# DYNAMICS OF TWO- AND THREE-WORKER "BUCKET BRIGADE" PRODUCTION LINES

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We describe all possible asymptotic behavior of "bucket brigade" production lines with two or three workers, each characterized by a constant work velocity. The results suggest wariness in interpreting simulation results. They also suggest a strategy for partitioning a workforce into effective teams to staff the lines.

*Bucket brigade* production is a way of organizing workers on a flow line, in which there are fewer workers than stations. Each worker carries a single item from station to station, waiting if necessary for that station to become available (workers are not allowed to pass each other). When the last worker completes an item, he walks back to take over the item of his predecessor, who relinquishes it and walks back to take over the item of *his* predecessor, and so on until the first worker walks back to start a new item.

Bucket brigades are in use in at least two commercial environments: apparel manufacturing (Bartholdi and Eisenstein 1996) and distribution warehousing (Bartholdi et al. 1999). In both environments, two- and three-worker teams are common.

Figure 1 summarizes all asymptotic (stable) behavior of a bucket brigade flow line with three workers. By "stable behavior" we mean qualitative structure that persists, even in the presence of perturbations. This is the behavior that will assert itself in practice. Most of the behavior we characterize is distinctive and can be easily recognized on the shop floor.

This categorization of behavior is based on the model of Theorem 3 of Bartholdi and Eisenstein (1996) in which the work to assemble an item is deterministic and is spread continuously and uniformly over a line (rather than concentrated at work stations).

This model has several important properties.

- It is simple enough to analyze.
- The behavior of this model underlies natural generalizations, such as when the amount of work within an interval of space is random and independent from that within disjoint intervals. In such cases the dynamics due to random work are merely superimposed upon the deterministic dynamics. (See, for example, Bartholdi et al. 1999.)
- It is normative: Most implementations of bucket brigades explicitly try to engineer the process to emulate this model because it reduces the chances of blocking.

It is also worth mentioning that the restriction to three or fewer workers is not severe. About half of the commercial lines we have seen are based on three-worker teams. One company, Riverside Fashions, Inc. has *only* three-worker teams.

## 1. THE DYNAMICS FUNCTION

Figure 1 classifies all three-worker lines based on the relative velocities of the workers: Letting $r_i = v_i/v_3$ be the ratio of the velocity of the $i$th worker to that of the third worker, any team of three workers on a bucket brigade flow line then corresponds to a point $(r_1, r_2)$ within Figure 1.

The dynamics of a three-worker line arise as follows. Let $x_i$ be the position of worker $i$ immediately after walkback, which we assume to be instantaneous. Then the time between completion of the $t$th and $(t + 1)$st items is $(1 - x_3^{(t)})/v_3$; and during that time each of the first and second workers can proceed no farther than allowed by their respective velocities or by their successors, whom they may not pass. This means that the dynamics function $\mathbf{x}^{(t+1)} = f(\mathbf{x}^{(t)})$ is piecewise-linear, where the exact form of $f$ depends on which worker, if any, will catch up with and be blocked by his successor during production of the next item. (See Devaney 1989 for an introduction to dynamical systems.)

The method of analysis is straightforward. To study cycles of length $j$, we enumerated all possible ways of composing $j$ of the various forms of the dynamics function; for each $j$-fold composition we solved simultaneous equations to find all points that were fixed with respect to the composition; from each such point, we generated the points of the corresponding $j$-cycle; and then we checked feasibility

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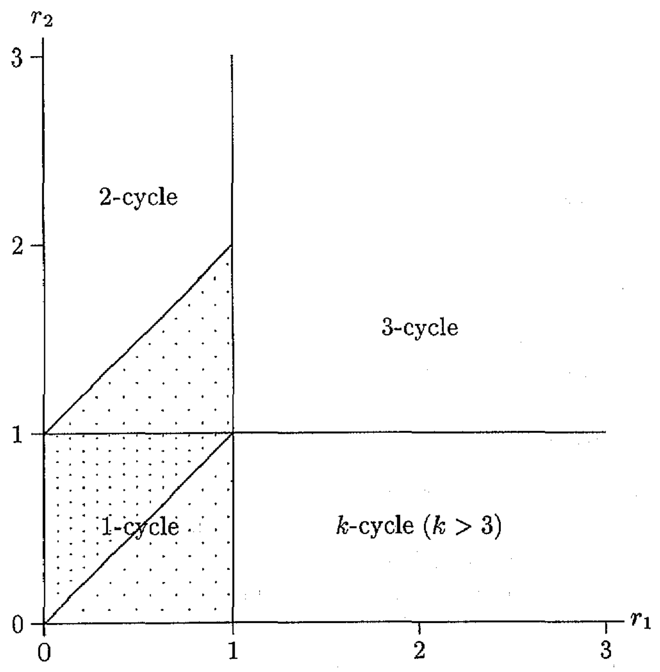


**Figure 1.** All possible asymptotic behavior of a three-worker bucket brigade production line. Behavior in any instance is determined by the ratios $r_i = v_i/v_3$ of the velocities of each worker to that of the last worker.

constraints on $r_1$ and $r_2$ to see where the points of the $j$-cycle were well-defined. This revealed fixed, but not necessarily asymptotic, behavior. Finally we identified the asymptotic behavior by computing the eigenvalues of each $j$-fold composition of the dynamics function. Because the number of forms assumed by the $j$-fold composition of the dynamics function increases as $4^j$, we carried out this program for cycles only up to $j = 6$ in length.

All of the analysis supporting Figure 1 is based on enormous amounts of straightforward algebra, most of which was performed using the software package *Mathematica* (Wolfram Research Inc. 1991). To conserve space and the patience of readers we omit these computational details but will send them to the curious on request.

## 2. REGION 1

For any bucket brigade line within Region 1 (shaded), the movements of the workers spontaneously converge to a fixed point corresponding to a perfectly balanced line and optimal production rate ($\Sigma_{i=1}^{n} v_i$ items per unit time).

Bartholdi and Eisenstein (1996) proved this self-balancing behavior for $n$-worker lines when $r_1 < \cdots < r_n$, which corresponds to the most heavily shaded subregion (that is, the central triangle) of Region 1. A new discovery is that for three-worker lines, self-balancing behavior asserts itself *anywhere* within Region 1. Note that for three-worker lines, it is sufficient for the last worker to be fastest for convergence to a fixed point. (This is not true, however, for lines of more than three workers, as examples to the contrary are easily found.)

There is, however, an important sense in which the most heavily shaded subregion of Region 1 (workers sequenced slowest-to-fastest) remains strictly preferable: The *rate of stability* (1 minus the modulus of the largest eigenvalue of the fixed point) is strictly larger when $r_1 < r_2 < r_3$ than it is anywhere else within Region 1. This means that when the workers are sequenced from slowest-to-fastest, the system can withstand perturbations of greater amplitude without changing qualitative behavior.

## 3. REGIONS 2 AND 3

When workers are sequenced other than slowest-to-fastest, then faster workers tend to catch up to and be blocked by slower workers. For example, in Region 2 the second worker is faster than the first and third workers together, and eventually he is repeatedly slowed by the third worker. As a result, the positions of the workers after walkbacks eventually alternates between $(0, r_1/(r_1 + r_3), 1)$ and $(0, 0, r_1/(r_1 + r_3))$, with suboptimal production rate $2(v_1 + v_3)$.

Similarly, in Region 3 the first two workers are both faster than the third and eventually both are repeatedly slowed by him. The positions of the workers after walkbacks eventually alternates between (0, 1, 1), (0, 0, 1), and (0, 0, 0) and the production rate collapses to $3v_3$, so that everyone is reduced to the velocity of the slowest worker.

## 4. REGION *K*

We cannot succinctly characterize asymptotic behavior of Region $k$, which is where the first worker is fastest and the second is slowest. The complication is that, unlike the other regions, the asymptotic behavior can depend not only on the value of $(r_1, r_2)$ but also on the initial positions of the workers. Consequently, systems with $(r_1, r_2)$ in Region $k$ display the most interesting and complex dynamics. Systems in this region converge to $k$-cycles for some value of $k > 3$. Through simulation we found some cycles of length greater than 30,000; and some simulations were halted after several days of searching without having detected a cycle.

The typical area for which the asymptotic behavior is a $k$-cycle is a contiguous band such as shown in Figure 2 for $k = 4, 5, 6$. Note that the bands can overlap. In such overlapping areas there are multiple attracting $k$-cycles and eventual behavior depends on the initial positions of the workers.

An interesting but difficult mathematical question is to describe the fine structure of Region $k$ between the bands. (The challenge is similar to that of describing the Mandelbrot set.) To get a glimpse of this structure we simulated a system with values of $r_1$ ranging from 1.2 to 10.0 in increments of 0.1 along the ray corresponding to $r_2 = 0.75$ and observed limit cycles of lengths 10, 20345, 24, 7, 7, 7, 25, 65, 18761, 211, 15, 19, 323, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 155, 95, 2919, 495, 156, 17, 17, 628, 142, 13, 13, 13, 13, 13, 35, 22, 31, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 32, 23, 60, 544, 14, 14, 14, 14, 14, 14, 14,

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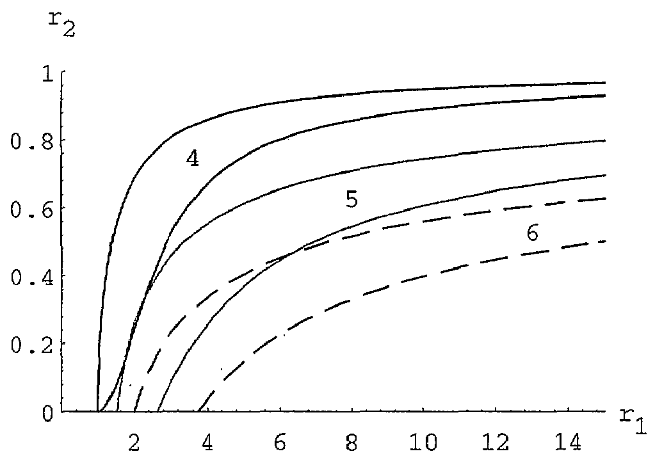


**Figure 2.** Region $k$ contains bands within which there exist stable cycles of apparently any length greater than 3.

75, 33, 19, 19, 19, respectively. The $k = 4$ band shown in Figure 2 is evident, as are what seem to be finer bands corresponding to longer cycles.

Interestingly, as $r_1$ gets large, each band of Figure 2 tends toward the $r_2 = 1$ boundary of Region $k$, so there are many attractors within a small region. Therefore, when $v_1 \gg v_3 \approx v_2$, a small perturbation in velocities can result in different qualitative behavior and so there is persistent uncertainty about how the system will evolve.

## 5. TWO WORKERS

Figure 1 also contains a description of all possible behavior of two-worker lines. We can imagine a two-worker line to be derived from a three-worker line by restricting the velocity of the first worker to be $v_1 = 0$. Such lines are described by the ray $(0, r_2)$, from which we conclude that only two modes of asymptotic behavior are available to them: a 1-cycle of optimal production rate; or a 2-cycle of suboptimal production rate twice that of the slower worker.

## 6. MORE THAN THREE WORKERS

We do not know useful conditions that are both necessary and sufficient for a general $n$-worker bucket brigade to balance itself (converge to a 1-cycle), but we have the following necessary condition: that the last worker must be faster than the first worker. Even this result is helpful because we frequently found lines in the apparel industry configured in violation of this, apparently because of a lingering fondness for the notion that work must be introduced quickly into the line.

**Lemma 1.** *For the line to balance itself it is necessary that* $r_1 < r_n$, *or equivalently,* $v_1 < v_n$.

**Proof.** For a bucket brigade production line to balance itself it is necessary that all eigenvalues of its dynamics function be of modulus no greater than 1 (see, for example, Mirsky 1990). Letting the matrix $A$ denote the dynamics function near the fixed point, the eigenvalues are the zeroes of

$$p(\lambda) = \det[\lambda I - A]$$
$$= \lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0,$$

for some set of coefficients $b_0, b_1, \ldots, b_{n-1}$. Letting $\lambda_i$ be the solutions to $p(\lambda)$ we have,

$$p(0) = b_0 = \det[-A] = -r_1 = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n.$$

Therefore, if $r_1 > 1$, then at least one of $|\lambda_i| > 1$ and the line fails to balance itself. □

## 7. CONCLUSIONS

All the types of behavior shown in Figure 1 are possible in practice: We measured worker speeds and found differences of a factor of 3 within each of several commercial sites. This suggests values of $r_1$ and $r_2$ lying within the interval from 1/3 to 3, which includes essentially all of Figure 1. Indeed, factory managers reported having seen all the types of asymptotic behavior we describe.

Figure 1 suggests wariness in interpreting some recent simulation results. Others have modeled three-worker bucket brigade lines by assuming that processing times are random and all workers are of identical velocity (Bischak 1996, Schroer et al. 1991, Zavadlav et al. 1996), which is roughly equivalent to a system in which the values of all $r_i$ begin at 1 and then change randomly over time. Figure 1 shows that such a system is poised at the cusp of several quite different asymptotic behaviors and will presumably wander among them. A system moving from one region to another of Figure 1 will experience an explosive bifurcation as the geometry of its asymptotic set changes suddenly.

Furthermore, it can be hard to tell how much of observed behavior is due to "real" randomness, such as in processing times, and how much is due to the dynamics, which, as in Region $k$, might be hard to distinguish from randomness.

Finally, our analysis suggests that bucket brigades work better when composed of workers of a wide spectrum of velocities, sequenced within each team from slowest to fastest. If a workforce is partitioned into teams in this way, then each production line will lie within Region 1 and so will achieve the maximum production rate. Furthermore, the greater rate of stability means that asymptotic behavior will assert itself more quickly and will be more resistant to disruption.

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