Chapter 14. Material Handling Models

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14.1. Automated Storage-Retrieval Systems (AS/RS) Cycle Times

Definitions

Unit Load

Automated storage and retrieval systems are primarily used for storing and handling unit loads. The *Effective Unit Load Dimensions* are used to size the system. They include the actual size of the unit load plus all the clearances between adjacent loads to accommodate access and fire prevention devices. The effective unit load dimensions are measured between the centroids of adjacent unit loads in the storage system.

Storage Policy

A storage policy is the set of rules that determine in which open location of the rack an arriving load will be stored. One of the most simple storage policies is random storage or RAN, which selects the storage location at random of the open locations. Usually this is modeled by assuming that the density function of the frequency of access is a uniform distribution over the face of the rack. Another policy often used is to store the arriving load in the "closest-open-location" or COL. It has been shown that this policy can be modeled by a uniform density function over the portion of the rack that is utilized, i.e., is filled with loads.

Chebyshev Travel

In many storage and retrieval systems the shuttle on the crane can move in the vertical direction while the crane itself is moving in the horizontal direction in the aisle. This is referred to as simultaneous travel and the actual travel time between two points in the rack can then be approximated by the Chebyshev travel time formula.

$$t_{CHEB} = \max\left\{ \frac{\Delta_x}{v_x}, \frac{\Delta_y}{v_y} \right\} \tag{14.1}$$
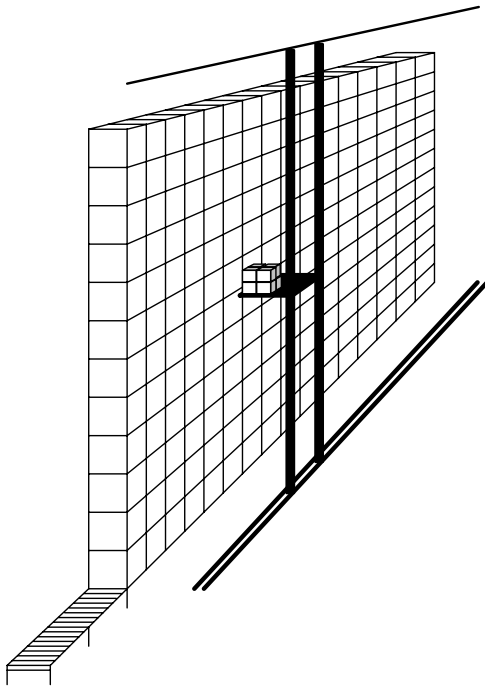

Figure 14.1. Unit Load AS/RS Illustration

Observe that in general there exist many different paths between two points which all take the same amount of time to traverse. The collection of these travel paths is a parallelogram based on the two points and where the angle of the parallelogram is determined by the speed ratio of the vertical shuttle speed and the horizontal crane speed. This parallelogram is illustrated in Figure 14.2
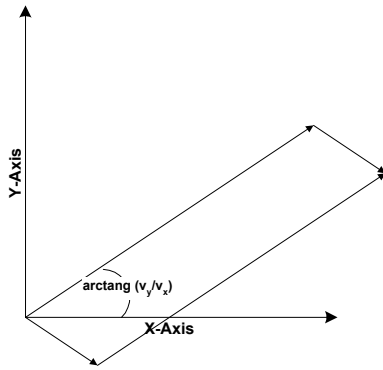
Figure 14.2. Parallelogram of Possible Chebyshev Travel Paths between Two Points

Command Cycle

The command cycle refers to the number of operations performed on a single trip of the crane in the automated storage and retrieval system. If the crane performs either a storage or retrieval, the number of operations is one and is called a single command. If the crane performs first a storage and then moves to another position in the rack and retrieves a load to the output point, then the number of operations performed per trip equals two and the crane is said to operate under dual command. By definition all dual command cycles consist of first a storage operation followed by a retrieval operation, unless explicitly stated otherwise. In person-aboard order picking systems, the crane might visit many locations where items are retrieved from the rack to the shuttle. This is called multiple command.

The expected or average time required to complete a single trip of the crane is called the cycle time. Depending on the crane cycle, we have single, dual, and multiple command cycle times. The cycle time includes the load pickup operation at the input point, the storage operation at the open rack location, the retrieval operation at the full rack location, and the deposit operation at the deposit point, if they are present in the cycle. The expected time to complete the travel of the crane during a cycle is called the single, dual, or multiple command travel time, respectively. It excludes the times required to execute the shuttle pick and place operations.

Dwell Point Policy

A dwell point policy specifies what the crane should do when it becomes idle, i.e., there are no further storage and retrieval request waiting to be executed. A dwell point policy might have two components corresponding to what to do after a storage and after a retrieval operation, respectively. Examples of dwell point policies are:

Always return to the input or pickup point.

Remain at the location of storage location or deposit point where the crane became idle.

Position the crane at the centroid location of the rack.

Cycle Time Computations for Automated Storage and Retrieval Systems

Introduction

Cycle times are very important performance measures both during design of the system and during operations. Cycle times are used to:

Determine the required number of aisles to satisfy throughput

Determine the number and location of the rack input and output station

Evaluate different crane operating policies such as the dwell point policy

Evaluate different storage policies such as random and class based storage

In addition to the rack sizes and the vehicle speeds and accelerations, the cycle times depend on the storage policy, dwell point strategy, command cycle mix, and pickup and deposit stations locations. Great care must be taken to apply the correct formula depending on all the above factors.

There exist basically three methods to determine the cycle times of a storage and retrieval system. If the system actually exists we can do time and motion sampling. Of course this is not possible for a planned system or to evaluate the effects of a new storage or dwell point policy. A second method is computer simulation, be either a simulation of the complete system or a simulation based upon random sampling. The third method estimates the cycle times based on analytical formulas for basis elements included in the cycle time. The amount of effort required determining the cycle time as well as the accuracy of the cycle time estimation increase sharply from low for the analytical formulas, to medium for simulation, to high for time and motion studies.

Random Storage Expected Travel Times

Random storage is the simplest storage policy to compute expected travel times for since the density function of the frequency of access of the rack locations is a uniform function.

The following notation will be used. For simplicity purposes all parameters and variables are expressed in compatible international standard units (ISO). In this section it is assumed that both input and output points are in the lower bottom corner of the rack unless otherwise indicated.

$L =$ horizontal rack size in meters (m), also called rack width

$H =$ vertical rack size in meters (m), also called rack height

$v_x =$ horizontal speed (m/s)

$v_y =$ vertical speed (m/s)

$l =$ effective storage location horizontal size (m), also called location width

$h =$ effective storage location vertical size (m), also called location height

$Q =$ total number of locations that the rack system must provide

T_{PD} $=$ execution time of a pickup or deposit operation

$a =$ fraction of total number of crane cycles that are single command

$T_x =$ horizontal rack size in the time domain (s)

$T_y =$ vertical rack size in the time domain (s)

$T =$ rack scale factor in the time domain (s)

$b =$ rack shape factor

$f_x() =$ probability density function of the horizontal travel time

$f_y() =$ probability density function of the vertical travel time

$f_z() =$ probability density function of the Chebyshev travel time

$F_x() =$ cumulative distribution function of the horizontal travel time

$F_y() =$ cumulative distribution function of the vertical travel time

$F_z() =$ cumulative distribution function of the Chebyshev travel time

$t_{IR} =$ one-way expected travel time from the input point to a random point in the unit rack

$t_{RO} =$ one-way expected travel time from a random point to the output point in the unit rack

$t_{RR} =$ expected travel time between two random points in the unit rack

$t_{OI} =$ travel time between output and input points in the unit rack

$t_{SC} =$ single command travel time in the unit rack (excluding pickup/deposit operations)

t_{DC} =   dual command travel time in the unit rack (excluding pickup/deposit operations)

t_C =   combined travel time in the unit rack (excluding pickup/deposit operations)

T_{SC} =   single command cycle time

T_{DC} =   dual command cycle time

T_C =   combined (single and dual command) cycle time

u =   expected number of operations per cycle

x =   number of vertical columns in the rack, also horizontal rack size in locations

y =   number of horizontal levels or rows in the rack, also vertical rack size in locations

z =   number of aisles in the rack system

To provide better insight in the travel and cycle time computations we will apply two transformations. Since we are interested in travel times, we will transform first the rack from the (original) space domain to the time domain. We will then transform the rack from the time domain to the normalized time domain. This allows us to split the travel time computations into factors related to the size (or scale) of the rack and the shape of the rack. The transformations are illustrated in the next figure.
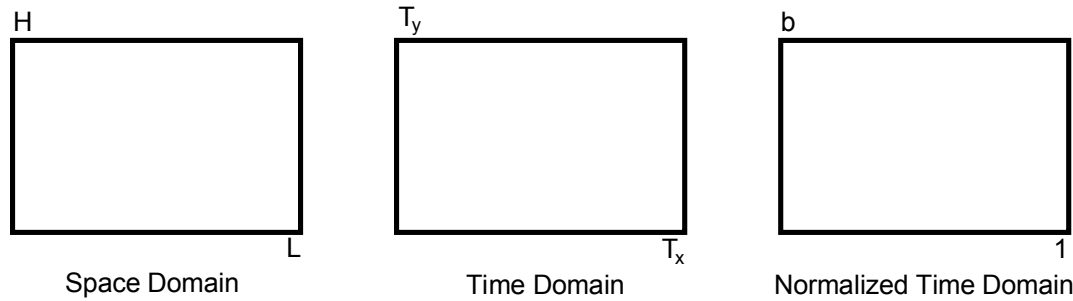

Figure 14.3. Rack Domain Transformations

The dimension of the rack in the time domain are given by:

$$T_x = \frac{l \cdot x}{v_x} = \frac{L}{v_x} \tag{14.2}$$

$$T_y = \frac{h \cdot y}{v_y} = \frac{H}{v_y} \tag{14.3}$$

The size of the rack in the time domain, also called its scale, is thus given by:

$$T = \max\{T_x, T_y\} \tag{14.4}$$

The ratio of the length to the width of the rack in the time domain is called its shape factor, which is computed as:

$$b = \frac{\min\{T_x, T_y\}}{T} \tag{14.5}$$

The dimensions of the rack in the normalized time domain are then 1 by b, where $0 \le b \le 1$. A rack with b equal to one is called "square in time". A square in time rack has the maximum amount of storage locations within a given distance to the pickup and deposit station. Obviously, it is advantageous to have racks be square in time. Even though in the space domain racks are long and not tall, in the time domain most shape factors fall between 0.8 and one, since the rack dimension and crane speeds have about the same ratio. Without loss of generality, it will be assumed from now on that the smaller (time) dimension is on the vertical axis.

Since the crane carriage and the crane shuttle move simultaneously, the time to travel between two points is given by the maximum of the horizontal and vertical travel time as computed by the Chebyshev travel time formula:

$$t_{CHEB} = \max\left\{\frac{\Delta_x}{v_x}, \frac{\Delta_y}{v_y}\right\} \tag{14.6}$$

The lines of equal travel or contour lines in the normalized time domain are illustrated in the next figure when the input/output point is in the lower left corner of the rack and the rack is longer than it is tall. In the left section of the rack the contour lines have an inverted L shape. In the right section of the rack the contour lines are vertical lines. These vertical lines can be thought of as inverted L shape lines where the horizontal part is clipped by the rack face.
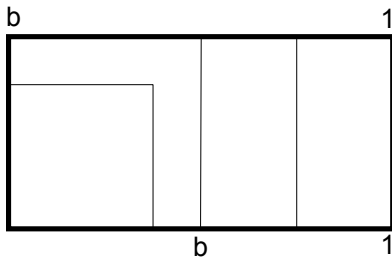

Figure 14.4. One Way Travel in an AS/RS Rack

The general formula for the computation of the expected travel time from the input/output point to a random point in the rack is given by

$$t_{IR} = \oint_{w \in \Omega} t(w) \cdot f(w) \, dw, \tag{14.7}$$

where $t(w)$ is the travel time to a point w in the rack Ω and $f(w)$ is the frequency of access density function to the point w. For random storage this frequency of access is equal to:

$$f(w) = \frac{1}{\Omega}.$$

(14.8)

The formula is complicated by the presence of the maximum operator in the Chebyshev travel time formula. Using the cumulative distribution functions of the travel times can eliminate this maximum operator. Since the horizontal and vertical travel times are independent, the probability that the maximum of the two is smaller than or equal to k is equal to the product of the probabilities that either is smaller than or equal to k, or

$$F_z(k) = \mathbf{P}\{z \le k\} = \mathbf{P}\{x \le k\} \cdot \mathbf{P}\{y \le k\} = F_x(k) \cdot F_y(k)$$

$$F_x(z) = \mathbf{P}\{t_x \le z\} = z$$

$$F_y(z) = \mathbf{P}\{t_y \le z\} = \begin{cases} z/b, & z \le b \\ 1, & z \ge b \end{cases}$$

$$F_z(z) = \mathbf{P}\{t_z \le z\} = \begin{cases} z^2/b, & z \le b \\ z, & z \ge b \end{cases}$$

The probability density function can then be found by taking the derivative, or

$$f_z(z) = \frac{dF_z(z)}{dz} = \begin{cases} 2z/b, & z \le b \\ 1, & z \ge b \end{cases}$$

The expected one-way travel time can then be found by integrating over the unit rack area with the above derived density function.

$$\begin{aligned} t_{IR} &= \oint_\Omega t_z(w) f_z(w) dw \\ &= \int_{z=0}^{z=b} \frac{2z^2}{b} dz + \int_{z=b}^{z=1} z dz \\ &= \frac{b^2}{6} + \frac{1}{2} \end{aligned}$$

(14.9)

The next step is to compute the expected travel time between two random points in the rack. In order to compute this, we need to compute the probability density function of the Chebyshev travel time. Again we will use the independence condition to derive the cumulative distribution function of the Chebyshev travel time in function of the cumulative distribution functions of the vertical and horizontal travel times between two points in the rack. We will call the travel time between two random points in the rack the

range. The probability density function and the cumulative distribution function of the vertical range are:

$$f_{yrange}(r) = 2 \int\limits_{v=0}^{v=b-r} f_y(v) f_y(v+r) dv$$

$$= 2 \int\limits_{v=0}^{v=b-r} \frac{1}{b^2} dv$$

$$= \frac{2}{b^2}(b-r)$$

$$F_{yrange}(z) = \int\limits_{r=0}^{r=z} f_{yrange}(r) dr$$

$$= \int\limits_{r=0}^{r=z} \frac{2}{b^2}(b-r) dr \qquad\qquad\qquad 0 \le z \le b$$

$$= \frac{2z}{b} - \frac{z^2}{b^2}$$

$$F_{yrange}(z) = 1 \qquad\qquad\qquad\qquad\qquad b \le z \le 1$$

The probability density function and the cumulative distribution function of the horizontal range are:

$$f_{xrange}(r) = 2 \int\limits_{v=0}^{v=1-r} f_x(v) f_x(v+r) dv$$

$$= 2 \int\limits_{v=0}^{v=1-r} dv$$

$$= 2(1-r)$$

$$F_{xrange}(z) = \int\limits_{r=0}^{r=z} f_{xrange}(r) dr$$

$$= \int\limits_{r=0}^{r=z} 2(1-r) dr$$

$$= 2z - z^2$$

The probability density function and the cumulative distribution function of the Chebyshev range are:

$$F_{zrange}(z) = F_{xrange}(z) \cdot F_{yrange}(z)$$

$$= \begin{cases} (2z - z^2)\left(\dfrac{2z}{b} - \dfrac{z^2}{b^2} \right), & 0 \le z \le b \\[2mm] 2z - z^2, & b \le z \le 1 \end{cases}$$

$$f_{zrange} = \frac{dF_{zrange}(z)}{dz}$$

$$= \begin{cases} \dfrac{4z^3}{b^2} - \dfrac{6z^2}{b^2} - \dfrac{6z^2}{b} + \dfrac{8z}{b}, & 0 \le z \le b \\ 2 - 2z, & b \le z \le 1 \end{cases}$$

The expected Chebyshev travel time between two uniformly distributed random points is then:

$$
\begin{aligned}
t_{RR} &= \oint_\Omega t_{zrange}(w) f_{zrange}(w) dw \\
&= \int_{z=0}^{z=b} \left(\frac{4z^4}{b^2} - \frac{6z^3}{b^2} - \frac{6z^3}{b} + \frac{8z^2}{b} \right) dz + \int_{z=b}^{z=1} (2 - 2z) dz \\
&= \frac{1}{3} + \frac{b^2}{6} - \frac{b^3}{30}
\end{aligned}
\qquad (14.10)
$$

Cycle Times for Random Storage (Single Corner P/D Point)

If the input and output station are both in the lower left corner of the rack and if the crane executes only one type of operation, i.e., only single commands or only dual commands, then we get the standard cycle time formulas. The single command cycle time is then composed of two one-way travels plus two operations, one to pick up the load and one to deposit the load.

$$
\begin{aligned}
T_{SC} &= T \cdot 2 \cdot t_{IR} + 2 \cdot T_{PD} \\
&= T \cdot \left(1 + \frac{b^2}{3} \right) + 2 \cdot T_{PD}
\end{aligned}
\qquad (14.11)
$$

The dual command cycle time is then composed of two one-way travels plus the travel between two random points, plus four operations, two pick up and two deposit operations.

$$
\begin{aligned}
T_{DC} &= T \cdot (2 \cdot t_{IR} + t_{RR}) + 4 \cdot T_{PD} \\
&= T \cdot \left(\frac{4}{3} + \frac{b^2}{2} - \frac{b^3}{30} \right) + 4 \cdot T_{PD}
\end{aligned}
\qquad (14.12)
$$

The average combined cycle time is then computed as the weighted average of the single and dual command cycle times:

$$T_C = a \cdot T_{SC} + (1 - a) \cdot T_{DC} \qquad (14.13)$$

Example of Simple Cycle Times for Random Storage

An AS/RS system is do be designed to store approximately 12,000 unit loads. Due to space limitations in the existing facility, it is know that the system will have eight tiers or levels of storage. The choice has been reduced to having either (a) 10 aisles of 75 columns (openings long), or (b) 9 aisles of 84 columns (openings long). The S/R machine will travel horizontally at an average speed of 450 feet per minute; simultaneously, it travels vertically an average speed of 80 feet per minute. The distance between the centroids of adjacent unit locations is 48 inches in the horizontal direction and 54 inches in the vertical direction. This centroid to centroid distance includes space for the structural elements of the rack and all fire and safety equipment. The P/D station is located at the end of the aisle at the floor level. The time required to pick up (P) or deposit (D) a load is 0.3 minutes. The throughput requirement on the system is to perform a total of 180 storage and 180 retrievals per hour. Randomized storage is used. It is anticipated that 40 % of the crane cycles are single command. We will determine for each of the alternatives listed above the overall expected cycle time and then determine if they will meet the throughput constraint and average crane utilization.

Since alternative (b) has fewer cranes and a longer aisle it will have the longest expected cycle times of the two. On the other hand, alternative (b) will be less expensive since it has the fewest cranes. So we will investigate this alternative first to see if it satisfies the throughput requirement.

$$H = y \cdot h = \frac{8 \cdot 54}{12} = 36 \ feet$$

$$L = x \cdot l = \frac{84 \cdot 48}{12} = 336 \ feet$$

$$T = \max\left\{\frac{L}{v_x}, \frac{H}{v_y}\right\} = \max\left\{\frac{336}{450}, \frac{36}{80}\right\} = \max\{0.7466, 0.45\} = 0.7466$$

$$b = \min\left\{\frac{L}{v_x}, \frac{H}{v_y}\right\} / T = \min\{0.7466, 0.45\} / 0.7466 = 0.603$$

$$a = 0.4$$

$$T_{PD} = 0.3$$

$$T_{SC} = T \cdot (1 + \frac{b^2}{3}) + 2 \cdot T_{PD} = 0.7466 \cdot (1 + \frac{0.603^2}{3}) + 2 \cdot 0.3 = 1.437$$

$$T_{DC} = T \cdot \left(\frac{4}{3} + \frac{b^2}{2} - \frac{b^3}{30}\right) + 4 \cdot T_{PD} = 0.7466 \cdot \left(\frac{4}{3} + \frac{0.603^2}{2} - \frac{0.603^3}{30}\right) + 4 \cdot 0.3 = 2.326$$

$$T_C = a \cdot T_{SC} + (1-a) \cdot T_{DC} = 0.4 \cdot 1.437 + 0.6 \cdot 2.326 = 1.970$$

$$u = a \cdot 1 + (1-a) \cdot 2 = 0.4 \cdot 1 + 0.6 \cdot 2 = 1.6$$

$$N = \left\lceil \frac{M \cdot S}{H} \right\rceil = \left\lceil \frac{360 \cdot \dfrac{1.970}{1.6}}{60} \right\rceil = \lceil 7.39 \rceil = 8$$

Since the required number of machines is 8 which is smaller than the designed number of machines, equal to 9, this alternative will meet the throughput requirements and the average machine utilization is 7.39 / 9 = 82.1 %. In practice, alternative (a) no longer has to be evaluated but we include it here for completeness sake.

$$H = y \cdot h = \frac{8 \cdot 54}{12} = 36 \ feet$$

$$L = x \cdot l = \frac{75 \cdot 48}{12} = 300 \ feet$$

$$T = \max\left\{ \frac{L}{v_x}, \frac{H}{v_y} \right\} = \max\left\{ \frac{300}{450}, \frac{36}{80} \right\} = \max\{0.667, 0.45\} = 0.667$$

$$b = \min\left\{ \frac{L}{v_x}, \frac{H}{v_y} \right\} / T = \min\{0.667, 0.45\} / 0.667 = 0.675$$

$$a = 0.4$$

$$T_{PD} = 0.3$$

$$T_{SC} = T \cdot (1 + \frac{b^2}{3}) + 2 \cdot T_{PD} = 0.667 \cdot (1 + \frac{0.675^2}{3}) + 2 \cdot 0.3 = 1.368$$

$$T_{DC} = T \cdot \left(\frac{4}{3} + \frac{b^2}{2} - \frac{b^3}{30} \right) + 4 \cdot T_{PD} = 0.667 \cdot \left(\frac{4}{3} + \frac{0.675^2}{2} - \frac{0.675^3}{30} \right) + 4 \cdot 0.3 = 2.234$$

$$T_C = a \cdot T_{SC} + (1-a) \cdot T_{DC} = 0.4 \cdot 1.368 + 0.6 \cdot 2.234 = 1.888$$

$$u = a \cdot 1 + (1-a) \cdot 2 = 0.4 \cdot 1 + 0.6 \cdot 2 = 1.6$$

$$N = \left\lceil \frac{M \cdot S}{H} \right\rceil = \left\lceil \frac{360 \cdot \dfrac{1.888}{1.6}}{60} \right\rceil = \lceil 7.08 \rceil = 8$$

Since the required number of machines is 8 which is smaller than the designed number of machines, equal to 10, this alternative will meet the throughput requirements and the average machine utilization is 7.08 / 10 = 70.8 %.

Cycle Times for Random Storage (General P/D Points)

Both pickup and deposit points are not always located in the lower left corner of the rack. The overall expected cycle time is then most easily computed based on a two-dimensional matrix which shows the previous and next operation, their associated probabilities and the expected cycle time for each combination. The fraction of single command storage operations is denoted by *a* in the following Table.

Table 14.1. Cycle Time Computation Matrix Template

	Next Operation	SC Storage	SC Retrieval	Dual
Prev. Operation	Probability	a/2	a/2	(1-a)
SC Storage	a/2			
SC Retrieval	a/2			
Dual	(1-a)			

We will compute the expected operation time in the following Automated Storage/Retrieval System using a continuous approximation. The horizontal dimension (L) equals 200 feet; the vertical dimension (H) equals 80 feet. The horizontal speed equals 250 feet per minute and the vertical speed equals 50 feet per minute. The input point is a the lower left corner with coordinates (0,0) and the output point is at the lower right corner with coordinates (200,0). All coordinates are expressed in feet. The products are stored in the rack following random storage and the rack is considered filled to capacity. Forty percent of the crane cycles are dual command cycles. The dwell point strategy of the crane is to remain at the point where it became idle, i.e., at the storage location after the completion of a storage operation and at the output point after the completion of a retrieval or dual command operation. We will assume that pickup and deposit times are equal to 30 seconds. We will execute all our computations to four significant digits and round intermediate results to four significant digits.

First, we will compute the shape and scale factor of the rack.

$$T = \max\left\{\frac{200}{250}, \frac{80}{50}\right\} = \max\{0.8, 1.6\} = 1.6$$

$$b = \min\left\{\frac{200}{250}, \frac{80}{50}\right\} \Big/ 1.6 = \min\{0.8, 1.6\}/1.6 = 0.5$$

$$a = 0.6$$

Observe that the vertical travel time is the longest, i.e. the rack is vertically dominated or taller than it is long (in time units).

Second, we will compute the expected travel times from the input/output point to a random point in the rack and between two random points in the rack for the normalized rack in the time domain. Since the output point is located symmetrically to the input point, the expected travel times from either point to a random point are the same.

$$t_{IR} = \left(\frac{1}{2} + \frac{0.5^2}{6}\right) = 0.5417$$

$$t_{RR} = \left(\frac{1}{3} + \frac{0.5^2}{6} - \frac{0.5^3}{30}\right) = 0.3708$$

$$t_{OI} = b = 0.5$$

Third, we compute the overall expected cycle time for the specified dwell point strategy based on the cycle time probability matrix, which is shown below. Observe that in the normalized time domain rack the length of the horizontal trip from output point to input point, denoted by t_{OI}, is equal to 0.5 for this example.

Table 14.2. Cycle Time Computation Matrix

	Next Operation	SC Storage	SC Retrieval	Dual
Prev. Operation	Probability	a/2	a/2	1-a
SC Storage	a/2	$2\,t_{IR}$	$2\,t_{IR}$	$2\,t_{IR} + t_{RR}$
SC Retrieval	a/2	$2\,t_{IR} + t_{OI}$	$2\,t_{IR}$	$2\,t_{IR} + t_{RR} + t_{OI}$
Dual	1-a	$2\,t_{IR} + t_{OI}$	$2\,t_{IR}$	$2\,t_{IR} + t_{RR} + t_{OI}$

Since the last two rows of the matrix are identical for this example, we can simplify the computation:

$$t_C = \frac{a}{2}\left[\frac{a}{2}2t_{IR} + \frac{a}{2}2t_{IR} + (1-a)(2t_{IR} + t_{RR})\right] +$$

$$\left(1-\frac{a}{2}\right)\left[\frac{a}{2}(2t_{IR} + 0.5) + \frac{a}{2}2t_{IR} + (1-a)(2t_{IR} + t_{RR} + t_{OI})\right]$$

$$= 2 \cdot t_{IR} + (1-a) \cdot t_{RR} + \left(1 - \frac{a}{2}\right)^2 \cdot t_{OI}$$

$$= 2 \cdot 0.5417 + (1 - 0.6) \cdot 0.3708 + (1 - 0.3)^2 \cdot 0.5 = 1.4767$$

Fourth, we transfer back the results from the normalized time rack to the original rack by multiplying with the rack scale factor and by adding the expected pickup and deposit time per cycle.

$$u = a \cdot 1 + (1-a) \cdot 2 = 0.6 \cdot 1 + (1 - 0.6) \cdot 2 = 1.4$$

$$T_C = t_C \cdot T + 2 \cdot u \cdot T_{PD} = 1.6 \cdot 1.4767 + 2 \cdot 1.4 \cdot 0.5 = 2.363 + 1.4 = 3.763$$

If the output point is moved to the coordinates (200, 20) while all other parameters remain the same, we can compute the expected one-way travel time to the output point and the expected time between two random points in the corresponding unit rack in the following way.

The difficult part is the computation of the expected time from a random point to the output point since we need to based this computation on two sub-racks created by the horizontal dividing line with vertical coordinate equal to 200. The expected time is then computed based on the relative size of these two sub racks. Observe that this computation has to be done in the time domain (not the normalized time domain) since the scale factors for each of the sub racks are different. Let A denote the area of a sub rack.

$$T_{TOP} = \max\left\{\frac{200}{250}, \frac{60}{50}\right\} = \max\{0.8, 1.2\} = 1.2$$

$$b_{TOP} = \frac{0.8}{1.2} = 0.667$$

$$T_{BOT} = \max\left\{\frac{200}{250}, \frac{20}{50}\right\} = \max\{0.8, 0.4\} = 0.8$$

$$b_{BOT} = \frac{0.4}{0.8} = 0.5$$

$$t_{OR,TOP} = \left(\frac{1}{2} + \frac{0.6667^2}{6}\right) = 0.5741$$

$$T_{OR,TOP} = 1.2 \cdot 0.5741 = 0.6889$$

$$t_{OR,BOT} = \left(\frac{1}{2} + \frac{0.5^2}{6}\right) = 0.5417$$

$$T_{OR,BOT} = 0.8 \cdot 0.5417 = 0.4333$$

$$T_{OR} = \frac{A_{TOP} \cdot T_{OR,TOP} + A_{BOT} \cdot T_{OR,BOT}}{A_{TOP} + A_{BOT}} = \frac{200 \cdot 0.4333 + 600 \cdot 0.6889}{800} = 0.6250$$

$$t_{OR} = \frac{T_{OR}}{T} = \frac{0.6250}{1.6} = 0.3906$$

Now we can compute the overall cycle time based on the probability matrix, which is shown below. The difference with before is the fact that t_{IR} is no longer equal to t_{RO}.

Table 14.3. Cycle Time Computation Matrix

	Next Operation	SC Storage	SC Retrieval	Dual
Prev. Operation	Probability	a/2	a/2	1-a
SC Storage	a/2	$2\,t_{IR}$	t_{IR+tRO}	$t_{IR} + t_{RO} + t_{RR}$
SC Retrieval	a/2	$2\,t_{IR} + t_{OI}$	$2\,t_{RO}$	$t_{IR} + t_{RO} + t_{RR} + t_{OI}$
Dual	1-a	$2\,t_{IR} + t_{OI}$	$2\,t_{RO}$	$t_{IR} + t_{RO} + t_{RR} + t_{OI}$

Since the last two rows of the matrix are identical for this example, we can simplify the computation:

$$
\begin{aligned}
t_C &= \frac{a}{2}\left[\frac{a}{2}2t_{IR} + \frac{a}{2}(t_{IR}+t_{RO})+(1-a)(t_{IR}+t_{RO}+t_{RR})\right] + \\
&\quad \left(1-\frac{a}{2}\right)\left[\frac{a}{2}(2t_{IR}+t_{OI})+\frac{a}{2}2t_{RO}+(1-a)(t_{IR}+t_{RO}+t_{RR}+t_{OI})\right] \\
&= 0.3\cdot\left[0.3\cdot2\cdot0.5417 + 0.3\cdot(0.5417+0.3906)+0.4(0.5417+0.3906+0.3708)\right] + \\
&\quad 0.7\cdot\left[0.3(2\cdot0.5417+0.5)+0.3(2\cdot0.3906)+0.4(0.5417+0.3906+0.3708+0.5)\right] \\
&= 1.3392
\end{aligned}
$$

Fourth, we transfer back the results from the normalized time rack to the original rack by multiplying with the rack scale factor and by adding the expected pickup and deposit time per cycle.

$$T_C = 1.6\cdot1.3392 + 2\cdot1.4\cdot0.5 = 2.1428 + 1.4 = 3.5428 \text{ minutes.}$$

Cycle Times for Random Storage for Order Picking Trucks

The order picking truck is another material handling equipment type operating in the aisles and is closely related to Automated Storage and Retrieval systems. Usually, the truck does not have a top rail to steady its vertical mast. In addition, safety rules usually prohibit the truck to move while a person is on the shuttle on the mast. As a consequence the movements of the order picking truck in the horizontal and vertical direction occur in a sequential fashion. The travel time can then be approximated with the rectilinear norm. Using the notations derived for the AS/RS, the cycle times can be derived as follows.

Since the horizontal and vertical travel occur sequentially and hence independent, they can be estimated separately. Assume without loss of generality that the unit rack is longer than it is tall, i.e., its horizontal dimension is equal to one and its vertical dimension is equal to b. In the unit or normalized rack the average horizontal travel is then 0.5 and the average vertical travel is then b/2. The expected one way travel time is then given by:

$$t_{IR} = \frac{1+b}{2} \tag{14.14}$$

The lines of equal travel or contour lines in the normalized time domain are illustrated in the next figure when the input/output point is in the lower left corner of the rack and the rack is longer than it is tall.
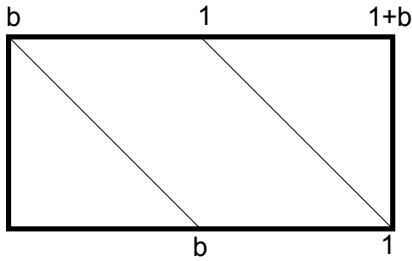

Figure 14.5. One Way Travel in an Order Picking Truck Rack

The expected one way travel time can also be derived by the following integration:

$$
\begin{aligned}
t_{IR} &= \frac{1}{b} \int_{x=0}^{1} \int_{y=0}^{b} (x+y)\,dy\,dx \\
&= \frac{1}{b} \int_{x=0}^{1} (bx + \frac{b^2}{2})\,dx \\
&= \frac{1+b}{2}
\end{aligned}
\tag{14.15}
$$

The single command cycle time is then given by:

$$
T_{SC} = 2 \cdot T \cdot t_{IR} + 2 \cdot T_{PD} = T(1+b) + 2 \cdot T_{PD}
\tag{14.16}
$$

Process Allocation and Operator Sizing

Unit Process Concept

A unit process is an elemental modification of material or process status done essentially without interruption.

A unit process specifies WHAT, not HOW it needs to be done. A unit process does not specify a machine but rather specifies the transformation, i.e. "*make a hole*" instead of "*drill a hole*". It can be represented as a black box, which in turn can be used as building blocks for the total manufacturing process.

Process Flow or Process Requirements

Transfer Diagrams and Transfer Equations

The process flows are determined based on the required quantity of finished product and the production routing which includes the data on scrap and rework rates. The input flows for each process are computed backwards from the final output flow based on the transfer diagram (and formulas) for each process.

The *defect rate* is the ratio of the number of defective parts to the number parts fed into the system. It is usually expressed as a percentage. The complement of the defect rate is the rate of good parts or *yield rate*.

The *rework rate* is the ratio of the number of parts that can be reprocessed to the total number of defective parts. It is usually expressed as a percentage. The complement of the rework rate is the *scrap rate*, i.e. the ratio of the number of parts leaving the manufacturing system to be scrapped to the number of defective parts. If the rework rate is equal to zero, i.e. all defective parts leave the manufacturing process, then the number of defective parts is equal to the number of scrapped parts and the scrap rate is 100 %.

The *transfer diagram* is a graphical representation of all the input and output flows and their relationships of a single process. The transfer formulas are the algebraic equivalent of the diagram.

Several typical cases arise depending on the number of times a product can be reworked before it has to be scrapped. The transfer diagram and formulas for these typical cases will be developed next. Extensions and modifications are the responsibility of the student.

The following notation will be used:

I_k	= input flow of process k
M_k	= manufacturing flow for process k, i.e. all parts to be processed
O_k	= output flow of process k, i.e. good parts
D_k	= defect flow of process k, i.e. defective parts
S_k	= scrap flow of process k, i.e. scrapped parts leaving the system
d_k	= defect rate of process k

r_k $\quad\quad\quad$ = rework rate of process k

s_k $\quad\quad\quad$ = scrap rate of process k

No rework allowed.

In this case the defect flow is equal to the scrap flow. The transfer diagram and formulas for a single process are given next.
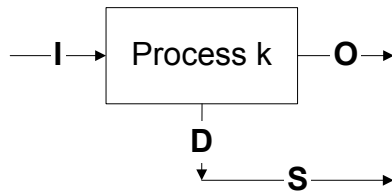

Figure 14.6. No Rework Transfer Diagram

$$O_k = I_k - D_k$$
$$D_k = d_k * I_k$$
$$O_k = (1 - d_k) * I_k \tag{14.1}$$
$$I_k = \frac{O_k}{(1 - d_k)}$$

For a completely serial system, where the output of process k is used as the input for process k + 1, a transfer formula for the complete system of N processes can be derived based upon the following formulas.

$$I_{k+1} = O_k$$
$$I_1 = O_N * \prod_{k=1}^{N} \frac{1}{(1 - d_k)} \tag{14.2}$$

Assume that there are three machines with the following defect rates (0.04, 0.01, 0.03). The required output from machine three is 97,000 parts per month. Working backwards from machine three, input for machine three is 97,000 / (1 - 0.03) = 100,000. Similarly the input for machines two and one is 101,010 and 105,219, respectively.

Infinite Rework Allowed.

In this case all the parts that can be reworked are added to the supply of new unprocessed parts. An example of such operation would be the turning of a cylindrical part to a desired diameter. Parts with a diameter within the tolerances are accepted. Parts with a diameter that is too large can be reworked. If on the other hand the diameter is too small, then the part has to be scrapped. The transfer diagram and formulas for a single process are given next.
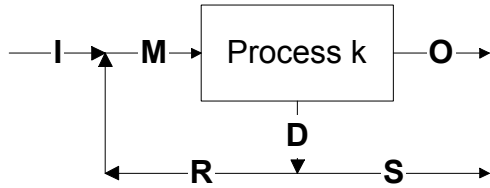
Figure 14.7. Infinite Rework Transfer Diagram

$$M_k = I_k + R_k$$
$$D_k = M_k * d_k$$
$$R_k = D_k * r_k$$
$$I_k = M_k - R_k = (1 - r_k d_k) * M_k \tag{14.3}$$
$$O_k = (1 - d_k) * M_k$$
$$I_k = \frac{O_k (1 - r_k d_k)}{(1 - d_k)}$$

For a completely serial system, where the output of process k is used as the input for process k+1, a transfer formula for the complete system of N processes can be derived based upon the following formulas.

$$I_{k+1} = O_k$$
$$I_1 = O_N \prod_{k=1}^{N} \frac{(1 - r_k d_k)}{(1 - d_k)} \tag{14.4}$$

Observe that formulas (14.3) and (14.4) reduce to formulas (14.1) and (14.2) respectively by setting the rework rate equal to 0, i.e. the no rework case.

Assume further in the example that the rework rate for all three machines is equal to 0.5. Working backwards from machine three, input for machine three is 97,000 · (1 - 0.015) / (1 - 0.03) = 98,500. Similarly the input for machines two and one is 98,998 and 101,060, respectively.

Limited Rework Allowed.
Sometimes a defective part can only be reprocessed a limited number of times. An example of such an operation would the turning as described above, but the turning operation hardens the outer layer of the cylindrical part so that after two turning operations the material becomes brittle and can no longer be turned. The transfer diagram is identical to the case of the unlimited rework case, but some of the paths can only be followed a finite number of times. The number of parts input into the machine on subsequent passes is equal to I, I(rd), I(rd)(rd).. If the number of rework passes is limited to L, then the transfer diagram and formulas are given by
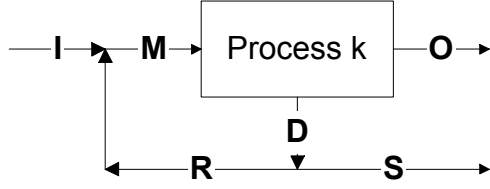
Figure 14.8. Limited Rework Transfer Diagram

$$O_k = (1 - d_k) * I_k * \sum_{i=0}^{L} (r_k d_k)^i$$

$$\sum_{i=0}^{\infty} (rd)^i = \frac{1}{1 - rd}, if |rd| < 1$$

$$\sum_{i=0}^{L} (rd)^i = \sum_{i=0}^{\infty} (rd)^i - \sum_{i=L+1}^{\infty} (rd)^i \qquad (14.5)$$

$$= \frac{1 - (rd)^{L+1}}{1 - rd}$$

$$I_k = O_k * \frac{(1 - r_k d_k)}{(1 - d_k)(1 - (r_k d_k)^{L+1})}$$

For a completely serial system, where the output of process k is used as the input for process k + 1, a transfer formula for the complete system of N processes can be derived based upon the following formulas.

$$I_{k+1} = O_k$$

$$I_1 = O_N \prod_{k=1}^{N} \frac{(1 - r_k d_k)}{(1 - d_k)(1 - (r_k d_k)^{L+1})} \qquad (14.6)$$

Assume further in the above example that the maximum number of times a part can be reworked is equal to three. Working backwards from machine three, input for machines three, two and one is 98,500, 98,998 and 101,060, respectively.

Assembly Operations

A more complicated example of the computation of process requirements is given in the examples.
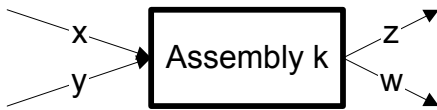

Figure 14.9. Assembly Operation Transfer Diagram

An example of the assembly transfer equations for a particular case of assembly operations is given next:

$$2x + y \Rightarrow z$$
$$x + 3y \Rightarrow w$$
$$\begin{cases} I_x = 2 \cdot O_z + O_w \\ I_y = O_z + 3 \cdot O_w \end{cases}$$

(14.7)

Machine and Department Space Requirements

Space requirements per machine are based on industry norms and health and safety rules. The number of machines required can be computed based on the required flows through that machine and on the manufacturing data included in the production routing sheet.

The following notation will be used:

N_j	=	required number of machines of type j
S_i	=	standard manufacturing time for product i
M_i	=	number of units to be processed per shift of product i
H	=	amount of time available in the planning period
R_j	=	reliability of machine j, expressed as percent up time
$\lceil x \rceil$	=	ceiling function which returns the smallest integer larger than or equal to x.
E_{ij}	=	efficiency of machine j for product i expressed as a percentage, i.e. it is the ratio of the standard time S_i divided by the real time for producing product i on machine j. Can the efficiency be larger than 100 %?

The number of machines required of type j is then given by the following formula

$$N_j = \left\lceil \frac{\sum_{i=1} \frac{S_i \cdot M_i}{E_{ij}}}{H \cdot R_j} \right\rceil$$

(14.8)

For example, 200 parts have to be produced per workday of 8 hours and each part requires a standard time of 2.8 minutes. The efficiency of the machine to be used to product the parts equals .80 % and this machine is 95 % reliable. The number of machines requires is then $\lceil (2.8 \cdot 200) / (480 \cdot 0.95 \cdot 0.80) \rceil =$ $\lceil 1.535 \rceil = 2$.

Basic Sizing and Allocation Model

Sizing and Allocation Problem Characteristics

Jobs and Processors

Sizing and Allocation

Minimize Costs

Capacitated Processors

Deterministic Parameters and Constraints

Notation

M Jobs or Customers (i)

N Processors or Machines (j)

Fixed (f) and Variable (c) Costs

Variable (r) Resource Requirements

Processor Capacities (s)

Required Completed Jobs (d)

Sizing (y) and Allocation (x) Variables

Basic Formulation

$$Min \quad \sum_{j=1}^{N} f_j y_j + \sum_{j=1}^{N} \sum_{i=1}^{M} c_{ij} x_{ij}$$

$$s.t. \quad \sum_{i=1}^{M} r_{ij} x_{ij} \leq s_j y_j \qquad \forall j$$

$$\sum_{j=1}^{N} x_{ij} \geq d_i \qquad \forall i \qquad\qquad (14.9)$$

$$x_{ij} \geq 0 \qquad \forall ij$$

$$y_j \in N^{\circ} = \{0,1,2,3...\} \qquad \forall j$$

Fixed Resource Requirement Model Extension

Assume that a fixed resource, denoted by the parameter g, is required as soon as any job of type i is processed on a machine of type j. The binary decision to process any job of type i on a machine of type j is represented by the variable z.

$$
\begin{aligned}
s.t \quad & x_{ij} \leq \min\{d_i, s_j/r_{ij}\}z_{ij} & \forall ij \\
& \sum_{i=1}^{M}\left(g_{ij}z_{ij} + r_{ij}x_{ij}\right) \leq s_j y_j & \forall j \\
& z_{ij} \leq y_j & \forall ij \\
& z_{ij} \in B = \{0,1\} & \forall ij
\end{aligned}
\tag{14.10}
$$

Strong Formulation

The allocation variables x represent the fraction of the total demand of job i allocated to machine type j.

$$
\begin{aligned}
Min \quad & \sum_{j=1}^{N}f_j y_j + \sum_{j=1}^{N}\sum_{i=1}^{M}c_{ij}d_i x_{ij} \\
s.t. \quad & \sum_{i=1}^{M}\left(g_{ij}z_{ij} + r_{ij}d_i x_{ij}\right) \leq s_j y_j & \forall j \\
& \sum_{j=1}^{N}x_{ij} \geq 1 & \forall i \\
& x_{ij} \leq y_j & \forall ij \\
& x_{ij} \leq \min\{1, s_j/d_i r_{ij}\}z_{ij} & \forall ij \\
& z_{ij} \leq y_j & \forall ij \\
& z_{ij} \in B = \{0,1\} & \forall ij \\
& 1 \geq x_{ij} \geq 0 & \forall ij \\
& y_j \in N^{\circ} = \{0,1,2,3...\} & \forall j
\end{aligned}
\tag{14.11}
$$

Basic Queuing Formulas

The following expressions describing the main characteristics of queuing systems are given in Ross (1993) and Giffin (1978).

Notation

W_q =   expected waiting time in the queue

L_q $\quad=\quad$ average length of the queue

W_s $\quad=\quad$ expected system residence time

L_s $\quad=\quad$ average number of customers in the system

P_n $\quad=\quad$ the probability that upon arrival of a customer there are exactly n customers already waiting in the queue

$P_{\geq n}$ $\quad=\quad$ the probability that upon arrival of a customer there are n or more customers already waiting in the queue

λ $\quad=\quad$ arrival rate

μ $\quad=\quad$ service rate per server

$E(x), \bar{x}, 1/\mu$ $\quad=\quad$ average service time (first moment of the distribution of the service time x)

$E(x^2), \overline{x^2}$ $\quad=\quad$ average squared service time (second moment of the distribution of the service time x)

$\sigma^2, E((x-1/\mu)^2)=$ variance of the service time distribution

ρ $\quad=\quad$ system utilization

The following equalities hold for all distributions

$$\overline{x^2} = (1/\mu)^2 + \sigma^2 \tag{14.12}$$

$$L_q = \lambda W_q \tag{14.13}$$

$$L_s = \lambda W_s \tag{14.14}$$

$$W_s = W_q + 1/\mu \tag{14.15}$$

$$L_s = L_q + \lambda/\mu \tag{14.16}$$

Equation (14.13) is called Little's Law.

M/M/1

In an M/M/1 queuing system the arrival process is a Poisson process with arrival rate λ, i.e., the interarrival times are independent exponentially distributed random variables with mean $1/\lambda$. The successive service times are assumed to be independent exponential random variables having a mean of $1/\mu$. The first M refers to the fact that the interarrival process is Markovian and the second M refers to

the fact that the service distribution is exponential and the thus Markovian. The 1 refers to the fact there is a single server.

The mean of an exponential distribution is equal to its standard deviation, or

$\sigma = 1/u$

$$\rho = \lambda / \mu \tag{14.17}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} \tag{14.18}$$

$$W_s = \frac{1}{(\mu - \lambda)} \tag{14.19}$$

$$L_s = L_q + \rho \tag{14.20}$$

$$P_0 = 1 - \lambda/\mu = 1 - \rho \tag{14.21}$$

$$P_n = (1 - \lambda/\mu)(\lambda/\mu)^n \tag{14.22}$$

$$P_{\geq 1} = \lambda/\mu \tag{14.23}$$

$$P_{\geq n} = (\lambda/\mu)^n \tag{14.24}$$

M/M/k

The number of servers is equal to k. There is a single waiting line and customers go to the first available server.

$$\rho = \lambda / k\mu \tag{14.25}$$

$$W_q = \frac{(\lambda/k\mu)(\lambda/\mu)^k}{k!\lambda(1 - \lambda/k\mu)^2 \left[\sum_{n=0}^{k-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^k}{k!(1 - \lambda/k\mu)} \right]} \tag{14.26}$$

$$W_s = W_q + 1/\mu \tag{14.27}$$

M/G/1

The successive service times are assumed to be independent random variables with a general distribution.

$$W_q = \frac{\lambda \cdot \left(\overline{x^2}\right)}{2(1 - \lambda/\mu)} = \frac{\lambda \cdot \left((1/\mu)^2 + \sigma^2\right)}{2(1 - \lambda/\mu)} \tag{14.28}$$

This Khintchine-Pollaczek formula for the expected waiting time in a M/G/1 queue is also given, among others, in Giffin (1978).

Note that the variance of uniformly distributed random variable between the boundary values a and b is equal to

$$\sigma^2 = \frac{(b-a)^2}{12} \tag{14.29}$$

M/D/1

For a discrete service time distribution, the service time has a constant value and the variance of the service time is equal to zero.

$$W_q = \frac{(\lambda/\mu)^2}{2\lambda(1 - \lambda/\mu)} \tag{14.30}$$

$$L_q = \frac{(\lambda/\mu)^2}{2(1 - \lambda/\mu)} \tag{14.31}$$

M/G/k

$$W_q \approx \frac{\left(\mu^2/2\right)\overline{x^2}\left(\lambda/k\mu\right)(\lambda/\mu)^k}{k!\lambda(1 - \lambda/k\mu)^2\left[\sum_{n=0}^{k-1}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^k}{k!(1 - \lambda/k\mu)}\right]} \tag{14.32}$$

This approximation is a very accurate approximation if the service time has a gamma distribution and it is exact if the service time has an exponential distribution.

Operations Examples

Machine Requirements Example

General Hospital needs to replace their outdated x-ray equipment in order to compete with other hospitals for a smaller patient population. A x-ray machine may be used for general x-rays as well as for special hip x-rays. The time to convert the x-ray machines from general x-ray to hip x-ray is 45 minutes, the time to convert from hip x-ray to general x-ray is 15 minutes. The x-ray machines cost $100,000 per machine but they are very reliable. The arrival of general and special hip x-ray patients is random and may not be scheduled. A general x-ray takes 12 minutes per patient and there are 11,000 patients per year for this x-ray, a special hip x-ray takes 15 minutes and there are 2,500 patients per year for this x-ray. The x-ray machines are in use 8 hours per day, 300 days per year and have a reliability factor of 98 %. The question is how many x-ray machines should be purchased? Based on the number of machines required we will discuss the organization of the x-ray department. Finally, we will estimate the expected waiting time for general x-ray patients, hip x-ray patients and all patients combined for your solution, assuming the x-ray machines have a reliability of 100 %. This example was adapted from Tompkins and White (1984).

We consider first the case where all patients join a single FIFO queue for set of homogeneous multipurpose x-ray machines. There are four distinct types of operations performed depending on the combination of the previous and current x-ray type. The total number of patients is 11,000 + 2,500 = 13,500. The fraction or probability of general x-ray requests is 11,000 / 13,500 = 0.815. The fraction or probability of special hip x-rays is 2,500 / 13,500 = 0.185. Since the patients arrive randomly and may not be scheduled or arranged in the queue, the probability of a general after general operation is then 0.815 × 0.815 = 0.664 and total number of operations of this type is 13,500 × 0.664 = 8963. The data for the four operations can be summarized in the following table.

Table 14.4. X-Ray Operations Data Summary

Operation Type	Probability	Operations	Unit Time	Total Time
general after general	0.664	8963	12 min.	107,556 min.
general after hip	0.151	2037	27 min.	54,999 min.
hip after hip	0.034	463	15 min.	6,945 min.
hip after general	0.151	2037	60 min.	122,220 min.

The required number of machines is then

$$N = \left\lceil \frac{107556 + 5499 + 6945 + 122220}{300 \cdot 8 \cdot 60 \cdot 0.98} \right\rceil = \left\lceil \frac{291720}{141120} \right\rceil = \lceil 2.067 \rceil = 3$$

Thus 3 machines are required and their average utilization rate is 68.9 %.

In the next section we will compute the expected waiting times for several configurations and operating policies of the hospital ward, but this derivation might be skipped at the undergraduate level. We assume that the machines are 100 % reliable. First we compute the expected waiting time for three non-dedicated machines and a single FIFO waiting line. The formula for the expected waiting time in queuing system with 3 servers is

$$W_q \approx \frac{\left(\mu^2/2\right)\overline{x^2}\left(\lambda/3\mu\right)\left(\lambda/\mu\right)^3}{3!\lambda(1-\lambda/3\mu)^2\left[\sum_{n=0}^{2}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^3}{3!(1-\lambda/3\mu)}\right]}$$

$$\lambda = \frac{13500}{300 \cdot 8 \cdot 60} = 0.094$$

$$\overline{x} = \frac{291720}{13500} = 21.6$$

$$\lambda\overline{x} = \lambda/\mu = 0.094 \cdot 21.6 = 2.026$$

$$\lambda/3\mu = 0.094 \cdot 21.6/3 = 0.675$$

$$\overline{x^2} = \frac{8963 \cdot 144 + 2037 \cdot 729 + 463 \cdot 225 + 2037 \cdot 3600}{13500} = 756.5$$

$$W_{com} = \frac{(1/(21.6^2 \cdot 2)) \cdot 756.5 \cdot 0.675 \cdot 2.026^3}{6 \cdot 0.094 \cdot (1-0.675)^2\left[1 + 2.026 + \frac{2.026^2}{2} + \frac{2.026^3}{6(1-0.675)}\right]} = 8.205$$

Observe that the average processing time for a general and hip x-ray patient, respectively, is equal to

$$\frac{8963 \cdot 12 + 2027 \cdot 27}{11000} = 14.78$$

$$\frac{463 \cdot 15 + 2037 \cdot 60}{2500} = 51.67$$

But if the x-ray department was organized differently, we might be able to reduce the number of machines by dedicating machines to operations and thus eliminating the setup times. Assuming that we dedicate a number of machines to general x-rays and a number of machines to hip x-rays, the required number of machines of each type are then:

$$N_{gen} = \left\lceil \frac{11000 \cdot 12}{141120} \right\rceil = \left\lceil \frac{132000}{141120} \right\rceil = \lceil 0.935 \rceil = 1$$

$$N_{hip} = \left\lceil \frac{2500 \cdot 15}{141120} \right\rceil = \left\lceil \frac{37500}{141120} \right\rceil = \lceil 0.266 \rceil = 1$$

In this case we need only two machines, but the expected utilization of the general x-ray machine is very high so the expected waiting times for that machine might not be acceptable. If the hospital decides to purchase three machines, then dedicating two machines to general x-rays and one machine to hip x-rays will reduce the utilization of all machines and improve patient service.

First we compute the expected waiting time if there are two dedicated machines with two independent FIFO queues. We can model this as two M/D/1 queues and . The Khintchine-Pollaczek formula for the expected waiting time in a M/G/1 queue is also given, among others, in Giffin (1978).

$$W_q = \frac{\lambda \cdot \left(\overline{x^2} \right)}{2(1 - \lambda/\mu)} = \frac{\lambda \cdot \left((1/\mu)^2 + \sigma^2 \right)}{2(1 - \lambda/\mu)}$$

For the x-ray machines with a discrete service time, i.e., the standard deviation of the service time is zero, the computations then yield the following expected waiting times:

$$\lambda_{gen} = \frac{11000}{300 \cdot 8 \cdot 60} = \frac{11000}{144000} = 0.076$$

$$\overline{x_{gen}} = 12$$

$$\overline{x_{gen}^2} = 144$$

$$W_{gen} = \frac{0.076 \cdot 144}{2(1 - 0.076 \cdot 12)} = 66$$

$$\lambda_{hip} = \frac{2500}{300 \cdot 8 \cdot 60} = \frac{2500}{144000} = 0.017$$

$$\overline{x_{hip}} = 15$$

$$\overline{x_{hip}^2} = 225$$

$$W_{hip} = \frac{0.017 \cdot 225}{2(1 - 0.017 \cdot 15)} = 2.641$$

The expected waiting time for the single general x-ray machine might not be acceptable to the hospital. We next compute the expected waiting for general x-rays if two machines are dedicated to general x-

rays. The expected waiting time in a M/G/2 queuing system with discrete service times is given in Ross (1993) as:

$$W \approx \frac{\lambda \overline{x^2}(\lambda \overline{x})}{2(2-\lambda \overline{x})^2 \left[1+\lambda \overline{x}+\frac{(\lambda \overline{x})^2}{2-\lambda \overline{x}}\right]} \tag{14.33}$$

The expected waiting time for general x-ray patients if there are two machines dedicated to general x-rays is then:

$$\lambda \overline{x} = 0.076 \cdot 12 = 0.917$$

$$W_{gen} = \frac{0.076 \cdot 144 \cdot 0.917}{2(2-0.917)^2 \left[1+0.917+\frac{0.917^2}{2-0.917}\right]} = 1.6$$

The waiting time computations illustrate again that if three machines are purchased, patient service is improved if two of them are dedicated to general x-rays and one of them is dedicated to hip x-rays.

Assume that the annualized cost of a x-ray machine is $75,000, $95,000, and $135,000 for a general x-ray, hip x-ray, and mixed use x-ray machine, respectively. Assume further, that the cost for performing a general x-ray on a general x-ray machine is $55 and is $75 on a mixed use x-ray machine, and the cost of performing a hip x-ray on a hip x-ray machine is $85 and $105 on a mixed use x-ray machine, respectively. Develop the sizing and allocation model for this system that will minimize the total yearly cost. Clearly define all set, variables, parameters, and constraints. Compute the parameter values. Solve the model to optimality. Discuss the optimal solution.

Process, Product, and Machine Requirements Example

Given the manufacturing data in the following production table, the object is to compute the product flows of products x, y, and z in the manufacturing facility, the required number of machines, and the required number of support materials u and v to produce parts x, y and z. Each time a machine processes a part it requires new support materials as indicated in the Table 14.5 below. Machine A can rework its products an unlimited number of times. Machine B can only rework its products twice, after the third pass all defective parts are scrapped. The required production for product z is 100,000 parts a year. The manufacturing plant is fully automatic. Two parts x plus one part y are assembled to one part

z on machine C. The assembly procedure behaves as a perfect process, i.e., it has a zero defect rate. The manufacturing plant operates 50 weeks per year, five days a week, one shift of eight hours a day. The standard production times are given in the Table 14.5 and are expressed in hours.
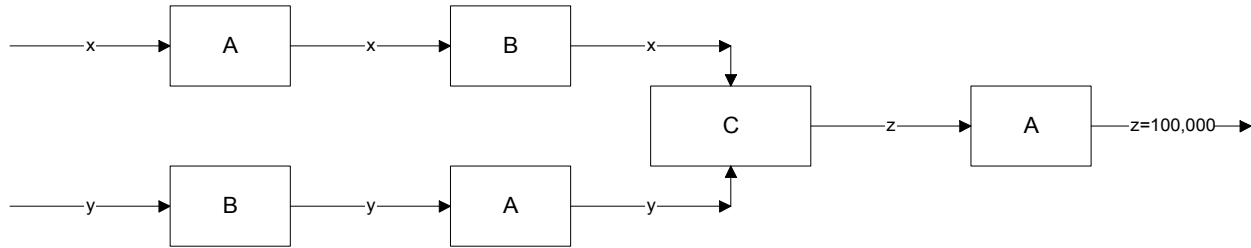

Figure 14.10. Assembly Chart For Process Design Example

Table 14.5. Production Characteristics

	Machine A	Machine B
part x standard time	0.005	0.015
part y standard time	0.01	0.025
part z standard time	0.025	
part x defect rate	10%	70%
part y defect rate	3%	50%
part z defect rate	5%	
part x rework rate	70%	40%
part y rework rate	80%	30%
part z rework rate	90%	
part x support materials	2u	v
part y support materials	2v	u
part z support materials	u + v	
part x efficiency	95%	90%
part y efficiency	92%	98%
part z efficiency	94%	
reliability factor	93%	91%

Using the formulas for infinite rework for machine A and for limited rework for machine B and the assembly equation, the input flows are equal to:

$$I_{Z_A} = \frac{100,000 \cdot (1 - 0.05 \cdot 0.9)}{(1 - 0.05)} = 100,527$$

$$I_{X_B} = \frac{2 \cdot 100,527}{(1 - 0.7)(1 + 0.7 \cdot 0.4 + (0.7 \cdot 04)^2)} = 493,360$$

$$I_{X_A} = \frac{493,360 \cdot (1 - 0.1 \cdot 0.7)}{(1 - 0.1)} = 509,806$$

$$I_{Y_A} = \frac{100,527 \cdot (1 - 0.03 \cdot 0.8)}{(1 - 0.03)} = 101,149$$

$$I_{Y_B} = \frac{101,149}{(1-0.5)(1+0.5 \cdot 0.3 + (0.5 \cdot 03)^2)} = 172,536$$

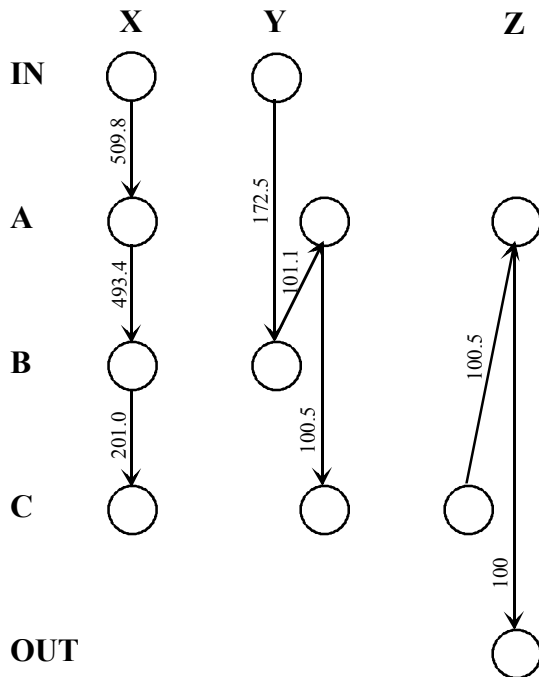The product flows can now be summarized in the following multiproduct process chart and in a from-to matrix.

Figure 14.11. Multiproduct Process Chart

Table 14.6. From-To Matrix

	IN	A	B	C	OUT
IN	-	509.8	172.5		
A		-	493.4	100.5	100
B		101.1	-	201	
C		100.5		-	
OUT					-

To compute the required support materials we have to use the manufacturing flows since each processing step requires support materials, whether it produces good or defective parts. The manufacturing flows in function of the input flows for machines A and B are given by

$$M_A = \frac{I}{(1-rd)}$$

$$M_B = \frac{I\left[1-(rd)^3\right]}{(1-rd)} = I(1+rd+r^2d^2)$$

$$M_{Z_A} = \frac{100527}{(1 - 0.05 \cdot 0.9)} = 105264$$

$$M_{X_B} = 493360 \cdot \left[1 + 0.7 \cdot 0.4 + (0.7 \cdot 0.4)^2\right] = 670181$$

$$M_{Y_A} = \frac{101149}{(1 - 0.03 \cdot 0.8)} = 103637$$

$$M_{X_A} = \frac{509806}{(1 - 0.1 \cdot 0.7)} = 548179$$

$$M_{Y_B} = 172536 \cdot \left[1 + 0.5 \cdot 034 + (0.5 \cdot 0.3)^2\right] = 202299$$

The required support materials can be easily computed based on the following table.

Table 14.7. Support Materials Computation

Flow Symbol	Flow Value	Support u	Support v
M_{ZA}	105264	105264	105264
M_{XB}	670181		670181
M_{YA}	103637		207274
M_{XA}	548179	1096358	
M_{YB}	202299	202299	
Total		1403921	982719

The required number of machines can now be computed based on the manufacturing flows and the production data.

$$N_A = \left\lceil \frac{\dfrac{548179 \cdot 0.005}{0.95} + \dfrac{103637 \cdot 0.10}{0.92} + \dfrac{105264 \cdot 0.025}{0.94}}{50 \cdot 5 \cdot 8 \cdot 0.93} \right\rceil = \left\lceil \frac{6811.2}{2000 \cdot 0.93} \right\rceil = \lceil 3.662 \rceil = 4$$

$$N_B = \left\lceil \frac{\dfrac{670181 \cdot 0.015}{0.90} + \dfrac{202299 \cdot 0.025}{0.98}}{50 \cdot 5 \cdot 8 \cdot 0.91} \right\rceil = \left\lceil \frac{16330.4}{2000 \cdot 0.91} \right\rceil = \lceil 8.973 \rceil = 9$$

If the required cost data are provided, the total production cost can now be computed from the required input parts, required support materials, and machine usage.

Assume that the annualized cost of a machine of type A, B, and C is $100,000, $50,000, and $500,000, respectively. Further assume that the purchase cost of a single unit of component u and v is $0.5 and $2, respectively. The production costs of a single unit of the different products on the different machines is

given in Table 14.8, where the assembly cost is expressed in units of product Z assembled on machine C.

Table 14.8. Marginal Production Costs

	A	B	C
X	$0.03	$0.15	
Y	$0.25	$0.10	
Z	$0.05		$0.50

Develop the sizing and allocation model for this system that will minimize the total yearly cost. Clearly define all set, variables, parameters, and constraints. Compute the parameter values. Solve the model to optimality. Discuss the optimal solution.

Simulation

Introduction

While material handling models are used for the preliminary design and sizing of material handling systems, detailed design, verification, and validation are most often based on digital simulation.

The more recent introduction of relatively inexpensive personal computers that have large internal memory and fast processor speed has allowed for the development of more interactive and graphical simulation systems. Several simulation systems even create three dimensional scale representations of the material handling systems complete with camera mounting and fly-throughs. Other systems allow the kinematic representation of material handling systems and their interaction with humans to test for ergonomic sound design. The graphical feedback allows the engineer a better overview of the system and thus enables faster design, analysis, and reporting and sales presentations to non-technical personnel.

A second recent development has been the inclusion of templates or smart objects for material handling systems. This relieves the user of the detailed coding of these complex objects and requires only that the user specifies certain parameters.

There still exists the widespread misconception that simulation is an optimization tool. While simulation lets you evaluate a number of alternative system configurations, it is intrinsically a descriptive evaluation tool. It cannot determine by itself the best value for a particular decision

parameter, as a normative optimizer could. This misconception is often further encouraged by the language used in the advertising and sales literature of the simulation systems.

Some of the simulation systems currently on the market are Automod from AutoSimulations, Quest from Deneb, Micro Saint from Micro Analysis Design, and VisFactory from EAI, and Promodel by Promodel Corp. The Institute of Industrial Engineers (www.iienet.org) publishes yearly a simulation software buyers guide in their Solutions magazine.

Exercises

DryGoods, Inc.

The Drygoods Corporation is a distributor of consumer products to a large number of drugstores in the southeastern region of the United States. Products are sold by the case. All the products for a single customer are stacked on one or more pallets and then delivered daily by a dedicated fleet of trucks. The order picking and consolidation occurs from 12 until 6 AM, the delivery to the stores from 6 until 8 AM. To reduce continuing labor shortage problems for this night shift operation, the company is planning to purchase a number of large automated pallet-wrapper machines.

Pallets are wrapped with a plastic film by connecting the film to the pallet and then rotating the pallet while the film spool is raised from the bottom of the pallet to the top of the boxes stacked on the pallet. Depending on the number of boxes stacked on the pallet, the pallets can be classified as low, medium, or high pallets and denoted with subscripts 1, 2, and 3, respectively. The processing times for the three different pallet types are 1.0, 1.5, and 2 minutes, respectively. The expected number of pallets that need to be shipped of each pallet type are 8, 24, and 16 pallets per hour, respectively. The required length of plastic wrap for wrapping each of the three different pallet styles is 12, 18, and 24 meters, respectively. The cost per meter of plastic wrap is $0.03. The total cost for one wrapper is $0.50 per minute. The tension of the plastic film during the wrapping process may cause the boxes to shift and extend beyond the pallet footprint. This is called overhang. In order for the pallets to be transported by forklift truck into the over-the-road trailers there is a limit on the overhang of the boxes on a pallet. The company estimates that 10 % of the pallets will have excessive overhang after being wrapped. If this occurs, the wrapping is cut away and discarded, the boxes rearranged, and then the pallet joins the waiting line to be wrapped again. If the pallet-wrapper is rotated 25 % slower, the company estimates that only 4 % of the

pallets will have excessive overhang. The new pallets and pallets that need to be wrapped again join a single waiting line in front of the one or more pallet wrappers. The company wants to know what the average number of pallets waiting to be wrapped will be, since it has to provide space for the waiting pallets. The equivalent hourly cost for one waiting space is $0.10. The company is requesting your assistance in computing the total cost of the wrapping operations for the fast and slow rotation speeds. The company is requesting a clear reporting of the various cost categories and a recommendation for the selection of the speed.

Friendly Fried Food, Inc.

The Friendly Fried Food Corporation uses a number of deep fryers to fry fish and French fries, which then are frozen and offered for sale in supermarkets. The oil in the deep fryer must be changed between the frying of fish and French fries. In addition, the oil in the deep fryer must also be changed after that oil has been used for an hour. The changeover times refer to the time required to clean and refill the deep fryer after a batch of that particular product type. The fish and French fries are produced by different and independent cleaning and preparation lines before they arrive at the common deep fryer. Fish and French fries are processed 4 hours a day, 200 days per year. Each fryer is 93 % reliable. Each batch waiting for the deep fryer requires a four feet by four feet staging area. This space includes all aisle and clearance space. The annual cost for a fryer is $100,000 and the annual cost per square foot of waiting area is $6.0. The company is trying to minimize the annual cost of the frying operation while satisfying throughput requirements.

Given the process data in the following table, how many deep fryers does the Friendly Food Facility need to fry the fish and the French fries? How should the Friendly Food Facility organize their frying operation? Compute the required waiting space for the operation of the deep-frying department. Do all computations to four significant digits and round intermediate results to four significant digits. Please answer in a clear and organized fashion. Show the formulas that you used for each intermediate result. Place the numerical intermediate results in a box. Justify the formulas that you have used. Show both formulas and numerical values clearly marked in a box. Finally, summarize your results in a clear table suitable for presentation to the executive committee of the Friendly Fried Food Corporation .

Table 14.9. Deep Frying Process Data

	Fish	French Fries
Standard Time	3 min	6 min
Annual Batches	4000	5000
Efficiency	105%	95%
Change Times	9 min	4 min

Material Handling Equipment Selection and Sizing

A company manufactures two part types using three machines. Six distinct operations are performed in the manufacturing plant. Two material handling devices are considered to transport the parts between the machines. Table 14.10 shows the cost and required time of a processing operation of a single part on a machine, where the time is given in parentheses. Table 14.11 shows the cost and required time of transporting a unit of a part type on a material handling device, where again the time is given in parentheses. Finally, Table 14.12 shows the required production volume and operations sequence of each part type. A missing value indicates that this particular combination of part type and material handling device or machine is not feasible, i.e. the operation cannot be performed on that machine or the part cannot be transported by that material handling device. Each machine and material handling device is available for 4000 minutes. The total budget for equipment fixed costs for the planning horizon is $1,400,000. The costs per machine and material handling device are given in Tables 14.1 and 14.2.

Table 14.10. Manufacturing Operation versus Machine Data

	MC1	MC2	MC3
Cost	$140,000	$188,000	$95,000
Operation 1	20 (18)	12 (17)	
Operation 2		17 (23)	23 (12)
Operation 3	15 (14)		12 (13)
Operation 4	8 (5)	9 (6)	10 (4)
Operation 5		12 (17)	
Operation 6	15 (21)		

Table 14.11. Material Handling Operation versus Device Data

	MH1	MH2
Cost	$255,000	$195,000
Part 1	15 (3)	13 (4)
Part 2	17 (8)	

Table 14.12. Part Type Production Data

	Volume	Operation Sequence
Part 1	240	O1-O2-O3-O6
Part 2	150	O2-O5-O4-O3

Draw a diagram of the production and material handling system, representing the material flows in the system. Draw this diagram separately for each part type. The nodes represent the different machines. The arcs represent the different material handling systems (use solid lines for material handling device one and dashed lines for material handling device two). Clearly mark each node. On each arc clearly indicate the required time per operation of executing that particular production or handling operation. Clearly define the necessary variables and display all material flow variables in their correct place in the diagrams. Write out the objective function and all the necessary constraints for this particular system using the defined variables and with numerical constants. Do not use any summation signs but write out each individual term. Solve this formulation with the mixed integer programming solver of your choice.

AS/RS Throughput Requirements

A unit load automated storage and retrieval system is being designed for the reserve storage area of a mail order retailer. The planned system has 8 aisles, each with an aisle-captive crane. The horizontal and vertical speed of the cranes is 380 and 60 feet per minute, respectively. The rack is 80 columns long and 12 rows high. The centroid to centroid distance between two adjacent columns is 56 inches and is 58 inches between two adjacent rows. The pickup and deposit point for each aisle is at the front of the rack and at ground level. It is estimated that 40 % of all crane cycles are single command cycles. The required throughput of the rack is 120 storage operations and 120 retrievals per hour. The pickup and deposit time of the shuttle is equal to 30 seconds. Products are stored randomly in each aisle and randomly between aisles. Storage unit loads are waiting on eight conveyor lines, one in front of each individual crane. To avoid the construction of long conveyors holding the unit loads to be stored the retailer has specified that the expected waiting for a storage request should be less than 8 minutes. It has been agreed upon that the crane cycle times can be approximated with sufficient accuracy by a distribution with a coefficient of variation equal to 2.

Compute the expected cycle time for the cranes, the expected utilization of the cranes, and the expected waiting time of the storage operations. Does this systems satisfy the throughput and waiting time requirements? Given your computations are all assumptions for this system valid?

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3.   Schwartz, L. B., S. C. Graves, and W. H. Hausman, (1978), "Scheduling Policies for Automatic Warehousing Systems: Simulation Results," *IIE Transactions*, Vol. 10, No. 3, pp. 260-270.