Logistics Systems Design: Continuous Point Location

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Continuous Point Location Overview

- Location Introduction
- Euclidean Minisum Location
- Euclidean Minimax Location
- Rectilinear Minisum Location
- Rectilinear Minimax Location

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Location Theory Definitions

- Location Problems:
  where to locate an object or facility
- Location Theory:
  a mathematical and algorithmic approach to solving location problems

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Location Theory Classification

1. Dimensionality of the Object
2. Structure of the Target Area
3. Location Costs
4. Location Constraints

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Dimensionality of the Object

- Object Dimensionality
- Object structure
- Number of Objects

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Object Dimensionality

- Point (0)
  - Supply Chain Configuration
- Line (1)
  - Picking Zones
- Area (2)
  - Facilities Layout
- Volume (3)
  - Truck Loading and Pallet Building
- Dynamic

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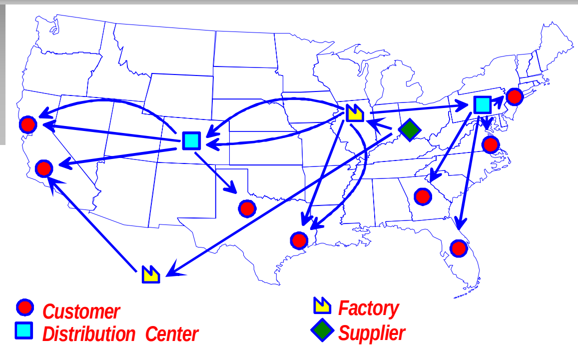
Point Location Example: Supply Chain Networks



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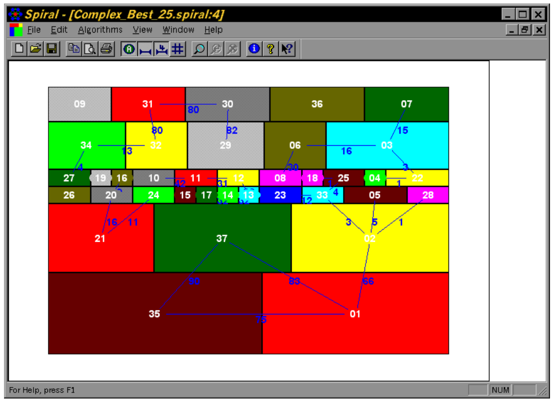
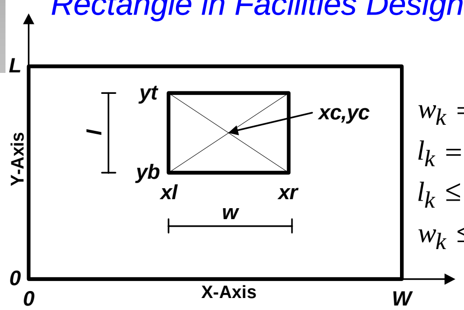
Line Location Example: Zones In Aisle Order Picking

Area Location Example: Block Layout



Object Structure Example

- Maximum Shape Ratio of Enclosing Rectangle in Facilities Design

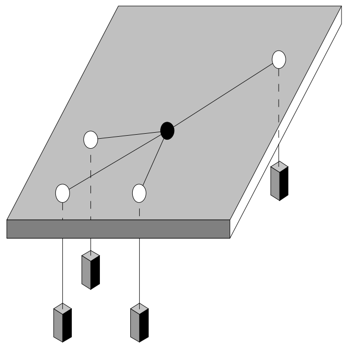


$$w_k = xr_k - xl_k$$
$$l_k = yt_k - yb_k$$
$$l_k \le S_k w_k$$
$$w_k \le S_k l_k$$

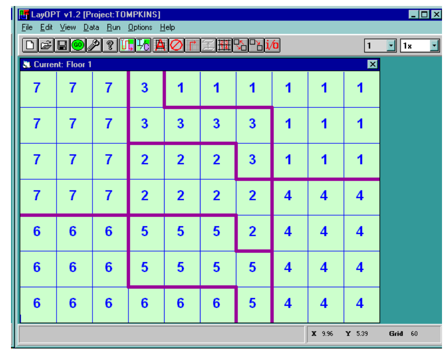
Structure of the Target Area

- Continuous
- Grid or Subdivision
- Network or Tree
- Discrete Candidates

Continuous Location Example: Supply Chain Design



Grid Location Example: Discrete Facilities Layout



Network Location Example: Supply Chain Design

Continuous Versus Discrete Candidate List

- Continuous
  - Infinite number of candidates
  - Distance norms
  - No obstacles or infeasible regions
- Discrete List of Candidates
  - Finite List
  - Actual Distances
  - Complex regions with obstacles

Location Costs

- Feasibility versus Optimization
- Minisum versus Minimax
- Interaction between to be Located Objects
- Fixed versus Variable Affinities
- Linear versus Concave costs
- Deterministic versus Stochastic
- Static versus Dynamic

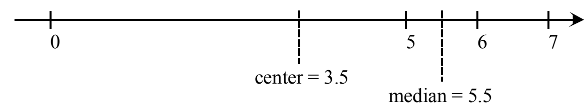
Minisum Versus Minimax

- Minisum (Median)
  - Minimize Total Cost
  - Economic Efficiency

$$\min_X \left\{ \sum_j C_j(X) \right\}$$

- Minimax (Center)
  - Minimize Worst Cost
  - Economic Equity

$$\min_X \left\{ \max_j C_j(X) \right\}$$



Minisum versus Minimax versus Maximin

- Minisum (Median)
  - Minimize Total Cost
  - Economic Efficiency

$$\min_X \left\{ \sum_j C_j(X) \right\}$$

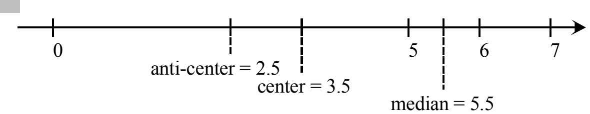
- Minimax (Center)
  - Minimize Worst Cost
  - Economic Equity

$$\min_X \left\{ \max_j C_j(X) \right\}$$

- Maximin
  - Max. Least Distance
  - Economic Equity

$$\max_X \left\{ \min_j C_j(X) \right\}$$

Minisum versus Minimax versus Maximin Example



Location Constraints

- Capacitated versus Uncapacitated
- Infeasible Regions

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Euclidean Point Location Introduction

- Locating of a Point Facility
- Costs (and Constraints) in Function of Euclidean Distance Norm
  - Rough Cut, Approximate Models
- Applications
  - Distribution System Reconfiguration
  - Emergency Facility Location

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Euclidean Minisum Location

- Problem Definition
- Function and Algorithm Properties
- Distance Norm
- Varignon Frame
- Location Properties
- Location Algorithm
- Location-Allocation Algorithm

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Euclidean Location Problem: Definition

- *Locate a given number of facilities to minimize interaction costs with fixed facilities*
- *Costs proportional to distance norm*
- *No site dependent costs*
- *No change in facility status*
- *Alternative generating algorithms*

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Euclidean Location Problem: Solution Algorithms

- *Approximate algorithms*
  - *Non-linear optimization, specialized heuristics*
  - *Low solution resolution*

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Euclidean Location Problem: Examples

- *Distribution system consolidation*
- *Blood processing system consolidation*
- *Bank branch relocation after a merger*

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Euclidean Distance Norm

$$d^E(X_i, P_j) = \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2}$$

- *Properties*
  - *Distance norm properties*
  - *Continuous*
  - *Convex*
  - *Differentiable except at $P_j(a_j,b_j)$*

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Distance Norm Properties

- Non-negativity $d^E(X) \geq 0 \quad \forall X$
- Equality to zero $d^E(X) = 0 \Leftrightarrow X = 0$
- Homogeneity $d^E(kX) = |k| d(X) \quad \forall X$
- Symmetry $d^E(-X) = d^E(X) \quad \forall X$
- Triangle inequality

$$d^E(X) + d^E(Y) \geq d^E(X+Y) \quad \forall X, \forall Y$$

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Continuous Function

$$\forall \delta \rightarrow \exists \varepsilon : \left| f(x+\varepsilon) - f(x) \right| \leq \delta \quad \forall x$$

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Convex Function

$$d^E(\lambda X_1 + (1-\lambda) X_2) \leq$$
$$\lambda d^E(X_1) + (1-\lambda) d^E(X_2) \quad \lambda \in [0,1]$$

- Local optimum is global optimum
- Convex level sets

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Hyperboloid Approximation

$$d^E(X_i, P_j) = \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon}$$

$$\frac{\partial d}{\partial x_i} = \frac{(x_i - a_j)}{\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon}}$$

- Differentiable everywhere
- Size of ε

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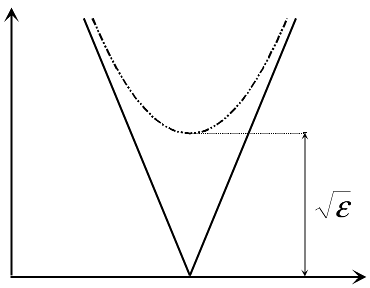
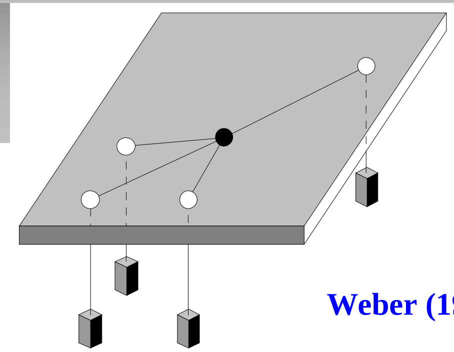
Hyperboloid Illustration



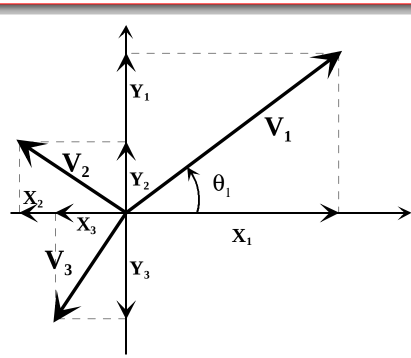
$$d^E(X_i, P_j) = \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon}$$

Varignon Frame

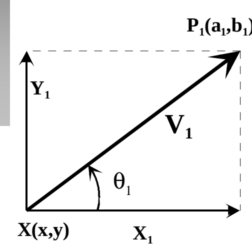


Weber (1906)

Varignon Forces Diagram



Forces Projections



$$X_1 = V_1 \cos\theta_1$$

$$Y_1 = V_1 \sin\theta_1$$

$$X_1 = \frac{w_1(a_1 - x)}{\sqrt{(x - a_1)^2 + (y - b_1)^2}}$$

$$Y_1 = \frac{w_1(b_1 - y)}{\sqrt{(x - a_1)^2 + (y - b_1)^2}}$$

Forces Equilibrium

$$\sum_j X_j = 0$$

$$\sum_j Y_j = 0$$

$$\sum_j \frac{w_j\left(x-a_j\right)}{\sqrt{\left(x-a_j\right)^2+\left(y-b_j\right)^2}} = 0$$

$$\sum_j \frac{w_j\left(y-b_j\right)}{\sqrt{\left(x-a_j\right)^2+\left(y-b_j\right)^2}} = 0$$

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Gradient Optimality Conditions

$$z = \sum_j w_j \sqrt{\left(x-a_j\right)^2+\left(y-b_j\right)^2}$$

$$\frac{\partial z}{\partial x} = \sum_j \frac{w_j\left(x-a_j\right)}{\sqrt{\left(x-a_j\right)^2+\left(y-b_j\right)^2}} = 0$$

$$\frac{\partial z}{dy} = \sum_j \frac{w_j\left(y-b_j\right)}{\sqrt{\left(x-a_j\right)^2+\left(y-b_j\right)^2}} = 0$$

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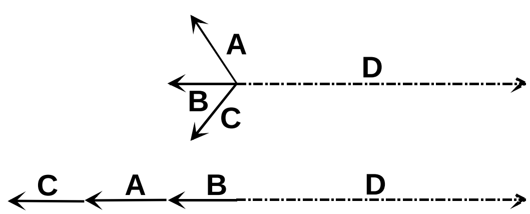
Scalar Sum Property Illustration



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Majority Property

- *Scalar Sum Property (Majority)*

$$\left.\begin{array}{l} w_k \ge \dfrac{W}{2} = \dfrac{\sum_{j=1}^{N} w_j}{2} \\ w_k \ge \sum_{j=1, j \ne k}^{N} w_j \end{array}\right\} \Rightarrow P_k = X^*$$

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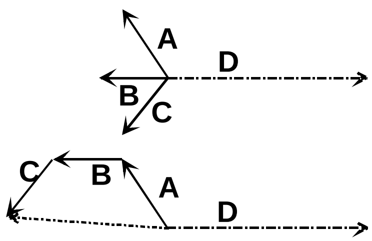
Vector Sum Property Illustration



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Vector Sum Property

$$\sqrt{\left(\sum_{j=1,j\neq k}^{N}\frac{w_j(a_k-a_j)}{\sqrt{(a_k-a_j)^2+(b_k-b_j)^2}}\right)^2+\left(\sum_{j=1,j\neq k}^{N}\frac{w_j(b_k-b_j)}{\sqrt{(a_k-a_j)^2+(b_k-b_j)^2}}\right)^2}\leq w_k$$

$$\Rightarrow P_k = X^*$$

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Weiszfeldís Iterative Procedure: Notation

$$g_{ij}(x_i,y_i)=\frac{w_{ij}}{\sqrt{\left(x_i-a_j\right)^2+\left(y_i-b_j\right)^2+\varepsilon}}$$

$$\sum_j (x_i-a_j)\cdot g_{ij}=0 \qquad x_i\sum_j g_{ij}=\sum_j a_j\cdot g_{ij}$$

$$\sum_j (y_i-b_j)\cdot g_{ij}=0 \qquad y_i\sum_j g_{ij}=\sum_j b_j\cdot g_{ij}$$

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Weiszfeldís Iterative Procedure: Convex Combination

$$\lambda_{ij}(x_i,y_i)=\frac{g_{ij}(x_i,y_i)}{\sum_j g_{ij}(x_i,y_i)} \qquad 0\leq\lambda_{ij}\leq 1 \qquad \sum_j \lambda_{ij}=1$$

$$x_i=\sum_j \lambda_{ij}(x_i,y_i)\cdot a_j$$

$$y_i=\sum_j \lambda_{ij}(x_i,y_i)\cdot b_j$$

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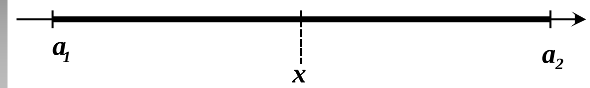
Weiszfeldís Iterative Procedure

$$\begin{cases} x_i^{k+1} = \sum_j \lambda_{ij}^k \left(x_i^k, y_i^k\right) \cdot a_j \\ y_i^{k+1} = \sum_j \lambda_{ij}^k \left(x_i^k, y_i^k\right) \cdot b_j \end{cases}$$

- Iterative Improvement Procedure
- k = Iteration Counter
- Weiszfeld Algorithm for Hyperboloid Approximation

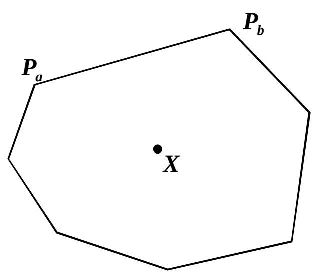
Line Segment Equation As a Convex Combination



$$x = \lambda a_1 + (1-\lambda) a_2$$
$$0 \le \lambda \le 1$$

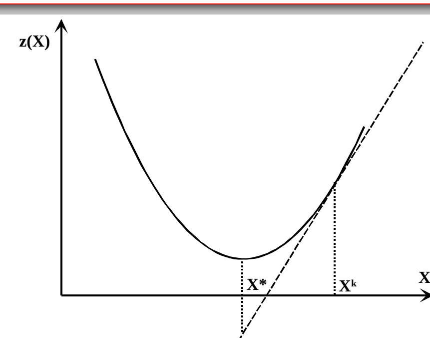
$$x = \lambda_1 a_1 + \lambda_2 a_2$$
$$\lambda_1 + \lambda_2 = 1$$
$$0 \le \lambda \le 1$$

Planar Convex Hull Property



$$X = \sum_j \lambda_j P_j$$
$$\sum_j \lambda_j = 1$$
$$0 \le \lambda \le 1$$

Convex Function Lower Bound

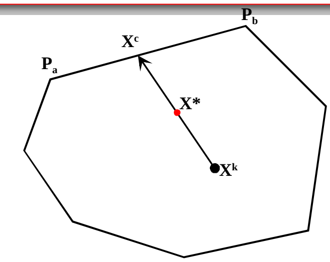


$$z(X) \ge z(X^k) + \nabla z(X^k) \cdot (X - X^k) \quad \forall X$$

Lower Bound Computation

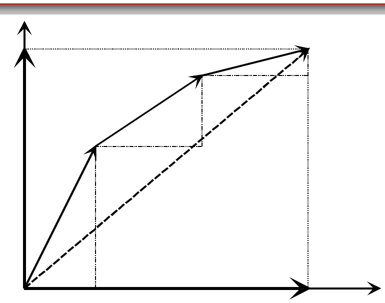
$$
\begin{aligned}
z(X^*) &\geq z(X^k) + \nabla z(X^k)\cdot(X^* - X^k) \\
&\geq z(X^k) - \left|\nabla z(X^k)\cdot(X^* - X^k)\right| \\
&\geq z(X^k) - \left\|\nabla z(X^k)\right\| \cdot \left\|X^* - X^k\right\| \\
&\geq z(X^k) - \left\|\nabla z(X^k)\right\| \cdot \max_j \left\|P_j - X^k\right\|
\end{aligned}
$$

Convex Hull Vector Sizes



$$
\begin{aligned}
\left\|X^* - X^k\right\| &\leq \left\|X^c - X^k\right\| \leq \max\left\{\left\|P_a - X^k\right\|, \left\|P_b - X^k\right\|\right\} \\
&\leq \max_j \left\|P_j - X^k\right\|
\end{aligned}
$$

Rectilinear Norm Based Lower Bound



$$
z_R(X_R^*) \geq z_E(X_E^*) \geq \sqrt{z_{RX}^2(X_{RX}^*) + z_{RY}^2(X_{RY}^*)}
$$

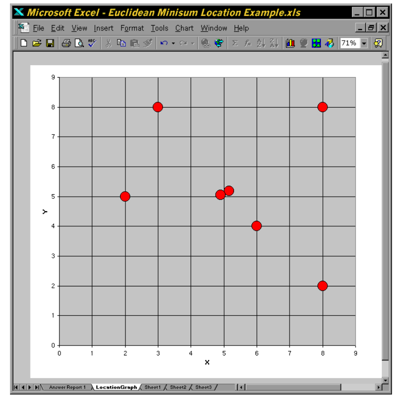
Location Algorithm

1. Problem Reduction
   - Majority Theorem
   - Vector Sum Theorem
2. Initial Starting Point
   - Center of Gravity
   - Largest Fixed Facility
   - Optimal Rectilinear Location
   - Center of Enclosing Rectangle

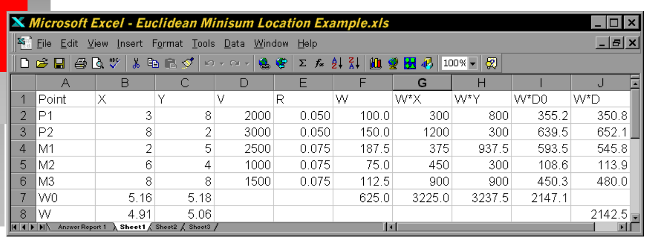
Location Algorithm cont.

3. Check Stopping Criteria
   - Maximum Number of Iterations
   - Compute Lower Bound
   - Stop if Gap within Tolerance
4. Compute Next Location
   - Weiszfeldís Method
   - Hyperboloid Approximation
   - Go to Step 3

Single Facility Location Example: Excel Location Graph



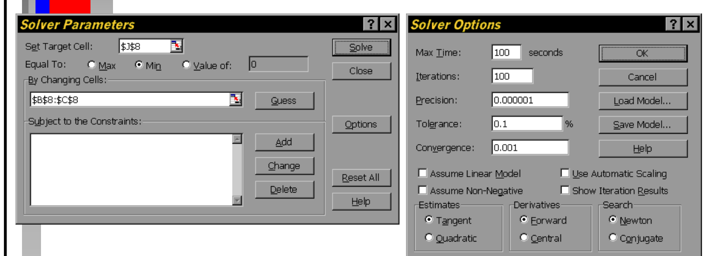
Single Facility Location Example with Excel



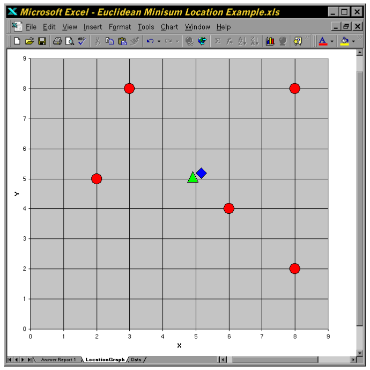
Microsoft Excel - Euclidean Minisum Location Example.xls

	A	B	C	D	E	F	G	H	I	J
1	Point	X	Y	V	R	W	W*X	W*Y	W*D0	W*D
2	P1	3	8	2000	0.050	100.0	300	800	355.2	350.8
3	P2	8	2	3000	0.050	150.0	1200	300	639.5	652.1
4	M1	2	5	2500	0.075	187.5	375	937.5	593.5	545.8
5	M2	6	4	1000	0.075	75.0	450	300	108.6	113.9
6	M3	8	8	1500	0.075	112.5	900	900	450.3	480.0
7	W0	5.16	5.18			625.0	3225.0	3237.5	2147.1	
8	W	4.91	5.06							2142.5

Single Facility Location Example: Excel Solver Options



Single Facility Location Example: Excel Location Graph



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Multiple New Facilities Objective Function

$$Min\ z = \sum_{i=1}^{H}\sum_{j=1}^{G} w_{ij}\sqrt{\left(x_i - a_j\right)^2 + \left(y_i - b_j\right)^2} + \sum_{i=1}^{H}\sum_{j=i+1}^{H} v_{ij}\sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2}$$

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Multiple New Facilities Notation

$$g_{ij}(x_i, y_i) = \frac{w_{ij}}{\sqrt{\left(x_i - a_j\right)^2 + \left(y_i - b_j\right)^2 + \varepsilon}}$$

$$h_{ij}(x_i, y_i) = \frac{v_{ij}}{\sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 + \varepsilon}} \quad i \neq j$$

$$h_{ii}(x_i, y_i) = 0$$

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Multiple New Facilities

$$x_i^k = \frac{\sum_{j=1}^{G} a_j g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} x_j h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)}{\sum_{j=1}^{G} g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)}$$

$$y_i^k = \frac{\sum_{j=1}^{G} b_j g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} y_j h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)}{\sum_{j=1}^{G} g_{ij}\left(x_i^{k-1}, y_i^{k-1}\right) + \sum_{j=1}^{H} h_{ij}\left(x_i^{k-1}, y_i^{k-1}\right)}$$

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Location-Allocation

- *Affinities or flows become variables*
- *Properties*
  - *Neither convex nor concave*
- *Heuristic algorithms*

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Location-Allocation Algorithm

- *Eilon-Watson-Ghandi Iterative Location-Allocation Algorithm*
  - *Pick Starting Locations*
  - *Allocate Fixed to Moveable Facilities*
    - *Multicommodity Network Flow*
  - *Relocate Moveable Facilities*
    - *Weiszfeld Hyperboloid Location*
  - *Iterate Until Converged*

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Allocation Phase Formulation

$$\min \quad \sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{m=1}^{L} c_{ijm} d_{ijm} w_{ijm}$$

$$s.t. \quad \sum_{i=1}^{M}\sum_{m=1}^{L} w_{ijm} = dem_k$$

$$\sum_{j=1}^{N}\sum_{m=1}^{L} w_{ijm} \le cap_i$$

$$\sum_{i=1}^{M}\sum_{m=1}^{L} w_{ijm} - \sum_{k=1}^{N}\sum_{m=1}^{L} w_{jkm} = 0$$

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Location Phase Objective

$$Min\ f(x,y) = \sum_{i=1}^{H}\sum_{j=1}^{G}\sum_{m=1}^{L} c_{ijm} w_{ijm} \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} + \sum_{i=1}^{H}\sum_{j=1}^{G}\sum_{m=1}^{L} c_{ijm} v_{ijm} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

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Simplified Location Objective

$$Min\ f(x,y) =$$

$$\sum_{i=1}^{H}\sum_{j=1}^{G} c_{ij} w_{ij} \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} +$$

$$\sum_{i=1}^{H}\sum_{j=1}^{H} c_{ij} v_{ij} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

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- **Euclidean Minimax Location**
- Rectilinear Minisum Location
- Rectilinear Minimax Location

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Euclidean Minimax Location

- Assumptions
  - Single moveable facility
  - Equal affinities
- Primal algorithm
- Dual algorithm

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Primal Algorithm

1. Pick initial radius z^0, set $k = 0$
2. Draw circles around P_j with radii z^k
3. Determine intersection Z of the circles
4. If Z is empty, increase z^k, k=k+1, go to 2
   if Z is more than single point, decrease z^k, k=k+1, go to 2
   if Z is single point, stop

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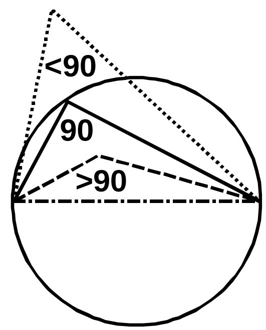
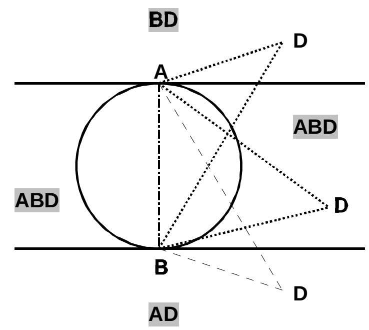
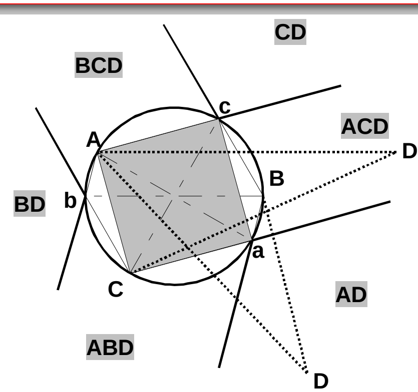
Angles in Triangles on a Circle



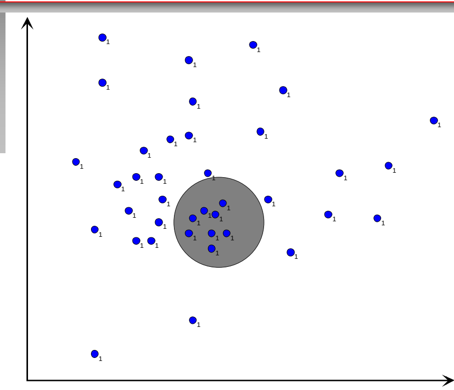
Dual Algorithm: Two Points



Dual Algorithm: Three Points



Euclidean Minimax Problem



Continuous Point Location Overview

- Location Introduction
- Euclidean Minisum Location
- Euclidean Minimax Location
- **Rectilinear Minisum Location**
- Rectilinear Minimax Location

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Rectilinear Minisum Location

- **Distance Norm and Properties**
- Minisum Location
- Minimax Location
- Rectilinear to Chebyshev Transformation

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Rectilinear Distance Norm

$$d_R = \left(|x - a_j| + |y - b_j|\right)$$

- Applications
  - Factories and Warehouses
  - Manhattan Grid Cities
  - Lower Bound for Euclidean

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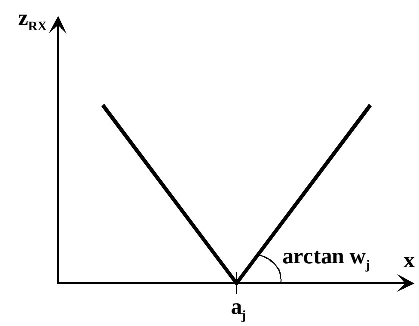
Rectilinear Distance Norm Component Graph



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Rectilinear Norm Properties

- Distance Norm Properties
- Continuous
- Convex
- Non-Differentiable
- Decomposable

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Rectilinear Minisum Location

- Distance Norm Properties
- Minisum Location
- Minimax Location
- Rectilinear to Chebyshev Transformation

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Rectilinear Minisum Location

- Minisum Properties
- Single Facility Median Conditions & Sequential Algorithm
- Single Facility Linear Programming Algorithm
- Multifacility Linear Programming Algorithm
- Multifacility Sequential Algorithm

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Rectilinear Minisum Problem

- Assumed Nonnegative Weights

$$w_j \geq 0 \quad \forall j$$

$$\min Z_R = \sum_{j=1}^{N} w_j \left(\left| x - a_j \right| + \left| y - b_j \right| \right)$$

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Rectilinear Minisum Problem Decomposition

$$\min Z_R = \sum_{j=1}^{N} w_j \left(|x - a_j| + |y - b_j| \right)$$

- Decomposable in Independent X and Y Components

$$\min Z_{RX}(X) = \sum_{j=1}^{N} w_j |x - a_j|$$

$$\min Z_{RY}(X) = \sum_{j=1}^{N} w_j |y - b_j|$$

Minisum Vector Notation

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad P_j = \begin{bmatrix} a_j \\ b_j \end{bmatrix}$$

$$W = \begin{bmatrix} w_1 \\ \ldots \\ w_N \end{bmatrix} \quad D_R = \begin{bmatrix} d_r(X, P_1) \\ \ldots \\ d_r(X, P_N) \end{bmatrix}$$

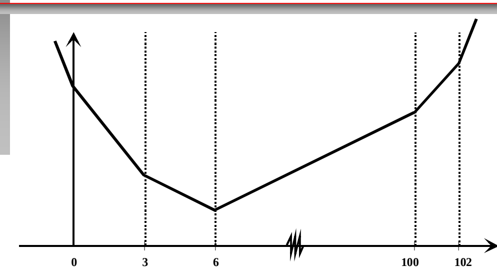
$$Z_R(X) = \sum_{j=1}^{N} w_j d_R(X, P_j) = W^T D_R$$

Majority Property

- Majority or Witzgall Property

$$w_k \geq \frac{W}{2} = \frac{\sum_{j=1}^{N} w_j}{2} \Rightarrow P_k = X^*$$

Rectilinear Minisum Function Example



a_j		0		3		6		100		102		
w_j		5		1		3		2		4		
L_j	0		5		6		9		11		15	
R_j	15		10		9		6		4		0	
s_x	-15		-5		-3		3		7		15	

Minisum Function Properties

- Piecewise Linear
- Breakpoints at Existing Facility Locations (Slope Changes)
- Optimal Location at Existing Facility Location (or Alternative Optima)
- Median Conditions

Median Conditions

- Renumber Existing Facilities by Increasing Coordinates

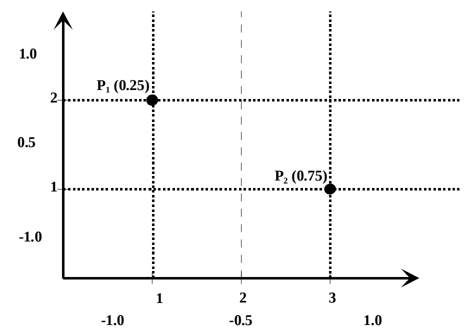
$$W = \sum_{j=1}^{N} w_j$$

$$j^* \Leftarrow \left\{ \sum_{j=1}^{j^*-1} w_j < \frac{W}{2}, \sum_{j=1}^{j^*} w_j \geq \frac{W}{2} \right\}$$

Grid Lines

- Horizontal and Vertical Lines through Existing Facilities
- Optimal Location on a Grid Line Intersection (or Alternative Optima)

Grid Lines Illustration



Contour Curve Properties

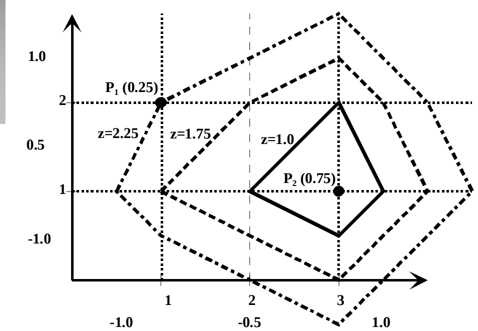
- *Slope of a Contour Line in a Box of the Grid*

$$\Delta x \cdot s_x + \Delta y \cdot s_y = 0$$

$$s_c = \frac{\Delta y}{\Delta x} = -\frac{s_x}{s_y}$$

- *All Contour Curves in a Box of the Grid are Parallel Lines*
- *Contour Curves are Closed Polygons*

Contour Lines Illustration



Linear Programming Transformation

$$\begin{cases} p_j^+ = |x - a_j| & \text{if } x \geq a_j, 0 \text{ otherwise} \\ p_j^- = |x - a_j| & \text{if } x < a_j, 0 \text{ otherwise} \end{cases}$$

$$|x - a_j| = p_j^+ + p_j^-$$

$$x - p_j^+ + p_j^- = a_j \quad \text{if } p_j^+ \cdot p_j^- = 0$$

Primal Formulation

$$\min \quad \sum_{j=1}^{N} w_j \left(p_j^+ + p_j^- \right)$$

$$s.t. \quad x - p_j^+ + p_j^- = a_j \quad \forall j \quad [u_j]$$

$$x \text{ unrestricted}$$

$$p \geq 0$$

Dual Formulation

$$\max \sum_{j=1}^{N} a_j u_j$$

$$\begin{aligned} s.t. \quad & \sum_{j=1}^{N} u_j = 0 & & [x] \\ & -w_j \le u_j & \forall j \quad & [p_j^+] \\ & u_j \le w_j & \forall j \quad & [p_j^-] \\ & u_j \ \ unrestricted & & \end{aligned}$$

Dual Variable Transformation

$$r_j = w_j - u_j \qquad \forall j$$

$$\sum_{j=1}^{N} w_j = f$$

Transformed Dual Formulation

$$\min \sum_{j=1}^{N} a_j r_j$$

$$\begin{aligned} s.t. \quad & \sum_{j} r_j = f & \\ & 0 \le r_j \le 2 w_j & \forall j \end{aligned}$$

Add Redundant Constraint

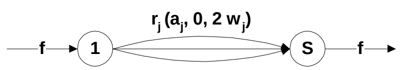
$$-\sum_{j=1}^{N} r_j = -f$$

Network Dual Formulation

$$\min \sum_{j=1}^{N} a_j r_j$$

$$\begin{aligned} s.t. \quad & \sum_{j} r_j = f & \\ & -\sum_{j} r_j = -f & \\ & 0 \le r_j \le 2 w_j & \forall j \end{aligned}$$

Dual Equivalent Network



Optimal Solution

$$f = \sum_{j=1}^{N} w_j$$

$$j^* \Leftarrow \left\{ \sum_{j=1}^{j^*-1} 2w_j < f, \sum_{j=1}^{j^*} 2w_j \geq f \right\}$$

Single Facility Algorithm

1. rank and renumber the existing facilities j by non-decreasing a_j
2. set j = 1, set F = f
3. if $2w_j \geq F$
   then set r_j=F, j*=j, and stop
   else set $r_j=2w_j$, F = F - $2w_j$, j = j+ 1, and go to step 3

Complementary Slackness

$$0 < r_{j^*} = F < 2w_{j^*}$$

$$0 < w_{j^*} - u_{j^*} < 2w_{j^*}$$

$$\begin{cases} (-w_{j^*} - u_{j^*}) \cdot p^+_{j^*} = 0 \\ (w_{j^*} - u_{j^*}) \cdot p^-_{j^*} = 0 \end{cases}$$

$$x^* - p^+_{j^*} + p^-_{j^*} = a_{j^*} \quad \Rightarrow \quad x^* = a_{j^*}$$

Multifacility Formulation

$$\min Z_{RX}(X) =$$

$$\sum_{i=1}^{M} \sum_{j=1}^{N} w_{ij} \left| x_i - a_j \right| + \sum_{i=1}^{M-1} \sum_{k>i}^{M} v_{ik} \left| x_i - x_k \right|$$

- Interactions between Moveable Facilities
- Decomposable

Variable Transformations

$$\begin{cases} q_{ik}^{+} = |x_i - x_k| & \text{if } x_i \ge x_k, 0 \text{ otherwise} \\ q_{ik}^{-} = |x_i - x_k| & \text{if } x_i < x_k, 0 \text{ otherwise} \end{cases}$$

$$|x_i - a_j| = p_{ij}^{+} + p_{ij}^{-}$$

$$x_i - p_{ij}^{+} + p_{ij}^{-} = a_j \qquad \text{if } p_{ij}^{+} \cdot p_{ij}^{-} = 0$$

$$|x_i - x_k| = q_{ik}^{+} + q_{ik}^{-}$$

$$x_i - x_k - q_{ik}^{+} + q_{ik}^{-} = 0 \qquad \text{if } q_{ik}^{+} \cdot q_{ik}^{-} = 0$$

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Primal Formulation

$$\min \sum_{i=1}^{M}\sum_{j=1}^{N} w_{ij}\left(p_{ij}^{+} + p_{ij}^{-}\right) + \sum_{i=1}^{M-1}\sum_{k>i}^{M} v_{ik}\left(q_{ik}^{+} + q_{ik}^{-}\right)$$

$$\begin{aligned} s.t. \quad & x_i - p_{ij}^{+} + p_{ij}^{-} = a_j & \forall i, \forall j & \quad [u_{ij}] \\ & x_i - x_k - q_{ik}^{+} + q_{ik}^{-} = 0 & \forall i, \forall k > i & \quad [l_{ik}] \\ & x \text{ unrestricted} & & \\ & p \ge 0,\ q \ge 0 & & \end{aligned}$$

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Dual Formulation

$$\max \sum_{i=1}^{M}\sum_{j=1}^{N} a_j u_{ij}$$

$$\begin{aligned} s.t. \quad & \sum_{j=1}^{N} u_{ij} + \sum_{k=i+1}^{M} l_{ik} - \sum_{k=1}^{i-1} l_{ki} = 0 & \forall i & \quad [x_i] \\ & -w_{ij} \le u_{ij} & \forall i, \forall j & \quad [p_{ij}^{+}] \\ & u_{ij} \le w_{ij} & \forall i, \forall j & \quad [p_{ij}^{-}] \\ & -v_{ik} \le l_{ik} & \forall i, \forall k > i & \quad [q_{ik}^{+}] \\ & v_{ik} \le l_{ik} & \forall i, \forall k > i & \quad [q_{ik}^{-}] \\ & u, l \text{ unrestricted} & & \end{aligned}$$

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Dual Variable Transformation

$$\begin{aligned} & r_{ij} = w_{ij} - u_{ij} & \forall i, \forall j \\ & s_{ik} = v_{ik} - l_{ik} & \forall i, \forall k > i \\ & \sum_{j} w_{ij} + \sum_{k>i} v_{ik} - \sum_{k<i} v_{ki} = f_i & \forall i \end{aligned}$$

Redundant Constraint

$$\sum_{i=1}^{M}\sum_{j=1}^{N} r_{ij} = \sum_{i=1}^{M} f_i = F$$

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Transformed Dual Formulation

$$\min \sum_{i=1}^{M}\sum_{j=1}^{N} a_j r_{ij}$$

$$s.t. \quad \sum_j r_{ij} + \sum_{k>i} s_{ik} - \sum_{k<i} s_{ki} = f_i \qquad \forall i$$

$$0 \le r_{ij} \le 2w_{ij} \qquad \forall i, \forall j$$

$$0 \le s_{ik} \le 2v_{ik} \qquad \forall i, \forall k > i$$

Network Dual Formulation

$$\min \sum_{i=1}^{M}\sum_{j=1}^{N} a_j r_{ij}$$

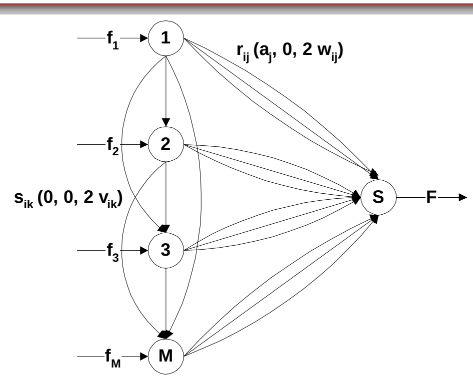
$$s.t. \quad \sum_j r_{ij} + \sum_{k>i} s_{ik} - \sum_{k<i} s_{ki} = f_i \qquad \forall i$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} r_{ij} = F$$

$$0 \le r_{ij} \le 2w_{ij} \qquad \forall i, \forall j$$

$$0 \le s_{ik} \le 2v_{ik} \qquad \forall i, \forall k > i$$

Multifacility Equivalent Network



Sequential Multifacility Algorithm

- *Optimal Location is Based on Sequence and Weights of Facilities, Not Interfacility Distances*
- *Sequential Algorithm Based on Cuts in Q-Locale Networks*

Picard, J., and H. D. Ratliff, 1978. "A Cut Approach to the Rectilinear Distance Facility Location Problem." ***Operations Research***, Vol. 26, No. 3, pp. 422-433.

Continuous Point Location Overview

- Location Introduction
- Euclidean Minisum Location
- Euclidean Minimax Location
- Rectilinear Minisum Location
- **Rectilinear Minimax Location**

Rectilinear Minimax Location

- Assumptions
  - Single Moveable Facility
  - Equal Affinities
- Graphical Algorithm
- Algebraic Solution
- Primal Algorithm

Graphical Algorithm

- Draw Smallest Enclosing 45-Degree Rectangle of Existing Points
- Extend Rectangle in One Direction to Form a Diamond (Center is Point A)
- Extend Rectangle in Opposite Direction to Form a Diamond (Center is Point B)
- Line Segment AB is Set of Optimal Locations

Algebraic Solution

$$c_1 = \min_j \{a_j + b_j\} \qquad c_2 = \max_j \{a_j + b_j\}$$

$$c_3 = \min_j \{-a_j + b_j\} \qquad c_4 = \max_j \{-a_j + b_j\}$$

$$c_5 = \max\{c_2 - c_1, c_4 - c_3\}$$

$$X^* = \lambda \begin{bmatrix} \frac{c_1 - c_3}{2} \\ \frac{c_1 + c_3 + c_5}{2} \end{bmatrix} + (1-\lambda) \begin{bmatrix} \frac{c_2 - c_4}{2} \\ \frac{c_2 + c_4 - c_5}{2} \end{bmatrix}$$

Primal Algorithm

1. Pick Initial Radius z^0, set k = 0
2. Draw Diamonds Around P_j with radii z^k
3. Determine Intersection Z of the Diamonds
4. If Z is Empty,
   Increase z^k, k=k+1, Go to 2
   If Z is More than Line Segment,
   Decrease z^k, k=k+1, Go to 2
   If Z is Line Segment (or Point), Stop

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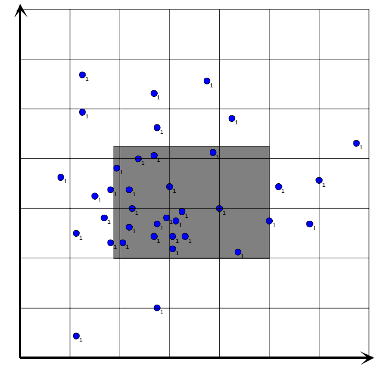
Minimax Location Problem



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Rectilinear Minisum Location

- Distance Norm Properties
- Minisum Location
- Minimax Location
- Rectilinear to Chebyshev Transformation

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Rectilinear and Chebyshev Equivalency

- 45 Degree Rotation and Scaling Transformation
  - Counter clockwise = positive angle
- Chebyshev to Rectilinear Norm Transformation
- Dwell Point Strategies

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45 Degree Rotation and Scaling Transformation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$S = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$Q = RS = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u \\ v \end{bmatrix} = QX = \begin{bmatrix} x+y \\ -x+y \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \qquad X = Q^{-1}U = \begin{bmatrix} (u-v)/2 \\ (u+v)/2 \end{bmatrix}$$

Chebyshev to Rectilinear Norm Equivalence

$$d_R(X,P_j) = |x-a_j| + |y-b_j| =$$
$$\max\{|u-c_j|,|v-d_j|\} = d_c(U,Q_j)$$

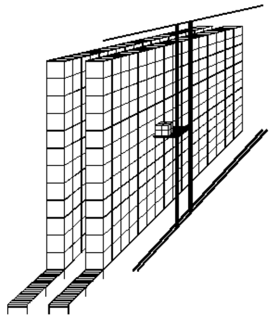
Rectilinear Minimax Equivalence

$$z = \min_X G(X,P) = \min_X \left\{ \max_j \{ w_j d_R(X,P_j) \} \right\} =$$
$$\max\left\{ \min_u G_u(u,c_j), \min_v G_v(v,d_j) \right\}$$

Chebyshev Minisum Equivalence

$$z = \min_U F(U,Q) = \min_U \left\{ \sum_j w_j d_C(U,Q_j) \right\} =$$
$$\min_x F_x(x,a_j) + \min_y F_y(y,b_j)$$

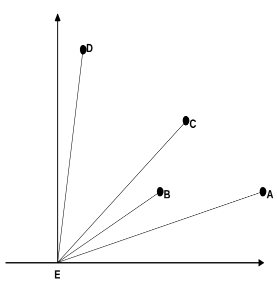
Chebyshev Travel Time AS/RS Illustration



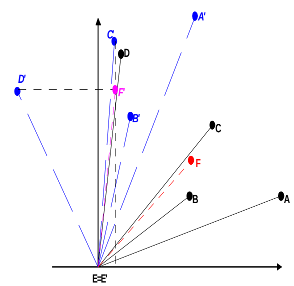
Dwell Point Policy

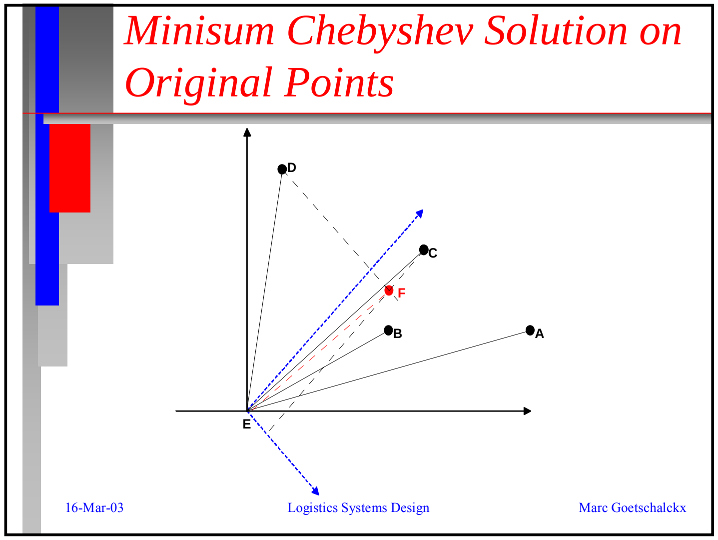
- Set of Rules Where to Position Crane When Idle
- Minimize Expected Distance to Next Service (Load) Request
- Minimize Maximum Distance to Next Service (Load) Request

Minisum Dwell Point in an AS/RS (Simultaneous Travel)



Minisum Rectilinear Solution on Rotated Points




Minisum Chebyshev Solution on Original Points
D
C
F
B
A
E
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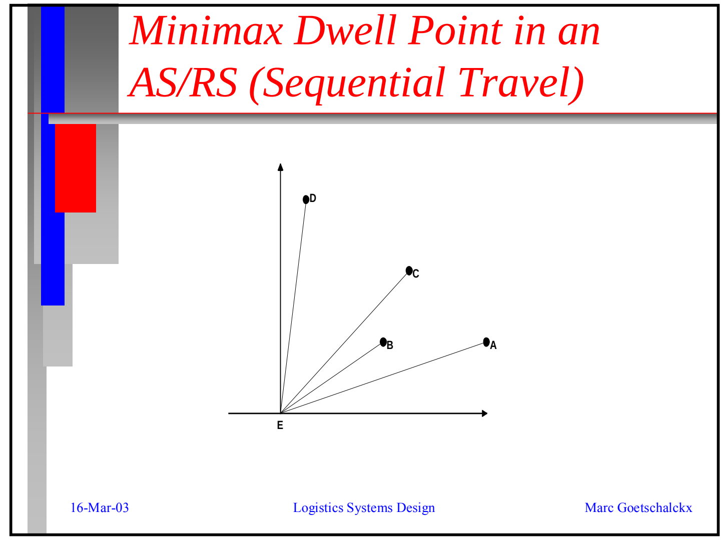
Minimax Dwell Point in an AS/RS (Sequential Travel)
D
C
B
A
E
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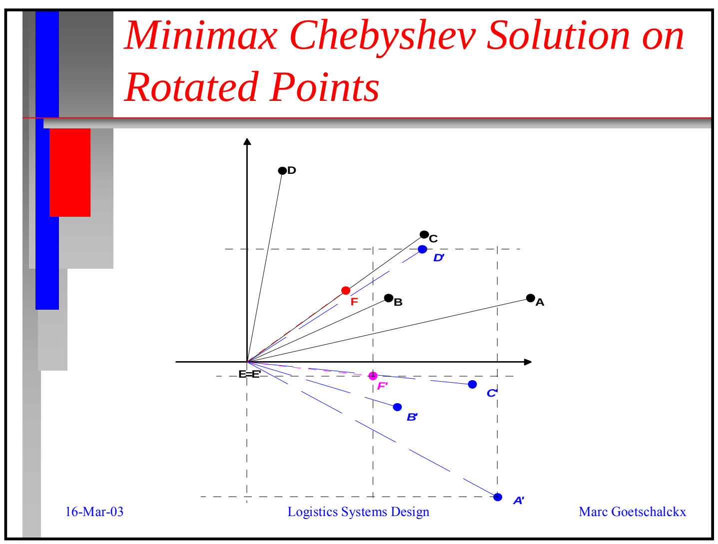
Minimax Chebyshev Solution on Rotated Points
D
C
D'
F
B
A
E=E'
F'
C'
B'
A'
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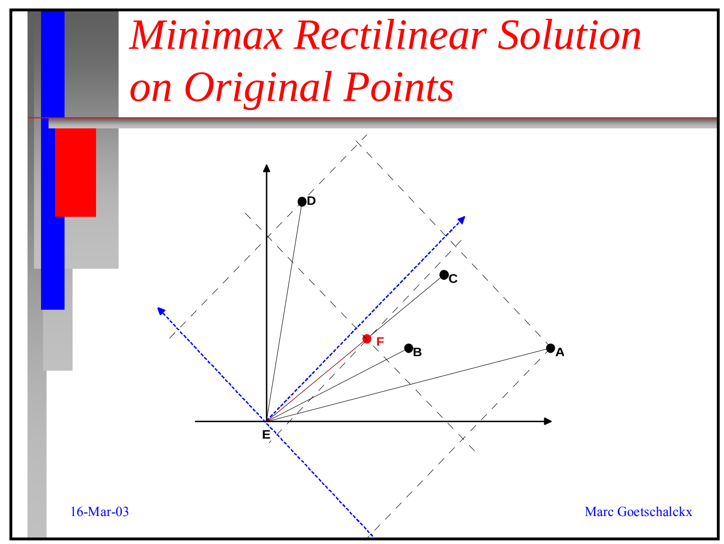
Minimax Rectilinear Solution on Original Points
D
C
F
B
A
E
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