Chapter 13. Manufacturing Models

13.1. Manufacturing Models Introduction

13.2. Manufacturing Operations Design

Unit Process Concept

A unit process is an elemental modification of material or process status done essentially without interruption.

A unit process specifies WHAT, not HOW it needs to be done. A unit process does not specify a machine but rather specifies the transformation, i.e. "*make a hole*" instead of "*drill a hole*". It can be represented as a black box, which in turn can be used as building blocks for the total manufacturing process.

Process Flow or Process Requirements

Transfer Diagrams and Transfer Equations

The process flows are determined based on the required quantity of finished product and the production routing which includes the data on scrap and rework rates. The input flows for each process are computed backwards from the final output flow based on the transfer diagram (and formulas) for each process.

The *defect rate* is the ratio of the number of defective parts to the number parts fed into the system. It is usually expressed as a percentage. The complement of the defect rate is the rate of good parts or *yield rate*.

The *rework rate* is the ratio of the number of parts that can be reprocessed to the total number of defective parts. It is usually expressed as a percentage. The complement of the rework rate is the *scrap rate*, i.e. the ratio of the number of parts leaving the manufacturing system to be scrapped to the number of defective parts. If the rework rate is equal to zero, i.e. all defective parts leave the manufacturing process, then the number of defective parts is equal to the number of scrapped parts and the scrap rate is 100 %.

The *transfer diagram* is a graphical representation of all the input and output flows and their relationships of a single process. The transfer formulas are the algebraic equivalent of the diagram.

Several typical cases arise depending on the number of times a product can be reworked before it has to be scrapped. The transfer diagram and formulas for these typical cases will be developed next. Extensions and modifications are the responsibility of the student.

The following notation will be used:

I_k = input flow of process *k*

M_k = manufacturing flow for process *k*, i.e. all parts to be processed

O_k = output flow of process *k*, i.e. good parts

D_k = defect flow of process *k*, i.e. defective parts

S_k = scrap flow of process *k*, i.e. scrapped parts leaving the system

d_k = defect rate of process *k*

r_k = rework rate of process *k*

s_k = scrap rate of process *k*

No rework allowed.

In this case the defect flow is equal to the scrap flow. The transfer diagram and formulas for a single process are given next.

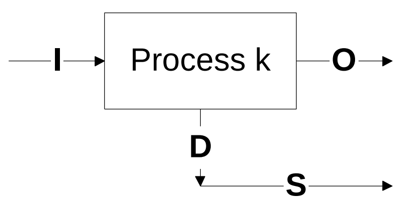

Figure 13.1. No Rework Transfer Diagram

$$\begin{aligned} O_k &= I_k - D_k \\ D_k &= d_k * I_k \\ O_k &= (1-d_k) * I_k \\ I_k &= \frac{O_k}{(1-d_k)} \end{aligned} \tag{13.1}$$

For a completely serial system, where the output of process k is used as the input for process k + 1, a transfer formula for the complete system of N processes can be derived based upon the following formulas.

$$\begin{aligned} I_{k+1} &= O_k \\ I_1 &= O_N * \prod_{k=1}^{N} \frac{1}{(1-d_k)} \end{aligned} \tag{13.2}$$

Assume that there are three machines with the following defect rates (0.04, 0.01, 0.03). The required output from machine three is 97,000 parts per month. Working backwards from machine three, input for machine three is 97,000 / (1 - 0.03) =

100,000. Similarly the input for machines two and one is 101,010 and 105,219, respectively.

Infinite Rework Allowed.

In this case all the parts that can be reworked are added to the supply of new unprocessed parts. An example of such operation would be the turning of a cylindrical part to a desired diameter. Parts with a diameter within the tolerances are accepted. Parts with a diameter that is too large can be reworked. If on the other hand the diameter is too small, then the part has to be scrapped. The transfer diagram and formulas for a single process are given next.

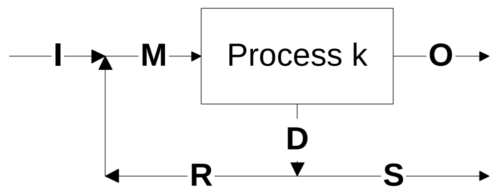

Figure 13.2. Infinite Rework Transfer Diagram

$$
\begin{aligned}
M_k &= I_k + R_k \\
D_k &= M_k * d_k \\
R_k &= D_k * r_k \\
I_k &= M_k - R_k = (1 - r_k d_k) * M_k \\
O_k &= (1 - d_k) * M_k \\
I_k &= \frac{O_k (1 - r_k d_k)}{(1 - d_k)}
\end{aligned}
\tag{13.3}
$$

For a completely serial system, where the output of process k is used as the input for process k+1, a transfer formula for the complete system of N processes can be derived based upon the following formulas.

$$
\begin{aligned}
I_{k+1} &= O_k \\
I_1 &= O_N \prod_{k=1}^{N} \frac{(1 - r_k d_k)}{(1 - d_k)}
\end{aligned}
\tag{13.4}
$$

Observe that formulas (13.3) and (13.4) reduce to formulas (13.1) and (13.2) respectively by setting the rework rate equal to 0, i.e. the no rework case.

Assume further in the example that the rework rate for all three machines is equal to 0.5. Working backwards from machine three, input for machine three is 97,000 · (1 - 0.015) / (1 - 0.03) = 98,500. Similarly the input for machines two and one is 98,998 and 101,060, respectively.

Limited Rework Allowed.

Sometimes a defective part can only be reprocessed a limited number of times. An example of such an operation would the turning as described above, but the turning operation hardens the outer layer of the cylindrical part so that after two turning operations the material becomes brittle and can no longer be turned. The transfer diagram is identical to the case of the unlimited rework case, but some of the paths can only be followed a finite number of times. The number of parts input into the machine on subsequent passes is equal to I, I(rd), I(rd)(rd).. If the number of rework passes is limited to L, then the transfer diagram and formulas are given by

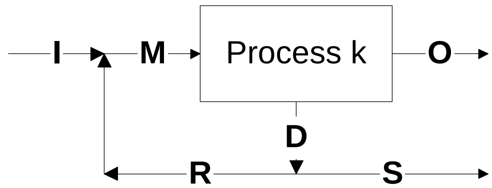

Figure 13.3. Limited Rework Transfer Diagram

$$O_k = (1-d_k) * I_k * \sum_{i=0}^{L} (r_k d_k)^i$$

$$\sum_{i=0}^{\infty} (rd)^i = \frac{1}{1-rd}, if\, |rd| < 1$$

$$\sum_{i=0}^{L} (rd)^i = \sum_{i=0}^{\infty} (rd)^i - \sum_{i=L+1}^{\infty} (rd)^i \tag{13.5}$$

$$= \frac{1-(rd)^{L+1}}{1-rd}$$

$$I_k = O_k * \frac{(1-r_k d_k)}{(1-d_k)(1-(r_k d_k)^{L+1})}$$

For a completely serial system, where the output of process k is used as the input for process k + 1, a transfer formula for the complete system of N processes can be derived based upon the following formulas.

$$I_{k+1} = O_k$$

$$I_1 = O_N \prod_{k=1}^{N} \frac{(1-r_k d_k)}{(1-d_k)(1-(r_k d_k)^{L+1})} \tag{13.6}$$

Assume further in the above example that the maximum number of times a part can be reworked is equal to three. Working backwards from machine three, input for machines three, two and one is 98,500, 98,998 and 101,060, respectively.

Assembly Operations

A more complicated example of the computation of process requirements is given in the examples.

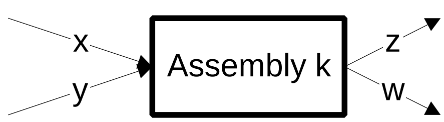

Figure 13.4. Assembly Operation Transfer Diagram

An example of the assembly transfer equations for a particular case of assembly operations is given next:

$$2x + y \Rightarrow z$$

$$x + 3y \Rightarrow w \tag{13.7}$$

$$\begin{cases} I_x = 2 \cdot O_z + O_w \\ I_y = O_z + 3 \cdot O_w \end{cases}$$

Machine and Department Space Requirements

Space requirements per machine are based on industry norms and health and safety rules. The number of machines required can be computed based on the required flows through that machine and on the manufacturing data included in the production routing sheet.

The following notation will be used:

N_j = required number of machines of type *j*

S_i = standard manufacturing time for product *i*

M_i = number of units to be processed per shift of product *i*

H = amount of time available in the planning period

R_j = reliability of machine *j*, expressed as percent up time

$\lceil x \rceil$ = ceiling function which returns the smallest integer larger than or equal to *x*.

E_{ij} = efficiency of machine *j* for product *i* expressed as a percentage, i.e. it is the ratio of the standard time S_i divided by the real time for producing product *i* on machine *j*. Can the efficiency be larger than 100 %?

The number of machines required of type j is then given by the following formula

$$N_j = \left\lceil \frac{\sum_{i=1} \frac{S_i \cdot M_i}{E_{ij}}}{H \cdot R_j} \right\rceil \tag{13.8}$$

For example, 200 parts have to be produced per workday of 8 hours and each part requires a standard time of 2.8 minutes. The efficiency of the machine to be used to product the parts equals .80 % and this machine is 95 % reliable. The number of machines requires is then $\lceil (2.8 \cdot 200) / (480 \cdot 0.95 \cdot 0.80) \rceil = \lceil 1.535 \rceil = 2$.

Basic Sizing and Allocation Model

Sizing and Allocation Problem Characteristics

Jobs and Processors

Sizing and Allocation

Minimize Costs

Capacitated Processors

Deterministic Parameters and Constraints

Notation

M Jobs or Customers (i)

N Processors or Machines (j)

Fixed (f) and Variable (c) Costs

Variable (r) Resource Requirements

Processor Capacities (s)

Required Completed Jobs (d)

Sizing (y) and Allocation (x) Variables

Basic Formulation

$$
\begin{aligned}
Min \quad & \sum_{j=1}^{N} f_j y_j + \sum_{j=1}^{N}\sum_{i=1}^{M} c_{ij} x_{ij} & & \\
s.t. \quad & \sum_{i=1}^{M} r_{ij} x_{ij} \le s_j y_j & \forall j & \\
& \sum_{j=1}^{N} x_{ij} \ge d_i & \forall i & \qquad (13.9) \\
& x_{ij} \ge 0 & \forall ij & \\
& y_j \in N^{\circ} = \{0,1,2,3\ldots\} & \forall j &
\end{aligned}
$$

Fixed Resource Requirement Model Extension

Assume that a fixed resource, denoted by the parameter *g*, is required as soon as any job of type *i* is processed on a machine of type *j*. The binary decision to process any job of type *i* on a machine of type *j* is represented by the variable *z*.

$$
\begin{aligned}
s.t \quad & x_{ij} \le \min\left\{d_i, s_j / r_{ij}\right\} z_{ij} & \forall ij & \\
& \sum_{i=1}^{M} \left(g_{ij} z_{ij} + r_{ij} x_{ij}\right) \le s_j y_j & \forall j & \qquad (13.10) \\
& z_{ij} \le y_j & \forall ij & \\
& z_{ij} \in B = \{0,1\} & \forall ij &
\end{aligned}
$$

Strong Formulation

The allocation variables *x* represent the fraction of the total demand of job *i* allocated to machine type *j*.

$$
\begin{array}{lll}
Min & \sum_{j=1}^{N} f_j y_j + \sum_{j=1}^{N}\sum_{i=1}^{M} c_{ij} d_i x_{ij} & \\
s.t. & \sum_{i=1}^{M}\left(g_{ij} z_{ij} + r_{ij} d_i x_{ij}\right) \le s_j y_j & \forall j \\
 & \sum_{j=1}^{N} x_{ij} \ge 1 & \forall i \\
 & x_{ij} \le y_j & \forall ij \\
 & x_{ij} \le \min\left\{1, s_j / d_i r_{ij}\right\} z_{ij} & \forall ij \\
 & z_{ij} \le y_j & \forall ij \\
 & z_{ij} \in B = \{0,1\} & \forall ij \\
 & 1 \ge x_{ij} \ge 0 & \forall ij \\
 & y_j \in N^{\circ} = \{0,1,2,3\ldots\} & \forall j
\end{array}
\tag{13.11}
$$

Basic Queuing Formulas

The following expressions describing the main characteristics of queuing systems are given in Ross (1993) and Giffin (1978).

Notation

W_q = expected waiting time in the queue

L_q = average length of the queue

W_s = expected system residence time

L_s = average number of customers in the system

P_n = the probability that upon arrival of a customer there are exactly n customers already waiting in the queue

$P_{\ge n}$ = the probability that upon arrival of a customer there are n or more customers already waiting in the queue

λ = arrival rate

m = service rate per server

$E(x), \overline{x}, 1/m$ = average service time (first moment of the distribution of the service time x)

$E\left(x^2\right), \overline{x^2}$ = average squared service time (second moment of the distribution of the service time x)

$s^2, E\left((x-1/m)^2\right)$= variance of the service time distribution

r = system utilization

The following equalities hold for all distributions

$$\overline{x^2} = \left(1/m\right)^2 + s^2 \tag{13.12}$$

$$L_q = lW_q \tag{13.13}$$

$$L_s = lW_s \tag{13.14}$$

$$W_s = W_q + 1/m \tag{13.15}$$

$$L_s = L_q + l/m \tag{13.16}$$

Equation (13.13) is called Little's Law.

M/M/1

In an M/M/1 queuing system the arrival process is a Poisson process with arrival rate λ, i.e., the interarrival times are independent exponentially distributed random variables with mean *1/λ*. The successive service times are assumed to be independent exponential random variables having a mean of *1/m*. The first M refers to the fact that the interarrival process is Markovian and the second M refers to the fact that the service distribution is exponential and the thus Markovian. The 1 refers to the fact there is a single server.

The mean of an exponential distribution is equal to its standard deviation, or

$$s = 1/u$$

$$r = l / m \tag{13.17}$$

$$W_q = \frac{l}{m(m - l)} \tag{13.18}$$

$$W_s = \frac{1}{(m - l)} \tag{13.19}$$

$$L_s = L_q + r \tag{13.20}$$

$$P_0 = 1 - l/m = 1 - r \tag{13.21}$$

$$P_n = (1 - l/m)(l/m)^n \tag{13.22}$$

$$P_{\geq 1} = l/m \tag{13.23}$$

$$P_{\geq n} = (l/m)^n \tag{13.24}$$

M/M/k

The number of servers is equal to *k*. There is a single waiting line and customers go to the first available server.

$$r = l / km \tag{13.25}$$

$$W_q = \frac{(l/km)(l/m)^k}{k!\, l(1 - l/km)^2 \left[\sum_{n=0}^{k-1} \frac{(l/m)^n}{n!} + \frac{(l/m)^k}{k!(1 - l/km)} \right]} \tag{13.26}$$

$$W_s = W_q + 1/m \tag{13.27}$$

M/G/1

The successive service times are assumed to be independent random variables with a general distribution.

$$W_q = \frac{l \cdot \left(\overline{x^2}\right)}{2(1 - l/m)} = \frac{l \cdot \left((1/m)^2 + s^2\right)}{2(1 - l/m)} \tag{13.28}$$

This Khintchine-Pollaczek formula for the expected waiting time in a M/G/1 queue is also given, among others, in Giffin (1978).

Note that the variance of uniformly distributed random variable between the boundary values *a* and *b* is equal to

$$s^2 = \frac{(b-a)^2}{12} \tag{13.29}$$

M/D/1

For a discrete service time distribution, the service time has a constant value and the variance of the service time is equal to zero.

$$W_q = \frac{(l/m)^2}{2\,l(1 - l/m)} \tag{13.30}$$

$$L_q = \frac{(l/m)^2}{2(1 - l/m)} \tag{13.31}$$

M/G/k

$$W_q \approx \frac{\left(m^2/2\right)\overline{x^2}\,(l/km)(l/m)^k}{k!\,l(1 - l/km)^2\left[\sum_{n=0}^{k-1}\frac{(l/m)^n}{n!} + \frac{(l/m)^k}{k!(1 - l/km)}\right]} \tag{13.32}$$

This approximation is a very accurate approximation if the service time has a gamma distribution and it is exact if the service time has an exponential distribution.

Manufacturing Operations Examples

Machine Requirements Example

General Hospital needs to replace their outdated x-ray equipment in order to compete with other hospitals for a smaller patient population. A x-ray machine may be used for general x-rays as well as for special hip x-rays. The time to convert the x-ray machines from general x-ray to hip x-ray is 45 minutes, the time to convert from hip x-ray to general x-ray is 15 minutes. The x-ray machines cost $100,000 per machine but they are very reliable. The arrival of general and special hip x-ray patients is random and may not be scheduled. A general x-ray takes 12 minutes per patient and there are 11,000 patients per year for this x-ray, a special hip x-ray takes 15 minutes and there are 2,500 patients per year for this x-ray. The x-ray machines are in use 8 hours per day, 300 days per year and have a reliability factor of 98 %. The question is how many x-ray machines should be purchased? Based on the number of machines required we will discuss the organization of the x-ray department. Finally, we will estimate the expected waiting time for general x-ray patients, hip x-ray patients and all patients combined for your solution, assuming the x-ray machines have a reliability of 100 %. This example was adapted from Tompkins and White (1984).

We consider first the case where all patients join a single FIFO queue for set of homogeneous multipurpose x-ray machines. There are four distinct type of operations performed depending on the combination of the previous and current x-ray type. The total number of patients is 11,000 + 2,500 = 13,500. The fraction or probability of general x-ray requests is 11,000 / 13,500 = 0.815. The fraction or probability of special hip x-rays is 2,500 / 13,500 = 0.185. Since the patients arrive randomly and may not be scheduled or arranged in the queue, the probability of a general after general operation is then 0.815   0.815 = 0.664 and total number of operations of this type is 13,500   0.664 = 8963. The data for the four operations can be summarized in the following table.

Table 13.1. X-Ray Operations Data Summary

Operation Type	Probability	Operations	Unit Time	Total Time
general after general	0.664	8963	12 min.	107,556 min.
general after hip	0.151	2037	27 min.	54,999 min.
hip after hip	0.034	463	15 min.	6,945 min.
hip after general	0.151	2037	60 min.	122,220 min.

The required number of machines is then

$$N = \left\lceil \frac{107556 + 5499 + 6945 + 122220}{300 \cdot 8 \cdot 60 \cdot 0.98} \right\rceil = \left\lceil \frac{291720}{141120} \right\rceil = \lceil 2.067 \rceil = 3$$

Thus 3 machines are required and their average utilization rate is 68.9 %.

In the next section we will compute the expected waiting times for several configurations and operating policies of the hospital ward, but this derivation might be skipped at the undergraduate level. We assume that the machines are 100 % reliable. First we compute the expected waiting time for three non-dedicated machines and a single FIFO waiting line. The formula for the expected waiting time in queuing system with 3 servers is

$$W_q \approx \frac{\left(m^2/2\right)\overline{x^2}\,(l/3m)(l/m)^3}{3!\, l(1 - l/3m)^2 \left[\sum_{n=0}^{2} \frac{(l/m)^n}{n!} + \frac{(l/m)^3}{3!(1 - l/3m)} \right]}$$

$$l = \frac{13500}{300 \cdot 8 \cdot 60} = 0.094$$

$$\bar{x} = \frac{291720}{13500} = 21.6$$

$$l\bar{x} = \ l/m = 0.094 \cdot 21.6 = 2.026$$

$$l/3m = 0.094 \cdot 21.6/3 = 0.675$$

$$\overline{x^2} = \frac{8963 \cdot 144 + 2037 \cdot 729 + 463 \cdot 225 + 2037 \cdot 3600}{13500} = 756.5$$

$$W_{com} = \frac{(1/(21.6^2 \cdot 2)) \cdot 756.5 \cdot 0.675 \cdot 2.026^3}{6 \cdot 0.094 \cdot (1-0.675)^2 \left[1 + 2.026 + \frac{2.026^2}{2} + \frac{2.026^3}{6(1-0.675)}\right]} = 8.205$$

Observe that the average processing time for a general and hip x-ray patient, respectively, is equal to

$$\frac{8963 \cdot 12 + 2027 \cdot 27}{11000} = 14.78$$

$$\frac{463 \cdot 15 + 2037 \cdot 60}{2500} = 51.67$$

But if the x-ray department was organized differently, we might be able to reduce the number of machines by dedicating machines to operations and thus eliminating the setup times. Assuming that we dedicate a number of machines to general x-rays and a number of machines to hip x-rays, the required number of machines of each type are then:

$$N_{gen} = \left\lceil \frac{11000 \cdot 12}{141120} \right\rceil = \left\lceil \frac{132000}{141120} \right\rceil = \lceil 0.935 \rceil = 1$$

$$N_{hip} = \left\lceil \frac{2500 \cdot 15}{141120} \right\rceil = \left\lceil \frac{37500}{141120} \right\rceil = \lceil 0.266 \rceil = 1$$

In this case we need only two machines, but the expected utilization of the general x-ray machine is very high so the expected waiting times for that machine might not be acceptable. If the hospital decides to purchase three machines, then dedicating two machines to general x-rays and one machine to hip x-rays will reduce the utilization of all machines and improve patient service.

First we compute the expected waiting time if there are two dedicated machines with two independent FIFO queues. We can model this as two M/D/1 queues and . The Khintchine-Pollaczek formula for the expected waiting time in a M/G/1 queue is also given, among others, in Giffin (1978).

$$W_q = \frac{l \cdot \left(\overline{x^2}\right)}{2(1 - \ l/m)} = \frac{l \cdot \left((1/m)^2 + s^2\right)}{2(1 - \ l/m)}$$

For the x-ray machines with a discrete service time, i.e., the standard deviation of the service time is zero, the computations then yield the following expected waiting times:

$$l_{gen} = \frac{11000}{300 \cdot 8 \cdot 60} = \frac{11000}{144000} = 0.076$$
$$\overline{x_{gen}} = 12$$
$$\overline{x_{gen}^2} = 144$$
$$W_{gen} = \frac{0.076 \cdot 144}{2(1 - 0.076 \cdot 12)} = 66$$

$$l_{hip} = \frac{2500}{300 \cdot 8 \cdot 60} = \frac{2500}{144000} = 0.017$$
$$\overline{x_{hip}} = 15$$
$$\overline{x_{hip}^2} = 225$$
$$W_{hip} = \frac{0.017 \cdot 225}{2(1 - 0.017 \cdot 15)} = 2.641$$

The expected waiting time for the single general x-ray machine might not be acceptable to the hospital. We next compute the expected waiting for general x-rays if two machines are dedicated to general x-rays. The expected waiting time in a M/G/2 queuing system with discrete service times is given in Ross (1993) as:

$$W \approx \frac{l\overline{x^2}(l\bar{x})}{2(2 - l\bar{x})^2\left[1 + l\bar{x} + \frac{\left(l\bar{x}\right)^2}{2 - l\bar{x}}\right]} \tag{13.33}$$

The expected waiting time for general x-ray patients if there are two machines dedicated to general x-rays is then:

$$l\bar{x} = 0.076 \cdot 12 = 0.917$$
$$W_{gen} = \frac{0.076 \cdot 144 \cdot 0.917}{2(2 - 0.917)^2\left[1 + 0.917 + \frac{0.917^2}{2 - 0.917}\right]} = 1.6$$

The waiting time computations illustrate again that if three machines are purchased, patient service is improved if two of them are dedicated to general x-rays and one of them is dedicated to hip x-rays.

Assume that the annualized cost of a x-ray machine is $75,000, $95,000, and $135,000 for a general x-ray, hip x-ray, and mixed use x-ray machine, respectively. Assume further, that the cost for performing a general x-ray on a general x-ray machine is $55 and is $75 on a mixed use x-ray machine, and the cost of performing a hip x-ray on a hip x-ray machine is $85 and $105 on a mixed use x-ray machine, respectively. Develop the sizing and allocation model for this system that will minimize the total yearly cost. Clearly define all set, variables, parameters, and constraints. Compute the parameter values. Solve the model to optimality. Discuss the optimal solution.

Process, Product, and Machine Requirements Example

Given the manufacturing data in the following production table, the object is to compute the product flows of products x, y, and z in the manufacturing facility, the required number of machines, and the required number of support materials u and v to produce parts x, y and z. Each time a machine processes a part it requires new support materials as indicated in the Table 13.2 below. Machine A can rework its products an unlimited number of times. Machine B can only rework its products twice, after the third pass all defective parts are scrapped. The required production for product z is 100,000 parts a year. The manufacturing plant is fully automatic. Two parts x plus one part y are assembled to one part z on machine C. The assembly procedure behaves as a perfect process, i.e., it has a zero defect rate. The manufacturing plant operates 50 weeks per year, five days a week, one shift of eight hours a day. The standard production times are given in the Table 13.2 and are expressed in hours.

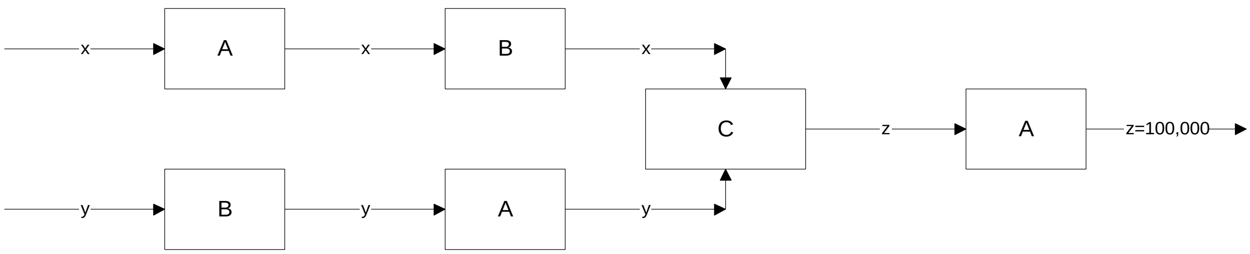

Figure 13.5. Assembly Chart For Process Design Example

Table 13.2. Production Characteristics

	Machine A	Machine B
part x standard time	0.005	0.015
part y standard time	0.01	0.025
part z standard time	0.025	
part x defect rate	10%	70%
part y defect rate	3%	50%
part z defect rate	5%	
part x rework rate	70%	40%
part y rework rate	80%	30%
part z rework rate	90%	
part x support materials	2u	v
part y support materials	2v	u
part z support materials	u + v	
part x efficiency	95%	90%
part y efficiency	92%	98%
part z efficiency	94%	
reliability factor	93%	91%

Using the formulas for infinite rework for machine A and for limited rework for machine B and the assembly equation, the input flows are equal to:

$$I_{Z_A} = \frac{100{,}000 \cdot (1 - 0.05 \cdot 0.9)}{(1 - 0.05)} = 100{,}527$$

$$I_{X_B} = \frac{2 \cdot 100{,}527}{(1-0.7)(1+0.7 \cdot 0.4 + (0.7 \cdot 04)^2)} = 493{,}360$$

$$I_{X_A} = \frac{493{,}360 \cdot (1-0.1 \cdot 0.7)}{(1-0.1)} = 509{,}806$$

$$I_{Y_A} = \frac{100{,}527 \cdot (1-0.03 \cdot 0.8)}{(1-0.03)} = 101{,}149$$

$$I_{Y_B} = \frac{101{,}149}{(1-0.5)(1+0.5 \cdot 0.3 + (0.5 \cdot 03)^2)} = 172{,}536$$

The product flows can now be summarized in the following multiproduct process chart and in a from-to matrix.

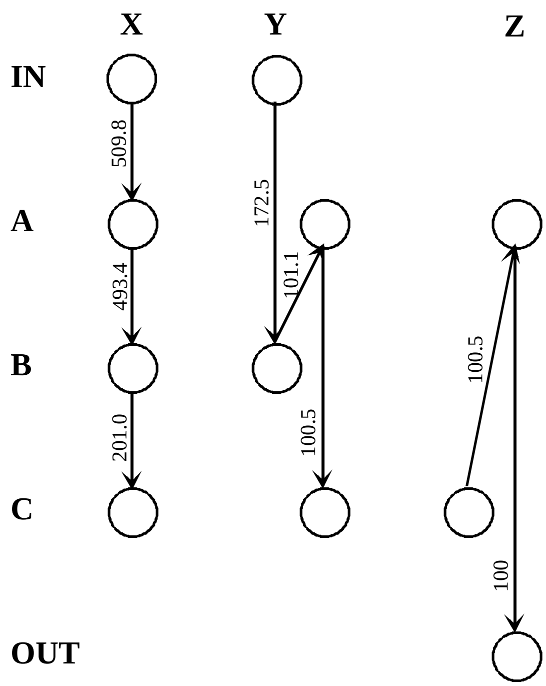

Figure 13.6. Multiproduct Process Chart

Table 13.3. From-To Matrix

	IN	A	B	C	OUT
IN	-	509.8	172.5		
A		-	493.4	100.5	100
B		101.1	-	201	
C		100.5		-	
OUT					-

To compute the required support materials we have to use the manufacturing flows since each processing step requires support materials, whether it produces good or defective parts. The manufacturing flows in function of the input flows for machines A and B are given by

$$M_A = \frac{I}{(1-rd)}$$

$$M_B = \frac{I\left[1-(rd)^3\right]}{(1-rd)} = I(1+rd+r^2d^2)$$

$$M_{Z_A} = \frac{100527}{(1-0.05\cdot 0.9)} = 105264$$

$$M_{X_B} = 493360\cdot\left[1+0.7\cdot 0.4+(0.7\cdot 0.4)^2\right] = 670181$$

$$M_{Y_A} = \frac{101149}{(1-0.03\cdot 0.8)} = 103637$$

$$M_{X_A} = \frac{509806}{(1-0.1\cdot 0.7)} = 548179$$

$$M_{Y_B} = 172536\cdot\left[1+0.5\cdot 034+(0.5\cdot 0.3)^2\right] = 202299$$

The required support materials can be easily computed based on the following table.

Table 13.4. Support Materials Computation

Flow Symbol	Flow Value	Support u	Support v
M_{ZA}	105264	105264	105264
M_{XB}	670181		670181
M_{YA}	103637		207274
M_{XA}	548179	1096358	
M_{YB}	202299	202299	
Total		1403921	982719

The required number of machines can now be computed based on the manufacturing flows and the production data.

$$N_A = \left\lceil \frac{\frac{548179\cdot 0.005}{0.95}+\frac{103637\cdot 0.10}{0.92}+\frac{105264\cdot 0.025}{0.94}}{50\cdot 5\cdot 8\cdot 0.93} \right\rceil = \left\lceil \frac{6811.2}{2000\cdot 0.93} \right\rceil = \lceil 3.662 \rceil = 4$$

$$N_B = \left\lceil \frac{\frac{670181\cdot 0.015}{0.90}+\frac{202299\cdot 0.025}{0.98}}{50\cdot 5\cdot 8\cdot 0.91} \right\rceil = \left\lceil \frac{16330.4}{2000\cdot 0.91} \right\rceil = \lceil 8.973 \rceil = 9$$

If the required cost data are provided, the total production cost can now be computed from the required input parts, required support materials, and machine usage.

Assume that the annualized cost of a machine of type A, B, and C is $100,000, $50,000, and $500,000, respectively. Further assume that the purchase cost of a single unit of component u and v is $0.5 and $2, respectively. The production costs of a single unit of the different products on the different machines is given in Table 13.5, where the assembly cost is expressed in units of product Z assembled on machine C.

Table 13.5. Marginal Production Costs

	A	B	C
X	$0.03	$0.15	
Y	$0.25	$0.10	
Z	$0.05		$0.50

Develop the sizing and allocation model for this system that will minimize the total yearly cost. Clearly define all set, variables, parameters, and constraints. Compute the parameter values. Solve the model to optimality. Discuss the optimal solution.

Exercises

***DryGoods, Inc*.**

The Drygoods Corporation is a distributor of consumer products to a large number of drugstores in the southeastern region of the United States. Products are sold by the case. All the products for a single customer are stacked on one or more pallets and then delivered daily by a dedicated fleet of trucks. The order picking and consolidation occurs from 12 until 6 AM, the delivery to the stores from 6 until 8 AM. To reduce continuing labor shortage problems for this night shift operation, the company is planning to purchase a number of large automated pallet-wrapper machines.

Pallets are wrapped with a plastic film by connecting the film to the pallet and then rotating the pallet while the film spool is raised from the bottom of the pallet to the top of the boxes stacked on the pallet. Depending on the number of boxes stacked on the pallet, the pallets can be classified as low, medium, or high pallets and denoted with subscripts 1, 2, and 3, respectively. The processing times for the three different pallet types are 1.0, 1.5, and 2 minutes, respectively. The expected number of pallets that need to be shipped of each pallet type are 8, 24, and 16 pallets per hour, respectively. The required length of plastic wrap for wrapping each of the three different pallet styles is 12, 18, and 24 meters, respectively. The cost per meter of plastic wrap is $0.03. The total cost for one wrapper is $0.50 per minute. The tension of the plastic film during the wrapping process may cause the boxes to shift and extend beyond the pallet footprint. This is called overhang. In order for the pallets to be transported by forklift truck into the over-the-road trailers there is a limit on the overhang of the boxes on a pallet. The company estimates that 10 % of the pallets will have excessive overhang after being wrapped. If this occurs, the wrapping is cut away and discarded, the boxes rearranged, and then the pallet joins the waiting line to be wrapped again. If the pallet-wrapper is rotated 25 % slower, the company estimates that only 4 % of the pallets will have excessive overhang. The new pallets and pallets that need to be wrapped again join a single waiting line in front of the one or more pallet wrappers. The company wants to know what the average number of pallets waiting to be wrapped will be, since it has to provide space for the waiting pallets. The equivalent hourly cost for one waiting space is $0.10. The company is requesting your assistance in computing the total cost of the wrapping operations for the fast and slow rotation speeds. The company is requesting a clear reporting of the various cost categories and a recommendation for the selection of the speed.

Friendly Fried Food, Inc.

The Friendly Fried Food Corporation uses a number of deep fryers to fry fish and French fries, which then are frozen and offered for sale in supermarkets. The oil in the deep fryer must be changed between the frying of fish and French fries. In addition, the oil in the deep fryer must also be changed after that oil has been used for an hour. The changeover times refer to the time required to clean and refill the deep fryer after a batch of that particular product type. The fish and French fries are produced by different and independent cleaning and preparation lines before they arrive at the common deep fryer. Fish and French fries are processed 4 hours a day, 200 days per year. Each fryer is 93 % reliable. Each batch waiting for the deep fryer requires a four feet by four feet staging area. This space includes all aisle and clearance space. The annual cost for a fryer is $100,000 and the annual cost per square foot of waiting area is $6.0. The company is trying to minimize the annual cost of the frying operation while satisfying throughput requirements.

Given the process data in the following table, how many deep fryers does the Friendly Food Facility need to fry the fish and the French fries? How should the Friendly Food Facility organize their frying operation? Compute the required waiting space for the operation of the deep frying department. Do all computations to four significant digits and round intermediate results to four significant digits. Please answer in a clear and organized fashion. Show the formulas that you used for each intermediate result. Place the numerical intermediate results in a box. Justify the formulas that you have used. Show both formulas and numerical values clearly marked in a box. Finally, summarize your results in a clear table suitable for presentation to the executive committee of the Friendly Fried Food Corporation .

Table 13.6. Deep Frying Process Data

	Fish	French Fries
Standard Time	3 min	6 min
Annual Batches	4000	5000
Efficiency	105%	95%
Change Times	9 min	4 min

Optimal Batch Size

From the marketing and sales department point of view, the best batch size is equal to one. This is equivalent to a make-to-order policy or demand driven production and gives the sales department the greatest flexibility. However, such small batch sizes might not be efficient for the manufacturing department if there are significant setup costs. For example, a customer could walk into a car dealership, "assemble" his own car from the available options. This would be acceptable to the customer with a manufacturing lead time of one week and a delivery lead time of a week. Current realistic lead times for this scenario are much longer. Business corporations have identified the capability to manufacture on demand rather than to inventory as a major competitive advantage. This manufacturing philosophy is called "mass customization".

An example of either extreme point of the spectrum of manufacturing technology is given next. Henry Ford is attributed the quote that “the customer could order a car in any color he desired, as long it was black” illustrating the state of the art in the automotive assembly process of the model T. This statement is a reflection that a single product is easier and more efficient to manufacture. On the other hand, captain Jean-Luc Picard of the starschip enterprise in “Star Trek: the Next Generation” can order a single cup of strong tea which is immediately delivered by the replicator in his quarters. This level of manufacturing flexibility and efficiency only exists in science fiction.

In computing the optimal batch size in production operations from the manufacturing point of view, we will use the following definitions:

T = Production cycle which repeats indefinitely.

R = Production time for the product being evaluated

Q = Production batch size for this product

M = Maximum inventory of this product during the production cycle

d = Product demand rate

p = Product production rate ($p > d$)

IC = Total inventory cost

MC = Total manufacturing cost

TC = Total cost

HC = Inventory holding cost per cycle per product unit

FC = Fixed costs for starting production of a batch of this product

VC = Variable (marginal) cost for production of one unit of this product

ic = Inventory cost per unit

mc = Manufacturing cost per unit

tc = Total unit cost

The inventory pattern over a cycle T is given in the next Figure.

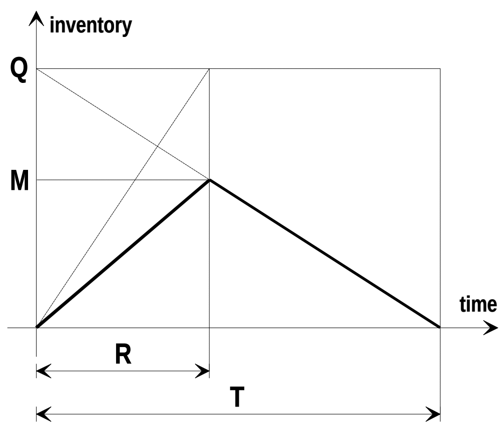

Figure 13.7. Inventory Pattern in Production Systems

We first compute the maximum product inventory:

$$M = Q - d \cdot R = Q - d\left(\frac{Q}{p}\right) = Q\left(1 - \frac{d}{p}\right) \qquad (13.34)$$

Next we compute the total costs over a full cycle:

$$
\begin{aligned}
TC &= IC + MC \\
&= \frac{HC \cdot M \cdot T}{2} + FC + VC \cdot Q \\
&= \frac{HC \cdot Q \cdot \left(1 - d/p\right) \cdot Q}{2 \cdot d} + FC + VC \cdot Q \\
&= FC + VC \cdot Q + \frac{HC(p-d)}{2 \cdot p \cdot d} Q^2
\end{aligned} \tag{13.35}
$$

We then compute the unit costs:

$$
tc = \frac{TC}{Q} = \frac{FC}{Q} + VC + \frac{HC(p-d)}{2pd} Q \tag{13.36}
$$

We find the optimal batch size by setting the first derivative equal to zero:

$$
\frac{d(tc)}{dQ} = -\frac{FC}{Q^2} + \frac{HC(p-d)}{2pd} Q = 0 \tag{13.37}
$$

$$
Q^* = \sqrt{\frac{2 \cdot FC \cdot d}{HC(1-\frac{d}{p})}} \tag{13.38}
$$

This is a generalization of the standard “Economic Order Quantity” or EOQ formula for which the production rate is infinite. The optimal batch size can then also be called the “Economic Production Quantity” or EPQ for a finite production rate.

The optimal total cost is then given by:

$$
\begin{aligned}
TC^* &= \frac{HC(p-d)\left(\frac{2pd \cdot FC}{HC(p-d)}\right)}{2pd} + FC + VC\sqrt{\frac{2pd \cdot FC}{HC(p-d)}} \\
&= 2 \cdot FC + VC\sqrt{\frac{2pd \cdot FC}{HC(p-d)}}
\end{aligned} \tag{13.39}
$$

Most of the factors in this equation are beyond the control of the production system. For example, the inventory holding cost HC is determined by the cost of capital and storage in the facility. The only way to reduce the optimal, efficient batch size is then to reduce the fixed or setup cost.

The minimization by taking the first derivative and setting it to zero is valid since the second derivative is positive and this proves that tc is convex with respect to Q.

$$
\frac{d^2(tc)}{dQ^2} = \frac{2 \cdot FC}{Q^3} + \frac{HC \cdot (p-d)}{2pd} > 0 \tag{13.40}
$$

Note that FC(Q^*) = IC(Q^*) and fc(Q^*) = ic (Q^*).

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