Chapter 18. Layout Models

18.1. Introduction

With the current state of the art, theoretical layout models can provide an optimal solution only to a significantly simplified version of the facilities design problem and for limited problem sizes. Their primary use is to provide a bound on what can be achieved by heuristic methods. However, since theoretical layout models are becoming more realistic and computer resources are becoming more inexpensive, the role of theoretical models is growing.

18.2. Model Classification and Hierarchy

The theoretical layout models can be classified and organized based upon the properties shown in the Figure 18.1. Graph based models generate an adjacency graph, while area based models generate a conceptual block layout or a material handling layout. Primal models maintain a feasible solution while attempting to obtain an optimal solution. Dual models maintain an optimal solution while attempting to obtain a feasible solution. In discrete area models the departments and building are composed of a number of equally sized unit squares. In continuous area models the dimensions of the building and departments can have fractional values.

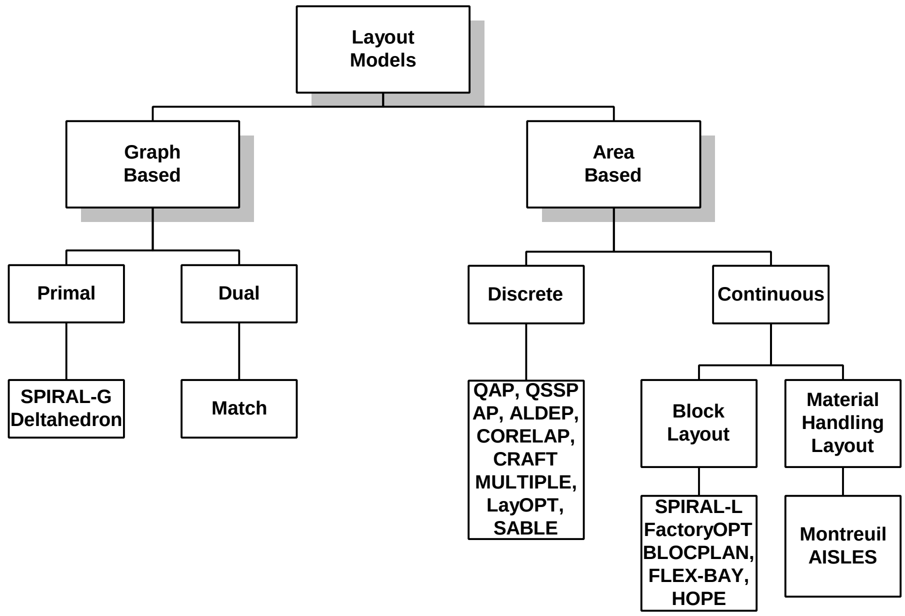

Figure 18.1. Theoretical Layout Models Hierarchy

18.3. Area Based Layout Models

Quadratic Assignment Formulation (QAP)

Problem Definition

One of the original modeling techniques for plant layouts with departments of equal size is the *Quadratic Assignment Problem* or QAP. The model was later extended to the case of departments with unequal size. If the department areas are unequal, then the departments are split up in a number of unit departments of equal area. Similarly, the total layout area is subdivided in a number of equal unit locations. See Figure 18.2 for a graphical illustration of the QAP.

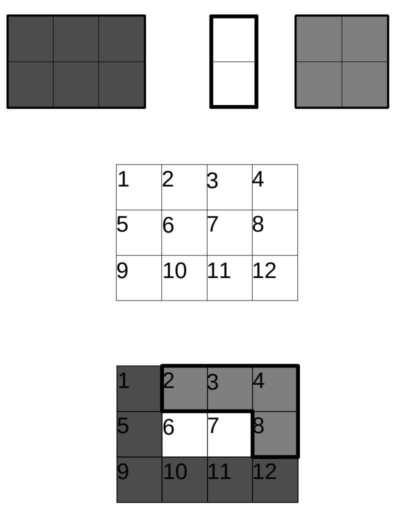

Figure 18.2. Quadratic Assignment Problem Illustration

Formulation

Parameters

M = number of unit departments

N = the number of unit locations

d_{jl} = distance between unit locations j and l

F_{pq} = total relation between departments p and q

f_{ik} = relationship between unit department i and k

a_p = area of department p

Usually M and N are equal. If necessary this always can be achieved through introduction of a dummy department with zero flow to all other departments.

f_{ik} can be computed as follows if unit department i belongs to department p and unit department k belongs to department q:

$$f_{ij} = \frac{F_{pq}}{a_p a_q} \tag{18.1}$$

Variables

x_{ij} = 1 if unit department i is assigned to unit location j, zero otherwise

Formulation

The QAP can then be formulated as:

$$\min \quad \sum_{i}^{M} \sum_{j}^{N} \sum_{k}^{M} \sum_{l}^{N} f_{ik} d_{jl} x_{ij} x_{kl} \tag{18.2}$$

$$s.t. \quad \sum_{j=1}^{N} x_{ij} = 1 \qquad i = 1..M \tag{18.3}$$

$$\sum_{i=1}^{M} x_{ij} = 1 \qquad j = 1..N \tag{18.4}$$

$$x_{ij} \in \{0,1\} \qquad i = 1..M, j = 1..N \tag{18.5}$$

Solution Algorithms

A large number of mathematical programming algorithms exist to solve the QAP, but current problem instances are limited to M around 20. A review of QAP algorithms is given in Hanan and Kurtzberg (1972).

A simple heuristic is to order the f_{ik} in non-increasing order and the d_{jl} in non-decreasing order and then assign unit departments to unit locations in these orders. This inner product of the f and d vectors provides a lower bound on the QAP solution value.

The QAP formulation determines the shape of departments and strangely shaped departments can be generated if no cohesion cost is incorporated. The ***cohesion cost*** is relationship cost between two unit squares that belong to the same department. The cohesion cost forces departments to have a compact shape. Usually, the QAP is run initially with a cohesion cost equal to zero. If a department shape is not compact enough, then the cohesion cost of this department is increased and the QAP is solved again until all shapes are acceptable. The inability to constrain or predict department shapes is a major weakness of the QAP method.

Quadratic Assignment Example

Consider the layout problem with three departments. The department areas are 3, 2, and 1 for departments A, B, and C respectively. The building is a 3 by 2 rectangle. The objective is the rectilinear centroid to centroid distance score. The departments have the following relationships with each other.

Table 18.1. Department Affinities in the QAP Example

	A	B	C	Area
A		12	18	3
B			6	2
C				1

The relationships between the unit squares belonging to the respective department are then given by:

$$f_{ab} = \frac{12}{3 \cdot 2} = 2$$

$$f_{ac} = \frac{18}{3 \cdot 1} = 6$$

$$f_{bc} = \frac{6}{2 \cdot 1} = 3$$

The first three unit squares belong to department A, the 4^{th} and 5^{th} unit square belong to department B, and the 6^{th} unit square belongs to department C. The building squares are numbered 1 through 6, as shown in the next figure.

1	**2**	**3**
4	**5**	**6**

Figure 18.3. Quadratic Assignment Example Building

The rectilinear centroid-to-centroid distances between the unit locations are shown in the next table.

Table 18.2. Rectilinear Distances between Unit Locations in the QAP Example

	1	2	3	4	5	6
1		1	2	1	2	3
2			1	2	1	2
3				3	2	1
4					1	2
5						1
6						

There are 6 times 6 or 36 variables, corresponding to every combination of unit square with unit location, and there are 6 plus 6 or 12 constraints, corresponding to six unit square assignment constraints and six unit location capacity constraints. The twelve constraints are given next.

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} = 1$$
$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} + x_{56} = 1$$
$$x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{66} = 1$$

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} \le 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} \le 1$$
$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} + x_{63} \le 1$$
$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} + x_{64} \le 1$$
$$x_{15} + x_{25} + x_{35} + x_{45} + x_{55} + x_{65} \le 1$$
$$x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{66} \le 1$$

Consider the number of terms in the quadratic objective function. There exist initially 36 possible combinations of locations in the objective function. But when the first unit square occupies a particular unit location, then the second unit square cannot occupy the same unit location, so there remain only 30 combinations of locations. If we represent all possible combinations of locations as a square matrix with N rows and N columns, then the elements on the main diagonal do not have to be considered. The number of remaining elements is then *N(N-1)*. We can also represent all possible combinations of unit squares as a square matrix with N rows and N columns. There exist initially 36 possible combinations of unit squares, but because of the symmetry of the problem we only have to consider the combinations in the upper triangular section of the matrix, i.e., all elements above the main diagonal. The number of remaining elements is then *N(N-1) /2*. For this particular example that leaves 15 combinations. In addition, if there is no cohesion cost then we do not have to consider unit square combinations where both unit squares belong to the same department. For this example, the combinations between unit squares of the same department are 1-2, 1-3, 2-3, and 4-5. This leaves 11 combinations for this particular example. The total number of terms in the objective function is then 11 times 30 or 330, which is fairly large even for this extremely small case. For instance, one term in the objective function is $2 \cdot 2 \cdot x_{11} x_{43}$, when unit square 1 is located in unit location 1 ***and*** unit square 4 is located in unit location 3.

Quadratic Set Packing Formulation (QSPP)

Bazaraa (1975) developed an optimal algorithm to construct a layout from departments with a predetermined shape. For each department all possible locations and orientations are enumerated. The resulting quadratic set packing problem is solved with branch and bound. A graphical illustration of the QSPP is given in Figure 18.4.

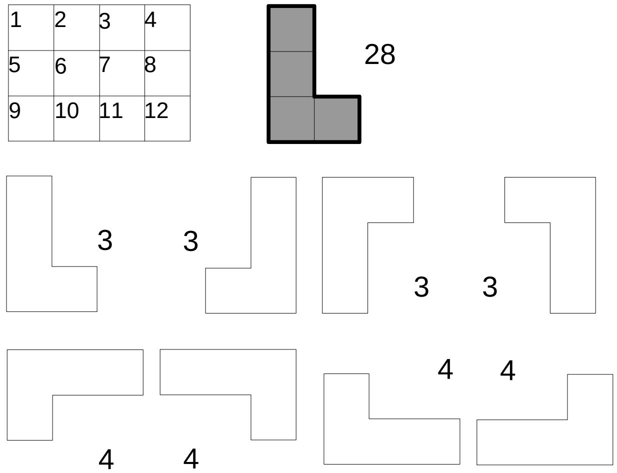

Figure 18.4. Quadratic Set Packing Problem

Let L_i be the list of candidate locations for department *i*. Let b_{ijn} = 1 if candidate location *j* of department *i* occupies unit square *n*. The other symbols are identical to the QAP formulation. The QSPP is then modeled as:

$$\min \quad \sum_{i=1}^{M}\sum_{j=1}^{L_i}\sum_{k=1}^{M}\sum_{l=1}^{L_k} f_{ik}\, d(j_i, l_k)\, x_{ij}\, x_{kl} \tag{18.6}$$

$$s.t. \quad \sum_{j=1}^{L_i} x_{ij} = 1 \qquad i = 1..M \tag{18.7}$$

$$\sum_{i=1}^{M}\sum_{j=1}^{L_i} b_{ijn} x_{ij} = 1 \qquad n = 1..N \tag{18.8}$$

$$x_{ij} \in \{0,1\} \qquad i = 1..M,\ j = 1..N \tag{18.9}$$

Additional side constraints can be added without making the problem any harder. Problems with 14 departments and 60 unit locations can be solved in seconds.

Three computation schemes for the lower bound are given in order of decreasing computational requirements. First, the bound on a partial layout can be computed by completing the layout while ignoring interaction among not yet located departments. The resulting formulation is a linear set covering problem (SCP). Second, the above linear SCP can be solved with linear programming. Finally, the best location for each unassigned department can be determined independently.

Linear Assignment Problems (AP)

Malette and Francis (1972) model the case where there is no interaction between the facilities, but there is interaction between facilities and fixed (interface) points with a linear Assignment Problem (AP). Typical examples of fixed interface points are doors and locks. Such models occur frequently in the layout of warehousing systems. An illustration of the AP is given in Figure 18.5.

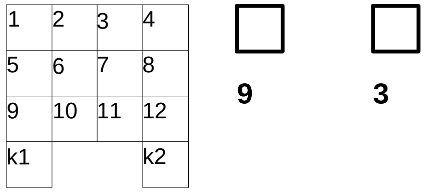

Figure 18.5. Linear Assignment Problem Illustration

Assume there are K fixed interface points. The problem can then be formulated as:

$$\min \quad \sum_{i=1}^{M}\sum_{j=1}^{N} x_{ij}\left(\sum_{k=1}^{K} f_{ik} d_{kj}\right) \tag{18.10}$$

$$s.t. \quad \sum_{j=1}^{N} x_{ij} = 1 \qquad \text{i=1..M} \tag{18.11}$$

$$\sum_{i=1}^{M} x_{ij} = 1 \qquad \text{j=1..N} \tag{18.12}$$

$$x_{ij} \in \{0,1\} \qquad \text{i=1..M, j=1..N} \tag{18.13}$$

If the factoring condition is satisfied, then the problem can be solved by a simple ranking procedure, see Malette and Francis (1972). The factoring condition is equivalent to stating that all departments have the same distribution (i.e. proportion) of interaction with the fixed interface points. The best layout can then be found by assigning the department with highest material flow to the locations with minimal weighted distance to the interface points.

Let F_i be the total material flow from department *i* to and from all interface points. Let *p* be the probability mass vector of interaction with the interface points, i.e. p_k is the proportion of the flow interacting with each interface point *k*. Let D_j be the weighted distance to a location *j*. Then the objective function of the AP can be transformed with:

$$\sum_{k=1}^{K} f_{ik} d_{kj} = F_i\left(\sum_{k=1}^{K} p_k d_{kj}\right) = F_i D_j \tag{18.14}$$

to

$$\min \sum_{i=1}^{M}\sum_{j=1}^{N} F_i D_j x_{ij} \tag{18.15}$$

If there are as many departments as locations, then the solution can be obtained by sorting F_i in decreasing order and D_j in increasing order and by assigning the corresponding elements. This equivalent to minimizing the innerproduct of two vectors by sorting the vectors in opposite directions.

18.4. Graph Theoretic Layout Models

The completed block layout can be represented as a graph. Similarly, the original relationship matrix can be represented as a complete undirected graph, where each department corresponds to a node and pair wise relationships form the edges. The ***adjacency graph*** is constructed from the relationship diagram by deleting all edges between non-adjacent departments, i.e. an edge is included in the adjacency graph if and only if the two departments are adjacent. The adjacency graph is thus a subgraph of the relationship diagram.

The block layout graph and adjacency graphs are related, in fact they are dual graphs. The relationship diagram can be constructed from the block layout by placing a node inside each department and by drawing the connecting relationship edge if and only if the two departments are adjacent, i.e. have a common wall. The resulting adjacency graph is ***planar***. A graph is planar if it can be drawn in a two-dimensional plane without crossing edges.

From an adjacency graph a block layout can be constructed if and only if the adjacency graph is planar, see Seppanen and Moore (1970). The concepts of layout graph, adjacency graph and their dual relationship are shown in Figure 18.6.

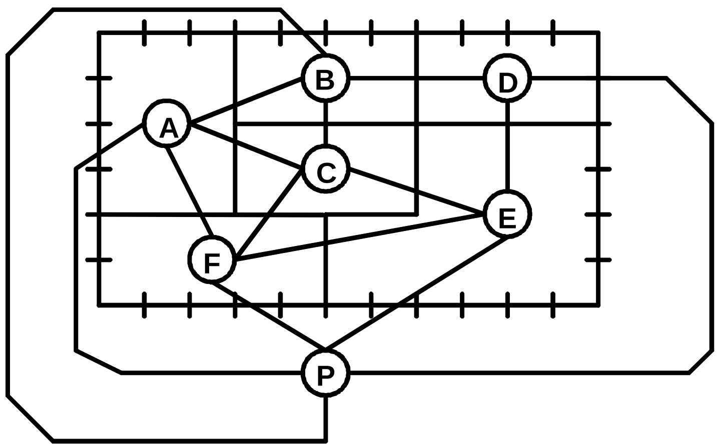

Figure 18.6. Adjacency Graph and Layout as Dual Graphs

Maximum Planar Subgraph Formulation (MPSP)

The graph theoretic models are concerned with determining the best relative location of departments, ignoring the complexities of department and building shape and area. Their objective is to maximize the adjacency score. But in order to construct a block layout, the adjacency graph must be planar. Hence the problem is to extract a planar subgraph out of the relationship diagram, so that its edge weight is maximized and the graph is planar. This problem is called the Maximum (Weight) Planar Subgraph Problem (MPSP). This problem is NP-Complete and cannot be solved for practical problem size in reasonable amount of computer time.

The following notation will be used. Let x_{ij} be 1 if department *i* is adjacent to department *j*, zero otherwise. Let *G(E)* be the adjacency graph consisting of edges *E*. Then the MPSP can be formulated as:

$$Max \quad \sum_{i=1}^{M+1} \sum_{j=1}^{M+1} f_{ij} x_{ij} \tag{18.16}$$

$$s.t. \quad \mathbf{G}\left(e_{ij} \middle| x_{ij} = 1\right) = planar \tag{18.17}$$

$$x_{ij} \in \{0,1\} \tag{18.18}$$

Observe that the perimeter or outside is represented as an additional department *M + 1*.

Foulds and Robison (1976) developed a branch and bound algorithm for the above formulation. It involves the testing at every node if the current partial graph is planar. Even though planarity testing is a polynomial procedure developed by Hopcroft and Tarjan (1974), it is still the bottleneck in the above algorithm. This algorithm has also the tendency to connect high weight nodes with many other departments, which yields an undesirable "umbrella effect".

It is known that a maximal planar graph can have at most 3(M+1)-6 edges. An upper bound to the MPSP can then be computed by adding the 3(M+1)-6 largest weights. This assumes there are at least so many positive weight edges in the relationship diagram.

Given that the original MPSP problem is unsolvable for realistic problem sizes, two heuristic approaches have been developed. The primal approach attempts to achieve a high adjacency score while always maintaining a planar and hence feasible graph. Heuristics are used to maximize the adjacency score. Examples of such primal algorithms are the Deltahedron and Hexagonal Adjacency Graphs. The dual approach always finds the maximum adjacency score while forcing the resulting graph closer and closer to planarity. The Perimeter Specified Adjacency Graph method is an example of a dual algorithm. One of the inherent advantages of a primal method is that a feasible planar graph exists if the method has to be terminated prematurely.

Deltahedron Heuristic

It is easy to construct either a maximum weight subgraph or a planar subgraph, but the combined MPSP is NP-complete, see Foulds (1983). On approach is to construct a guaranteed planar graph with a very high edge weight.

Foulds and Robison (1978) presented such a heuristic procedure to solve the MPSP which avoid planarity testing. The algorithm constructs a graph out of triangles. Such a triangulated graph is always planar. The algorithm attempts to achieve a high adjacency score. The algorithm starts by constructing a tetrahedron, i.e. a complete triangular graph with four vertices. The remaining vertices are then inserted one at a time in the triangle which maximizes the their insertion gain. One variant inserts the vertices by decreasing total edge weight. A second variant inserts the vertices by decreasing difference (regret) between their first and second candidate triangle.

Observe that if relationships with the outside have to be incorporated, one of the initial four vertices must correspond with the outside department. The algorithm is illustrated in Figure 18.7.

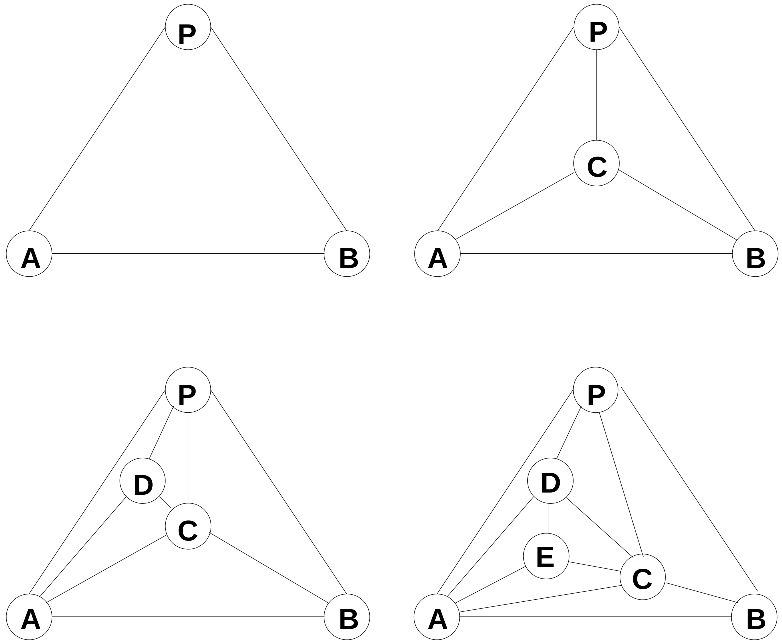

Figure 18.7. Deltahedron Heuristic Illustration

Hexagonal Adjacency Graph Heuristic

Another heuristic to construct a maximum weight planar subgraph is given by the SPIRAL technique, which is discussed in the section on Computer Aided Layout. The spiral technique was originally introduced by Reed (1967) and adapted to a computer implementation by Goetschalckx (1986, 1991). The SPIRAL technique constructs a guaranteed planar graph with high adjacency score based on an underlying hexagonal grid.

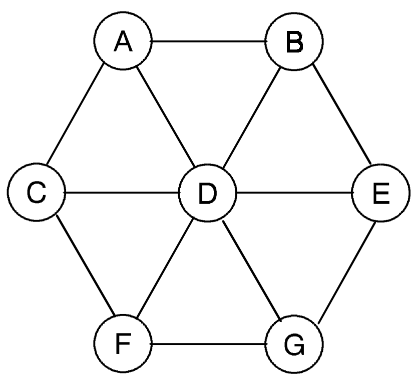

Figure 18.8. Hexagonal Grid Illustration

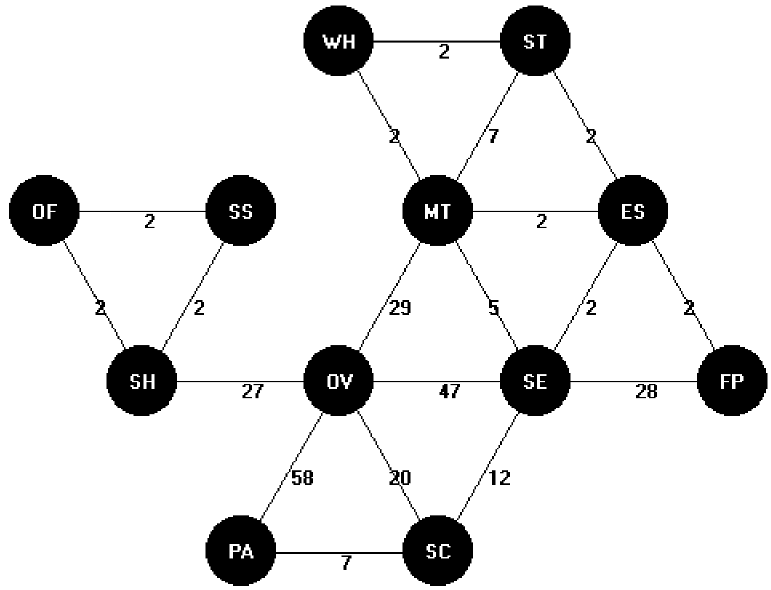

Figure 18.9. SPIRAL Adjacency Graph Illustration

In the case of the hexagonal grid, an upper bound on the adjacency score of the graph can be computed based on the following integer programming formulation, where *0* represents the artificial outside department:

$$\max \quad \sum_{i=1}^{N-1} \sum_{j>i}^{N} r_{ij} x_{ij} + \sum_{i=1}^{N} r_{io} y_i \tag{18.19}$$

$$s.t. \quad \sum_{j=0}^{N} x_{ij} = 6 \qquad i = 1..N \tag{18.20}$$

$$y_i \le x_{io} \qquad i = 1..N \tag{18.21}$$

$$x_{io} \le 6 y_i \qquad i = 1..N \tag{18.22}$$

$$y_i \in \{0,1\} \qquad i = 1..N \tag{18.23}$$

$$x_{ij} \in \{0,1\} \qquad i = 1..N-1,\ j = i+1..N \tag{18.24}$$

$$x_{i0} \ge 0 \qquad i = 1..N \tag{18.25}$$

The following notation is used:

x_{ij} is 1 if departments *i* and *j* are adjacent, 0 otherwise. For the artificial outside department, with $j = 0$, x_{ij} represents the number of times department *i* is adjacent to the outside perimeter.

y_i is 1 if department *i* is adjacent to the outside, 0 otherwise.

r_{ij} is the binary relationship between departments *i* and *j*.

The formulation maximizes the adjacencies between departments and between a department and the outside (18.19), subject to the constraint that each department must have exactly six neighbors (18.20). These neighboring departments can be the artificial outside department. If a department has at least one outside department as a neighbor the relationship with the outside is satisfied and included in the objective function (18.21) and (18.22). All the adjacencies are binary variables, except the adjacencies with the outside department that are integer (18.23), (18.24) and (18.25).

The value of the formulation lies in the fact that it provides an upper bound on the optimal solution quality and on the maximum deviation of the heuristic solutions.

Perimeter Specified Adjacency Matching

The Perimeter Specified Adjacency Matching algorithm by Montreuil et al. (1987) creates a maximum weight subgraph and imposes incrementally additional constraints to impose planarity. In addition the model is more specific since departments can be one or more times adjacent depending on the length of their perimeter.

The perimeter length b_i of each department *i* is computed, based on the suggested department shape. Each department has to be adjacent exactly b_i times with other departments. The value of each of those adjacencies is stored in the relationship matrix. Based on the shape of the two departments, there exists an upper and lower bound on the number of times those departments can be adjacent. Let x_{ij} be the number of times departments *i* and *j* are adjacent, let l_{ij} and u_{ij} be the lower and upper bound on the number times they can be adjacent and let c_{ij} be value for each time they are adjacent. The following matching formulation is then solved:

$$Max \quad \sum_{i=1}^{M} \sum_{j=i+1}^{M+1} c_{ij} x_{ij} \tag{18.26}$$

$$s.t. \quad \sum_{j=1}^{M+1} x_{ij} = b_i \qquad i = 1..M \tag{18.27}$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad i = 1..M,\ j = 1..M+1 \tag{18.28}$$

$$x_{ij} \in \mathbf{N}^{+} \qquad i = 1..M,\ j = 1..M+1 \tag{18.29}$$

The resulting solution is usually not a feasible layout, because the graph is not planar or the perimeter is not feasible. It is the task of the human operator to impose additional constraints (by modifying l_{ij} and u_{ij}) that will force the solution to a feasible layout. An example of the perimeter specified adjacency graph is given in Figure 18.10.

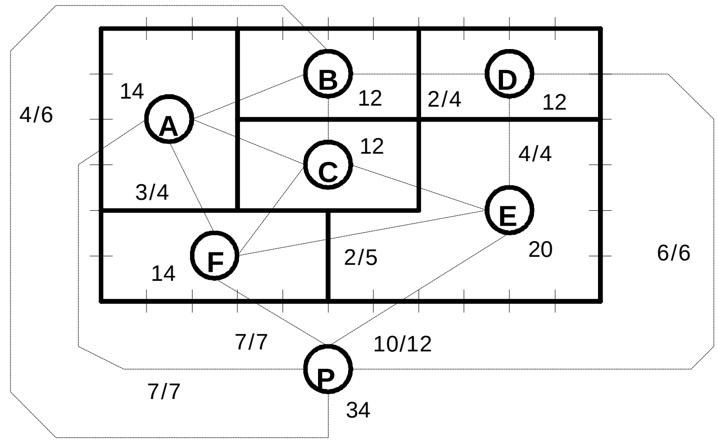

Figure 18.10. Perimeter Specified Adjacency Graph

Montreuil et al. (1987) have incorporated this algorithm in an interactive layout system called MATCH. The location of the departments must be done interactively by the human operator on the CRT screen. As such there is no location procedure build into the algorithm. It is possible to have a heuristic drawing method for the matching graph based on the expanded spiral technique.

18.5. Area Layout Models with Shape Constraints

Problem Definition

The objective is to design a conceptual block layout for a number of departments with unequal areas. All departments have to fit inside the confines of a rectangular building and the departments cannot overlap. In addition, the departments must satisfy a shape ratio constraint. The departments have pairwise affinities. The objective function is to minimize the affinity-weighted centroid-to-centroid rectilinear distance score.

The following notation will be used:

Parameters

K = number of departments, indexed by k

W = width along the x axis of the building

L = length along the y axis of the building

A = area of the building

a_k = area of department k

S_k = maximum value of the shape ratio of department k

F_{kl} = affinity between departments k and l

p_k = shape penalty for department k

The parameter and variable definitions are illustrated in Figure (18.11).

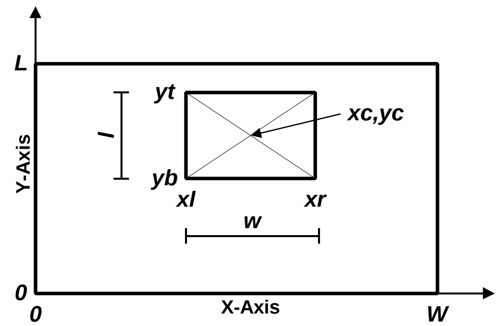

Figure 18.11. Variable and Parameter Illustration in Block Layout with Shape Constraints

Variables

s_k = shape ratio of department k

xl_k = leftmost x coordinate of the smallest rectangle enclosing department k

xr_k = rightmost x coordinate of the smallest rectangle enclosing department k

yt_k = topmost y coordinate of the smallest rectangle enclosing department k

yb_k = bottommost y coordinate of the smallest rectangle enclosing department k

w_k = width along the x axis of the smallest rectangle enclosing department k

l_k = length along the y axis of the smallest rectangle enclosing department k

xc_k = x coordinate of the centroid of department k

yc_k = y coordinate of the centroid of department k

dy_{kl}^{+}, dy_{kl}^{-} = y components of the rectilinear distance between departments k and l

dx_{kl}^{+}, dx_{kl}^{-} = x components of the rectilinear distance between departments k and l

Notice that in the discrete formulation, when a department is composed of a number of unit squares, the shape of the department is not necessarily rectangular and in that case the centroid of the department may not coincide with the centroid of the enclosing rectangle.

Definitions

The length and width of a department are defined as

$$w_k = xr_k - xl_k \tag{18.30}$$

$$l_k = yt_k - yb_k \tag{18.31}$$

The shape ratio of a department is defined as the maximum of the length to width or width to length ratios of the smallest rectangle complete enclosing the department, or

$$s_k = \max\left\{\frac{l_k}{w_k}, \frac{w_k}{l_k}\right\} \tag{18.32}$$

For rectangular buildings, the building area is equal to the product of the building width and length, or

$$A = L \cdot W \tag{18.33}$$

Feasibility Constraints

Based on the input parameters, the following feasibility constraints can be immediately tested

$$\sum_{k=1}^{K} a_k \le A \tag{18.34}$$

Discrete Assignment Formulation

The first decision in the discrete assignment formulation is the determination of the unit square size. Large unit squares tend to produce more regular department shapes but limit the flexibility of the algorithm and thus tend to yield higher distance scores. Small unit squares allow more irregularly shaped departments and make the problem size larger, but they tend to yield lower distance scores. Based on the size of the unit square, the building is divided into R rows by C columns of unit locations and each department is divided into a number of unit squares. All areas are then normalized by dividing them by the unit square size and all distances are normalized by dividing them by the length of the side of the unit square.

Notation

Parameters

u = length of the side of the unit square

R = number of rows along the y axis in the building

C = number of columns along the x axis in the building

b_k = number of unit squares for department k

I_k = set of unit squares belonging to department k

M = total number of unit squares for all departments, index by i

N = number of unit locations in the building, indexed by j

c_j = column index along the x axis of unit location j

r_j = row index along the y axis of unit location j

Variables

x_{ij} = 1 if unit square i is located at unit location j, 0 otherwise

Definitions

$$N = R \cdot C \tag{18.35}$$

$$M = \sum_{k=1}^{K} b_k \tag{18.36}$$

$$b_k = \left| I_k \right| \tag{18.37}$$

Feasibility Constraints

Based on the input parameters, the following feasibility constraints can be immediately tested

$$\sum_{k=1}^{K} b_k \le R \cdot C \tag{18.38}$$

Assignment Constraints

Just as in the case of the Quadratic Assignment Problem, we have two sets of constraints. The first set ensures that every unit square is assigned to exactly one unit location. The second set ensures that every unit location holds at most one unit square and thus ensures that department will not overlap.

$$\sum_{j=1}^{N} x_{ij} = 1 \qquad i = 1..M \tag{18.39}$$

$$\sum_{i=1}^{M} x_{ij} \le 1 \qquad j = 1..N \tag{18.40}$$

Shape Ratio Constraints

The definition of the shape ratio contains a maximum operator, which prevents this constraint to be included into a linear programming solver. However, there exists an easy transformation of (18.32) into two linear constraints.

$$\begin{aligned} l_k &\le S_k w_k \\ w_k &\le S_k l_k \end{aligned} \tag{18.41}$$

Using the definitions of the length and width of a department we then get the following constraint set, with two constraints per department.

$$xr_k - xl_k \le S_k \left(yt_k - yb_k \right) \qquad k = 1..K \tag{18.42}$$

$$yt_k - yb_k \le S_k \left(xr_k - xl_k \right) \qquad k = 1..K \tag{18.43}$$

The boundary values of each department have to be derived from the assignment variables. Adding the following constraint sets determines the upper bounds on the coordinates.

$$c_j x_{ij} \le xr_k \qquad k = 1..K, i \in I_k, j = 1..N \tag{18.44}$$

$$r_j x_{ij} \le yt_k \qquad k = 1..K, i \in I_k, j = 1..N \tag{18.45}$$

The lower bounds can be determined with the following constraints.

$$xl_k \le \left(c_j - 1\right) x_{ij} + C\left(1 - x_{ij}\right) \qquad k = 1..K, i \in I_k, j = 1..N$$

$$yb_k \le \left(r_j - 1\right) x_{ij} + R(1 - x_{ij}) \qquad k = 1..K, i \in I_k, j = 1..N$$

These constraints can then be simplified to the following constraints.

$$xl_k \le \left(c_j - 1 - C\right) x_{ij} + C \qquad k = 1..K, i \in I_k, j = 1..N \tag{18.46}$$

$$yb_k \le \left(r_j - 1 - R\right) x_{ij} + R \qquad k = 1..K, i \in I_k, j = 1..N \tag{18.47}$$

Centroid Constraints

The coordinates of the centroid of each department are computed as the average of the centroids of the unit squares belonging to that department. The centroid coordinates are then linked to the assignment variables with the following constraints

$$xc_k = \frac{\sum_{i \in I_k} \sum_{j=1}^{N} c_j x_{ij}}{b_k} - 0.5 \qquad k = 1..K$$

$$yc_k = \frac{\sum_{i \in I_k} \sum_{j=1}^{N} r_j x_{ij}}{b_k} - 0.5 \qquad k = 1..K$$

Since the centroid coordinates of each department only will be used in the distance calculations, we can ignore the 0.5 offset to yield the following simplified constraints.

$$b_k xc_k = \sum_{i \in I_k} \sum_{j=1}^{N} c_j x_{ij} \qquad k = 1..K \tag{18.48}$$

$$b_k yc_k = \sum_{i \in I_k} \sum_{j=1}^{N} r_j x_{ij} \qquad k = 1..K \tag{18.49}$$

Distance Terms in the Objective Function

For each pair of departments there exists a term in the objective function which multiplies the centroid to centroid rectilinear distance with the corresponding affinity.

$$F_{kl}\left(\left|xc_k - xc_l\right| + \left|yc_k - yc_l\right|\right) \qquad k = 1..K, l = 1..K$$

The absolute value operator cannot be directly used in a linear programming formulation. We use the following standard definitions and substitutions.

$$dx_{kl}^{+} = \begin{cases} xc_k - xc_l & if\ xc_k \geq xc_l \\ 0 & otherwise \end{cases}$$
$$dx_{kl}^{-} = \begin{cases} xc_l - xc_k & if\ xc_k \leq xc_l \\ 0 & otherwise \end{cases} \tag{18.50}$$

We can then substitute for each absolute value term in the objective function based on the following equality

$$\left|xc_k - xc_l\right| = dx_{kl}^{+} + dx_{kl}^{-} \tag{18.51}$$

provided we add the following two constraints.

$$xc_k - xc_l = dx_{kl}^{+} - dx_{kl}^{-} \tag{18.52}$$

$$dx_{kl}^{+} \cdot dx_{kl}^{-} = 0 \tag{18.53}$$

Observe that the columns in the constraint matrix corresponding to the dx_{kl}^{+} and dx_{kl}^{-} variables are identical but with opposite signs. Those columns and the corresponding variables are thus linearly dependent and they cannot appear together in the optimal solution basis. This implies that at least one of the two variables must be non-basic and thus equal to zero, and so we do not have to add explicitly the nonlinear constraint (18.53) to the formulation. The terms in the objective function are then

$$F_{kl}\left(dx_{kl}^{+} + dx_{kl}^{-} + dy_{kl}^{+} + dy_{kl}^{-}\right) \qquad k = 1..K, l = 1..K \tag{18.54}$$

Provided we add the following constraint sets

$$xc_k - xc_l = dx_{kl}^{+} - dx_{kl}^{-} \qquad k = 1..K, l = 1..K \tag{18.55}$$

$$yc_k - yc_l = dy_{kl}^{+} - dy_{kl}^{-} \qquad k = 1..K, l = 1..K \tag{18.56}$$

The evaluation of location of departments with respect to relations with the outside requires different formulas, since the centroid of the outside department is not defined. We define the distance of a department to the outside as the smallest distance of the centroid of the department to any of the four borders of the building.

$$d_{ko} = \min\left\{xc_k, W - xc_k, yc_k, L - yc_k\right\} \qquad k = 1..K \tag{18.57}$$

We can then add the weighted distance score to the objective functions and add the following constraints for each department. Observe that different constraints must me added depending on the sign of the relationship of the department with the outside.

$$\begin{array}{llll} if\ F_{ko} > 0 & \Rightarrow & Min. & \sum_k F_{ko} \cdot d_{ko} \\ & & s.t. & d_{ko} \geq xc_k \cdot dl_k \\ & & & d_{ko} \geq \left(W - xc_k\right) \cdot dr_k \\ & & & d_{ko} \geq yc_k \cdot db_k \\ & & & d_{ko} \geq \left(L - yc_k\right) \cdot dt_k \\ & & & dl_k + dr_k + db_k + dt_k \geq 1 \\ & & & dl_k, dr_k, db_k, dt_k \in \left\{0,1\right\} \end{array} \tag{18.58}$$

The *dl, dr, db,* and *dt* variables are logical OR variables indicating that one the relationships has to be satisfied.

$$\begin{array}{llll} if\ F_{ko} < 0 & \Rightarrow & Min. & \sum_k F_{ko} \cdot d_{ko} \\ & & s.t. & d_{ko} \leq xc_k \\ & & & d_{ko} \leq W - xc_k \\ & & & d_{ko} \leq yc_k \\ & & & d_{ko} \leq L - yc_k \end{array} \tag{18.59}$$

Discrete Formulation

$$
\begin{array}{lll}
Min. & \sum_{k=1}^{K-1}\sum_{l=k+1}^{K} F_{kl}\left(dx_{kl}^{+}+dx_{kl}^{-}+dy_{kl}^{+}+dy_{kl}^{-}\right)+\sum_{k=1}^{K} F_{ko}\cdot d_{ko} & \\
s.t. & \sum_{j=1}^{N} x_{ij}=1 & i=1..M \\
 & \sum_{i=1}^{M} x_{ij}\le 1 & j=1..N \\
 & c_j x_{ij}\le xr_k & k=1..K, i\in I_k, j=1..N \\
 & r_j x_{ij}\le yt_k & k=1..K, i\in I_k, j=1..N \\
 & xl_k\le\left(c_j-1-C\right)x_{ij}+C & k=1..K, i\in I_k, j=1..N \\
 & yb_k\le\left(r_j-1-R\right)x_{ij}+R & k=1..K, i\in I_k, j=1..N \\
 & xr_k-xl_k\le S_k\left(yt_k-yb_k\right) \quad k=1..K & \\
 & yt_k-yb_k\le S_k\left(xr_k-xl_k\right) \quad k=1..K & \\
 & b_k xc_k=\sum_{i\in I_k}\sum_{j=1}^{N} c_j x_{ij} & k=1..K \\
 & b_k yc_k=\sum_{i\in I_k}\sum_{j=1}^{N} r_j x_{ij} & k=1..K \\
 & xc_k-xc_l=dx_{kl}^{+}-dx_{kl}^{-} & k=1..K-1, l=k+1..K \\
 & yc_k-yc_l=dy_{kl}^{+}-dy_{kl}^{-} & k=1..K-1, l=k+1..K \\
 & d_{ko}\ge xc_k\cdot dl_k & k=1..K \\
 & d_{ko}\ge\left(W-xc_k\right)\cdot dr_k & k=1..K \\
 & d_{ko}\ge yc_k\cdot db_k & k=1..K \\
 & d_{ko}\ge\left(L-yc_k\right)\cdot dt_k & k=1..K \\
 & dl_k+dr_k+db_k+dt_k\ge 1 & k=1..K \\
 & dl_k, dr_k, db_k, dt_k\in\{0,1\} & k=1..K \\
 & x_{ij}\in\{0,1\} & i=1..M, j=1..N \\
 & xr_k, xl_k, yt_k, yb_k, xc_k, yc_c\ge 0 & k=1..K \\
 & dx_{kl}^{+}, dx_{kl}^{-}, dy_{kl}^{+}, dy_{kl}^{-}\ge 0 & k=1..K, l=1..K
\end{array}
$$

For notational simplicity, only the constraints for the case of positive relationships with the outside are shown in this formulation. If the relationship of a department with the outside is negative, then the constraint set (18.11) should be used instead.

Example

We expand the example given in the section on the quadratic assignment formulation. Consider again the layout problem with three departments. The department areas are 3, 2, and 1 for departments A, B, and C respectively. The building is a 3 by 2 rectangle. The objective is the rectilinear centroid-to-centroid distance score. The departments have the relationships with each other as shown in Table 18.3. In addition, the maximum shape ratio for each department is equal to 3. The first three unit squares belong to department A, the 4th and 5th unit square belong to department B, and the 6th unit square belongs to department C. The building squares are numbered 1 through 6, as shown in the Figure (18.12).

Table 18.3. Department Data for Discrete Layout Formulation

	A	B	C	Area	Max Shape	I
A		12	18	3	3	1,2,3
B			6	2	3	4,5
C				1	3	6

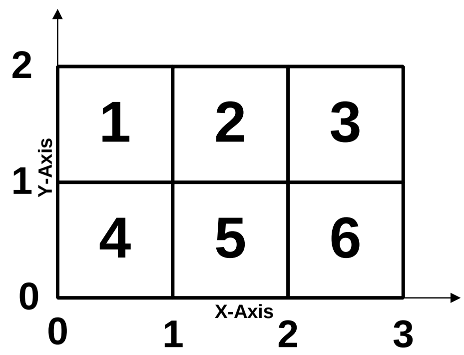

Figure 18.12. Discrete Layout Formulation Example Building

The rectilinear centroid-to-centroid distances between the unit locations as well as the row and column indices are shown in Table 18.4.

Table 18.4. Distance Data for Discrete Layout Formulation

	1	2	3	4	5	6	r	c
1		1	2	1	2	3	2	1
2			1	2	1	2	2	2
3				3	2	1	2	3
4					1	2	1	1
5						1	1	2
6							1	3

Exercise

Consider the shape and location of a department, as shown in the Figure (18.13), that constitute the partial solution to the discrete formulation for the block layout with shape constraints. Write all the equations in algebraic (symbolic) form that 1) compute the centroid location of this department, 2) determine the boundary coordinates of this department, 3) ensure that the shape ratio of this department is less than *S*. Use the notation developed in class. Do not include any other constraints or objectives. The area of the department is four unit squares and there are 50 unit locations in the layout problem. Clearly define all variables and parameters that you are using and label the constraints. Indicate how many constraints of each type there are for this department and explain how you derived this number.

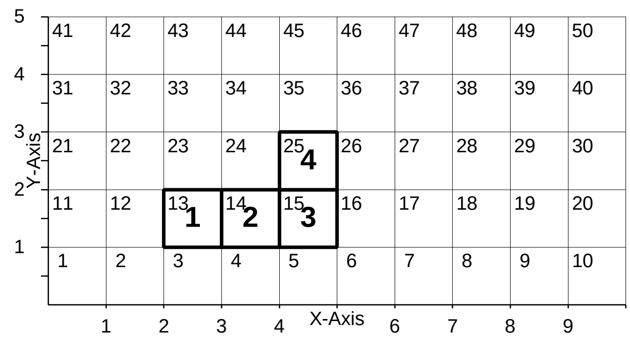

Figure 18.13. Discrete Block Layout Exercise

Write in numerical format, i.e. with all parameters expressed as numbers, all the equations of the above three sets when the corresponding assignment variable is equal to one. In other words, write those constraints and only those in which $x_{ij} = 1$ for this department *k* with a set of unit squares $I_k = \{1,2,3,4\}$ and all possible locations *j*. This implies that you should not include the variables $x_{ij} = 0$ in those constraints nor should include constraints that only use variables $x_{ij} = 0$. Assume that *S=2*. Based on these numerical constraints, compute the numerical value for the centroid and boundary coordinates of this department. Do the shape and location of the department satisfy the shape constraints?

Continuous Formulation

The continuous formulation is based on the assumption that only rectangular department shapes are allowed.

Variables

zr_{kl} = 1 if department *k* is located completely to the right of department *l*

zb_{kl} = 1 if department *k* is located completely to the bottom of department *l*

Building Constraints

To ensure that all departments lay within the building enclosure, we add the following constraint set, with constraints for each department.

$$0 \le yt_k \le L \qquad k = 1..K \qquad (18.60)$$

$$0 \le xr_k \le W \qquad k = 1..K \qquad (18.61)$$

To avoid negative length and width of a department we add the following constraint set, with two constraints for each department.

$$0 \le xl_k \le xr_k \qquad k = 1..K \qquad (18.62)$$

$$0 \le yb_k \le yt_k \qquad k = 1..K \qquad (18.63)$$

Centroid Coordinates and Distances

Given the rectangular department shape, the computation of the centroid coordinates is straightforward as the middle between the boundary coordinates.

$$xc_k = \frac{xr_k + xl_k}{2} \qquad k = 1..K \tag{18.64}$$

$$yc_k = \frac{yt_k + yb_k}{2} \qquad k = 1..K \tag{18.65}$$

The transformation of the objective function is also identical to discrete formulation. Substituting the definition of the centroid coordinates, we get the following equations for the centroid to centroid distances.

$$xr_k + xl_k - xr_l - xl_l = 2dx_{kl}^{+} - 2dx_{kl}^{-} \qquad k = 1..K-1, l = k+1..K \tag{18.66}$$

$$yt_k + yb_k - yt_l - yb_l = 2dy_{kl}^{+} - 2dy_{kl}^{-} \qquad k = 1..K-1, l = k+1..K \tag{18.67}$$

If the relationship between two departments is negative, any optimal solution will make the corresponding distances as large as possible. Without any additional constraints, the problem becomes unbounded. The original nonlinear constraint (18.53) is required for the optimal solution. Alternatively, two new binary variables can be used to ensure that either the positive or negative component of the distance between two departments is positive, but not both. For every pair of departments with a negative relationship between them the following set of constraints is added to the formulation.

$$\begin{aligned}
&xr_k + xl_k - xr_l - xl_l = 2dx_{kl}^{+} \cdot zpx_{kl} - 2dx_{kl}^{-} \cdot znx_{kl} \\
&yt_k + yb_k - yt_l - yb_l = 2dy_{kl}^{+} \cdot zpy_{kl} - 2dy_{kl}^{-} \cdot zny_{kl} \\
&zpx_{kl} + znx_{kl} = 1 \\
&zpy_{kl} + zny_{kl} = 1 \\
&zpx_{kl}, znx_{kl}, zpy_{kl}, zny_{kl} \in \{0,1\}
\end{aligned} \tag{18.68}$$

$$\begin{aligned}
&xr_k + xl_k - xr_l - xl_l = 2dx_{kl}^{+} \cdot zpx_{kl} - 2dx_{kl}^{-} \cdot znx_{kl} \Rightarrow \\
&xr_k + xl_k - xr_l - xl_l = 2vpx_{kl} - 2vnx_{kl}
\end{aligned} \tag{18.69}$$

$$dx_{kl}^{+} zpx_{kl} = vpx_{kl} \quad \Leftrightarrow \quad \begin{aligned} &dx_{kl}^{+} \le W zpx_{kl} \\ &-dx_{kl}^{+} + vpx_{kl} \le 0 \\ &dx_{kl}^{+} - vpx_{kl} + W zpx_{kl} \le W \end{aligned} \tag{18.70}$$

For every pair of departments with a negative relationship between them, this adds 14 extra constraints and four extra binary variables.

Department Non-Overlap Constraints

The constraints to ensure that a department is to the right or to the bottom of another department can be formulated as

$$xr_l - xl_k \le W(1 - zr_{kl}) \qquad k = 1..K, l = 1..L$$

$$yt_k - yb_l \le L(1 - zb_{kl}) \qquad k = 1..K, l = 1..L$$

They can be transformed into

$$xr_l - xl_k + W \cdot zr_{kl} \le W \qquad k = 1..K, l = 1..L \tag{18.71}$$

$$yt_k - yb_l + L \cdot zb_{kl} \le L \qquad k = 1..K, l = 1..L \tag{18.72}$$

$$zb_{kl} + zr_{kl} + zb_{lk} + zr_{lk} \ge 1 \qquad k = 1..K, l = 1..K \tag{18.73}$$

$$zr_{kl} + zr_{lk} \le 1 \qquad k = 1..K, l = 1..K \tag{18.74}$$

$$zb_{kl} + zb_{lk} \le 1 \qquad k = 1..K, l = 1..K \tag{18.75}$$

Just like in the case of the discrete formulation, we add the constraint set, with two constraints per department, to limit the shape ratio of each department.

Department Area Constraints

The area constraint for each department is the following nonlinear equation.

$$l_k = \frac{a_k}{w_k} \tag{18.76}$$

The gradient is given by

$$\frac{dl_k}{dw_k} = -\frac{a_k}{w_k^2} \tag{18.77}$$

The second derivative is given by

$$\frac{d^2 l_k}{dw_k^2} = \frac{2a_k}{w_k^3} \tag{18.78}$$

The second derivative is positive everywhere and thus the area constraint is a convex curve and the tangent to the curve in any point p on the curve will be a lower support of the curve. The equation of the support in point p is then

$$(l_k - l_k^p) \ge -\frac{a_k}{\left(w_k^p\right)^2}\left(w_k - w_k^p\right) \tag{18.79}$$

A number of these supports can be added to the formulation to construct a linear approximation of the nonlinear area constraint. The more supports are added the closer the approximation, but the larger the problem size. Three to five supports usually suffice to satisfy the area constraint with a relative precision of 0.001. P will indicate the number of supports and is a parameter determined by the user.

Since the distance score tends to be smaller for long and narrow departments, one of the supports at the endpoints of the feasible interval of w_k is often binding. It is recommended to add at least the two supports at the endpoints of the interval. The feasible interval of w_k is determined by the maximum shape ratio for the department and the building dimensions. The values of w_k and l_k at the endpoints of the feasible interval are indexed by *p=1* and *p=2* and are given by

$$l_k^1 = \min\left\{\sqrt{a_k S_k}, L\right\}, \qquad w_k^1 = \frac{a_k}{l_k^1} \tag{18.80}$$

$$w_k^2 = \min\left\{\sqrt{a_k S_k}, W\right\}, \qquad l_k^2 = \frac{a_k}{w_k^2} \tag{18.81}$$

Continuous Formulation

$$
\begin{array}{lll}
Min. & \sum_{k=1}^{K-1}\sum_{l=k+1}^{K} F_{kl}\left(dx_{kl}^{+}+dx_{kl}^{-}+dy_{kl}^{+}+dy_{kl}^{-}\right)+\sum_{k=1}^{K} F_{ko} d_{ko} & \\
s.t. & xl_k \le xr_k & k=1..K \\
 & yb_k \le yt_k & k=1..K \\
 & xr_k \le W & k=1..K \\
 & yt_k \le L & k=1..K \\
 & xr_k - xl_k \le S_k\left(yt_k - yb_k\right) & k=1..K \\
 & yt_k - yb_k \le S_k\left(xr_k - xl_k\right) & k=1..K \\
 & xr_l - xl_k + W\cdot zr_{kl} \le W & k=1..K, l=1..L \\
 & yt_k - yb_l + L\cdot zb_{kl} \le L & k=1..K, l=1..L \\
 & zb_{kl} + zr_{kl} + zb_{lk} + zr_{lk} \ge 1 & k=1..K, l=1..K \\
 & zr_{kl} + zr_{lk} \le 1 & k=1..K, l=1..K \\
 & zb_{kl} + zb_{lk} \le 1 & k=1..K, l=1..K \\
 & (yt_k - yb_k - l_k^p) \ge -\frac{a_k}{\left(w_k^p\right)^2}\left(xr_k - xl_k - w_k^p\right) & k=1..K, p=1..P \\
 & xr_k + xl_k - xr_l - xl_l = 2dx_{kl}^{+} - 2dx_{kl}^{-} & k=1..K-1, l=k+1..K \\
 & yt_k + yb_k - yt_l - yb_l = 2dy_{kl}^{+} - 2dy_{kl}^{-} & k=1..K-1, l=k+1..K \\
 & dx_{kl}^{+} \le W & k=1..K, l=1..K, F_{kl}<0 \\
 & dx_{kl}^{-} \le W & k=1..K, l=1..K, F_{kl}<0 \\
 & dy_{kl}^{+} \le L & k=1..K, l=1..K, F_{kl}<0 \\
 & dy_{kl}^{-} \le L & k=1..K, l=1..K, F_{kl}<0 \\
 & d_{ko} \ge xc_k \cdot dl_k & k=1..K \\
 & d_{ko} \ge \left(W - xc_k\right)\cdot dr_k & k=1..K \\
 & d_{ko} \ge yc_k \cdot db_k & k=1..K \\
 & d_{ko} \ge \left(L - yc_k\right)\cdot dt_k & k=1..K \\
 & dl_k + dr_k + db_k + dt_k \ge 1 & k=1..K \\
 & dl_k, dr_k, db_k, dt_k \in \left\{0,1\right\} & k=1..K \\
 & zr_{kl}, zb_{kl} \in \{0,1\} & k=1..K, l=1..K \\
 & xr_k, xl_k, yt_k, yb_k \ge 0 & k=1..K \\
 & dx_{kl}^{+}, dx_{kl}^{-}, dy_{kl}^{+}, dy_{kl}^{-} \ge 0 & k=1..K, l=1..K
\end{array}
$$

Again for notational simplicity, only the constraints for the case of positive relationships with the outside are shown in this formulation. If the relationship of a department with the outside is negative, then the constraint set (18.11) should be used instead.

Example

For the same small example as in the case of the discrete formulation, the continuous formulation is given next with five supports for the nonlinear area constraint for each department.

Formulation 18.1. Continuous Formulation for the Tiny Example

```
MIN
 +12 XP001002 +12 XN001002 +12 YP001002 +12 YN001002
 +6 XP001003 +6 XN001003 +6 YP001003 +6 YN001003
 +18 XP002003 +18 XN002003 +18 YP002003 +18 YN002003
S.T.
XR001 - XL001 >= 0
YB001 - YT001 >= 0
XR001 <= 3.000000
YB001 <= 2.000000
XR001 - XL001 - 3.000000 YB001 + 3.000000 YT001 <= 0
- 3.000000 XR001 + 3.000000 XL001 + YB001 - YT001 <= 0
1.333333 XR001 - 1.333333 XL001 + YB001 - YT001 >= 4.000000
0.925926 XR001 - 0.925926 XL001 + YB001 - YT001 >= 3.333333
0.680272 XR001 - 0.680272 XL001 + YB001 - YT001 >= 2.857143
0.520833 XR001 - 0.520833 XL001 + YB001 - YT001 >= 2.500000
0.411523 XR001 - 0.411523 XL001 + YB001 - YT001 >= 2.222222
0.333333 XR001 - 0.333333 XL001 + YB001 - YT001 >= 2.000000
XR002 - XL002 >= 0
YB002 - YT002 >= 0
XR002 <= 3.000000
YB002 <= 2.000000
XR002 - XL002 - 3.000000 YB002 + 3.000000 YT002 <= 0
- 3.000000 XR002 + 3.000000 XL002 + YB002 - YT002 <= 0
2.000000 XR002 - 2.000000 XL002 + YB002 - YT002 >= 4.000000
1.202041 XR002 - 1.202041 XL002 + YB002 - YT002 >= 3.101021
0.801361 XR002 - 0.801361 XL002 + YB002 - YT002 >= 2.531973
0.572122 XR002 - 0.572122 XL002 + YB002 - YT002 >= 2.139388
0.428831 XR002 - 0.428831 XL002 + YB002 - YT002 >= 1.852202
0.333333 XR002 - 0.333333 XL002 + YB002 - YT002 >= 1.632993
XR003 - XL003 >= 0
YB003 - YT003 >= 0
XR003 <= 3.000000
YB003 <= 2.000000
XR003 - XL003 - 3.000000 YB003 + 3.000000 YT003 <= 0
- 3.000000 XR003 + 3.000000 XL003 + YB003 - YT003 <= 0
3.000000 XR003 - 3.000000 XL003 + YB003 - YT003 >= 3.464102
1.530612 XR003 - 1.530612 XL003 + YB003 - YT003 >= 2.474358
0.925926 XR003 - 0.925926 XL003 + YB003 - YT003 >= 1.924501
0.619835 XR003 - 0.619835 XL003 + YB003 - YT003 >= 1.574592
0.443787 XR003 - 0.443787 XL003 + YB003 - YT003 >= 1.332347
0.333333 XR003 - 0.333333 XL003 + YB003 - YT003 >= 1.154701
XR001 + XL001 - XR002 - XL002 - 2 XP001002 + 2 XN001002 = 0
YB001 + YT001 - YB002 - YT002 - 2 YP001002 + 2 YN001002 = 0
XR001 + XL001 - XR003 - XL003 - 2 XP001003 + 2 XN001003 = 0
YB001 + YT001 - YB003 - YT003 - 2 YP001003 + 2 YN001003 = 0
XR002 + XL002 - XR003 - XL003 - 2 XP002003 + 2 XN002003 = 0
YB002 + YT002 - YB003 - YT003 - 2 YP002003 + 2 YN002003 = 0
XR002 - XL001 + 3.000000 ZR001002 <= 3.000000
YB002 - YT001 + 2.000000 ZB001002 <= 2.000000
XR003 - XL001 + 3.000000 ZR001003 <= 3.000000
YB003 - YT001 + 2.000000 ZB001003 <= 2.000000
XR001 - XL002 + 3.000000 ZR002001 <= 3.000000
YB001 - YT002 + 2.000000 ZB002001 <= 2.000000
XR003 - XL002 + 3.000000 ZR002003 <= 3.000000
YB003 - YT002 + 2.000000 ZB002003 <= 2.000000
XR001 - XL003 + 3.000000 ZR003001 <= 3.000000
YB001 - YT003 + 2.000000 ZB003001 <= 2.000000
XR002 - XL003 + 3.000000 ZR003002 <= 3.000000
YB002 - YT003 + 2.000000 ZB003002 <= 2.000000
ZR001002 + ZR002001 + ZB001002 + ZB002001 >= 1
ZR001002 + ZR002001 <= 1
ZB001002 + ZB002001 <= 1
ZR001003 + ZR003001 + ZB001003 + ZB003001 >= 1
ZR001003 + ZR003001 <= 1
ZB001003 + ZB003001 <= 1
ZR002003 + ZR003002 + ZB002003 + ZB003002 >= 1
ZR002003 + ZR003002 <= 1
ZB002003 + ZB003002 <= 1
integer
ZR001002
ZB001002
ZR001003
ZB001003
ZR002001
ZB002001
ZR002003
```

```
ZB002003
ZR003001
ZB003001
ZR003002
ZB003002
End
```

The optimal solution for the continuous formulation is shown in Figure (18.14) and has an objective function value of 53.

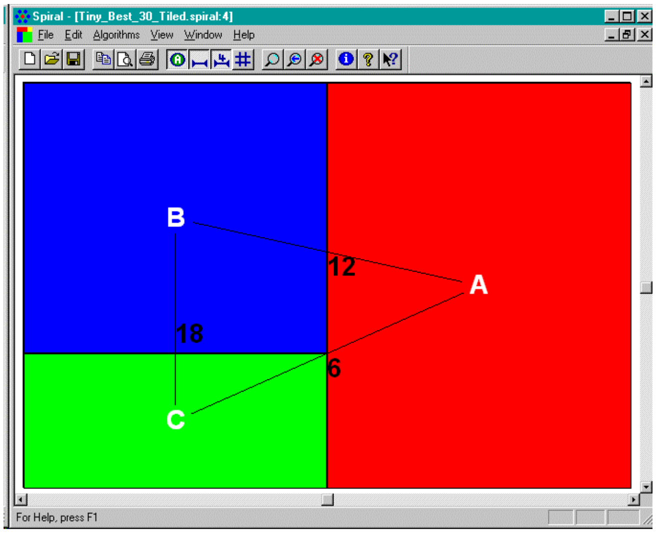

Figure 18.14. Optimal Continuous Solution for the Tiny Layout Example

Alternative Continuous Formulation

The absolute values in the objective function can also be eliminated by substitution based on the following constraint.

$$|x| = \max\{x, -x\} \tag{18.82}$$

or

$$\begin{aligned} |x| &\geq x \\ |x| &\geq -x \end{aligned} \tag{18.83}$$

or in terms of the centroid coordinates

$$\begin{aligned} 2dx_{kl} &\geq xr_k + xl_k - xr_l - xl_l \\ 2dx_{kl} &\geq -xr_k - xl_k + xr_l + xl_l \end{aligned} \tag{18.84}$$

$$\begin{aligned} 2dy_{kl} &\geq yt_k + yb_k - yt_l - yb_l \\ 2dy_{kl} &\geq -yt_k - yb_k + yt_l + yb_l \end{aligned} \tag{18.85}$$

This formulation has *N(N-1)* more constraints but *N(N-1)* fewer variables. This formulation also has difficulties with negative relationships between two departments, since there is no upper bound on the distance variable and it will grow to infinity in any optimal solution. To avoid this the lower bound constraints can be converted to equalities and two new binary variables are introduced in a manner similar to the original formulation.

Shape Penalty Based Relaxation

It is possible that there does not exist a conceptual block layout satisfying the shape constraints. We can relax the continuous and discrete formulation in the following way. The shape constraints in the original model are equivalent to

$$xr_k - xl_k \le s_k\left(yt_k - yb_k\right) \qquad k = 1..K$$
$$yt_k - yb_k \le s_k\left(xr_k - xl_k\right) \qquad k = 1..K$$

and

$$s_k \le S_k \qquad k = 1..K \qquad (18.86)$$

We can relax this last constraint set by including it in the objective function with a shape penalty p_i for each department *i*. The shape penalty is set by the user.

$$Min. \quad \sum_{k=1}^{K-1}\sum_{l=k+1}^{K} F_{kl} d_{kl} + \sum_{k=1}^{K} F_{k0} d_{k0} + \sum_{k=1}^{K} \max\left\{0, s_k - S_k\right\} \cdot p_k \qquad (18.87)$$

Observe that if the realized shape ratio of a department is less than the maximum allowed shape ratio for that department no penalty is added. If the realized shape ratio is larger than the maximum allowed value, a penalty proportional to the violation of the shape ratio is added to the objective function. This corresponds to a penalty function for each individual department as illustrated in Figure 18.15.

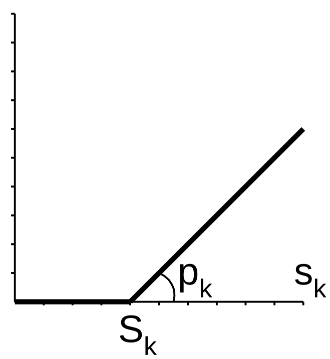

Figure 18.15. Shape Penalty Function for a Single Department

18.6. Optimal Block Layout with Shape Constraints Using Benders Decomposition

Introduction

18.7. Optimal Shape Penalties Using Lagrangean Relaxation

Introduction

In the formulation (18.87) the shape constraints were relaxed and incorporated into the objective function with a shape penalty determined by the user. This method is an example of a more general class of relaxations, where one or more constraints are relaxed and the amount of violation of the constraint is multiplied by a penalty factor and added to the objective function. This class of relaxation is called Lagrangean relaxation and the penalty factor is called the Lagrangean multiplier. Lagrangean relaxation is often used in the solution of difficult integer and mixed integer programming problems either by itself or as the algorithm used to compute the bounds in a branch-and-bound algorithm. This method is also denoted as dual decomposition, since the constraints are split into two sets, namely the relaxed and non-relaxed constraints.

We will first develop the generic Lagrangean Relaxation algorithm and then apply it to the block layout problem with shape constraints. The remainder of this section requires a thorough knowledge of linear and integer programming and may be more suitable for advanced undergraduate or graduate courses.