lower bounds on $C_1(\hat{x}) - C_2(\bar{x})$ given in Proposition 4.2 are continuous functions of $\varepsilon$ which go to 0 as $\varepsilon$ goes to 0. Also these bounds are respectively positive and negative for positive $\varepsilon$. This shows that they are both of good quality for small values of $\varepsilon$ and this analysis does not ensure that one of these two is always better (i.e., has a larger intercept at $\bar{x}$) than the other.

The analysis above (the proof of Proposition 4.2) is also interesting per-se since it offers ways of characterizing $\varepsilon$-optimal primal-dual solutions and allows us to derive bounds on the two quantities $\langle \hat{\mu}, g(\hat{\mu}, \bar{x}) \rangle$ and $\max_{y \in Y} \langle \nabla_y L_\varepsilon(\hat{y}, \hat{\lambda}, \hat{\mu} - \bar{y}) \rangle$ which, by the first order optimality conditions, are null if $\hat{y}$ and $(\hat{\lambda}, \hat{\mu})$ are respectively optimal primal and dual solutions. More precisely, if $\hat{y}$ (resp. $(\hat{\lambda}, \hat{\mu})$) is an $\varepsilon$-optimal feasible primal (resp. dual) solution, then we have shown that $-2\varepsilon \leq \langle \hat{\mu}, g(\hat{\mu}, \bar{x}) \rangle \leq 0$ and $0 \leq \max_{y \in Y} \langle \nabla_y L_\varepsilon(\hat{y}, \hat{\lambda}, \hat{\mu} - \bar{y}) \rangle \leq 2D_Y \sqrt{L_\varepsilon} + 2\varepsilon$.

5. ISDDP algorithm for nondifferentiable problems. The objective of this section is to introduce and study new variants of ISDDP which use the inexact cuts built in the previous sections.

We consider multistage stochastic nonlinear optimization problems of the form
\begin{equation}
\min_{x_t \in X_t(x_{t-1}, \xi_t)} f_t(x_t, x_{t-1}, \xi_t) + \mathbb{E}\left[ \min_{x_{t+1} \in X_{t+1}(x_{t}, \xi_{t})} f_{t+1}(x_{t+1}, x_{t}, \xi_{t}) + \mathbb{E}\left[ \ldots + \mathbb{E}\left[ \min_{x_{T} \in X_{T}(x_{T-1}, \xi_{T})} f_{T}(x_{T-1}, x_{T-1}, \xi_{T}) \right] \right] \right],
\end{equation}
where $x_0$ is given, $(\xi_t)_{t=0}^T$ is a stochastic process, $\xi_1$ is deterministic, and
\begin{equation*}
X_t(x_{t-1}, \xi_t) = \{ x_t \in \mathbb{R}^n : A_t x_t + B_t x_{t-1} = b_t, g_t(x_t, x_{t-1}, \xi_t) \leq 0, x_t \in X_t \}.
\end{equation*}

We make the following assumption on $(\xi_t)$:

(H) $(\xi_t)$ is interstage independent and for $t = 2, \ldots, T$, $\xi_t$ is a random vector taking values in $\mathbb{R}^{K_t}$ with a discrete distribution and a finite support $\Theta_t = \{ \xi_{t1}, \ldots, \xi_{tN_t} \}$ with $p_{ti} = \mathbb{P}(\xi_t = \xi_{ti}) > 0, i = 1, \ldots, N_t$, while $\xi_1$ is deterministic.

In the sequel, we will denote by $A_{tij}$, $B_{tij}$, and $b_{tij}$ the realizations of $A_t$, $B_t$, and $b_t$ in $\xi_{tij}$.

For this problem, we can write Dynamic Programming equations: the first stage problem is
\begin{equation}
Q_1(x_0) = \left\{ \begin{array}{l}
\min_{x_1 \in \mathbb{R}^n} f_1(x_1, x_0, \xi_1) + Q_2(x_1) \\
x_1 \in X_1(x_0, \xi_1)
\end{array} \right.
\end{equation}
for $x_0$ given and for $t = 2, \ldots, T$, $Q_t(x_{t-1}) = \mathbb{E}_{\xi_t}[Q_t(x_{t-1}, \xi_t)]$ with
\begin{equation}
Q_t(x_{t-1}, \xi_t) = \left\{ \begin{array}{l}
\min_{x_t \in \mathbb{R}^n} f_t(x_t, x_{t-1}, \xi_t) + Q_{t+1}(x_t) \\
x_t \in X_t(x_{t-1}, \xi_t)
\end{array} \right.
\end{equation}
with the convention that $Q_{T+1}$ is null.

We set $X_0 = \{ x_0 \}$ and make the following assumptions (H1) on the problem data:

(H1): there exists $\varepsilon > 0$ such that for $t = 1, \ldots, T$,
1) $X_t$ is a nonempty, compact, and convex set.
2) For every \( j = 1, \ldots, N_t \), the function \( f_t(\cdot, \xi_{ij}) \) is convex, proper, lower semicontinuous on \( X_t^* \times X_{t-1} \) and for every \( x_{t-1} \in X_{t-1}^* \) we have

\[ X_t \subset \text{dom}(f_t(\cdot, x_{t-1}, \xi_{ij})). \]

3) For every \( j = 1, \ldots, N_t \), each component \( g_t(\cdot, \xi_{ij}), i = 1, \ldots, p \), of function \( g_t(\cdot, \xi_{ij}) \) is convex, lower semicontinuous and finite on \( X_t \times X_{t-1} \).

4) \( X_1(x_0, \xi_{ij}) \neq \emptyset \) and for every \( t = 2, \ldots, T \), for every \( j = 1, \ldots, N_t \), for every \( x_{t-1} \in X_{t-1}^* \), the set \( \text{ri}(X_t) \cap X_t(x_{t-1}, \xi_{ij}) \) is nonempty.

5) for every \( t \geq 2 \), for every \( j = 1, \ldots, N_t \), there is \( (x_{tj}, x_{t-1j}) \in \text{ri}(X_t) \times X_{t-1}^* \) such that \( g_t(x_{tj}, x_{t-1j}, \xi_{ij}) < 0 \).

We are now in a position to describe the ISDDP algorithm for nondifferentiable optimization problems of form (5.45). The ISDDP algorithm given below combines SDDP with the inexact cuts derived in Section 2.2:

**ISDDP algorithm.**

**Step 0** \textbf{Initialization.} Let \( Q^0_t : X_{t-1}^* \to \mathbb{R}, t = 2, \ldots, T + 1 \), be affine functions satisfying \( Q^0_t \leq Q_t \). Set \( k = 1 \).

**Step 1** \textbf{Forward pass.} Setting \( x^0_t = x_0 \), generate a sample \((\tilde{\xi}^1_t, \tilde{\xi}^2_t, \ldots, \tilde{\xi}^k_t)\) from the distribution of \((\xi_1, \xi_2, \ldots, \xi_T)\) and for \( t = 1, 2, \ldots, T \), compute a \( \delta^k_t \)-optimal solution \( x^k_t \) of

\[
\min \left\{ f_t(x_t, x^k_{t-1}, \tilde{\xi}^k_t) + Q^k_{t+1}(x_t) : x_t \in X_t(x^k_{t-1}, \tilde{\xi}^k_t) \right\}.
\]

**Step 2** \textbf{Backward pass.}

For \( t = T, T - 1, \ldots, 2 \),

For \( j = 1, \ldots, N_t \),

Compute an \( \varepsilon^k_t \)-optimal solution \( x^k_{tj} \) of

\[
\begin{align*}
Q^k_t(x^k_{t-1}, \xi_{ij}) = \min_{x_t, z} & f_t(x_t, z, \xi_{ij}) + Q^k_{t+1}(x_t) \\
& A_{tj} x_t + B_{tj} z = b_{tj}, \\
& g_t(x_t, z, \xi_{ij}) \leq 0, \\
& x_t \in X_t, \\
& z = x^k_{t-1},
\end{align*}
\]

and an \( \varepsilon^k_t \)-optimal dual solution \( \lambda^k_{tj} \) of the dual of problem (5.49) obtained dualizing constraints \( z = x^k_{t-1} \).

End For

Compute

\[
\begin{align*}
\beta^k_t &= \sum_{j=1}^{N_t} p_{tj} \lambda^k_{tj}, \\
\theta^k_t &= \sum_{j=1}^{N_t} p_{tj} \left( f_t(x^k_{tj}, x^k_{t-1j}, \xi_{ij}) + Q^k_{t+1}(x^k_{tj}) - \langle \lambda^k_{tj}, x^k_{t-1j} \rangle \right)
\end{align*}
\]

and store the new cut

\[
C^k_t(x_{t-1}) := \theta^k_t - 2\varepsilon^k_t + \langle \beta^k_t, x_{t-1} \rangle
\]

for \( Q_t \), making up the new approximation \( Q^k_t = \max\{Q^{k-1}_t, C^k_t\} \).

End For
Step 4) Do $k \leftarrow k + 1$ and go to Step 1).

**Remark 5.1.** ISDDP algorithm given above applies both to differentiable and nondifferentiable problems. In the differentiable case (when all functions $f_t(\cdot, \cdot, \xi_t)$ and $g_t(\cdot, \cdot, \xi_t)$ are differentiable), compared to ISDDP introduced in [13], the variant of ISDDP given above does not need to solve an additional optimization problem to obtain the intercept of the cut. However, all subproblems solved in the forward and backward passes have additional variables and constraints; the number of additional variables and constraints being the size of $x_{t-1}$ for stage $t$.

When objective functions $f_t(\cdot, \cdot, \xi_t)$ have Fenchel-type representations (which is the case of all “well structured” convex functions), we can also derive another variant of ISDDP that combines SDDP with the inexact cuts given in Section 3. For instance, assuming to alleviate notation that $f_t$ is deterministic of the form $f_t(x_t, x_{t-1})$ with Fenchel-type representation

\begin{equation}
\tag{5.50}
f_t(x_t, x_{t-1}) = x_{t-1}^T a_{t,1} + x_t^T a_{t,2} + \max_{w \in \mathcal{W}_t} x_t^T A_t w + x_{t-1}^T B_t w - \Psi_t(w),
\end{equation}

setting

\[
\Delta_{k+1} = \{ \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k) \in \mathbb{R}^{k+1} : \lambda \geq 0, \sum_{i=0}^k \lambda_i = 1 \},
\]
\[
\theta^0_{t,k} = [\theta^0_t, \theta^1_t - 2 \varepsilon_t^1; \ldots; \theta^k_t - 2 \varepsilon_t^k], \quad \theta^0_{t,k} = [\beta^0_t, \beta^1_t, \ldots, \beta^k_t],
\]
\[
\phi_{t,k}(\lambda) = -\lambda^T \theta^0_{t,k},
\]

from the Fenchel-type representation

\[
Q^k_{t+1}(x_t) = \max_{\lambda \in \Delta_{k+1}} \sum_{i=0}^k \lambda_i (\theta^i_t - 2 \varepsilon^i_t + (\beta^i_t, x_t)) = \max_{\lambda_2 \in \Delta_{k+1}} x_t^T \beta^0_{t,k} \lambda_2 - \phi_{t,k}(\lambda_2),
\]

of $Q^k_{t+1}$ where $\varepsilon^0_t = 0$, we deduce the Fenchel-type representation

\begin{equation}
\tag{5.51}
x_{t-1}^T a_{t,1} + x_t^T a_{t,2} + \max_{\lambda \in \Lambda} x_t^T A^k_t \lambda + x_{t-1}^T B_t \lambda - \tilde{\phi}_{t,k}(\lambda)
\end{equation}

of $f_t(x_t, x_{t-1}) + Q^k_{t+1}(x_t)$ where

\[
A^k_t = [\tilde{A}_t, \beta^0_{t,k}], \quad B_t = [\tilde{B}_t, 0], \quad \tilde{\phi}_{t,k}(\lambda_1, \lambda_2) = \Psi_t(\lambda_1) + \phi_{t,k}(\lambda_2),
\]
\[
\Lambda = \{ \lambda = (\lambda_1, \lambda_2) : \lambda_1 \in \mathcal{W}_t, \lambda_2 \in \Delta_{k+1} \}.
\]

In this situation, (5.51) provides a Fenchel-type representation of the objective functions of problems (5.49) solved in the backward passes which allows us to build, using Section 3, inexact cuts of controlled accuracy for value functions $Q^k_t(\cdot, \xi_t)$ and therefore for $Q_t$.

We now study the convergence of ISDDP and start introducing more notation. Due to Assumption (H), the realizations of $(\xi_t)_{t=1}^T$ form a scenario tree of depth $T + 1$ where the root node $n_0$ associated to a stage 0 (with decision $x_0$ taken at that node) has one child node $n_1$ associated to the first stage (with $\xi_1$ deterministic). We denote by $N$ the set of nodes and for a node $n$ of the tree, we define:
- $C(n)$: the set of children nodes (the empty set for the leaves);
- $x_n$: a decision taken at that node;
- $p_n$: the transition probability from the parent node of $n$ to $n$;
-
• \(\xi_n\): the realization of process \((\xi_t)\) at node \(n\): for a node \(n\) of stage \(t\), this realization \(\xi_n\) contains in particular the realizations \(b_n\) of \(b_t\), \(A_n\) of \(A_t\), and \(B_n\) of \(B_t\).

Next, we define for iteration \(k\) decisions \(x^k_n\) for all node \(n\) of the scenario tree simulating the policy obtained in the end of iteration \(k - 1\) replacing cost-to-go function \(Q_t\) by \(Q^{k-1}_t\) for \(t = 2, \ldots, T + 1\):

Simulation of ISDDP policy in the end of iteration \(k - 1\).

\[
\begin{align*}
\text{For } t = 1, \ldots, T, \\
\quad \text{For every node } n \text{ of stage } t - 1, \\
\quad \quad \text{For every child node } m \text{ of node } n, \\
\quad \quad \quad \inf_{x_m} f_t(x_m, x^k_n, \xi_m) + Q^{k-1}_{t+1}(x_m) \\
\quad \quad \quad A_m x_m + B_m x^k_n = b_m, \\
\quad \quad \quad g_t(x_m, x^k_n, \xi_m) \leq 0, \\
\quad \quad \quad x_m \in X_t, \\
\end{align*}
\]

(5.52) \(Q^{k-1}_t(x^k_n, \xi_m) = \)

where \(x^k_{n_0} = x_0\).

End For
End For
End For

We will assume that the sampling procedure in ISDDP satisfies the following property:

(H2) The samples in the backward passes are independent: \((\xi^k_2, \ldots, \xi^k_T)\) is a realization of \((\xi^k_2, \ldots, \xi^k_T) \sim (\xi_2, \ldots, \xi_T)\) and \(\xi^1, \xi^2, \ldots\) are independent.

As said in the introduction, a useful tool for the convergence analysis of SDDP and ISDDP is Lemma 5.2 in [9] for vanishing errors and Lemma 4.1 in [13] for bounded errors. We provide different proofs of these lemmas with slightly different assumptions, one of them being stronger (the continuity of \(f\) which is satisfied when the lemmas are applied to study the convergence of ISDDP) and two being weaker. More precisely, in these lemmas we do not assume \(f^n \leq f\) and take equicontinuous sequences \(f^n\) instead of sequences of Lipschitz continuous functions. If we assumed \(f^n \leq f\), the proof would be a little shorter, because boundedness of \(\{f^n\}\) would be immediate. From these assumptions, we also derive a stronger conclusion, used in the convergence analysis.

**Lemma 5.1.** Let \((X, d)\) be a compact metric space. If \(\{x_n\}_{n \in \mathbb{N}}\) is a sequence in \(X\), \(\{f^n\}_{n \in \mathbb{N}}\) is an equicontinuous sequence of real functions on \(X\), \(f^1 \leq f^2 \leq f^3 \leq \ldots\), and \(f\) is a continuous real function on \(X\) then the following conditions are equivalent:

(a) \(\lim_{m,n \to \infty} f^n(x_n) - f(x_n) = 0\).
(b) \(\lim_{n \to \infty} f^n(x_n) - f(x_n) = 0\).

Moreover, if (a) or (b) holds then \(f^n\) converges uniformly to a continuous function which coincides with \(f\) on the set

\[
Y_\ast = \left\{ y \in X : y = \lim_{j \to \infty} x_{n_j} \text{ for some subsequence } \{x_{n_j}\}_{j \in \mathbb{N}} \right\}.
\]
Proof. See the Appendix.

The proof of the previous lemma can be adapted to prove Lemma 5.2 which will be used in the convergence analysis of ISDDP with bounded errors.

**Lemma 5.2.** Let $(X, d)$ be a compact metric space, let $f : X \to \mathbb{R}$ be continuous and suppose that the sequence of equicontinuous functions $f^k, k \in \mathbb{N}$ satisfies $f^k(x) \leq f^{k+1}(x)$ for all $x \in X, k \in \mathbb{N}$. Let $(x^k)_{k \in \mathbb{N}}$ be a sequence in $X$ and assume that

\[
\lim_{k \to +\infty} f(x^k) = S
\]

for some finite $S \geq 0$. Then

\[
\lim_{k \to +\infty} f(x^k) - f(x^k) \leq S.
\]

Moreover, $f^n$ converges uniformly to a continuous function $g$ such that $|f(y) - g(y)| \leq S$ for every $y$ in the set

\[
Y_* = \left\{ y \in X : y = \lim_{j \to \infty} x_{n_j} \text{ for some subsequence } \{x_{n_j}\}_{j \in \mathbb{N}} \right\}.
\]

Proof. See the Appendix.

We are now in a position to state our first convergence theorem for ISDDP.

**Theorem 5.3.** (Convergence of ISDDP with bounded errors). Consider the sequences of decisions $(x^k_n)_{n \in \mathbb{N}}$ and of functions $(Q^k_t)$ generated in the simulation of ISDDP. Assume that (H), (H1), and (H2) hold, and that errors $\varepsilon^k_t$ and $\delta^k_t$ are bounded: $0 \leq \varepsilon^k_t \leq \varepsilon, 0 \leq \delta^k_t \leq \delta$ for finite $\delta, \varepsilon$. Then the following holds:

(i) for $t = 2, \ldots, T + 1$, for all node $n$ of stage $t - 1$, almost surely

\[
0 \leq \lim_{k \to +\infty} Q_t(x^k_n) - Q^k_t(x^k_n) \leq \lim_{k \to +\infty} Q_t(x^k_n) - Q^k_t(x^k_n) \leq (\delta + 2\varepsilon)(T - t + 1);
\]

(ii) for every $t = 2, \ldots, T$, for all node $n$ of stage $t - 1$, the limit superior and limit inferior of the sequence of upper bounds

\[
\left( \sum_{m \in C(n)} p_m f_t(x^k_m, x^k_n, \xi_m) + Q_{t+1}(x^k_m) \right)_{k}
\]

satisfy almost surely

\[
0 \leq \lim_{k \to +\infty} \sum_{m \in C(n)} p_m \left[ f_t(x^k_m, x^k_n, \xi_m) + Q_{t+1}(x^k_m) \right] - Q_t(x^k_n),
\]

\[
\lim_{k \to +\infty} \sum_{m \in C(n)} p_m \left[ f_t(x^k_m, x^k_n, \xi_m) + Q_{t+1}(x^k_m) \right] - Q_t(x^k_n) \leq (\delta + 2\varepsilon)(T - t + 1);
\]

(iii) the limit superior and limit inferior of the sequence $Q^{k-1}_1(x_0, \xi_1)$ of lower bounds on the optimal value $Q_1(x_0)$ of (5.45) satisfy almost surely

\[
Q_1(x_0) - \delta T - 2\varepsilon(T - 1) \leq \lim_{k \to +\infty} Q^{k-1}_1(x_0, \xi_1) \leq \lim_{k \to +\infty} Q^{k-1}_1(x_0, \xi_1) \leq Q_1(x_0);
\]

(iv) for $t = 2, \ldots, T$, almost surely the sequence of functions $(Q^k_t)_{k}$ converges uniformly to a continuous function $Q^*_t$ which is at most at distance $(\delta + 2\varepsilon)(T - t + 1)$ from $Q_t$ on every accumulation point $x_n$ of the sequences $(x^k_n)_{k}$ for every node $n$ of stage $t - 1$. 

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Proof. (i) We show (5.55) for \( t = 2, \ldots, T + 1 \), and all node \( n \) of stage \( t - 1 \) by backward induction on \( t \). The relation holds for \( t = T + 1 \). Now assume that it holds for \( t + 1 \) for some \( t \in \{2, \ldots, T\} \). Let us show that it holds for \( t \). Take a node \( n \) of stage \( t - 1 \). Let \( S_n \) be the iterations where the sampled scenario passes through node \( n \) and take an iteration \( k \in S_n \). It was shown in Lemma 5.2 in [13] that for the classes of problems we consider, Assumptions (H1)-3,5) imply that almost surely for every \( j, k \) of iterations we consider, Assumptions (H1)-3,5) imply that almost surely for every \( j, k \), there exists \( x_t \) satisfying

\[
x_t \in \text{ri}(\mathcal{X}_t), A_{tj}x_t + B_{tj}x_{t-1}^k = b_{tj} \quad \text{and} \quad g_t(x_t, x_{t-1}^k, \xi_{tj}) < 0.
\]

Recalling that \( \mathcal{X}_t \times \mathcal{X}_{t-1} \subset \text{dom}(g_t) \) for all \( i \), we can reproduce the reasoning used just after Proposition 4.1 in Section 4 to deduce that for every \( j, t \), there exists

\[
(x_t, z) \in \text{ri}(S_{tj})
\]

where

\[
S_{tj} = \{ (x_t, z) : A_{tj}x_t + B_{tj}z = b_{tj}, g_t(x_t, z, \xi_{tj}) \leq 0, x_t \in \mathcal{X}_t \}.
\]

Condition (5.58) is exactly Slater condition (2.12) (from Proposition 2.3) written for problem (5.49) solved in the backward pass of iteration \( k \) for scenario \( j \). Therefore, we can apply Proposition 2.3 to value function \( \Omega^k_t(\cdot, \xi_{tj}) \) to obtain a \( 2\varepsilon^k_t \)-inexact cut for this function for stage \( t \) and iteration \( k \) of ISDDP. More precisely, fix \( j \in \{1, \ldots, N_t\} \) and take \( m \) such that \( \xi_{tj} = \xi_m \). Recalling that \( \lambda^k_m \) is defined in (5.52) and setting

\[
C^k_{tm}(x_n) = f_t(x_n^k, x_n, \xi_m) + \Omega^k_{t+1}(x_n^k) - 2\varepsilon^k_t + \langle \lambda^k_m, x_n - x_n^k \rangle,
\]

using Proposition 2.3, we get for all \( x_n \in \mathcal{X}_{t-1} \) and \( k \in S_n \):

\[
C^k_{tm}(x_n) \leq \Omega^k_t(x_n, \xi_m)
\]

and

\[
\Omega^k_t(x_n^k, \xi_m) - C^k_{tm}(x_n^k) \leq 2\varepsilon^k_t.
\]

This implies that \( \Omega^k_t \) is indeed a valid cut for \( \Omega_t \); for \( x_n \in \mathcal{X}_{t-1} \) and \( k \in S_n \), we have

\[
\Omega_t(x_n) = \sum_{m \in C(n)} p_m \Omega_t(x_n^k, \xi_m) \geq \sum_{m \in C(n)} p_m \Omega^k_t(x_n^k, \xi_m) \geq \sum_{m \in C(n)} p_m C^k_{tm}(x_n^k) = C^k_t(x_n).
\]

Also by definition of \( x_n^k \), computed in the simulation of iteration \( k \) we get

\[
f_t(x_n^k, x_n^k, \xi_m) + \Omega^k_{t+1}(x_n^k) \leq \Omega^k_{t+1}(x_n^k, \xi_m) + \delta^k_t.
\]

Therefore, for \( k \in S_n \):

\[
C^k_t(x_n^k) = \sum_{m \in C(n)} p_m C^k_{tm}(x_n^k),
\]

\[
\geq \sum_{m \in C(n)} p_m \left[ \Omega^k_t(x_n^k, \xi_m) - 2\varepsilon^k_t \right],
\]

\[
\geq -2\varepsilon + \sum_{m \in C(n)} p_m \Omega^k_{t+1}(x_n^k, \xi_m),
\]

\[
\geq -2\varepsilon - \sum_{m \in C(n)} p_m \left[ f_t(x_n^k, x_n^k, \xi_m) + \Omega^k_{t+1}(x_n^k, \xi_m) - \delta^k_t \right],
\]

\[
\geq -2\varepsilon - \delta + \sum_{m \in C(n)} p_m \left[ f_t(x_n^k, x_n^k, \xi_m) + \Omega^k_{t+1}(x_n^k) \right].
\]
It follows that for $k \in S_n$

\[
0 \leq (5.61) \quad Q_t(x^n_k) - Q^k_t(x^n_k) \leq Q_t(x^n_k) - C^k_t(x^n_k)
\]

\[
\leq 2\delta + \bar{\delta} + \sum_{m \in C(n)} p_m \left[ Q_t(x^n_m, \xi_m) - f_t(x^n_m, x^n_k, \xi_m) - Q_{t+1}^k(x^n_k) \right] (5.63)
\]

\[
\leq 2\delta + \bar{\delta} + \sum_{m \in C(n)} p_m \left[ Q_t(x^n_m, \xi_m) - f_t(x^n_m, x^n_k, \xi_m) - Q_{t+1}^k(x^n_k) \right] \leq 0 \text{ by definition of } \Omega_t \text{ and } x^n_k.
\]

\[
0 \leq 2\delta + \bar{\delta} + \sum_{m \in C(n)} p_m \left[ Q_{t+1}^k(x^n_m) - Q_{t+1}^k(x^n_k) \right] (5.64)
\]

Using the induction hypothesis, we have for every $m \in C(n)$ that \(\lim_{k \to +\infty} Q_{t+1}^k(x^n_m) - Q_{t+1}^k(x^n_m) \leq (\delta + 2\bar{\delta})(T - t)\). Following the proof of Lemma 4.2 in [14], we obtain that sequence \((\beta^k_t)\) is almost surely bounded and that functions \((Q^k_t)\) are $L$-Lipschitz continuous and therefore sequence \((Q^k_t)\) is monotone and equicontinuous. Since \(Q_t\) is continuous on \(X_{t-1}\), we can apply Lemma 5.2 to obtain \(\lim_{k \to +\infty} Q_{t+1}^k(x^n_m) - Q_{t+1}^k(x^n_m) \leq (\delta + 2\bar{\delta})(T - t)\), which, plugged into (5.64), gives

\[
(5.65) \quad \lim_{k \to +\infty, k \in S_n} Q_t(x^n_k) - Q^k_t(x^n_k) \leq (\delta + 2\bar{\delta})(T - t + 1).
\]

Finally, to conclude the proof of (i), it remains to show that

\[
(5.66) \quad \lim_{k \to +\infty, k \notin S_n} Q_t(x^n_k) - Q^k_t(x^n_k) \leq (\delta + 2\bar{\delta})(T - t + 1),
\]

and with relation (5.65) at hand, relation (5.66) can be shown by contradiction following the end of the proof of Theorem 4.2 in [13].

(ii) and (iii) can be shown using (i) and following the proof of Theorem 4.2-(ii), (iii) in [13].

(iv) is an immediate consequence of (i) and Lemma 5.2.

We can now state our second convergence theorem for ISDDP:

**Theorem 5.4 (Convergence of ISDPP with vanishing errors).** Consider the sequences of decisions \((x^n_k)_{n \in \mathbb{N}}\) and of functions \((Q^k_t)\) generated in the simulation of ISDDP. Assume that (H), (H1), and (H2) hold, and that for all $t$ we have \(\lim_{k \to +\infty} \varepsilon^k_t = \lim_{k \to +\infty} \delta^k_t = 0\). Then almost surely the limit of the sequence \((Q^k_t)\) is the optimal value \(Q_1(x_0)\) of (5.45). Moreover, for \(t = 2, \ldots, T\), almost surely the sequence of functions \((Q^k_t)\) converges uniformly to a continuous function \(Q^*_t\) which coincides with \(Q_t\) on every accumulation point \(\bar{x}_n\) of the sequences \((x^n_k)\) for every node $n$ of stage $t - 1$.

**Proof.** It suffices to follow the proof of Theorem 5.3 and to use Lemma 5.1 instead of Lemma 5.2.

If instead of the inexact cuts from Section 2 we use in ISDDP the inexact cuts from Section 3 based on Fenchel-type representations of the objective, we obtain similar convergence results, due to the fact that the error terms in both the cuts from Section 2 and from Section 3 linearly depend on $\delta^k_t$ and $\varepsilon^k_t$.

**6. Conclusion.** In [13], an inexact variant of SDDP called ISDDP was introduced. Two variants of the method were described in [13]: one for linear problems and one for nonlinear differentiable problems. In this paper, we explained how to extend
ISDDP for nondifferentiable multistage stochastic programs. We provided formulas to compute inexact cuts for value functions of possibly nondifferentiable optimization problems and combined these cuts with SDDP to describe two new inexact variants of SDDP, one for each of the classes of cuts derived (the cuts from Section 2 and the cuts from Section 3).

Several comments are in order:

- the variants of ISDDP presented in this paper can be used both for nonlinear differentiable and nonlinear nondifferentiable optimization problems.
- For errors bounded from above by $\varepsilon$, same as ISDDP for linear programs introduced in [13], ISDDP variants of this paper provide $3\varepsilon T$-optimal first stage solutions. Using the analysis of Section 4, it is easy to check that ISDDP for nonlinear stochastic programs from [13] provides for bounded errors a $O(T\sqrt{\varepsilon})$-optimal first stage solution. However, all subproblems solved in the forward and backward passes of the variant of ISDDP that uses the cuts from Section 2 have additional variables and constraints; the number of additional variables and constraints being the size of $x_{t-1}$ for stage $t$.
- All variants of ISDDP from [13] and from this paper converge to an optimal policy for vanishing noises. The convergence analysis of ISDDP applied to nonlinear programs in [13] was however more technical due to the fact that the error terms in the inexact cuts were not a linear function of $\delta_k$ and $\epsilon_k$ (see Proposition 5.4 in [13]).

As a future work, it would be interesting to compare the performance of the variants of ISDDP presented in this paper with, for instance, SDDP, and the other variants of inexact SDDP, namely ISDDP from [13] and inexact StoDCuP from [14].

7. Appendix.

**Lemma 7.1.** Assume that $F_i : \mathbb{R}^m \to \mathbb{R}^m$ is $L_i$-co-coercive for $i = 1, \ldots, n$. Then $\sum_{i=1}^n F_i$ is $(\sum_{i=1}^n L_i)$-co-coercive.

**Proof.** We can assume w.l.o.g that all $L_i$ are positive. Let $S(x) = \sum_{i=1}^n F_i(x)$, $L = \sum_{i=1}^n L_i > 0$, and $\alpha_i = \frac{L_i}{L}$. Observing that $\sum_{i=1}^n \alpha_i = 1$ and using the convexity of $\|\cdot\|^2$ we get:

\[
\langle y - x, S(y) - S(x) \rangle \geq \sum_{i=1}^n \frac{1}{\alpha_i} \|F_i(x) - F_i(y)\|^2
\]

\[
(\text{7.67}) \quad = \frac{1}{L} \sum_{i=1}^n \alpha_i \|\frac{1}{\alpha_i} (F_i(x) - F_i(y))\|^2
\]

\[
\geq \frac{1}{L} \|S(y) - S(x)\|^2,
\]

which achieves the proof of the lemma. \(\square\)

**Proof of Lemma 5.1.** Implication (a) $\Rightarrow$ (b) holds trivially. Suppose (b) holds. Since $X$ is compact and $f$ is continuous, the sequence $\{f^n(x_n)\}$ is bounded. Combining this result with the compactness of $X$ and the equicontinuity of $\{f^n\}$ we conclude that this sequence is pointwise uniformly bounded. Hence the monotone sequence $\{f^n(x)\}$ converges for any $x \in X$. Recall that $Y_s$ is the set of limit points of $\{x_n\}$ and let $g : X \to \mathbb{R}$ be the pointwise limit of $\{f^n\}$ that is,

\[
g(x) = \lim_{n \to \infty} f^n(x) \quad (x \in X).
\]

We claim that

1. $g$ is continuous;
2. $\{f^n\}$ converges uniformly to $g$;
3. $g(y) = f(y)$ for any $y \in Y_s$.  

We have already established that $\|f^n(x_n) - f^n(y)\| \to 0$ for any $n$. Since $f$ is Lipschitz,

\[
\|f^n(x_n) - f^n(y)\| \leq L(x_n - y) \to 0.
\]
Continuity of $g$ follows from the equicontinuity of $\{f^n\}$ and its convergence to $g$. Since $\{f^n\}$ is a sequence of equicontinuous functions converging monotonically in a compact set to a continuous function $g$, this convergence is uniform. To prove item 3, suppose that $\lim_{j \to \infty} x_n = y$. Direct use of the triangle inequality yields

$$|f^n(y) - f(y)| \leq |f^n(y) - f(x_n^j)| + |f(x_n^j) - f(x_n)| + |f(x_n) - f(y)|.$$ 

It follows from the equicontinuity of $\{f^n\}$, the continuity of $f$, and the convergence of $\{x_n\}$ to $y$ that the first and third terms in the right-hand side of the above inequality converge to 0, while it follows from Assumption (b) that the middle term also converges to 0. Since $\{f^n(y)\}$ converges to $g(y)$, we have $g(y) = f(y)$.

To end the proof, take $\varepsilon > 0$. There exists $M_0 \in \mathbb{N}$ such that

$$m \geq M_0 \Rightarrow |f^m(x) - g(x)| < \varepsilon \quad \forall x \in X.$$ 

It follows from the definition of $Y_*$ and the compactness of $X$ that there is $N_0 \in \mathbb{N}$ such that $d(x^n, Y_*) < \delta$ for $n \geq N_0$. Suppose that $m \geq M_0$ and $n \geq N_0$. There is $y \in Y_*$ such that $d(x^n, y) < \delta$. Therefore

$$|f^n(x_n) - f(x_n)| \leq |f^n(x_n) - g(x_n)| + |g(x_n) - g(y)| + |g(y) - f(x_n)|$$

which achieves the proof of the lemma. \hfill \qed

**Proof of Lemma 5.2.** The proof is a simple extension of the proof of Lemma 5.1. We outline the changes in the proof below. Since the sequence $f^n(x_n) - f(x_n)$ is bounded from above and $f$ is continuous on the compact set $X$, the sequence $f^n(x_n)$ is bounded from above. Same as in Lemma 5.1, together with the equicontinuity, the monotonicity of $f^n$, and the compactness of $X$, this implies that the sequence $f^n(x)$ converges for every $x \in X$ uniformly to a continuous function $g$. For every $y \in Y_*$, taking $\{x_n\}$ satisfying $y = \lim_{j \to \infty} x_n$, we get

$$|g(y) - f(y)| = |\lim_{j \to \infty} f^n_j(y) - f(\lim_{j \to \infty} x_n)| = |\lim_{j \to \infty} f^n_j(y) - f(x_n)|$$

To conclude, it suffices to modify the last inequality (7.68) in Lemma 5.1 by

$$|f^n(y) - f(x_n)| \leq |f^n(y) - g(x_n)| + |g(x_n) - g(y)| + |g(y) - f(x_n)|$$

which concludes the proof of the lemma. \hfill \qed

REFERENCES


