Quality Engineering
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lqen20

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Version of record first published: 19 Oct 2007.

To cite this article: William Li & C. F. J. Wu (1999): AN INTEGRATED METHOD OF PARAMETER DESIGN AND TOLERANCE DESIGN, Quality Engineering, 11:3, 417-425

To link to this article: http://dx.doi.org/10.1080/08982119908919258

Quality Engineering, 11(3), 417–425 (1999)

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Key Words

Design optimization; Internal noise; Quality cost; Tolerance cost; Variation reduction.

Introduction

Parameter design is a methodology that builds on ingenious engineering and statistical ideas to improve a system (i.e., product or process) by making its performance insensitive to noise or hard-to-control variation. It classifies the factors in a system into two types: control factors and noise factors. By using statistical design of experiments and data analysis tools, it identifies settings of the control factors to make the quality characteristic of the system less sensitive to the variation in the noise factors. For a comprehensive review on parameter design, see Refs. 1 and 2.

We now use a simple example to illustrate Taguchi's approach to parameter design and tolerance design. The output voltage y of a circuit is a nonlinear function of its transistor gain, which is the design variable denoted by x in Figure 1. A design engineer can specify or choose the nominal value of x in the circuit design. In practice, because of the variation in manufacturing and environmental conditions, and degradation over time, the actual value of the transistor gain may be different from the nominal value. Suppose that in the current design, a in Figure 1 denotes the nominal value and the bell-shaped curve around a represents the variation of x around a. Through the nonlinear function f, the transmitted variation on the response y is a curve around $f(a)$. To reduce the variation in y, *parameter design* will move the nominal value a to b. Because f is flatter around b, the transmitted variation around $f(b)$ is much smaller. Here, we treat the nominal value of x as a control factor and the deviation of x from the nominal value as a noise factor. The advantage of parameter design is that it can achieve reduction in the variation of y by simply changing the nominal value of x without reducing the variation of x, which would usually incur an extra cost. If the variation around $f(b)$ is still too large, Taguchi (3) advocated the use of *tolerance design* to further reduce this variation by tightening the tolerance of x around b. In Figure 1, this is represented by the shaded arrow. A tighter tolerance (i.e., smaller variation) around the nominal value results in a

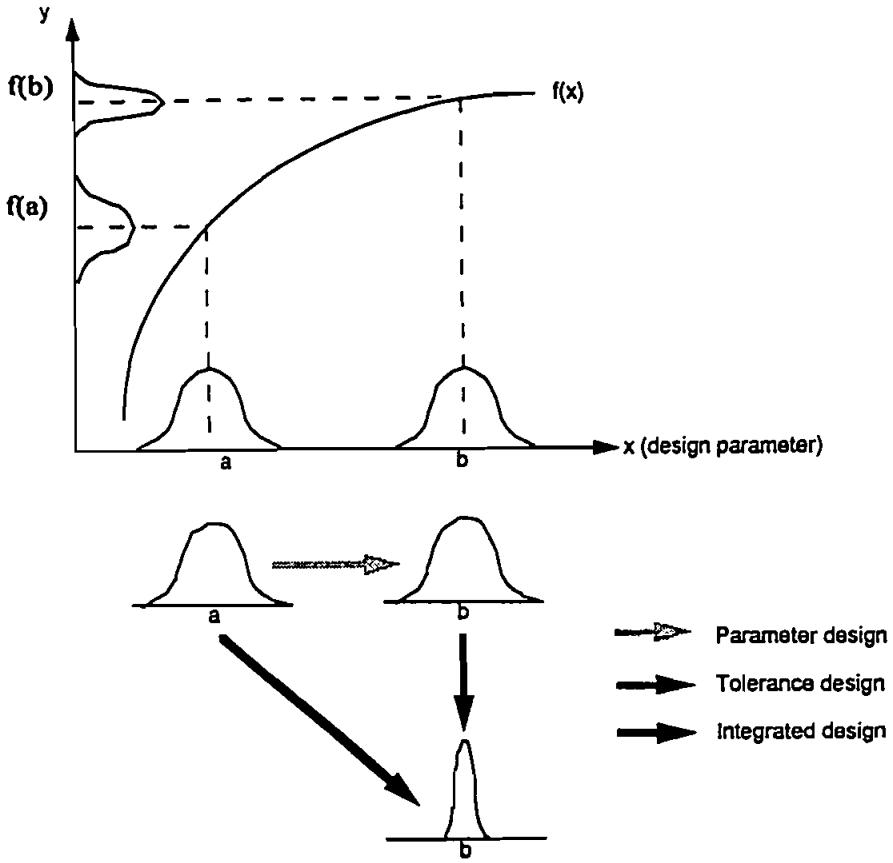

Figure 1. Parameter design, tolerance design, and integrated design.

smaller transmitted variation on y but incurs an extra cost. We refer to Taguchi's methodology as a two-stage approach to quality improvement:

1. Perform a parameter design on the nominal value x of a design variable
2. (if necessary) Perform a tolerance design by suitably narrowing the tolerance t around the nominal value x

The two-stage approach can be viewed as an optimization over the two variables x and t separately. It is clear that a joint optimization in x and t is a superior approach. In Figure 1, we can choose the nominal value and its tolerance simultaneously and achieve design optimization in a single stage (as represented by the solid arrow). We refer to this new approach as an *integrated method of parameter design and tolerance design.*

Throughout the article, we use a chemical engineering example from Mori (4), which involves the design of cyclones, to illustrate the two approaches. First, we give a rigorous formulation of Taguchi's two-stage approach by defining a total cost function F in Eq. (7) as the sum of a quality loss Q and a tolerance cost C. The two-stage approach leads to a much better design (in terms of a smaller F-value) of cyclones than the original design reported by Mori (4). It is then argued that the integrated approach is superior because it allows two sets of variables in the F function to be simultaneously optimized. To implement the integrated approach, a good approximation to the quality loss Q [see Eqs. (3) and (6)] is required. Three methods (Taylor expansion, Monte Carlo simulations, and OA-based Monte Carlo) are considered and their performances are compared. By using the Taylor expansion method in the optimization procedure of the integrated approach, the cyclone design based on the two-stage approach can be improved further. The reduction in total cost is substantial. Some concluding remarks are given at the end.

A Chemical Engineering Example

An example involving the design and manufacturing of cyclones was discussed by Mori (4). The objective is to choose optimum settings of seven design variables so that the overall cost is minimized. In chemical engineering, cyclones are used to separate solid mass and gaseous mass. Seven variables of the cyclone have been chosen for optimization (see Fig. 2). Among the seven variables, D_0, D_1, D_2, and D_3 are different diameters for different parts of the cyclone, r is a physical coefficient, V_0 is the velocity of the gas going to the cyclone, and H_t is the height of the cyclone. According to the pertinent theory, the response (denoted by x_{ca}), the critical parameter of particles, is related to the seven variables by

$$x_{ca} = \left(\frac{2}{2+n}\right)\left(\frac{1+n}{2+n}\right)^n\left(\frac{4D_2}{3D_1(1-D_0/D_1)}\right)^n \times \sqrt{\frac{\phi 9\mu A_0\{1-(Q_{bm}/Q_0)\}}{\pi n_1' r^2 \rho_p H_t V_0}}, \quad (1)$$

where $\phi = 1.3$, $n_1' = \frac{1}{3}$, $n = 0.85$, $\rho_p = 2 \times 10^3$ kg/m^3, $\mu = 1.8 \times 10^{-5}$ N/m^2, $A_0 = \frac{1}{4}\pi D_0^2$, and

$$\frac{Q_{bm}}{Q_0} = 2.62\left[1 - 0.36\left(\frac{D_3}{D_1}\right)^{-0.56}\right]^{-3/2}\left(\frac{D_3}{D_1}\right)^{1.16}.$$

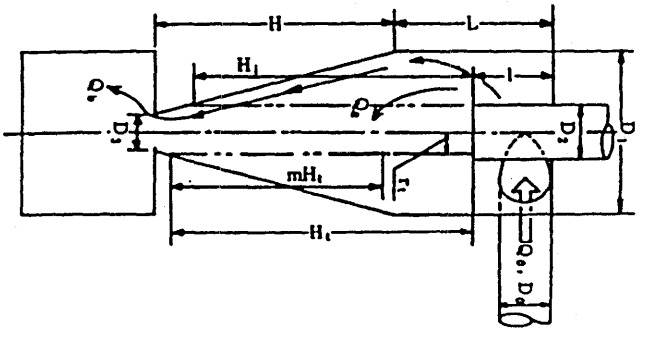

Figure 2. Diagram of cyclone.

The nominal value for x_{ca} is 1.5 μm. To conform to standard notation, we use $x_1, \ldots, x_7$ and y to represent the input variables D_0, D_1, D_2, D_3, r, V_0, H_t, and the response x_{ca} in Eq. (1). After simplification, Eq. (1) becomes

$$y = 174.42\left(\frac{x_1}{x_5}\right)\left(\frac{x_3}{x_2 - x_1}\right)^{0.85} \times \sqrt{\frac{1 - 2.62[1 - 0.36(x_4/x_2)^{-0.56}]^{3/2}(x_4/x_2)^{1.16}}{x_6 x_7}}.$$

The original values of the variables are shown in Table 1. The goal is to choose both the nominal values and tolerances of these variables to optimize some objective function which will be discussed later. The tolerance of each variable has three grades: classes A, B, and C. The lowest grade, C, has t_i values within 25% of the nominal values x_{i_0} and is least costly. (Explanation for grades A and B is similar and omitted.) The cost function $C_i(t_i)$ is shown in Table 2. For simplicity, Mori (4) assumed that the cost for grade C is zero. Then, the cost for grades B and A were adjusted accordingly. For example, for x_1, the cost for C, B, and A is respectively 0, 15, and 90 instead of 10, 25, and 100. In the following discussion, we will adopt the same assumption for consistency.

Taguchi's Two-Stage Approach: Parameter Design Followed by Tolerance Design

In this section, we describe Taguchi's two-stage approach and illustrate its application to the design of cyclones. First, we need to develop a cost function that serves as a driving force to link parameter design and tolerance design. Suppose y is the quality characteristic of a product or process and y_0 is the target value according to design specifications. The quality loss, although complicated in many practical situations, can be reasonably assumed to be a nondecreasing function of the deviation $|y - y_0|$. Taguchi (3), among others, advocated the use of the quadratic loss function

$$L(y) = k(y - y_0)^2 \quad (2)$$

Table 1. Original Settings of the Variables of the Cyclone Example

x_1 D_0	x_2 D_1	x_3 D_2	x_4 D_3	x_5 r	x_6 V_0	x_7 H_t
0.10 m	0.30 m	0.10 m	0.10 m	1.50	16.0 m/sec	0.75 m

Table 2. Cost Function for the Cyclone Example

VARIABLES	CLASS C (±25%)	CLASS B (±12.5%)	CLASS A (±2.5%)
x_1	10	25	100
x_2	20	50	200
x_3	20	50	200
x_4	50	100	500
x_5	50	200	1000
x_6	10	25	100
x_7	10	25	100

Note: Costs given in Japanese yen.

as a measure of quality loss. The main advantage is that it allows the average quality loss Q to be decomposed into two terms,

$$Q = \mathrm{E}(k(y-y_0)^2) = k[(\mu_y - y_0)^2 + \sigma_y^2], \tag{3}$$

where μ_y and σ_y^2 are the mean and variance of y, respectively, and E( ) is the expectation. The simplicity rendered by Eq. (3) makes it easier to compute Q both theoretically and numerically. The quadratic loss is also a good approximation to real quality loss if the probability of large $|y-y_0|$ values is small and the actual cost for y near y_0 is not far from being quadratic. Although there are questions about the practicability of choosing an appropriate k value to capture real quality loss, for the reasons stated above we adopt Eqs. (2) and (3) in our formulation.

Suppose $x_1, \ldots, x_n$ represent the components or variables of a system (i.e., product or process) and y is related to x_i through the relationship

$$y = f(x_1, \ldots, x_n). \tag{4}$$

For example, in the design of an output transformerless pull-push circuit (5), the response y, the midpoint voltage of the circuit, is a function of the resistances of five resistors and the current gain of a transistor. Assume that each x_i follows a Normal distribution $N(x_{i_0}, \sigma_i^2)$. In this context, the purpose of parameter design is to find values of x_{i_0}, given σ_i^2, to minimize Q in Eq. (3). In design engineering, x_{i_0} is called the nominal value of the ith component (or input variable) of x_i. It is common to define the tolerance t_i of x_i to be the 3σ value of x_i; that is,

$$t_i = 3\sigma_i. \tag{5}$$

For its intuitive appeal and mathematical tractability, we will adopt Eq. (5) as our definition of tolerance. Many other definitions of tolerance can be found in Ref. 6. Because the normally distributed x_i is determined by its mean x_{i_0} and its standard deviation σ_i, it is easy to see that Q is a function of x_{i_0} and σ_i, or equivalently of x_{i_0} and t_i $(= 3\sigma_i)$. Notationally, we can write

$$Q = Q(\mathbf{x}_0, \mathbf{t}), \tag{6}$$

where $\mathbf{x}_0 = (x_{i_0}, \ldots, x_{n_0})$ and $\mathbf{t} = (t_1, \ldots, t_n)$. Parameter design can now be restated as the minimization of $Q(\mathbf{x}_0, \mathbf{t})$ over $\mathbf{x}_0$ with fixed $\mathbf{t}$.

Ideally, one would like to choose smaller $\mathbf{t}$ values to reduce the quality loss Q, but this would also lead to a higher cost (e.g., more expensive material, higher-priced supplier, more attention or manpower assigned to the process). Thus, there is a trade-off between the quality loss Q and the cost C. For the different stages of production, different requirements may lead to different choices of tolerance. At the design stage, functionality, performance, and reliability are the main concerns. Tolerance should be set as tight as possible. However, this leads to higher cost and more difficulties in manufacturing. At the manufacturing stage, loose tolerance is preferred. At the assembly stage, for the interchangeability of components, tight tolerance is again desirable. An ideal objective function should reflect the total cost, which results in an overall balance over these three stages of production.

Let $C_i(t_i)$ represent the cost associated with t_i, where C_i is a nonincreasing function in t_i, and $C(\mathbf{t}) = \sum_{i=1}^{n} C_i(t_i)$. Then, we define the total cost to be

$$F(\mathbf{x}_0, \mathbf{t}) = Q(\mathbf{x}_0, \mathbf{t}) + C(\mathbf{t}). \tag{7}$$

After $\mathbf{x}_0$ are identified in parameter design, if the quality of y is still unsatisfactory, the only recourse is to choose smaller values of $\mathbf{t}$ in Eq. (7) to reduce the quality loss Q further. This second step is called *tolerance design* in Taguchi's terminology.

We now show how the two-stage approach can be used to improve the original design whose values were given in the previous section. In order to evaluate the total quality cost F, we need to evaluate the coefficient k. By following

Taguchi's method, Mori (4) determined k as follows. Suppose $y_0 \pm \Delta_0$ are the functional limits—the values of y at which the product would fail in half of the applications. In the cyclone example, Mori chose $\Delta_0 = 0.3$. The loss at the functional limits, denoted by A_0, is 1000 yen. From Eq. (2), we have $k = A_0/\Delta_0^2 = 1000/(0.3)^2 \approx \frac{1}{9} \times 10^5$.

According to Eq. (3), the remaining work is to evaluate $E(y - y_0)^2$. Mori used an orthogonal-array (OA)-based Monte Carlo method, which will be discussed later, to calculate the total quality cost for the original design, according to which all variables have grade C. Then, $E(y - y_0)^2 = 0.1742$. Because there are 10,000 units produced each year, the total quality loss is

$$\begin{aligned} Q &= k[E((y - y_0)^2)](10{,}000) \\ &= (\tfrac{1}{9} \times 10^5)(0.1742 \times 10^4) \\ &= 19.35 \text{ million yen/year.} \end{aligned}$$

Because, in the original design, all the variables are from the lowest grade C, the tolerance cost $C(\mathbf{t})$ is 0. (Note that Mori's computation assumed that the cost for grade C is 0.) Thus, the total quality cost $F = Q + C = 19.35$. Unless otherwise stated, the unit for Q is million yen/year for the rest of the article. The results of the initial design are summarized in Table 3. The same table also contains results given by Taguchi's two-stage approach as performed by Mori (4).

In the original settings, the average value of y is 1.7632. The large difference between this value and the target 1.5 results in the high cost of 19.35. Parameter design can easily fix this problem and reduce the cost to 7.58. The use of tolerance design further reduces the cost to 4.69, which is done by upgrading the components 1, 2, 3 from C to B.

An Integrated Approach: Simultaneous Application of Parameter Design and Tolerance Design

Taguchi's two-stage approach has made a great impact in practice and is viewed as a valuable tool in quality engineering. It can, however, be improved further. By viewing F in Eq. (7) as a joint function of the nominal values $\mathbf{x}_0$ and the tolerances $\mathbf{t}$, doing a simultaneous optimization of F in $\mathbf{x}_0$ and $\mathbf{t}$ is superior to optimizing F over $\mathbf{x}_0$ with $\mathbf{t}$ fixed and then optimizing F over $\mathbf{t}$ with $\mathbf{x}_0^*$ found in the previous step. The former is what we called an integrated approach of parameter design and tolerance design. Consider the contour plots in Figure 3. In Figure 3a, the influence of $\mathbf{x}_0$ on F is independent of the values of $\mathbf{t}$; the two-stage approach leads to an optimal solution. On the other hand, the scenario depicted in Figure 3b, which is more common and realistic, would require a joint optimization of $\mathbf{x}_0$ and $\mathbf{t}$ to find an optimal solution.

In practice, there are two types of components or input variables: (1) those with tolerance requirement and (2) those without tolerance requirement. Examples of type 1 include resistances of a circuit and diameters of a mechanical part. Both the nominal values of resistances (diameters) and their tolerances are of interest and need to be determined. Examples of type 2 include variables such as position, machine type, bake time, and spin speed. Once the values of these variables are chosen, they do not change in manufacturing or product usage. Thus, they do not have any tolerance requirement. Only the nominal values are of interest and need to be determined. Keeping the two types of variables in mind, the total cost should be more clearly defined as

$$F(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n; t_1, \ldots, t_m), \tag{8}$$

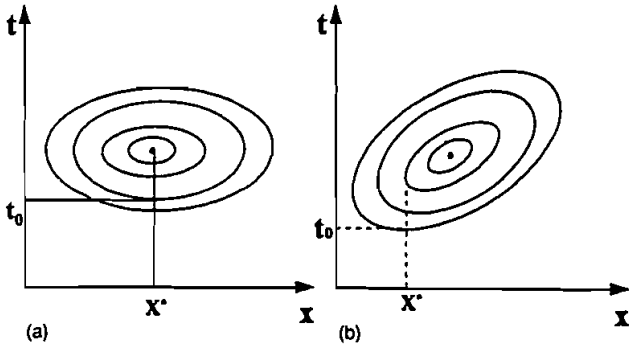

Figure 3. Contour plots of $F(\mathbf{x}_0, \mathbf{t})$.

Table 3. Summary of Results from Using the Two-Stage Approach and the Integrated Approach

DESIGN	$\mathbf{x}_0$	t	Q	C	$Q + C$
Original	(0.1, 0.3, 0.1, 0.1, 1.5, 16, 0.75)	(C, C, C, C, C, C, C)	19.35	0	19.35
After parameter design	(0.075, 0.3, 0.1, 0.115, 1.125, 16, 0.75)	(C, C, C, C, C, C, C)	7.58	0	7.58
After tolerance design	(0.075, 0.3, 0.1, 0.115, 1.125, 16, 0.75)	(B, B, B, C, C, C, C)	3.94	0.75	4.69
Integrated approach	(0.075, 0.375, 0.12, 0.12, 1.125, 20, 0.6)	(B, B, B, C, C, B, B)	3.11	1.05	4.16

where the first m variables are of type 1, the remaining of type 2, and $t_i = 3\sigma_i$ is the tolerance associated with x_i. Obviously, when m in Eq. (8) is larger, there is a greater need for joint optimization in x_i and t_i. Thus, the proposed integrated approach is more powerful when more variables are of type 1. The approach is, however, general enough to accommodate variables of type 2.

A type-1 variable has two distinct but related aspects. Its nominal value x_{i_0} can be viewed as a control factor in parameter design. The deviation $x_i - x_{i_0}$ of x_i from the nominal value x_{i_0} can be viewed as a noise factor because it is not controllable or hard to control. Taguchi (3) called this type of noise factor an *internal noise*.

A similar attempt to improve Taguchi's two-stage approach was made by Chan and Xiao (7). Recognizing the potential to improve upon Taguchi's approach, they proposed performing the two-stage design iteratively until the $\mathbf{t}$ value stops changing. Formally, at the kth stage, given $\mathbf{x}_0^{(k-1)}$ and $\mathbf{t}^{(k-1)}$ from the previous stage, find $\mathbf{x}_0^{(k)}$ to optimize $F(\mathbf{x}_0, \mathbf{t}^{(k-1)})$, and then find $\mathbf{t}^{(k)}$ to optimize $F(\mathbf{x}_0^{(k)}, \mathbf{t})$, and repeat until $\mathbf{t}^{(k)} = \mathbf{t}^{(k-1)}$. It requires extra time and material to iteratively perform the two-stage design, whereas our proposed method only requires a *single* stage to perform design optimization. It is more efficient in terms of time and total cost. The need to do parameter design and tolerance design simultaneously or in fewer iterations was also noted and advocated by Bisgaard and Ankenman (8), but no systematic methodology was developed.

Methods of Implementation for the Integrated Approach

The integrated approach is easy to grasp conceptually but its implementation needs a careful development. The main difficulty is to evaluate the Q function efficiently. Three methods are discussed here.

Taylor Expansion Method

By combining Eqs. (3) and (7), we can rewrite

$$F(\mathbf{x}_0, \mathbf{t}) = k\sigma_y^2 + k(\mu_y - y_0)^2 + \sum_{i=1}^{m} C_i(t_i). \tag{9}$$

A simple method is to use the Taylor series expansion to do *tolerance analysis*, which allows one to study the effect of the component tolerances on the variability of y. The basic idea in tolerance analysis is to linearize the functional relation f by using a Taylor series expansion:

$$\begin{aligned} y &= f(x_1, x_2, \ldots, x_n) \\ &\approx f(x_{1_0}, x_{2_0}, \ldots, x_{n_0}) + \sum_{i=1}^{m} d_i(x_i - x_{i_0}), \end{aligned}$$

where $d_i = \partial f/\partial x_i\big|_{x_i = x_{i_0}}$ is the first derivative of f evaluated at x_{i_0}. Note that in the general formulation of Eq. (9), x_{m+1} to x_n are fixed at their nominal values. The next step is to find the relationship between (σ_y, μ_y) and (x_{i_0}, t_i). The right side of Eq. (9) has three terms. The first term, σ_y^2, can be evaluated through the *first-order* Taylor expansion:

$$\sigma_y^2 \approx \sum_{i=1}^{m} d_i^2\sigma_i^2 = \frac{1}{9}\sum_{i=1}^{m} d_i^2 t_i^2. \tag{10}$$

The second term can be computed by using the *second-order* expansion:

$$y \approx f(x_{1_0}, \ldots, x_{n_0}) + \sum_{i=1}^{m} d_i(x_i - x_{i_0}) + \sum_{i=1}^{m} d_{ii}(x_i - x_{i_0})^2,$$

where $d_{ii} = \tfrac{1}{2}\,\partial^2 f/\partial x_i^2\big|_{x_i = x_{i_0}}$. Then

$$\mu_y \approx f(x_{1_0}, \ldots, x_{n_0}) + \sum_{i=1}^{m} d_{ii}\mathrm{E}((x_i - x_{i_0})^2),$$

and

$$\begin{aligned} \mu_y - y_0 &\approx f(x_{1_0}, \ldots, x_{n_0}) - y_0 + \sum_{i=1}^{m} d_{ii}\sigma_i^2 \\ &= f(x_{1_0}, \ldots, x_{n_0}) - y_0 + \frac{1}{9}\sum_{i=1}^{m} d_{ii}t_i^2. \end{aligned} \tag{11}$$

Even if $\mathbf{x}_0$ is chosen to satisfy $f(x_{1_0}, \ldots, x_{n_0}) = y_0$, there is still a difference between μ_y and y_0 unless f is strictly linear. We call this difference the *nominal-value shift*. Note that the existing approaches to tolerance analysis do not consider nominal-value shift.

By combining Eqs. (9)–(11), we obtain the approximation

$$\begin{aligned} F(\mathbf{x}_0, \mathbf{t}) \approx{}& \frac{k}{9}\sum_{i=1}^{m} d_i^2 t_i^2 \\ &+ k\left(\frac{1}{9}\sum_{i=1}^{m} d_{ii}t_i^2 + f(x_{1_0}, \ldots, x_{n_0}) - y_0\right)^2 + \sum_{i=1}^{m} C_i(t_i). \end{aligned} \tag{12}$$

The cost of computing the right expression of Eq. (12) is largely determined by the cost of evaluating d_i and d_{ii}. If

f has an explicit functional form and its first two derivatives d_i and d_{ii} can be obtained by straight differentiation or symbolic manipulation (e.g., MAPLE or MATHEMATICA), then the optimization of the right-hand expression of Eq. (12) can be done by using standard mathematical programming methods (9). Otherwise, a finite-difference approximation to d_i and d_{ii} or a Monte Carlo simulation should be employed.

Monte Carlo Method

If f does not have an explicit form or it is costly or difficult to evaluate d_i and d_{ii}, the Taylor expansion method will not be practical. An alternative is to use Monte Carlo simulation to evaluate $F(\mathbf{x}_0, \mathbf{t})$ in Eq. (9). Noting that $Q(\mathbf{x}_0, \mathbf{t})$ in $F(\mathbf{x}_0, \mathbf{t})$ is the expectation of $(f(x_1, \ldots, x_n) - y_0)^2$ with respect to the distribution of x_i, $x_i \sim N(x_{i_0}, \sigma_i^2)$, $\sigma_i = t_i/3$, $i = 1, \ldots, m$, we can approximate $Q(\mathbf{x}_0, \mathbf{t})$ by a finite sum

$$Q_N(\mathbf{x}_0, \mathbf{t}) = \frac{1}{N} \sum_{j=1}^{N} (f(\mathbf{x}_0^{(j)}) - y_0)^2, \tag{13}$$

where $\mathbf{x}_0^{(j)}$ is the jth simulation sample from the distribution of $\mathbf{x} = (x_1, \ldots, x_n)$ and N is the total number of simulations. Specifically, because x_i, $i = m + 1, \ldots, n$, are fixed at their nominal values x_{i_0}, we need only to draw simulation samples for $x_1, \ldots, x_m$. Let $x_i^{(j)}$ be the jth random sample from $N(x_{i_0}, \sigma_i^2)$, $j = 1, \ldots, N$. Then, $\mathbf{x}_0^{(j)} = (x_1^{(j)}, \ldots, x_m^{(j)}, x_{(m+1)_0}, \ldots, x_{n_0})$. On top of the Monte Carlo approximation Q_N to Q, we then need to apply some mathematical programming method to optimize $Q_N(\mathbf{x}_0, \mathbf{t}) + C(\mathbf{t})$.

The main advantage of this method is that it does not require an explicit form of f, or the computation of d_i and d_{ii}. There are some problems, however, with the direct simulation method. It often involves too many computations. This is because N has to be sufficiently large to achieve a reasonable precision, and for each simulation, many iterations are required for the optimization procedure.

OA-Based Monte Carlo Method

To reduce the number of simulations, an orthogonal-array (OA)-based Monte Carlo method is recommended. It consists of two steps:

1. Use a few representative points to simulate a normally distributed variable.
2. Use an orthogonal array to reduce the number of combinations that are used for simulations over the variables.

An effective method for Step 1 is to choose the three representative points (3)

$$x_{i_0} - \sqrt{\frac{3}{2}}\sigma_i \quad x_{i_0}, \quad x_{i_0} + \sqrt{\frac{3}{2}}\sigma_i \tag{14}$$

to simulate $N(x_{i_0}, \sigma_i^2)$. Then, Monte Carlo simulation [see Eq. (13)] is performed at the 3^n combinations of the n variables. Consider a simple example where $y = f(x_1, x_2)$. The quality loss Q can be approximated by

$$Q_9(\mathbf{x}_0, \mathbf{t}) = \frac{1}{9} \sum_{j=1}^{9} (f(\mathbf{x}_0^{(j)}) - y_0)^2,$$

where $\mathbf{x}_0^{(j)}$ are the nine combinations of $(x_1 - \sqrt{\frac{3}{2}}\sigma_1, x_1, x_1 + \sqrt{\frac{3}{2}}\sigma_1)$, and $(x_2 - \sqrt{\frac{3}{2}}\sigma_2, x_2, x_2 + \sqrt{\frac{3}{2}}\sigma_2)$. Take, for instance, the class C components in the cyclone example. Because $t_i = 25\% x_{i_0}$, $\sqrt{\frac{3}{2}}\sigma_i = \sqrt{\frac{3}{2}}(t_i/3) \approx 10\% x_{i_0}$, the three representative points are $((1 - 10\%)x_{i_0}, x_{i_0}, (1 + 10\%)x_{i_0})$. Another choice of representative points is $x_{i_0} - \sqrt{3}\sigma_i$, x_{i_0}, $x_{i_0} + \sqrt{3}\sigma_i$ (10). Although this systematic method is an effective method of simulation, the number of simulations 3^n can be prohibitively large for large n. In the cyclone example, there are seven variables and $3^7 = 2187$ points for simulation. To further reduce the number of simulations, we can use the 36-run orthogonal array OA(36, 3^7) given in Table 4. Each row of the matrix in Table 4 corresponds to a simulation run. The seven columns represent the seven variables in Eq. (1). The levels 1, 2, and 3 for each row and column denote respectively the low, middle, and high points in Eq. (14) for the given run and variables. The same OA(36, 3^7) was also used by Mori (4) to evaluate the quality loss. If 36 runs are too small to give a good precision, we can use several replicates of OA(36, 3^7) [by permuting columns and levels or using other columns in the larger OA(36, 3^{12})] to multiply the simulation size, which will still be much smaller than 2187. A general discussion on the use of orthogonal arrays for parameter design optimization can be found in Ref. 5.

Comparison of Different Methods

In this section, we first compare the three methods discussed in the last section by applying them to performing function evaluations and parameter designs. One of the methods, the Taylor expansion, is then used in the integrated approach. The result of the integrated approach is then compared with that given by the two-stage approach.

Table 4. A 36-Run OA(36, 3^7)

RUN	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	1	1	1	1	2	2	2
5	2	2	2	2	3	3	3
6	3	3	3	3	1	1	1
7	1	1	2	3	1	2	3
8	2	2	3	1	2	3	1
9	3	3	1	2	3	1	2
10	1	1	3	2	1	3	2
11	2	2	1	3	2	1	3
12	3	3	2	1	3	2	1
13	1	2	3	1	3	2	1
14	2	3	1	2	1	3	2
15	3	1	2	3	2	1	3
16	1	2	3	2	1	1	3
17	2	3	1	3	2	2	1
18	3	1	2	1	3	3	2
19	1	2	1	3	3	3	1
20	2	3	2	1	1	1	2
21	3	1	3	2	2	2	3
22	1	2	2	3	3	1	2
23	2	3	3	1	1	2	3
24	3	1	1	2	2	3	1
25	1	3	2	1	2	3	3
26	2	1	3	2	3	1	1
27	3	2	1	3	1	2	2
28	1	3	2	2	2	1	1
29	2	1	3	3	3	2	2
30	3	2	1	1	1	3	3
31	1	3	3	3	2	3	2
32	2	1	1	1	3	1	3
33	3	2	2	2	1	2	1
34	1	3	1	2	3	2	3
35	2	1	2	3	1	3	1
36	3	2	3	1	2	1	2

Table 5 compares the three methods in terms of the mean, variance, and mean squared error for function evaluations in the cyclone example. The orthogonal array OA(36, 3^7) was used in the OA-based Monte Carlo method. The results from this method (0.1742) and the Taylor expansion method (0.1713) are quite close. Both are confirmed by the result given by the Monte Carlo method with $N = 2000$ (0.1823), which is used as the benchmark.

In the cyclone example, the objective function has both continuous variables $(x_1, \ldots, x_7)$ and discrete variables $(t_1, \ldots, t_7)$: Each t_i has three possible values, so there are $3^7 = 2187$ different values of the discrete variables. In the following, we solve the optimization problem by investigating each of the 2187 cases. In each case, the objective function only depends on $\mathbf{x}_0$. Thus, the key step in the example is the optimization procedure in parameter design. To further distinguish the three methods, we apply them to parameter design at two tolerance combinations: (i) the initial choice (C, C, C, C, C, C, C) and (ii) the choice after the two-stage approach (B, B, B, C, C, C, C). The results are given in Tables 6 and 7. The subroutine E04JAF is used for the computation. It is found that the Taylor expansion method performs the best among the three. It is as accurate as the Monte Carlo method but takes much less CPU time. The Monte Carlo method is not stable, as evidenced by its lack of convergence in the case of (ii) (see Table 7). The OA-based Monte Carlo method is not CPU-intensive, but it does not perform well in finding the optimum.

Table 5. Comparison of Three Methods on Function Evaluations for the Cyclone Example

METHODS	μ	Var	$E((y - y_0)^2)$
OA-Based Monte Carlo	1.763	0.1050	0.1742
Monte Carlo	1.765	0.1120	0.1823
Taylor expansion	1.762	0.1028	0.1713

The Taylor expansion method is applied to the integrated approach in the cyclone example. To evaluate $F(\mathbf{x}_0) = E(k(y - y_0)^2) = k\sigma_y^2 + k(\mu_y - y_0)^2$, a second-order Taylor series expansion [see Eq. (11)] is used, where d_i and d_{ii} are acquired by a MAPLE program. The resulting approximation [see Eq. (12)] is optimized over $\mathbf{x}_0$ by using the subroutine E04UCF of NAG. The result is given in Table 3.

The integrated approach improves upon the two-stage approach and reduces further the total cost from 4.69 to 4.11 by upgrading the components 6 and 7 from C to B and by choosing different values of $\mathbf{x}_0$, particularly the value of x_6 from 16 to 20. This example demonstrates the effectiveness of the proposed approach in reducing cost. In the computation, the Taylor expansion method was used in the optimization procedure, whereas the OA-based Monte Carlo method was used to evaluate the quality loss at $\mathbf{x}_0^*$ so that the result is comparable to those given by Mori (4).

Conclusion

Taguchi proposed a two-stage approach to quality improvement: perform parameter design and follow by tolerance design. We propose a novel approach to integrate parameter design and tolerance design into a single stage of design optimization. It enjoys a definite advantage over the

Table 6. Comparison of Three Methods on Parameter Design at t = (C, C, C, C, C, C, C)

METHODS	$\mathbf{x}_0^*$	$E((y-y_0)^2)$	CPU
Taylor expansion	(0.079, 0.291, 0.112, 0.091, 1.489, 17.036, 0.944)	0.0665	2.09
OA-Based Monte Carlo	(0.101, 0.300, 0.086, 0.125, 1.500, 16.000, 0.750)	0.0708	0.32
Monte Carlo	(0.092, 0.291, 0.092, 0.109, 1.508, 15.996, 0.744)	0.0662	8.40

Table 7. Comparison of Three Methods on Parameter Design at t = (B, B, B, C, C, C, C)

METHODS	$\mathbf{x}_0^*$	$E((y-y_0)^2)$	CPU
Taylor expansion	(0.095, 0.294, 0.095, 0.125, 1.494, 15.995, 0.744)	0.0349	0.16
OA-Based Monte Carlo	(0.095, 0.294, 0.095, 0.125, 1.494, 15.995, 0.740)	0.0358	0.20
Monte Carlo	Fail to converge		

two-stage approach because it allows the design optimization to be done jointly in two sets of variables (i.e., the nominal values and their tolerances). The method is flexible enough to accommodate different cost structures for the component tolerances. The methodology is illustrated with a successful application to the design of cyclone.

Acknowledgments

This research was supported by AFOSR/ARPA grant no. F49620-95-1-0524 and the National Science Foundation DMS 9704649.

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