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SIGNAL-TO-NOISE RATIO AND RELATED MEASURES IN PARAMETER DESIGN OPTIMIZATION: AN OVERVIEW

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SUMMARY. Robust parameter design works by identifying factor settings to reduce variation in products or processes. One of its key elements is the use of the Signal-to-Noise (SN) ratio for parameter design optimization, which has stirred some controversies in the past. In this article, modeling and analysis of the SN ratio and related measures are considered along with their validity under various models. Results show that the performance of the SN ratio is very much model-dependent and its validity deteriorates as the true model deviates from the assumed model. Use of the log sample variance is less model-dependent.

1. Introduction and Review of SN Ratio

Robust parameter design is one of the most creative and effective tools in quality engineering. It had been practiced in Japan for many years before it was introduced to the US by its originator G. Taguchi in the mid-1980's. Since then it has been widely adopted in industries as evidenced by many industrial case studies. It has also attracted the attention of researchers in statistics as well as in industrial engineering. One of the central ideas in Taguchi's (1991) approach to parameter design is the use of the Signal-to-Noise(SN) ratio for variation reduction and parameter design optimization. The SN ratio has generated a lot of controversies as seen by the discussions on Box's (1988) paper and the panel discussions edited by Nair (1992). The main purpose of this paper is to provide a systematic and objective study of the performance of the SN ratio and related measures in a variety of situations represented by a wide spectrum of statistical models.

In parameter design, factors are divided into two types: *control factors* x, which are easily controllable by the experimenter and *noise factors* z, which are difficult or expensive to control during manufacturing or operation. In this paper, the "static" (i.e., simple response) problem is considered, where there is a single characteristic of interest, y, and a particular value of y is considered optimal. The variation in z

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is transmitted to the output y through f. More specifically, if z represents the "identified noise factors," i.e., noise factors which are included and varied systematically in the experiment and δ represent the "unidentified noise factors," then $y = f(x; z, \delta)$.

In most cases, $f(\cdot)$ is unknown and statistical methods have to be used to implement parameter designs and analyze the results. We consider the *nominal-the-best problem*, where the ideal value or "target" for the response is fixed and denoted by t. A loss is incurred if the output deviates from t and, as recommended by Taguchi, that loss is measured using the quadratic loss function $L(y,t) = K(y-t)^2$, where K is some constant. The ultimate objective is to identify levels of the control factors which will, on average, keep the response on target while making it insensitive to the variation in the noise factors, i.e., to choose control factor levels to minimize the expected loss

$$R(x) = E_z L(y,t) = KE(y-t)^2 = K[\sigma^2 + (\mu - t)^2], \qquad \mu = E(y)$$

which is a function of both the variance and the deviation of the mean from target. Taguchi (1991) recommends the following strategy for parameter design optimization. The set-up of the experiment is a cross array (also called inner-outer array), where the control factors are varied according to an orthogonal array called "control array" (or "inner array") and for each setting of the control factors, the noise factors are varied systematically according to an orthogonal array called "noise array" (or "outer array"). The noise array provides "replications" for each control setting. These "replications" are used to calculate the mean, $\bar{y}$ and the standard deviation, s for each control setting. For a static problem with positive continuous output y, the SN ratio proposed by Taguchi is,

$$\mathrm{SN} = \log \frac{\mu^2}{\sigma^2}, \tag{1}$$

and estimates are obtained as $\log \bar{y}^2/s^2$. The purpose of the SN ratio is to estimate the effect of the noise factors on the response and to minimize that effect. The estimated SN ratios are analyzed to determine which control factors have an effect. The control factors are then divided into:

a: *adjustment factors*: factors which affect the mean but not (or barely) the SN ratio. These may be known in advance or identified from data analysis.

d: *non-adjustment factors*: factors which affect the variability as measured by the SN ratio.

Factors that influence neither the mean nor the SN ratio could be used to reduce cost, but are of no relevance in the parameter design problem itself.

Once the sets of adjustment and non-adjustment factors have been identified, a two-step procedure is then performed to find the optimal settings of the control factors, $x = (a, d)$, which minimize the average loss $R(x)$:

Step 1: Select the levels of d to maximize the SN ratio while ignoring the mean.

Step 2: Select the levels of a to adjust the mean on target with the levels of d fixed as in Step 1.

Some reasons and advantages given in the literature for using the SN ratio as opposed to working with the standard deviation, the variance or the loss are as follows (see Phadke, 1989):

- Justified in electric circuit theory where it was first used.

- The standard deviation decreases or increases when the mean decreases or increases so the 2-step optimization procedure cannot be done in terms of the standard deviation.

- It evaluates quality in terms of decibels(dB) and serves as an ideal characteristic for measuring quality improvements in engineering terms.

- Interactions among control factors due to the presence of the squared deviation of the mean from targets are eliminated when using the SN ratio.

- Optimum settings remain valid for different target, except for adjustment of the mean.

Based on the research findings in the literature, there are limited situations in which use of SN ratio is justified or seems to work:

- When the variance is proportional to the square of the mean, i.e., $\sigma_y^2 = \mu^2\phi$, (Nair and Pregibon, 1986; León, Shoemaker and Kackar, 1987; Box, 1988), y follows a multiplicative model $y = \mu\epsilon$. Since the mean is required to be close to target, reduction of variation in the response is done by reducing ϕ through an appropriate selection of the levels of the non-adjustment factors d. The SN ratio is justifiable since in this situation, μ^2/σ_y^2 (or the SN ratio) does not depend on the adjustment factors.

- When the variation in the $\log s^2$ component is much larger than the variation in the $\log \bar{y}^2$ component, the SN ratio is dominated by the variation in $\log s^2$. Therefore an analysis of the SN ratio essentially reduces to an analysis of $\log s^2$.

- They work for the special case of linear input-output systems, where the standard deviation increases with mean and for each additional unit increase in the mean there is a corresponding increase in the standard deviation, e.g., volume control of a radio.

Criticisms of the SN ratio abound (Nair, 1992; Box, 1988; Hunter, 1985):

- In many systems the standard deviation is not linearly proportional to the mean (e.g., Nair and Pregibon, 1986) and there is no guarantee that a factor will have a multiplicative effect on y.

- It fails to use all of the information and might hide important relationships in the data since the original data are compressed into one or two summarizing quantities (SN ratio and $\bar{y}$). Some information in the data is being discarded as the SN ratio is not an efficient measure (Box, 1988).

- A non-monotonic and many-to-one transformation of the data, $(y_1, \ldots, y_n) \to \log \bar{y}^2/s^2$, complicates the analysis and may create spurious curvature and interaction effects (Hou, 1998). Additivity of the control factors in the response is not carried through to the SN ratio and modeling the responses is usually simpler than modeling summary quantities.

• The use of squared-error loss is not always appropriate (León and Wu, 1992; Moorhead and Wu, 1998). Loss functions cannot be independent of the metric in which the data are observed.

Since the goal of parameter design is to move the mean on target while simultaneously minimizing the variance, an alternative to the SN ratio is to model the mean and $\log s^2$ separately. Another strategy, first introduced by León, Shoemaker and Kacker (1987) (cf. also León and Wu, 1992), is to optimize a quantity called "Performance Measure Independent of Adjustment (PerMIA)". The PerMIA can be thought of as an extension of the SN ratio.

Although these methods have been compared on real data, no general study of comparison using analytic and simulation studies seems to be available. There is a need to know how well each approach performs and the important characteristics of the underlying model that make one approach superior to another. Also of interest is to find out in what situations these approaches have comparable performances. In conducting the study, one must recognize that these approaches are motivated by different models. In Section 2 we compare the performance of the SN ratio and $\log s^2$ analyses for a general additive model with a single noise factor, and we give a simulated example to illustrate some of the results. We also look at a model where the variance of the response is proportional to various powers of the mean. In Section 3 we consider models where the noise factor has a multiplicative effect on the error term and again compare the performance of the SN ratio and $\log s^2$ analyses. Finally, in Section 4 we perform a simulation study to illustrate the various results obtained in the previous sections.

2. Performance of the SN Ratio and $\log s^2$

The performance of the SN ratio will be studied and contrasted with that of the $\log s^2$ in various special cases of the following general model:

$$y = \mu(a,d) + V(\mu(a,d))[\phi_1(d)z_1 + \ldots + \phi_k(d)z_k + \theta(d)\delta],$$

where $\mathrm{E}(y) = \mu(a,d)$ is a function of (a,d) measuring the location effects of the control factors, $\phi_j(d)$ is a function of the non-adjustment factors measuring the dispersion effects associated with the noise variable z_j and $V(\mu(a,d))$ is some function of the mean. For simplicity, assume $\mathrm{E}(z_j) = \mathrm{E}(\delta) = 0, \mathrm{Var}(z_j) = 1, j = 1,\ldots,k$, $\mathrm{Cov}(z_j, z_{j'}) = 0$, where δ is the remaining variation from unidentified noise variables including measurement error. Under this model,

$$\sigma_y^2 = V^2(\mu(a,d))[P(d) + \theta^2(d)\sigma_\delta^2], \tag{2}$$

where $P(d) = \sum_{j=1}^{k} \phi_j^2(d)$. It is assumed that all major sources of noise are accounted for and the remaining variation is small, and therefore the term involving σ_δ^2 becomes negligible. We restrict our study to the common case where $\phi_j(d)$ is either linear or log-linear. In the response modeling approach, it is sometimes assumed that $\theta(d)$ is a constant (Welch, Yu, Kang and Sacks, 1990; Shoemaker, Tsui and Wu, 1991).

2.1. *Additive model.* We first look at the case where the noise factors enter in an additive way in the model. Consider the model

$$y = \beta_0 + \beta_1 x_1 + \phi_0 z + \phi_2 x_2 z + \delta, \tag{3}$$

where $\beta_0, \phi_0 > 0$. There is one adjustment factor (x_1) (not known as an adjustment factor before the analysis) and one non-adjustment factor (x_2), standardized so that $\mathrm{E}(x_1) = \mathrm{E}(x_2)$ and one noise factor (z). The true spaces of adjustment and non-adjustment factors are $A = x_1$ and $D = x_2$ with optimal settings given by (a^*, d^*). Let the spaces of adjustment factors and non-adjustment factors identified through the SN ratio analysis be denoted by A_{SN} and D_{SN} respectively and the corresponding optimal settings by $(a^*_{\mathrm{SN}}, d^*_{\mathrm{SN}})$. Obviously, A_{SN} and D_{SN} are not necessarily equivalent to A and D. The following equalities:

$$A_{\mathrm{SN}} = A, D_{\mathrm{SN}} = D, a^*_{\mathrm{SN}} = a^*, d^*_{\mathrm{SN}} = d^*,$$

will indicate the validity and effectiveness of the SN ratio analysis; violation of some of the equalities would indicate that the SN ratio analysis gives suboptimal results. The interest lies in knowing how the coefficients of the factors in (3) affect the identification of the adjustment and non-adjustment factors when an SN ratio analysis is used. The remaining variation δ in (3) is assumed negligible and is set to zero. Assuming $\mathrm{Var}(z) = 1$,

$$\mathrm{Var}(y) = (\phi_0 + \phi_2 x_2)^2 \tag{4}$$

and variation reduction is obtained by choosing an appropriate setting for x_2. The SN ratio in (1) is given by

$$\mathrm{SN} = \log \frac{(\beta_0 + \beta_1 x_1)^2}{(\phi_0 + \phi_2 x_2)^2} = 2\{\log(\frac{\beta_0}{\phi_0}) + \log[(1 + \frac{\beta_1}{\beta_0} x_1)/(1 + \frac{\phi_2}{\phi_0} x_2)]\}.$$

It is clear from (5) that the SN ratio depends on both x_1 and x_2, i.e., $D_{\mathrm{SN}} = (x_1, x_2)$. How much influence x_1 and x_2 have on the SN ratio depends on the relative magnitudes of β_1/β_0 and ϕ_2/ϕ_0. To keep the analysis simple, we assume that β_1 is large enough so that x_1 is always found significant in the analysis of the mean. If the factors take on the values $+1$ or -1, optimal settings for x_2 for variation reduction, determined from (4) and (5) are:

$$x_2 = +1 \text{ if } sign(\frac{\phi_0}{\phi_2}) = - \quad \text{and} \quad x_2 = -1 \text{ if } sign(\frac{\phi_0}{\phi_2}) = +.$$

Optimal settings, d^*_{SN}, for the factors classified in D_{SN} are selected so as to maximize the SN ratio. Based on the relative magnitudes of the coefficients of x_1 and x_2 in the last expression in (5), qualitative results may be inferred and three possible cases arise:

CASE 1: $|\phi_2/\phi_0| > |\beta_1/\beta_0|$. One expects x_2 to have more influence on the SN ratio and an SN ratio analysis is close to a $\log s^2$ analysis. There are two cases:

(i) β_1/β_0 is small so that x_1 is found to be insignificant in the SN ratio analysis, and ϕ_2/ϕ_0 is large enough so that x_2 is found to be significant; (ii) both β_1/β_0 and ϕ_2/ϕ_0 are large and therefore both x_1 and x_2 are found significant in the SN ratio analysis. In the first case, $D_{\text{SN}} = D = \{x_2\}$ and consequently, $A_{\text{SN}} = A = \{x_1\}$. Analysis of the SN ratio indeed leads to optimal settings since only the levels of x_2 are selected to maximize the SN ratio and the same level minimizes the variance in (4) and $d^*_{\text{SN}} = d^*$. Because ϕ_2 dominates the other terms in (3), this case is very close to the model $\sigma_y^2 = \mu^2\phi^2$ in Section 1 for which the SN ratio analysis is justifiable. In the second case, $D_{\text{SN}} = \{x_1, x_2\}$ and $A_{\text{SN}} = \emptyset$ so that levels for both x_1 and x_2 are selected to maximize the SN ratio. Optimal settings for the non-adjustment factors identified through the SN ratio involve settings for both the true adjustment and non-adjustment factors, i.e., $d^*_{\text{SN}} = (a, d^*)$. In general, the setting a in $d^*_{\text{SN}} = (a, d^*)$ will not correspond to the optimal setting a^*. The adjustment step cannot be performed and the 2-step procedure fails.

CASE 2: $|\beta_1/\beta_0| > |\phi_2/\phi_0|$. The adjustment factor x_1 has more influence on the SN ratio and the SN ratio analysis is close to the $\bar{y}$ analysis; x_1 is incorrectly identified as the most important factor for reducing variation. There are two cases: (i) ϕ_2/ϕ_0 is small and only x_1 is found significant in the SN ratio analysis; (ii) both β_1/β_0 and ϕ_2/ϕ_0 are large enough so that x_1 and x_2 are found significant in the SN ratio analysis. In the first case, $D_{\text{SN}} = \{x_1\}$ and consequently $A_{\text{SN}} = \emptyset$. Neither factor is correctly classified and an optimal solution cannot be found. In the second case, $D_{\text{SN}} = \{x_1, x_2\}$ and $A_{\text{SN}} = \emptyset$. The optimal setting d^* of x_2 is found but d^*_{SN} also involves a setting for x_1. The second step of the 2-step procedure cannot be performed.

CASE 3: $|\phi_2/\phi_0| \approx |\beta_1/\beta_0|$. Both x_1 and x_2 are equally likely to be detected with an SN ratio analysis. If the ratios are large enough, both factors are found significant, i.e., $D_{\text{SN}} = \{x_1, x_2\}$ and $A_{\text{SN}} = \emptyset$; otherwise, $D_{\text{SN}} = \emptyset$ and $A_{\text{SN}} = A = \{x_1\}$. The consequences in the first case are the same as discussed in Cases 1 and 2, while in the second case, variation reduction appears impossible.

Under (3), $\log(\sigma_y^2) = 2\log\phi_0 + 2\log(1 + \phi_2 x_2/\phi_0)$ depends on neither β_0 nor β_1. It is clear that in all three cases above, analysis in terms of the $\log s^2$ should result in correctly detecting x_2 as a non-adjustment factor and setting d^* as an optimal level.

A simulated example follows to illustrate and support the analytic results in Case 2 above.

EXAMPLE 1. Let the model be $y = 10 + 3x_1 + 3z + .2x_2 z + 0x_3 + \delta$, where $(x_1, x_2, x_3, z) = \pm 1$ and the error, $\delta \sim$N$(0, .04)$. In this case, $\phi_2/\phi_0 = .2/3 = .067 < \beta_1/\beta_0 = 3/10 = .3$, which corresponds to Case 2 above. The control factors are varied according to a 2^3 design (the coefficient for x_3 in the model is 0) and the noise array corresponds to the two levels of the noise factor. Three replications are made for each level of the noise factor. ANOVA tables for the estimated SN ratio and $\log s^2$ are given in Tables 1 and 2. Both control factors appear significant in the SN ratio analysis but x_1 is incorrectly declared as more significant for reducing variation. By contrast the $\log s^2$ analysis correctly identifies x_2 as a significant non-adjustment factor.

Table 1. ANOVA TABLE FOR SN RATIO IN EXAMPLE 1.

	Df	Sum of Sq	Mean Sq	F Value	Pr(F)
x1	1	61.325	61.325	647.005	0.0000142
x2	1	1.842	1.842	19.434	0.0116146
x3	1	0.069	0.069	0.728	0.4415243
Residuals	4	0.379	0.095		

Table 2. ANOVA TABLE FOR $\log s^2$ IN EXAMPLE 1.

	Df	Sum of Sq	Mean Sq	F Value	Pr(F)
x1	1	0.000966	0.000966	0.297	0.6145
x2	1	0.095243	0.095243	29.335	0.0056
x3	1	0.001724	0.001724	0.531	0.5066
Residuals	4	0.01298701	0.003247		

If x_1 is very easy to control, then what is lost from this analysis is a better understanding of the process, but if there is a cost to controlling x_1, this analysis could result in a more expensive process.

A simple model with a log-linear structure for the dispersion effects is

$$y = \mu(a, d) + e^{\phi(d)} z + \delta, \tag{6}$$

where $\phi(d)$ is a linear function of d. Because the log variances is approximately linear in d, one would expect the $\log s^2$ analysis to work well. If the noise δ in (6) is negligible, we have approximately $\log \sigma_y^2 \approx 2\phi(d)$. The following particular model with one adjustment factor (x_1), one non-adjustment factor (x_2), and one noise factor (z) is considered,

$$y = \beta_0 + \beta_1 x_1 + e^{\phi_0 + \phi_2 x_2} z + \delta.$$

The log-variance and SN ratio are respectively

$$\log \sigma_y^2 = 2(\phi_0 + \phi_2 x_2) \quad \text{and} \quad \text{SN} = \log \frac{(\beta_0 + \beta_1 x_1)^2}{e^{2(\phi_0 + \phi_2 x_2)}}.$$

The SN ratio can be roughly approximated as in (7) using a Taylor series when $|\beta_1| < |\beta_0|$,

$$\text{SN} = 2\left[\log\left(1 + \frac{\beta_1}{\beta_0} x_1\right) - (\log)\phi_2 x_2 + constant\right] \approx constant + 2\log\left(\frac{\beta_1}{\beta_0} x_1 - \phi_2 x_2\right). \tag{7}$$

In most practical applications, we expect $|\beta_1| < |\beta_0|$ to hold because the overall factor effect should be greater than individual factor effects. For both analyses in terms of $\log s^2$ and SN ratio, identification of x_2 as a non-adjustment factor depends on the coefficient ϕ_2. When an SN ratio analysis is performed, we see from

(7) that incorrect classification of x_1 as a non-adjustment factor could occur when β_1/β_0 is large enough. On the other hand, the $\log s^2$ analysis identifies x_2 as a non-adjustment factor if ϕ_2 is large, regardless of the values of β_0 and β_1.

There are situations where more than one factor might not be controlled during production. In theses cases it was found that, for a non-adjustment factor to be detected with a $\log s^2$ analysis there needs to be noise factor main effects (cf. Bérubé, 1997). Also, there is the possibility that even though individual coefficients may be large, they may cancel each other and the non-adjustment factor not detected. If an SN ratio analysis is used, then identification of a non-adjustment factor depends on the relative magnitude of the coefficients.

2.2. *Mean-variance relationship.* The following models have an extra level of complication: a relationship between the variance and the mean. Consider models with one noise factor of the form

$$y = \mu(a, d) + \mu^{\gamma}(a, d)\phi(d)z + \delta, \tag{8}$$

or

$$y = \mu(a, d) + e^{\gamma\mu(a,d)+\phi(d)}z + \delta. \tag{9}$$

Separation of the mean and variance can be achieved by using an appropriate transformation of the data (Nair and Pregibon, 1986; Box, 1988; Grize, 1991; Engel, 1992). An alternative recommended by Nelder and Lee (1991) and Hamada and Nelder (1997) is the use of generalized linear models. For model (8), some monotone function of the statistic $\bar{y}^{2\gamma}/s^2$ could be used, since $\mu^{2\gamma}/\text{Var}(y) = 1/\phi^2$ is a PerMIA. For model (9), one might use some function of $e^{2\gamma\bar{y}}/s^2$.

Consider the following special case of (8) with the mean and dispersion functions being linear,

$$y = (\beta_0 + \beta_1 x_1) + (\beta_0 + \beta_1 x_1)^{\gamma}(\phi_0 + \phi_2 x_2)z + \delta. \tag{10}$$

There is one adjustment factor (x_1), one non-adjustment factor (x_2), standardized so that $E(x_1) = E(x_2)$ and one noise factor (z). Assume that δ is negligible and is set to be zero. Then $\text{Var}(y) = (\beta_0 + \beta_1 x_1)^{2\gamma}(\phi_0 + \phi_2 x_2)^2$. The SN ratio is given by

$$\text{SN} = \log \frac{(\beta_0 + \beta_1 x_1)^2}{(\beta_0 + \beta_1 x_1)^{2\gamma}(\phi_0 + \phi_2 x_2)^2} = -2\log[(\beta_0 + \beta_1 x_1)^{\gamma-1}(\phi_0 + \phi_2 x_2)]. \tag{11}$$

It can be seen from this expression that for $\gamma < 1$, the effect of x_2 on the SN ratio is diminished by the effect of x_1 so that x_2 may become insignificant. For $\gamma > 1$, the effect of x_2 on the SN ratio is amplified by the effect of x_1 so that x_2 should remain significant. For $|\beta_1| < |\beta_0|$ and $|\phi_2| < |\phi_0|$, a Taylor series expansion can be used as a rough approximation,

$$\text{SN} \approx -2\log\left[constant + \frac{(\gamma - 1)\beta_1}{\beta_0}x_1 + \frac{\phi_2}{\phi_0}x_2\right].$$

Detection of x_1 as a significant factor on the variation depends on both β_1/β_0 and γ.

Under (10), a $\log s^2$ analysis could also lead to incorrect classification of the factors. Consider the following

$$\log \sigma_y^2 = 2\gamma \log(\beta_0 + \beta_1 x_1) + 2\log(\phi_0 + \phi_2 x_2) \approx 2\left(constant + \frac{\gamma\beta_1}{\beta_0}x_1 + \frac{\phi_2}{\phi_0}x_2\right)$$

for $|\beta_1| < |\beta_0|$ and $|\phi_2| < |\phi_0|$. Here as in the SN ratio analysis, detection of x_1 as a significant factor on the variation depends on both β_1/β_0 and γ.

A simulation is conducted using 500 simulations under (10) with the coefficients for x_1 and x_2 satisfying $1/10 = \beta_1/\beta_0 < \phi_2/\phi_0 = 2/2.5$ (the coefficient of x_3 is kept at 0). With the coefficients fixed, the value of γ is varied and both the SN ratio and $\log s^2$ are analyzed using ANOVA procedures. Figure 1 summarizes the results, plotting the p-values for the three factors against γ. We see that the chance of falsely detecting x_1 decreases (p-value increases) as γ approaches 1, in the case of the SN ratio analysis, or 0, in the case of the $\log s^2$ analysis. The SN ratio analysis would lead to the correct classification of x_1 as an adjustment factor only in a very small interval around $\gamma = 1$ (i.e., when the variance of the response is proportional to the mean square) or as γ becomes more negative. The $\log s^2$ analysis performs better as x_1 is found to be insignificant for a wider range of γ values. Different values of β_1/β_0 and ϕ_2/ϕ_0 would result in x_1 being correctly classified for a slightly narrower or wider range of γ values (Bérubé, 1997).

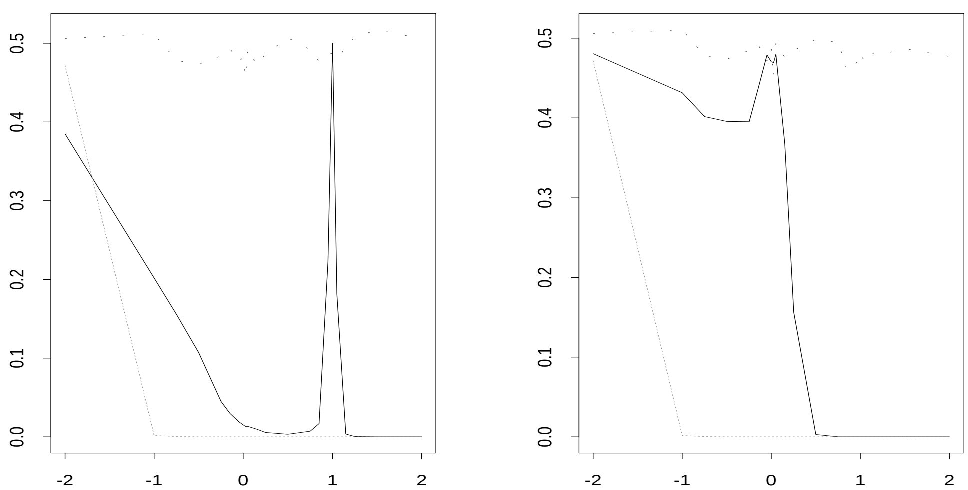

Figure 1. P-values for factors x_1: ——, x_2: ----- and x_3: - - -, with $1/10 = \beta_1/\beta_0 < \phi_2/\phi_0 = 2/2.5$. (The left plot for the SN ratio analysis, the right plot for the $\log s^2$ analysis).

In the models specified above, neither factor has an effect on both the mean and the dispersion. In the model (12), the performance of the SN ratio depends largely on the combined effect of factors that affect both the mean and the dispersion. Here, the non-adjustment factor x_2 entering linearly into the mean and dispersion functions,

$$y = (\beta_0 + \beta_1 x_1 + \beta_2 x_2) + (\beta_0 + \beta_1 x_1 + \beta_2 x_2)^{\gamma}(\phi_0 + \phi_2 x_2)z + \delta, \tag{12}$$

with variance $\mathrm{Var}(y) = (\beta_0 + \beta_1 x_1 + \beta_2 x_2)^{2\gamma}(\phi_0 + \phi_2 x_2)^2$, where $\beta_0, \phi_0 > 0$ and δ is assumed negligible. Here $A = x_1$ and $D = x_2$, and there is one noise factor, z. Variation reduction is achieved by choosing the levels of x_2 to minimize $(\phi_0+\phi_2 x_2)^2$. The SN ratio is

$$\begin{aligned} \mathrm{SN} &= \log \frac{(\beta_0 + \beta_1 x_1 + \beta_2 x_2)^2}{(\beta_0 + \beta_1 x_1 + \beta_2 x_2)^{2\gamma}(\phi_0 + \phi_2 x_2)^2} \\ &\approx -2\log\left[constant + \frac{(\gamma-1)\beta_1}{\beta_0} x_1 + \left(\frac{(\gamma-1)\beta_2}{\beta_0} + \frac{\phi_2}{\phi_0}\right) x_2 \right. \\ &\qquad \left. - \frac{(\gamma-1)\beta_1\beta_2}{\beta_0^2} x_1 x_2 \right] \end{aligned} \tag{13}$$

if $|\beta_1| + |\beta_2| < |\beta_0|$. To simplify the analysis, it is assumed under (12) that x_1 and x_2 are always detected as being significant in the analysis of $\bar{y}$, so once D_{SN} is determined, the space of adjustment factors identified through the SN ratio analysis follows directly. Ignoring the $x_1 x_2$ term in (13), we can use the relative magnitude of the coefficients of x_1 and x_2 in (13) to distinguish four cases.

CASE 1: $D_{\mathrm{SN}} = D = x_2$, $A_{\mathrm{SN}} = A = x_1$. The SN ratio correctly partitions the adjustment and non-adjustment factors if x_2 is found significant and x_1 not significant. Qualitatively, this means that in (13),

$$\left| \frac{(\gamma-1)\beta_2}{\beta_0} + \frac{\phi_2}{\phi_0} \right| >> \left| \frac{(\gamma-1)\beta_1}{\beta_0} \right| \approx 0.$$

Even though the SN ratio analysis may have correctly identified the factors, optimal settings still need to be selected. Once the setting for x_2 is chosen, the setting for x_1 is selected to bring the mean on target. We see from (13) that $d^*_{\mathrm{SN}} = d^*$ if and only if

$$sign\left(\frac{(\gamma-1)\beta_2}{\beta_0} + \frac{\phi_2}{\phi_0}\right) = sign\left(\frac{\phi_2}{\phi_0}\right). \tag{14}$$

CASE 2: $D_{\mathrm{SN}} = \emptyset$, $A_{\mathrm{SN}} = (A, D)$. This occurs if both coefficients of x_1 and x_2 in (13) are small. Variation reduction appears impossible, and adjustment using the incorrectly classified factor x_1 might result in an increases in the variance.

CASE 3: $D_{\mathrm{SN}} = (A, D)$, $A_{\mathrm{SN}} = \emptyset$. This occurs if both coefficients of x_1 and x_2 are large in (13). No adjustment factors are detected and the adjustment step in the 2-step procedure cannot be performed.

CASE 4: $D_{\mathrm{SN}} = A$ and $A_{\mathrm{SN}} = D$. Both the true a and d are incorrectly classified (see simulation in Section 4), if the coefficient of x_1 in (13) is large, and the coefficient of x_2 is close to 0.

3. Multiplicative Noise Model

Additive models assume that the inability or unwillingness to control noise factors results in uncontrollable shifts in the response which translate into increased variability over all noise levels. Thus the variation in the response is the *same* for each level of the noise factor.

One can think of situations where the noise factor might affect the variability of the response directly by having a multiplicative effect on the error term. For example, consider a situation in which there are two suppliers (noise factor z) and two temperature settings (control factor, x). Assume now that supplier 1's product results in responses that are more variable than supplier 2's product at the higher temperature but both are equivalent at the low temperature. Here we would need a model in which the noise factor directly influences the magnitude of the variation. One possible model is

$$y = \mu + e^{\alpha x + (\lambda_0 + \lambda_1 x) z} \delta. \tag{15}$$

If $\alpha \approx \lambda_0 \approx \lambda_1$, at $x = -1$, there is no supplier effect on the variation but at $x = 1$, supplier 1 results in increased variability. For each level of the noise factor, there is a difference in variation.

In the response model (Shoemaker, Tsui and Wu, 1991), the objective is to identify important control-by-noise interactions but only those entering the model in an additive way are considered. It has been mentioned in the literature that the residual error may have non-constant variance and that one should try to identify any dispersion effects associated with unobserved noise variables (Freeny and Nair, 1992; Engel and Huele, 1996). This corresponds to a model with the error term δ having non-constant variance depending on the level of one or more control factors.

First we study how the SN ratio analysis performs in identifying A and D for a simple model with a multiplicative noise effect. Consider the following special case of (15),

$$y = \beta_0 + \beta_1 x + e^{(\lambda_0 + \lambda_1 x) z} \delta, \tag{16}$$

where x is a 2-level factor and $z \sim N(0,1)$ and $\delta \sim N(0, \sigma_\delta^2)$ are independent. Its response variance is given by

$$\mathrm{Var}(y) = \sigma_\delta^2 e^{2(\lambda_0 + \lambda_1 x)^2}. \tag{17}$$

It is assumed that the noise array consists of random settings of the noise factor so that $\mathrm{E}(s_i^2)$, where s_i^2 is the variance calculated over the noise array settings, is equal to (17). A reduction in variation is obtained by choosing an appropriate setting for x to minimize $(\lambda_0 + \lambda_1 x)$, i.e., $x = -1$ if $sign(\lambda_1/\lambda_0) = +$ and $x = 1$ if $sign(\lambda_1/\lambda_0) = -$. The SN ratio is given by

$$\mathrm{SN} = 2\log(\beta_0 + \beta_1 x) - 2(\lambda_0 + \lambda_1 x)^2 - \log \sigma_\delta^2 \approx constant + 2\left(\frac{\beta_1}{\beta_0} - 2\lambda_0\lambda_1\right) x, \tag{18}$$

for $|\beta_1| < |\beta_0|$. The classification of x in D_{SN} or A_{SN} depends on the relative magnitude of the various coefficients:

CASE 1. $D_{\rm SN} = D$. This occurs if $|(\beta_1/\beta_0) - 2\lambda_1\lambda_0| >> 0$. The optimal level for x is correctly identified if and only if $sign\left(\frac{\beta_1}{\beta_0} - 2\lambda_1\lambda_0\right) = sign\left(\frac{\lambda_1}{\lambda_0}\right)$.

Case 2: $D_{\rm SN} = \emptyset$. This occurs if $\beta_1/\beta_0 \approx 2\lambda_1\lambda_0$. Here, variation reduction through x appears impossible, and x will be used as an adjustment factor.

On the other hand, since $\log \sigma_y^2 = constant + 4\lambda_0\lambda_1 x$, a $\log s^2$ analysis using only an estimate of the variance as a performance measure should identify x as a non-adjustment factor along with its optimal setting.

Depending on the nature of the noise factor, both an uncontrollable additive and/or multiplicative noise effect might affect the variability of the response. For example in a model like $y = \mu + (\phi_0 + \phi_1 x)z + e^{\alpha x + (\lambda_0 + \lambda_1 x)z}\delta$, both the SN ratio analysis and the $\log s^2$ analysis combine the additive and multiplicative terms in an overall measure of variability. One consequence of this is that dispersion effects might not be detected if the additive and multiplicative contributions cancel each other (Bérubé, 1997). An optimal strategy should include steps for identification of control-by-noise interactions in the additive term as well as the multiplicative term.

4. Simulation Study

The purpose of the simulation study is to compare the following three modeling techniques: (i) SN ratio, (ii) PerMIA and (iii) $\log s^2$, with respect to their ability to detect factors that affect the variation through an additive or multiplicative term. Both control and noise factors are studied at two levels in a cross array setting. Estimates of the SN ratio, the PerMIA and $\log s^2$ are obtained for each run of the control factors. Analysis of variance and F tests can be used to determine the significant factors, assuming an estimate of error exists. In practice, half-normal plots or Lenth's method (Wu and Hamada, 2000) can be used to determine important effects, but these would be difficult to study in a simulation setting.

4.1. *Additive noise effect.* The first set of simulations was done for a location-scale model with linear functions. There is one adjustment factor (x_1), one non-adjustment factor (x_2) and one noise factor (z),

$$y = \beta_0 + \beta_1 x_1 + \phi_0 z + \phi_2 x_2 z + \delta, \tag{19}$$

where the error is $N(0, (0.2)^2)$. The coefficients are varied according to the three cases listed in Section 2.1. In each case, 1000 simulations were generated for different values of $|\phi_2/\phi_0|$ and $|\beta_1/\beta_0|$, with a 2^3 control array (with coefficient of x_3 being 0), a noise array with two settings of the noise factor and three replications for each noise setting. Table 3 contains the means and standard deviations of the p-values, for factors x_1, x_2 and x_3 generated by these simulations (the standard deviation is given in parenthesis after the mean). It appears that for both the SN ratio analysis and $\log s^2$ analysis, x_2 is always correctly classified as such. When using an SN ratio analysis, x_1 is consistently misclassified as a non-adjustment factor. The $\log s^2$ analysis leads to the correct classification of x_1. These simulations show that the following relationships hold for model (19): $D_{\rm SN} \subset D$ and $D_{\log s^2} = D$.

Table 3. Results of simulations for model (19)

case	parameters		adjustment fac. x_1	non-adj. fac. x_2	x_3
$\lvert\phi_2/\phi_0\rvert = \lvert\beta_1/\beta_0\rvert$	$\beta_0 = 10$, $\beta_1 = 5$	SN	0.0004(0.0009)	0.0004(0.0006)	0.484(0.291)
	$\phi_0 = 1$, $\phi_2 = .5$	$\log s^2$	0.489(0.287)	0.0004(0.0006)	0.486(0.291)
$\lvert\phi_2/\phi_0\rvert = \lvert\beta_1/\beta_0\rvert$	$\beta_0 = 10$, $\beta_1 = 1$	SN	0.044(0.077)	0.046(0.074)	0.505(0.287)
	$\phi_0 = 1$, $\phi_2 = .1$	$\log s^2$	0.490(0.288)	0.045(0.074)	0.504(0.285)
$\lvert\phi_2/\phi_0\rvert > \lvert\beta_1/\beta_0\rvert$	$\beta_0 = 10$, $\beta_1 = 1$	SN	0.0471(0.0750)	0.0124(0.0195)	0.495(0.297)
	$\phi_0 = 1$, $\phi_2 = .15$	$\log s^2$	0.506(0.290)	0.0122(0.0192)	0.497(0.299)
$\lvert\phi_2/\phi_0\rvert < \lvert\beta_1/\beta_0\rvert$	$\beta_0 = 10$, $\beta_1 = 1.5$	SN	0.0114(0.0192)	0.0479(0.0744)	0.506(0.285)
	$\phi_0 = 1$, $\phi_2 = .1$	$\log s^2$	0.527(0.285)	0.0473(0.0752)	0.503(0.283)

The next set of simulations is based on model (12), where there is one adjustment factor (x_1), one non-adjustment factor (x_2), one noise factor (z) and the variance is proportional to a power of the mean. We first obtain p-values for each control factor from an SN ratio analysis and a $\log s^2$ analysis on the simulated data. Next, we perform a least squares regression on $\log s^2 = 2\gamma \log \bar{y} + \log \phi$. If the slope is found significantly different from 0 at the 5% level, a PerMIA is obtained as $\log\left(\bar{y}^{2\hat{\gamma}}/s^2\right)$ and values for $\log s^2$ of the transformed data (denoted by $\log s_t^2$) are calculated (if the slope is not significantly different from 0, the $\log s^2$ calculated from the original data is used). We get p-values from ANOVAs of the PerMIA and $\log s_t^2$. This procedure is repeated for each set of simulated data and we report the means and standard deviations of the p-values. Table 4 summarizes these results for 5000 simulations (with the same cross array setup as before) performed in the context of Case 4 for model (12) with $\beta_0 = 8, \beta_1 = 2, \beta_2 = 1, \phi_0 = .008, \phi_2 = -.002$ and $\gamma = 2$.

Table 4. Mean and standard deviation of p-values for model (12)

	adjustment fac. x_1	non-adj. fac. x_2	x_3
SN	0.0049(0.0084)	0.0592(0.1075)	0.405(0.304)
$\log s^2$	0.0003(0.0007)	0.480(0.299)	0.222(0.242)
PerMIA	0.067(0.082)	0.011(0.021)	0.483(0.299)
$\log s_t^2$	0.066(0.081)	0.011(0.021)	0.482(0.299)

One sees from the table that both the SN ratio analysis and the $\log s^2$ analysis for the original data incorrectly identify x_1 as a very significant non-adjustment factor. The factor x_2 is not found to be significant on average and therefore may not be classified as a non-adjustment factor as it should. With both factors being incorrectly classified, parameter design optimization cannot be performed. Analysis of the estimated PerMIA or analysis of $\log s^2$ of the transformed data will on average lead to correct classification of the control factors.

4.2. *Multiplicative noise effects.* Now we consider model (16). Four situations are considered:

(i) x_1 is an adjustment factor,

(ii) x_1 is a non-adjustment factor and affects only the variance,

(iii) x_1 is a non-adjustment factor affecting both the mean and variance, effects canceling each other and (iv) x_1 is a non-adjustment factor affecting both the mean and variance, effects reinforcing each other.

Coefficients $\beta_0 = 10$ and $\lambda_0 = 0.5$ are fixed while β_1 and λ_1 are varied according to the four cases above. From (18), the SN ratio depends on x_1 through the mean relative to β_1/β_0 and through the variance relative to $2\lambda_0\lambda_1$. Table 5 summarizes p-value results from 2000 simulations in the context of a cross array setup with a 2^{6-1} control array (defined by I= 123456), and a noise array with two random settings of the noise factor and three replications for each noise setting.

Table 5. MEAN AND STANDARD DEVIATION OF P-VALUES FOR MODEL

				p-value for x_1
CASE 1	$\beta_1/\beta_0 = .4$	$2\lambda_1\lambda_0 = 0$	SN	0.003(0.082)
			$\log s^2$	0.504(0.084)
CASE 2	$\beta_1/\beta_0 = 0$	$2\lambda_1\lambda_0 = .4$	SN	0.040(0.082)
			$\log s^2$	0.039(0.082)
CASE 3	$\beta_1/\beta_0 = .4$	$2\lambda_1\lambda_0 = .4$	SN	0.102(0.083)
			$\log s^2$	0.042(0.082)
CASE 4	$\beta_1/\beta_0 = .4$	$2\lambda_1\lambda_0 = -.4$	SN	0.0003(0.086)
			$\log s^2$	0.039(0.083)

In case 1, the SN ratio analysis incorrectly identifies x_1 as a non-adjustment factor and in case 2, the SN ratio analysis correctly identifies x_1 as a non-adjustment factor. This indicates that the SN ratio is unable to detect whether the variations in y are due to x_1 through the mean or through the variance. In case 3, the SN ratio analysis fails to detect the variance effect of x_1 when x_1 affects both the mean and the variance equally in the same direction. In case 4, the SN ratio analysis finds x_1 very significant as a variance reducing factor, when in fact the effect might not be as large as indicated. In all four cases, the $\log s^2$ analysis correctly classifies x_1. These results are similar to those found in an additive model of the form $y = \beta_0 + \beta_1 x_1 + (\phi_0 + \phi_1 x_1)z + \delta$ where the same four situations considered above are possible for x_1 (cf. Bérubé, 1997).

Summary

Results from the previous sections clearly show that the performance of the SN ratio is very much model-dependent. As pointed out earlier, using an SN ratio analysis for models where the standard deviation of the response is linearly proportional to the mean will result in a correct identification of factors and optimal levels. This holds irrespectively of the coefficients of the factors in the model. For most other models, however, its performance depends on the values of the coefficients of the control and noise factors in the model, as well as the form of the mean-variance relationship. Its validity deteriorates as the true model deviates from the assumed model that supports the SN ratio analysis. Even small changes in the coefficients can make a difference in the SN ratio analysis and affect the outcome. Obviously, in practice both the model and the coefficients are unknown so that blanket use of the SN ratio cannot be recommended. Indiscriminate use of the $\log s^2$ analysis can also lead to incorrect results and its success is again model-dependent, though to a lesser degree. For power-of-the-mean models, its success depends as much on the particular power as the SN ratio analysis does. On the other hand, the $\log s^2$ analysis performs well for simple exponential variance models as in (6) and simple multiplicative noise models as in (16) while the SN ratio analysis does poorly. When an adjustment factor is known in advance, the consequences of analyzing either the SN ratio or $\log s^2$ might be less severe, as for models in (3), (6), (8) and (9). Variation reduction might be attained but at a cost of controlling more factors than necessary. Advanced knowledge of an adjustment factor does not, however, guarantee that the SN ratio analysis will lead to variation reduction as was shown with model (12). Despite the grim picture of SN ratio and $\log s^2$ analysis presented here, better analyses of parameter designs will result from sounder statistical approaches. For alternative approaches to the SN ratio analysis, see among others, Engel and Huele (1996) (alternate fitting of a mean model and variance model), Hamada and Nelder (1997) (generalized linear model), Shoemaker, Tsui and Wu (1991) (direct response modeling associated with examination of control-by-noise interaction plot). A summary of sound methods for parameter design can be found in Wu and Hamada (2000).

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