MINIMIZING DEFLECTION AND BENDING MOMENT IN A BEAM WITH END SUPPORTS

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Abstract

We give heuristics to sequence blocks on a beam, like books on a bookshelf, to minimize simultaneously the maximum deflection and the maximum bending moment of the beam. For a beam with simple supports at the ends, one heuristic places the blocks so that the maximum deflection is no more than $16/9\sqrt{3} \approx 1.027$ times the theoretical minimum and the maximum bending moment is within 4 times the minimum. Another heuristic allows maximum deflection up to 2.054 times the theoretical minimum but restricts the maximum bending moment to within 2 times the minimum. Similar results hold for beams with fixed supports at the ends.

Key words: combinatorial mechanics, heuristics, sequencing, beam, deflection, bending moment

1 Introduction

The limiting factor in the design of beams loaded by weights is often either the permissible deflection [6] or the bending moment [8]. For given loads, the positions at which they are placed determine the deflection and bending moment along the beam; therefore deflection and bending moment can be controlled to some extent by judicious sequencing of the loads. Unfortunately, as we shall show, it can be computationally difficult to determine the best sequence of loads on the beam; however we give fast heuristics that position the loads so that both the maximum deflection and the maximum bending moment are guaranteed to be not "too much" larger than the minimum possible. The guarantees are analytical, not experimental, and so might be useful in certifying performance of beams.

We model the problem of minimizing deflection of a beam as that of sequencing n homogeneous "blocks" (intervals) on a beam of length L so that the maximum deflection at any point along the beam is as small as possible. We make the simplifying assumption that any interaction between the blocks is negligible (as would be the case if the beam deflected only slightly or if the blocks were not very high). We use the same model for the problem of minimizing bending moment, except that the objective is to make the maximum bending moment as small as possible. The jth block is characterized by its length l_j and weight w_j. We assume that the blocks fill the beam exactly, possibly by the artifice of including some imaginary blocks of zero weight.

The beam is initially straight, with a uniform cross section of area moment of inertia I. We also make the usual assumptions of engineering design that the beam material is isotropic and homogeneous, and that it obeys Hooke's law with a modulus of elasticity E [8].

We will discuss the case in which the beam has simple supports at both ends, but our analysis also applies, with differences only in detail, when the beam has fixed supports.

The objective of minimizing the maximum deflection is *not* identical to that of minimizing the maximum bending moment. For example, consider a beam of length 1 on which are to be placed two blocks of weight $w_1 = w_2 = 1$ and length $l_1 = l_2 = 0.1$ and a third block of weight $w_3 = 1.02$ and length $l_3 = 0.2$. Then to minimize the maximum deflection one must place blocks 1 and 3 at one end of the beam and block 2 at the opposite end; but to minimize the maximum bending moment one must place blocks 1 and 2 at one end of the beam and block 3 at the opposite end. Thus the sequence that minimizes one objective can fail to minimize the other. Nevertheless, these two objectives appear to be highly coincident. Evidence of this is that each of our heuristics is guaranteed to perform "well" with respect to both objectives simultaneously—even though the point at which maximimum deflection is achieved can be distance nearly $L/2$ from the point at which maximum bending moment is achieved. Furthermore, for every result we prove about deflection, there is a similar result about bending moment that is provable by a similar argument. (Accordingly we give detailed arguments only for deflection.)

All of our heuristics work by reducing the deflection or the bending moment at the center of the beam. Fortunately, as we show, neither the maximum deflection nor maximum bending moment can be much greater than that at the center, regardless of the placement of the blocks.

2 Deflection and bending moment

Standard engineering design textbooks catalogue equations for the deflection and bending moment of a beam with different types of supports [7, 8]. For example, consider a beam of length L on which is superimposed a coordinate axis with the origin at the leftmost end as in Figure 1. If the beam has simple supports at the ends, then the deflection at any point x due to a point force of magnitude F applied at x_F is

$$D(x, F, x_F) = \begin{cases} \frac{F(L-x_F)x\left(2Lx_F - x^2 - x_F^2\right)}{6EIL}, & \text{for } x < x_F; \\ D(L-x, F, L-x_F), & \text{for } x \geq x_F, \end{cases} \tag{1}$$

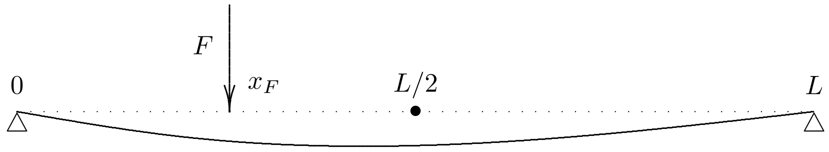

Figure 1: Application of a point force to a beam.

where the second term follows by symmetry of the beam and supports. The bending moment is

$$M(x,F,x_F) = \begin{cases} \frac{F(L-x_F)x}{L} & \text{for } x < x_F; \\ M(L-x,F,L-x_F) & \text{for } x \geq x_F. \end{cases} \tag{2}$$

(Strictly speaking we have written the negative of the deflection. We take this liberty for convenience of presentation so that we can speak of "minimizing the maximum" for both deflection or bending moment. The alternative is to speak of "maximizing the minimum" deflection and "minimizing the maximum" bending moment, with obvious opportunities for confusion.)

We can compute the deflection and bending moment due to a block of length $l \leq L$ and with homogeneous weight distribution of w/l units of weight per unit length by invoking the following.

The Principle of Superposition The deflection (bending moment) at any point in a beam subject to multiple loading is equal to the sum of the deflections (bending moments) caused by each load acting separately [8].

We use Equation 1 to calculate the deflection due to an infinitesimal section of length dl, and then, by the Principle of Superposition, we integrate that expression over the length of the block (Figure 2). Then, when the block is placed with its center at r, the deflection at any point x is

$$D(x,w,l,r) =$$
$$\begin{cases} \frac{(-L+r)wx\left(4r^2+l^2-8Lr+4x^2\right)}{24EIL}, & \text{for } x \leq r-l/2; \\ D(L-x,w,l,L-r), & \text{for } x \geq r+l/2; \\ D(x,\frac{w(x-r+l/2)}{l},x-r+l/2,\frac{x+r-l/2}{2}) & \\ \quad + D(x,\frac{w(r+l/2-x)}{l},r+l/2-x,\frac{r+l/2+x}{2}), & \text{otherwise} \end{cases}$$

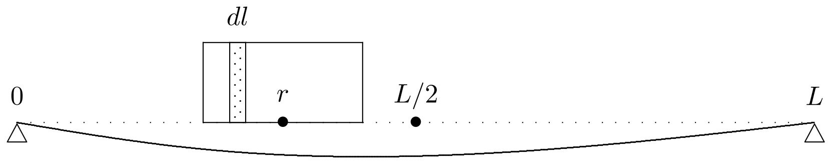

Figure 2: Deflection at the center of a beam of length L due to a homogeneous block.

where the second expression follows by symmetry of the beam and supports, and the last expression follows from the Principle of Superposition, which allows us to write the deflection due to a point underneath the block as the sum of the deflections due to the portions of the block to the right and to the left of the point. Note that we have written the deflection in such a way as to emphasize that it is a function of the weight, length, and placement of the block. Also, rather than simply writing its algebraic form, we have written the function recursively to show its structure. Finally, we use D to refer to deflection due to either point forces or blocks and rely on context to make the distinction clear.

A similar argument shows that the bending moment due to a homogeneous block is

$$M(x, w, l, r) = \begin{cases} \frac{(L-r)wx}{L}, & \text{for } x \leq r - l/2; \\ M(L-x, w, l, L-r), & \text{for } x \geq r + l/2; \\ M(x, \frac{w(x-r+l/2)}{l}, x-r+l/2, \frac{x+r-l/2}{2}) & \\ \quad + M(x, \frac{w(r+l/2-x)}{l}, r+l/2-x, \frac{r+l/2+x}{2}) & \text{otherwise.} \end{cases}$$

It is straightforward to show that, for both point forces and blocks, deflection is a concave function of x. Therefore, by the Principle of Superposition, and because sums of concave functions are concave, the deflection due to a *set* of point forces or blocks must be a concave function of x. Similarly, bending moment is a concave function. Accordingly, for a set of point forces or blocks whose positions are fixed, the point of greatest deflection or of greatest bending moment can be found by an efficient one-dimensional search procedure such as Fibonacci search [2].

3 The center of the beam

When a force is applied to a beam, the maximum deflection in the beam generally occurs elsewhere than at the point of application. Intuitively, one expects the deflection at the center of the beam to be large, though not maximal. In fact, for any placement of the blocks, neither the maximum deflection nor the maximum bending moment can exceed that at the center by much:

Theorem 1. *For a simply supported beam, the deflection at any point is at most* $16/9\sqrt{3} \approx 1.027$ *times the deflection at the center of the beam and the bending moment is at most 2 times that at the center.*

Proof. We first prove the theorem for point forces and then argue that it must hold for continuous loads as well.

Assume without loss of generality that a point force F acts to the right of the center of the beam, so that $x_F \geq L/2$. Differentiating the first part of Equation 1 with respect to x and setting the derivative to zero, we get the point of maximum deflection in the simply supported beam. Substituting back in the original equation, we get the value of the maximum deflection:

$$\max_x D(x, F, x_F) = \frac{FL^2 x_F}{9EI}\left(1 - \frac{x_F}{L}\right)\left(2 - \frac{x_F}{L}\right)\sqrt{\frac{x_F}{3L}\left(2 - \frac{x_F}{L}\right)}.$$

Taking the derivative of $\max_x D(x, F, x_F)/D(L/2, F, x_F)$ with respect to x_F, we find that this ratio is minimal at $x_F = L/2$, where it assumes the value 1. Furthermore, $\max_x D(x, F, x_F)/D(L/2, F, x_F)$ increases with x_F, so that the ratio approaches its maximum value as the point of application of F approaches the end of the beam. Evaluating this limit gives a value of $16/9\sqrt{3} \approx 1.027$.

Consider now a collection of point forces $F_1, \ldots, F_n$ acting at (possibly) different points on the beam, and let x^* be the point of maximum deflection in the beam. By the Principle of Superposition, the total deflection at the center is

$\sum_{i=1}^{n} D(L/2, F_i, x_i)$, and the maximum deflection is $\sum_{i=1}^{n} D(x^*, F_i, x_i)$. Since $\max_x D(x, F_i, x_i)$ is the maximum deflection in the beam due to F_i alone, then obviously $D(x^*, F_i, x_i) \leq \max_x D(x, F_i, x_i)$. Summing the last inequality over all the forces, we get

$$\sum_{i=1}^{n} D(x^*, F_i, x_i) \leq \sum_{i=1}^{n} \max_x D(x, F_i, x_i) \leq \frac{16}{9\sqrt{3}} \sum_{i=1}^{n} D(L/2, F_i, x_i).$$

Finally, when the beam is loaded with blocks rather than point forces, the same argument holds with summation replaced by integration.

To establish the result for bending moment, we use a similar argument, but base it on the fact that for a single force, the maximum bending moment occurs at the point of application of the force, and has a magnitude of

$$\max_x M(x, F, x_F) = F x_F \left(1 - \frac{x_F}{L}\right).$$

In this case the ratio $\max_x M(x, F, x_F)/M(L/2, F, x_F)$ approaches its maximum value of 2 as x_F approaches L. □

4 V-shaped sequences

A simple class of sequences reduces both deflection and bending moment at the center of the beam and therefore tends to reduce the maximum values along the entire beam. We call this class the *V-shaped* sequences: Each such sequence has the property that all the blocks whose centers fall on the same side of the center of the beam are arranged in non-decreasing order of average density w_i/l_i from the center towards the ends of the beam (and if the center of a block is coincident with the center of the beam, then that block must be less dense than one of its adjacent neighbors). We will prove that such sequences do not cause "too much" deflection or bending moment at the center; but first we need the following technical result.

Lemma 1. *The farther a block is from the center of the beam, the smaller is the deflection and the bending moment at the center due to that block.*

Proof. When the block is completely to the left of the center of the beam, the derivative of $D(L/2, w, l, r)$ with respect to the distance r is, by simple algebra, positive for $0 \leq r \leq L/2$. Thus decreasing r, or by symmetry increasing r, away from $L/2$ will decrease the deflection at the center.

If the block overlaps the center of the beam, we can assume without loss of generality that the center of the block is to the right of that of the beam. Then moving the block to the right a distance δ away from the center is equivalent by the Principle of Superposition to cutting an imaginary section of length δ from the left side of the block and placing at its right end. By the previous discussion, this will reduce the deflection at the center.

A similar argument establishes the result for bending moment. □

The following result says that the class of V-shaped sequences includes any sequence that minimizes deflection or bending moment at the center of the beam. This will be useful when we bound the quality of a V-shaped sequence.

Lemma 2. *Any sequence that minimizes deflection at the center or that minimizes bending moment at the center must be V-shaped.*

Proof. By a simple interchange argument any sequence that is not V-shaped can be improved: Within any sequence of blocks that is not V-shaped, there must be an adjacent pair of blocks B_i and B_j that are in strictly decreasing order of density; that is, $w_i/l_i > w_j/l_j$ and either $r_i < r_j < L/2$ or $L/2 < r_i < r_j$ (without loss of generality we assume the second case). By the Principle of Superposition, the deflection at the center due to blocks B_i and B_j is equal to that of two imaginary blocks B_u and B_v, shown in Figure 3, with $l_u = l_i + l_j$, $l_v = l_i$, $w_u/l_u = w_j/l_j$, $w_v/l_v = w_i/l_i - w_j/l_j$, $r_u = (r_i + r_j)/2$, and $r_v = r_i$. Interchanging B_i and B_j changes the deflection of the beam in exactly the same way as would keeping B_u fixed and moving B_v outward by a distance l_j. By Lemma 1, this reduces the deflection at the center.

A similar argument establishes the claim for bending moment. □

The following shows that no V-shaped sequence can cause "too much" de-

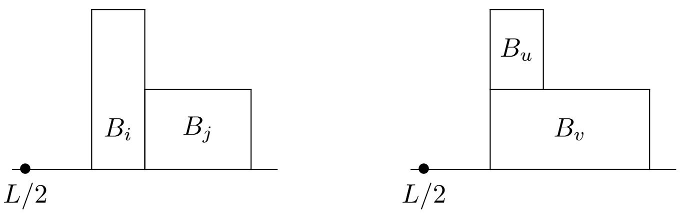

Figure 3: The deflection in the beam due to blocks i and j, with heights proportional to their densities, is the same as that caused by blocks u and v.

flection or bending moment at the center of the beam. This will form the basis of our heuristics.

Theorem 2. *For any V-shaped sequence of a given set of blocks, the deflection at the center of a beam is never more than twice the minimum possible and the bending moment is never more than twice the minimum.*

Proof. First we show that it is sufficient to consider only those cases in which all blocks are of equal length. To see this, consider a set of n blocks for which the worst V-shaped sequence produces a deflection D_1^V at the center of a given beam, and for which the optimal sequence produces a deflection D_1^* at the center of that beam. Now imagine cutting those blocks into a set of equal length pieces, using, for example, $\gcd(l_1, l_2, \ldots, l_n)$ as the common length (where gcd is the greatest common divisor function). Let D_2^V and D_2^* be the deflections at the center of the beam produced by the worst V-shaped sequence and by an optimal sequence of the new set of blocks, respectively. Since the sequence that produced D_1^V remains V-shaped when the blocks are cut, $D_1^V \leq D_2^V$; and since the imaginary blocks can be arranged more freely, $D_1^* \geq D_2^*$. This implies $D_1^V/D_1^* \leq D_2^V/D_2^*$, and the worst V-shaped sequence for the imaginary set of equal length blocks has no better performance than the worst V-shaped sequence for the original set of blocks.

Now consider a set of n blocks of equal length L/n. For convenience, assume the blocks are indexed so that $w_1 \geq \cdots \geq w_n$. Then the V-shaped sequence

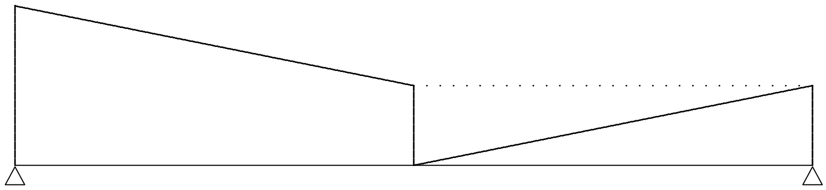

Figure 4: The distribution of mass density in the worst V-shaped sequence.

that produces the greatest deflection at the center of the beam is

$$S^V = \begin{cases} (B_1, B_2, \ldots, B_{n/2}, B_n, B_{n-1}, \ldots, B_{n/2+1}) & \text{for } n \text{ even;} \\ (B_1, B_2, \ldots, B_{(n+1)/2}, B_n, B_{n-1}, \ldots, B_{(n+3)/2}) & \text{for } n \text{ odd.} \end{cases}$$

These sequences are V-shaped, as suggested by Figure 4. That they are the worst V-shaped sequences follows by an interchange argument: In any other V-shaped sequence S of the blocks, B_1 has to be at one of the ends of the sequence S. Compare S with S^V by comparing the positions of the successive blocks B_2, $B_3, \ldots$; let B_j be the first of these blocks out of sequence in S and let B_k be the block in S in the position of B_j in S^V. Then $j < k$ so block B_k is lighter than block B_j and block B_j must be the block at the opposite end of S from B_1. In fact the blocks B_{j+1}, B_{j+2}, $\ldots$, B_{k-1} that are lighter than B_j but heavier than B_k must also be on the same side as B_j so that S has the following structure.

$$S = (B_1, B_2, \ldots, B_{j-1}, B_k, B_{k-1}, B_{k-2}, \ldots, B_{j+1}, B_j).$$

Now consider two cases.

Case 1. $j > k - j$ In this case, block B_k is closer to the center of the sequence S than any of the blocks B_{k-1}, B_{k-2}, $\ldots$, B_j. If we remove from each block B_{k-i}, $i = 1, 2, \ldots, k - j$ weight $B_{k-i} - B_{k-i+1}$ and place it on block B_k, we create the V-shaped sequence:

$$S' = (B_1, B_2, \ldots, B_{j-1}, B_j, B_k, B_{k-1}, \ldots, B_{j+1}),$$

and since all the weight has been moved closer to the center, S' produces a larger deflection at the center than S according to Lemma 1.

Case 2. $j \le k - j$ In this case, block B_k is farther from the center of the sequence S than some of the blocks B_{k-1}, B_{k-2}, $\ldots$, B_j. We exchange

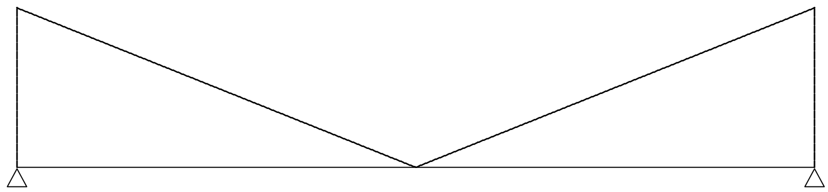

Figure 5: The distribution of mass density after the blocks have been split in two and arranged in an optimal sequence.

blocks B_1, B_2, …, B_{j-1} with blocks B_j, B_{j+1}, …, B_{2j-2} to create the V-shaped sequence:

$S' = (B_j, B_{j+1}, \ldots, B_{2j-2}, B_k, B_{k-1}, B_{k-2}, \ldots, B_{2j-1}, B_{j-1}, B_{j-2}, \ldots, B_1)$

that produces the same deflection at the center as S. Now, B_j is the first block that is out of sequence in S' and block B_{2j-1} is the block that is in its place. So, we may apply the arguments of Case 1 to the sequence S' to create a V-shaped sequence S^* that produces a larger deflection at the center than S' and hence also larger than S.

Now the problem of bounding the performance of a V-shaped sequence is reduced to that of bounding the ratio of the deflection at the center of the beam due to sequence S^V to that due to sequence S^*. In sequence S^V, block j weighs w_j and is distance $r_j = (2j-1)L/(2n)$ for $j \leq n/2$ and $r_j = (n+2j-1)L/(2n)$ for $j > n/2$ from the center of the beam. Substituting these values in the expression for deflection gives a linear form in the w_j, which we write as $\sum_{j=1}^{n} b_j w_j$.

Rather than evaluate the deflection for the optimum sequence S^*, it is convenient to use the lower bound on deflection that we get by splitting each block into two equal parts, indexed as j and $n+j$, and placing the pieces symmetrically about the center in a V-shape as suggested by Figure 5. Blocks j and $n+j$ each weigh $w_j/2$ and are centered at $r_j = (2j-1)L/(4n)$ and $r_{n+j} = (4n-2j+1)L/(2n)$. Substituting these values in the expression for deflection gives a linear form in the w_j, which we write as $\sum_{j=1}^{n} c_j w_j$. The ratio R of the deflection at the center of the beam due to sequence S^V to that due to sequence S^* has the form

$$R = \frac{\sum_{j=1}^{n} b_j w_j}{\sum_{j=1}^{n} c_j w_j} \leq \max_j \frac{b_j}{c_j},$$

and by tedious but simple algebra

$$\max_j \frac{b_j}{c_j} \leq \frac{12n^2 - 16j^2 + 16j - 8}{6n^2 - 2j^2 + 2j - 1}$$

Taking the derivative of the last expression with respect to j, we see that it is decreasing in j for $j \geq 1$, and therefore is largest at $j = 1$ where

$$R \leq \frac{12n^2 - 8}{6n^2 - 1} \leq 2 \text{ for } n \geq 1.$$

A similar argument establishes the bound for bending moment. □

The bounds of 2 on the performance of arbitrary V-shaped sequences are tight as can be seen from the following example: Assume $L = 1$ and consider three blocks, of lengths $l_1 = l_2 = l < 1/2, l_3 = 1 - 2l$ and of weights $w_1 = w_2 = 1, w_3 = 0$. Then the ratios of deflection and bending moment at the center due to the V-shaped sequence (B_1, B_2, B_3) to those of the sequence (B_1, B_3, B_2) approach 2 as l approaches 0.

Invoking Theorem 1, we have the following guarantee of the quality of any V-shaped sequence.

Corollary 1. *Any V-shaped sequence produces a maximum deflection no more than* $32/9\sqrt{3} \approx 2.054$ *times the theoretical minimum and a maximum bending moment no more than 4 times the theoretical minimum.*

Any algorithm that generates V-shaped sequences will inherit the corresponding performance guarantees. There are several natural, simple algorithms to generate V-shaped sequences. Among the more interesting is to sort the blocks by density and then iteratively place the next densest block as far as possible from the center of the beam. This requires $O(n \log n)$ effort due to the sorting. While we have proved only the bounds 2.054 and 4, we suspect this heuristic in fact has a stronger performance guarantee. We conjecture that any sequence of blocks constructed by this heuristic produces deflection and bending moment at the center that is no more than 5/4 times minimum. This would mean that the maximum deflection would be no more than $20/9\sqrt{3} \approx 1.283$ and the maximum bending moment would be no more than 5/2 times optimum.

5 Minimizing deflection at the center

A V-shaped sequence has the advantage of being easy to compute, but it does not have a strong guarantee of quality because it only reduces deflection and bending moment at the center of the beam; it does not minimize either of them. With more effort we can compute sequences that *exactly* minimize the deflection or the bending moment at the center of the beam. Such an algorithm will capitalize more effectively on the bound of Theorem 1.

Our solution is via a dynamic programming recursion based on Lemma 2. For convenience we assume that the lengths of the beam and of the blocks are integral. We begin by sorting the blocks and relabelling them in non-decreasing order of average density, so that $w_1/l_1 \leq \cdots \leq w_n/l_n$. Now let $D_j^*(x)$ denote the minimum deflection at the center of the beam due to blocks $1, \ldots, j$ placed completely within the intervals $[0, x]$ and $\left[x + \sum_{i=j+1}^{n} l_i, L\right]$. Initially

$$D_j^*(x) = \begin{cases} 0 & \text{for } x = 0, j = 0; \\ \infty & \text{otherwise} \end{cases}$$

and the recursion for $j = 1$ to n is given by

$$D_j^*(x) = \min \begin{cases} D_{j-1}^*(x - l_j) + D(L/2, w_j, l_j, x - l_j/2) \\ D_{j-1}^*(x) + D(L/2, w_j, l_j, x + \sum_{i=j}^{n} l_j - l_j/2). \end{cases}$$

The optimal solution is then the sequence of blocks that minimizes $D_n^*(x)$ for $0 \leq x \leq L$.

A similar dynamic programming recursion determines a sequence of blocks that exactly minimizes the bending moment at the center of the beam, with $D_j^*(x)$ replaced by $M_j^*(x)$, the minimum bending moment at the center due to blocks $1, \ldots, j$; and with $D(L/2, w_j, l_j, r_j)$ replaced by $M(L/2, w_j, l_j, r_j)$. The optimal solution is the sequence of blocks that minimizes $M_n^*(x)$ for $0 \leq x \leq L$.

Theorem 3. *The dynamic programming recursions determine sequences of the n blocks that minimize deflection or bending moment at the center of a beam of length L. Furthermore, the recursions can be evaluated within $O(nL)$ steps.*

Proof. Each block j corresponds to a stage in the dynamic programming recursion. The state of the process is given by the variable x $(0 \leq x \leq L)$. For any value of x, we consider two possible decisions: either place block B_j on the left side (in the interval $[x - l_j, x]$) or on the right side (in the interval $[x + (\sum_{i=j+1}^{n} l_i), x + (\sum_{i=j}^{n} l_i)]$). This determines the $O(nL)$ time complexity, and it dominates the $O(n \log n)$ time required to sort the blocks initially. The correctness of the recursions follow from a straightforward application of the Principle of Optimality [2]. □

By Lemma 2 any sequence that minimizes one of the criteria at the center of the beam must be V-shaped and therefore cannot be "too bad" with respect to the other criteria. Therefore, by Theorems 1 and 2 we have the following.

Corollary 2. *Dynamic programming to minimize deflection at the center gives a sequence that produces maximum deflection no greater than* $16/9\sqrt{3} \approx 1.027$ *times the theoretical minimum and bending moment no greater than 4 times minimum.*

Corollary 3. *Dynamic programming to minimize bending moment at the center gives a sequence that produces maximum deflection no greater than* $32/9\sqrt{3} \approx 2.054$ *times the theoretical minimum and bending moment no greater than 2 times minimum.*

To evaluate the recursion technically requires *pseudo-polynomial* time because the number of computational steps is a polynomial function of the length L of the beam (rather than a binary encoding of L). In practice, when the block lengths are not integral, the problem is converted to a discrete state space problem by choosing an appropriate scale. For example, one can measure all lengths in units of size $\gcd(l_1, l_2, \ldots, l_n)$ and the actual time complexity of the dynamic program is $O(nL/\gcd(l_1, l_2, \ldots, l_n))$.

In the special case in which all blocks have equal length, then any two blocks of a sequence can be interchanged without changing the position of any other block. This allows a fully polynomial-time algorithm,with worst-case running time independent of L. A simple interchange argument establishes that the deflection and the bending moment at the center of the beam are both minimized by sorting the blocks into non-decreasing order of weight and repeatedly placing the next heaviest block as far from the center of the beam as possible. This means that the maximum deflection will be within $16/9\sqrt{3} \approx 1.027$ times the theoretical minimum and the maximum deflection will be within 2 times minimum. This heuristic requires only $O(n \log n)$ time (for sorting the blocks).

6 Complexity

The following result shows that it is unlikely that either deflection or bending moment can be minimized at the center of the beam any more quickly (in the worst case) than in pseudo-polynomial time.

Theorem 4. *The problems of minimizing the deflection and of minimizing the bending moment at the center of a beam with homogeneous discrete loads are NP-hard.*

Proof. We will show that the decision problem for deflection is NP-complete. The Deflection Problem can be stated as follows: Given a set of homogeneous blocks, a beam, and a threshold value D_0, is there a sequence of the blocks that causes a deflection of at most D_0 at the center of the beam? The reduction is from the Partition Problem, which is known to be NP-complete [3]. An instance of the Partition Problem is given by a set of indices $J = 1, 2, \ldots, n$ and a set of positive integers $\{l_j\}_{j \in J}$; the question is whether there exists a partition J_1, J_2 such that $\sum_{j \in J_1} l_j = \sum_{j \in J_2} l_j$. Given such an instance, create an instance of the Deflection Problem as follows. There are n blocks, with block j of length l_j and weight l_j, and there is an additional block of length $l_{n+1} = 1$ and weight 0. The beam is of length $L = \sum_{j=1}^{n+1} l_j$; the values of E and I are irrelevant, so we take $EI = 1$ for simplicity. The threshold value is $D_0 = (L-1)(5L^3 - 3L^2 - 3L + 1)/384$. The value of D_0 is equal to the deflection at the center of the beam due to 2 homogenous blocks of length $(L-1)/2$ each

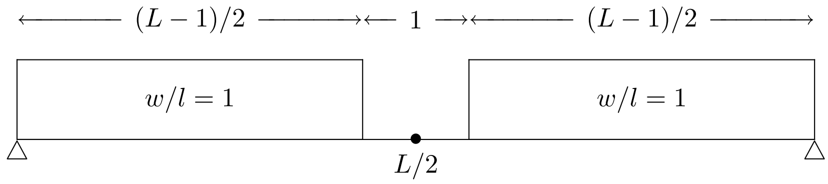

Figure 6: Partition Problem recast as a deflection problem

and of density $w/l = 1$ placed on each end of the beam as in Figure 6. If there exists a partition J_1, J_2 of the set of indices, then the answer to the Deflection Problem is affirmative: it suffices to place all the blocks corresponding to J_1 next to each other on one end of the beam, and those corresponding to J_2 on the other end. The resulting placement is equivalent to that of Figure 6, and the deflection at the center is exactly D_0. If the answer to the Deflection Problem is affirmative, the deflection at the center of the beam will necessarily be exactly D_0 and the blocks would have to be placed as in Figure 6, with block $n + 1$ exactly at the center of the beam. To see this, we consider any other placement of the blocks in which block $n+1$ has its center offset from the center of the beam by a distance δ and the other blocks are fitted in the intervals $[0, (L - 1)/2 + \delta]$ and $[(L + 1)/2 + \delta, L]$. We can compute the total deflection as if due to three homogenous blocks corresponding to the regions of equal density. By simple algebra the deflection at the center is larger than D_0.

A similar reduction from the Partition Problem establishes the formal difficulty of minimizing bending moment. In this case we ask whether there is a sequence of blocks with bending moment no greater than $(L - 1)^2/4$. □

Notice that this leaves open the question of whether maximum deflection or maximum bending moment can be exactly minimized in pseudo-polynomial time or whether these problems are "strongly" NP-hard [3]. The first alternative would seem more likely if there is always an optimal sequence that is V-shaped about some point (possibly not the center); however, we do not know whether this is true.

7 Related work

One-dimensional problems of sequencing blocks have the same flavor as problems of machine scheduling, only with an objective that is determined by physical law rather than economics. For example, the special problems of minimizing the deflection and bending moment at the center of a beam are similar to that of scheduling n jobs on a single machine to minimize weighted absolute deviation from a restrictive or small common due date [4, 5]. In the latter problem, earliness costs are assessed against all jobs completed before the common due date, and tardiness costs are assessed against all those completed after it. The problem is to minimize the sum of these costs. The problems are analogous, with the 1-dimensional beam corresponding to the line of time, the center of the beam corresponding to the common due date, and deflection or bending moment corresponding to earliness/tardiness costs. The lengths and weights of the blocks correspond to the processing times and the economic weights of the jobs, respectively. Lemma 2 establishes what Hall, Kubiak, and Sethi (1991) refer to as the "weakly V-shaped property" of an optimal schedule [4]; and the dynamic programming algorithm is a modification of the one presented by Hoogeveen and van de Velde [5]. The difference between the problems is that for the "earliness/tardiness" problem, an optimal schedule need not start at time 0, while for the deflection and the bending moment problems, the blocks are confined to the interval $[0, \sum_{i=1}^{n} l_i]$.

8 Conclusions

We have suggested three heuristics to reduce maximum deflection and maximum bending moment in a beam. These heuristics do not exactly minimize either deflection or bending moment; but each heuristic has a performance guarantee that says that neither deflection nor bending moment can be "too much" larger than the minimum possible. Furthermore, the stronger the guarantee for one objective, the weaker the guarantee for the other, as summarized in Table 1.

Heuristic	*Error bound: max deflection*	*Error bound: max bending moment*	*Effort*
Arbitrary sequence	∞	∞	$O(1)$
V-shaped sequence	2.054	4	$O(n \log n)$
DP-deflection	1.017	4	$O(nL)$
DP-bending moment	2.054	2	$O(nL)$

Table 1: Comparison of performance guarantees and computational effort for three heuristics. The error bound is the largest possible ratio of the maximum deflection (bending moment) to the smallest maximum deflection (bending moment) possible.

To put these guarantees in perspective, note that for arbitrary sequences of blocks there is no finite upper bound on the ratio of maximum deflection to the minimum possible nor on the ratio of maximum bending moment to the minimum possible. To see this, compare the sequences (B_1, B_3, B_2) and (B_1, B_2, B_3), where B_1 and B_2 are both of length $(L - \epsilon)/2$ and weight ϵ, and B_3 is of length ϵ and weight 1. As ϵ approaches 0, the ratios of maximum deflections and of maximum bending moments become arbitrarily large.

We have given detailed analysis for the case of a beam with simple supports; however our arguments apply when the beam has fixed supports at both ends. Using the appropriately modified equations of deflection and bending moment, we can show that for a beam with fixed supports at the ends, the deflection at any point is at most $32/27 \approx 1.185$ times the deflection at the center of the beam and the maximum bending moment is at most 4 times that at the center; furthermore, these bounds are tight. Our previous analysis can be continued to show that any V-shaped sequence causes deflection at most $64/27 \approx 2.37$ times the theoretical minimum and bending moment at most 8 times the minimum. Similarly, the dynamic program to minimize exactly deflection at the center gives a sequence that causes deflection no more than $32/27 \approx 1.185$ times minimum; and the dynamic program to minimize exactly bending moment at the center gives a sequence that causes deflection no more than 4 times minimum.

It is worth remarking that an easily-solved special case with fixed supports is the loading of a cantilever beam: The maximum deflection always occurs at the free end of the beam, and the maximum bending moment at its fixed end. A proof similar to that of Lemma 2 allows us to establish that the maximum deflection and the maximum bending moment of a cantilever beam are both minimized by sorting the blocks in non-decreasing order of average density and then repeatedly placing the next densest block as far from the free end as possible. This requires only $O(n \log n)$ time, again for sorting the blocks.

The performance guarantees for our heuristics are weaker for the problem of bending moment than for the problem of deflection, which suggests that the problem of bending moment is in some sense more difficult. Unfortunately the problem of bending moment is also probably the more keenly felt as a practical problem. It would be useful as well as interesting to design heuristics with improved performance guarantees for bending moment.

We have only considered the case of homogeneous blocks, for which deflection and bending moment are each minimized at the center of the beam by some sequence that has a V-shaped profile in the weight per unit length of the blocks. For non-homogeneous blocks, the V-shape property does not hold, and no special structure of the optimal solution is apparent. It is possible to use the same heuristics to sequence a set of imaginary homogeneous blocks of the same weights and lengths as the real blocks, then sequence the actual blocks in the same way and orient them such that each block has its center of gravity farther from the center of the beam. The worst-case performance of this procedure is not known to the authors.

We have not considered other interesting structures such as beams with differing end supports (for example, one simple and one fixed) or beams whose supports are not at their ends. Also of interest are the 2-dimensional versions of the problems, where it is desired to find an arrangement of blocks that minimizes the deflection or the bending moment of an elastic plate.

The problems of minimizing deflection and bending moment in a beam are examples of a more general class of problems that asks how a load should be distributed on a given structure. This is complementary to the traditional question of mechanical design, which asks for the structure to bear a given load.

Elsewhere we have suggested the name "combinatorial mechanics" for this apparently new class of problems [1].

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