On Some Ordering Properties of the Generalized Inverses of Nonnegative Definite Matrices

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ABSTRACT

For any positive definite matrices A and B, it is known that $A \geqslant B$ iff $B^{-1} \geqslant A^{-1}$. This paper investigates the extensions of the above result to any two real nonnegative definite matrices A and B.

1. INTRODUCTION

For any real positive definite matrices A and B, it is known that $A \geqslant B$ iff $B^{-1} \geqslant A^{-1}$. The extension of this result to real nonnegative definite (n.n.d.) matrices A and B is not trivial, since the "inverse" of A or B may not be uniquely defined. In this paper we consider the following question: for any n.n.d. matrices A and B with $A \geqslant B$ and any given generalized inverse A^- of A (or B^- of B), does there exist a generalized inverse B^- of B (or A^- of A) such that $B^- \geqslant A^-$? This question is answered for every possible combination of the ranks of A, B, A^-, B^-, and A^-, B^- can be interpreted as any of the $\{1\}$-, $\{2\}$-, $\{1,2\}$-, or $\{1,2,3,4\}$-inverses of A, B. *All the matrices considered are real.* Before stating the main results of the paper, we need to give some basic facts about generalized inverses.

For any matrix A of real elements, a generalized inverse X is defined by one or several of the following conditions due to Penrose [5]:

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)' = AX, \tag{3}$$

$$(XA)' = XA, \tag{4}$$

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where A' denotes the transpose of A. Following Ben-Israel and Greville [2], a matrix X is called a $\{i,j,\dots,l\}$-inverse of A if it satisfies the equations $(i),(j),\dots,(l)$ from among the equations (1), (2), (3), (4), and it is also denoted by $A^{(i,j,\dots,l)}$. For example, a $\{1\}$-inverse is called a generalized inverse and a $\{1,2\}$-inverse is called a reflexive generalized inverse in Rao and Mitra [7]; the $\{1,2,3,4\}$-inverse (which is unique) is usually called the pseudo or Moore-Penrose inverse and also denoted by A^+. For any $\{1\}$-, $\{2\}$-, or $\{1,2\}$-inverse, it is well known that rank $A^{(1)} \geqslant$ rank A, rank $A^{(2)} \leqslant$ rank A, and rank $A^{(1,2)} =$ rank A. For any n.n.d. matrix A, an $A^{(1)}$ or $A^{(2)}$ may not even be symmetric. From Theorem 1 of Sec. 2, *any symmetric* $\{2\}$*-inverse of an n.n.d. matrix A must be n.n.d.* However, the corresponding result for $\{1\}$-inverse does not hold. A characterization of the n.n.d. $\{1\}$-inverses of any n.n.d. matrix A is given in Theorem 2. With this in mind, we will *assume that all the generalized inverses of an n.n.d. matrix considered in this paper are n.n.d.*

For any two n.n.d. matrices A and B, we define $A \geqslant B$ iff $A - B$ is n.n.d. Let the rank of $A, B, A^{(1)}, B^{(1)}$ be denoted by $r, s, \bar{r}, \bar{s}$. In order that $A \geqslant B$ and $B^{(1)} \geqslant A^{(1)}$ hold, it is necessary that

$$\bar{s} \geqslant \bar{r} \geqslant r \geqslant s. \tag{5}$$

In Theorem 5, we show that: (i) for any $(\bar{r}, \bar{s})$ satisfying (5) and any $A^{(1)}$ of rank $\bar{r}$, there exists a $B^{(1)}$ of rank $\bar{s}$ such that $B^{(1)} \geqslant A^{(1)}$; (ii) if $r = s$, then for any $(\bar{r}, \bar{s})$ satisfying (5) and any $B^{(1)}$ of rank $\bar{s}$, there exists an $A^{(1)}$ of rank $\bar{r}$ such that $B^{(1)} \geqslant A^{(1)}$. However, (ii) is not true for $r > s$. A class of counterexamples to (ii) is given in Theorem 5A. Similarly, a necessary condition for $A \geqslant B$ and $B^{(2)} \geqslant A^{(2)}$ is

$$r \geqslant s \geqslant \bar{s} \geqslant \bar{r}, \tag{6}$$

where $\bar{r} = \text{rank} A^{(2)}$, $\bar{s} = \text{rank} B^{(2)}$. In Theorem 6, we show that: (i) for any $(\bar{r}, \bar{s})$ satisfying (6) and any $B^{(2)}$ of rank $\bar{s}$, there exists an $A^{(2)}$ of rank $\bar{r}$ such that $B^{(2)} \geqslant A^{(2)}$, (ii) if $r = s$, then for any $(\bar{r}, \bar{s})$ satisfying (6) and any $A^{(2)}$ of rank $\bar{r}$, there exists a $B^{(2)}$ of rank $\bar{s}$ such that $B^{(2)} \geqslant A^{(2)}$. Again, (ii) is not true for $r > s$. For the special case $A = B + xx'$, a necessary and sufficient condition for (ii) to hold is $x'A^{(2)}x < 1$ (see Theorem 6A and Lemma 5). For a $\{1,2\}$- or $\{1,2,3,4\}$-inverse, a necessary condition on the ranks becomes $\bar{r} = \bar{s} = r = s$. Results analogous to Theorems 5(ii) and 6(ii) can be established. These are summarized in Theorems 7 and 8.

In the course of proving these theorems, we have also obtained some other results of independent interest. In Theorem 3, we show that for any $\{1\}$-inverse $A^{(1)}$ of A and any $u \geqslant \bar{r} \geqslant v \geqslant r$, $r = \text{rank } A$, $\bar{r} = \text{rank } A^{(1)}$, there exist $\{1\}$-inverses $A_u^{(1)}$ and $A_v^{(1)}$ of A with ranks u and v such that $A_u^{(1)} \geqslant A^{(1)}$

$\geqslant A_v^{(1)}$. A similar result for $\{2\}$-inverses is stated in Theorem 4. Lemmas 5 and 6 may also be useful in other contexts. They do not seem to exist in the literature.

The question originally arose in the context of a statistical problem. For its applications in statistics, see Cheng and Wu [3] and Milliken and Akdeniz [4].

2. SOME RESULTS CONCERNING THE NONNEGATIVE DEFINITE GENERALIZED INVERSES OF A NONNEGATIVE DEFINITE MATRIX

For any $n\times n$ matrix A, we denote the column space of A by $\mathfrak{M}(A)$. A rank (or full rank) factorization of A with rank r is $A=FG'$, where F and G are both $n\times r$ matrices of rank r. The identity matrix of order r is denoted by I_r.

LEMMA 1. *Let $A=QQ'$ be a rank factorization of an $n\times n$ n.n.d. matrix A with rank r. Then G is a symmetric $\{1,2\}$-inverse of A iff G can be expressed as $G=HH'$, where H is an $n\times r$ matrix and $H'Q=I_r$. In particular, any symmetric $\{1,2\}$-inverse of an n.n.d. matrix is n.n.d.*

Proof. This can be proved simply by modifying the argument of Lemma 2.5.2 of Rao and Mitra [7, p. 28]. It can also be found in Pringle and Rayner [6, p. 25]. ■

THEOREM 1. *Any symmetric $\{2\}$-inverse $A^{(2)}$ of an n.n.d. matrix A is n.n.d.*

Proof. Since $A=PP'$ for some matrix P and $A^{(2)\prime}=A^{(2)}$, $A^{(2)}=A^{(2)}AA^{(2)}=(A^{(2)}P)(A^{(2)}P)'$ shows that $A^{(2)}$ is n.n.d. ■

Since any $n\times n$ n.n.d. matrix A of rank r, $r<n$, can always be expressed as

$$P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}P', \tag{7}$$

where P is an $n\times n$ nonsingular matrix, it follows from a result due to R. C. Bose [6, p. 8] that any symmetric $\{1\}$-inverse $A^{(1)}$ of A can be expressed as

$$(P')^{-1}\begin{bmatrix} I_r & U \\ U' & W \end{bmatrix}P^{-1}, \tag{8}$$

where U and W are arbitrary $r\times(n-r)$ and $(n-r)\times(n-r)$ matrices, and W is symmetric. It is clear that the symmetric $A^{(1)}$ is n.n.d. iff the middle matrix in (8) is n.n.d., which is not always true. Therefore, no result analogous to Theorem 1 holds for $\{1\}$-inverse. Later on, a different characterization of the n.n.d. $\{1\}$-inverses will be needed. This is stated as

THEOREM 2. *For any n.n.d. matrix A of rank r, $A^{(1)}$ is n.n.d. with rank s iff $A^{(1)}$ is a symmetric $\{1,2\}$-inverse of $A+XX'$, where X is of order $n\times(s-r)$ and* $\operatorname{rank}(A:X)=s$.

Proof. The "if" part follows from Theorem 2.7.1 of Rao and Mitra [7, p. 31].

For the "only if" part, we give an algebraic proof. From (8), any n.n.d. $A^{(1)}$ of rank s can be expressed as

$$(P')^{-1}\begin{pmatrix} I_r & U \\ U' & W \end{pmatrix}P^{-1},$$

where

$$\begin{pmatrix} I_r & U \\ U' & W \end{pmatrix}=\begin{pmatrix} F' \\ G' \end{pmatrix}(F:G) \tag{9}$$

is a rank factorization with $F'F=I_r$, and F and G are $s\times r$ and $s\times(n-r)$ matrices with $\operatorname{rank}(F:G)=s$. Let F_2 be an $s\times(s-r)$ matrix such that $F_2'F_2=I_{s-r}$ and $F'F_2=0$. Then, from $(F:F_2)'(F:F_2)=I_s$, we have

$$(F:F_2)(F:F_2)'=I_s=FF'+F_2F_2'. \tag{10}$$

(i) First we want to show that there exists an $n\times(s-r)$ matrix X such that

$$\begin{pmatrix} F' \\ G' \end{pmatrix}(F:G)=\left(\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}+XX'\right)^{(1,2)} \quad \text{and} \quad \operatorname{rank}\left[\begin{pmatrix} I_r \\ 0 \end{pmatrix} \vdots\ X\right]=s. \tag{11}$$

Since rank $(F:G)=s$, the equation

$$F_2=(F:G)X \tag{12}$$

is consistent. Any solution X can be expressed as

$$X=(F:G)^{(1)}F_2. \tag{13}$$

From (12), (13), and $F'F_2=0$, it follows that

$$X'\begin{pmatrix} F' \\ G' \end{pmatrix}(F:G)X=I_{s-r}, \tag{14}$$

and

$$(I_r:F'G)X=F'(F:G)X=0. \tag{15}$$

From (10), (12), (14), (15), it can be easily verified that the X defined in (13) satisfies (11).

(ii) From Theorem 1.8 of Pringle and Rayner [6, p. 13], if P_1 and P_2 are nonsingular, then $P_2^{-1}A^{(1,2)}P_1^{-1}$ is also a $\{1,2\}$-inverse of P_1AP_2. Therefore, from (9) and (11), the n.n.d. matrix $A^{(1)}$ is a $\{1,2\}$-inverse of

$$P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}P'+PXX'P'=A+(PX)(PX)',$$

completing the proof. ■

REMARK. Rao and Mitra [7, p. 31] have proved that $A^{(1)}$ is a $\{1\}$-inverse with rank s, $s\geqslant r=\text{rank } A$, iff it is a $\{1,2\}$-inverse of $A+XY'$, where X and Y are of order $n\times(s-r)$ and

$$\text{rank}(A:X)=\text{rank}\begin{pmatrix} A \\ Y \end{pmatrix}=s.$$

Their result does not include the "only if" part of our Theorem 2 as special case.

THEOREM 3. *Let A be an $n\times n$ n.n.d. matrix of rank r. For any n.n.d. $A^{(1)}$ of rank t and any $r\leqslant s\leqslant t\leqslant u$, there exist n.n.d. $\{1\}$-inverses $A_s^{(1)}$ of rank s and $A_u^{(1)}$ of rank u such that $A_s^{(1)}\leqslant A^{(1)}\leqslant A_u^{(1)}$.*

Proof. Let $A=WW'$ be a rank factorization of A, where W is an $n\times r$ matrix of rank r. From Lemma 1 and Theorem 2, there exists an $n\times(t-r)$ matrix X with $\text{rank}(W:X)=t$ and $A^{(1)}=Z'Z$, where Z is a $t\times n$ matrix satisfying

$$Z[W:X]=I_t. \tag{16}$$

Let Z_s be the submatrix of Z consisting of the first s rows of Z, and X_1 be the submatrix of X consisting of the first $s-r$ columns of X. Then (16) implies $Z_s[W:X_1]=I_s$, and from Lemma 1 and Theorem 2 $Z_s'Z_s$ is a $\{1\}$-inverse of A with rank s and $Z_s'Z_s \leqslant A^{(1)}$. On the other hand, let $\{v_{t+1},\dots,v_u\}$ be $u-t$ orthonormal $n\times 1$ vectors which are orthogonal to $\mathfrak{M}(W:X)$. Define $X_2=[X:v_{t+1},\dots,v_u]$ and $Z_u'=[Z':v_{t+1},\dots,v_u]$. Then (16) implies $Z_u[W:X_2]=I_u$, and $Z_u'Z_u$ is a $\{1\}$-inverse of A with rank u and $Z_u'Z_u \geqslant A^{(1)}$. ∎

THEOREM 4. *Let A be an $n\times n$ n.n.d. matrix of rank r. For any n.n.d. $A^{(2)}$ of rank t and any $r \geqslant s \geqslant t \geqslant u$, there exist n.n.d. $\{2\}$-inverses $A_s^{(2)}$ of rank s and $A_u^{(2)}$ of rank u such that $A_s^{(2)} \geqslant A^{(2)} \geqslant A_u^{(2)}$.*

Proof. Let $A^{(2)}=YY'$ be a rank factorization of $A^{(2)}$, where Y is an $n\times t$ matrix of rank t. By multiplying a left inverse of Y and a right inverse of Y' by $YY'AYY'=YY'$, we get $Y'AY=I_t$. Let Y_u be the submatrix of Y consisting of the first u columns. Then $Y_u'AY_u=I_u$ and $Y_uY_u'AY_uY_u'=Y_uY_u'$. Y_uY_u' is a $\{2\}$-inverse of A with rank u, and $Y_uY_u' \leqslant A^{(2)}$. On the other hand, let $A=WW'$ be a rank factorization of A, where W is an $n\times r$ matrix of rank r. $Y'AY=(Y'W)(Y'W)'=I_t$, and the t column vectors $\{v_1,\dots,v_t\}$ of $W'Y$ are orthonormal vectors in $\mathfrak{M}(W')$. Since $\dim \mathfrak{M}(W')=r$, we can choose $\{v_{t+1},\dots,v_s\}$ from $\mathfrak{M}(W')$ such that $\{v_i\}_{i=1}^{s}$ are orthonormal. Then $[v_1,\dots,v_s]=[W'Y:W'Z]$ for some $n\times(s-t)$ matrix Z, and

$$\begin{bmatrix} Y' \\ Z' \end{bmatrix} WW'[Y:Z]=I_s.$$

Therefore, $YY'+ZZ' \geqslant YY'=A^{(2)}$ and is a $\{2\}$-inverse of A with rank s. ∎

COROLLARY 1. *Let A be an $n\times n$ n.n.d. matrix of rank r and $r_2 \leqslant r \leqslant r_1$. For any $A^{(1)}$ of rank r_1, there exists an $a^{(2)}$ of rank r_2 such that $A^{(2)} \leqslant A^{(1)}$. Conversely, for any $a^{(2)}$ of rank r_2, there exists an $A^{(1)}$ of rank r_1 such that $A^{(1)} \geqslant A^{(2)}$.*

LEMMA 2.

(i) *For any $\{1\}$-inverse $A^{(1)}$ of A and any $x \in \mathfrak{M}(A)$,*

$$A^{(1)} - \frac{A^{(1)}xx'A^{(1)}}{1+x'A^{(1)}x} \tag{17}$$

is a $\{1\}$*-inverse of* $A+xx'$*. A similar result holds for* $\{1,2\}$*- and* $\{1,2,3,4\}$*-inverses.*

(ii) *For any* $\{2\}$*-inverse* $A^{(2)}$ *of* A *and any* x,

$$A^{(2)}-\frac{A^{(2)}xx'A^{(2)}}{1+x'A^{(2)}x} \tag{17'}$$

is a $\{2\}$*-inverse of* $A+xx'$.

The proof can be obtained by straightforward matrix multiplication. Note that in (ii) x can be arbitrary. See also Albert [1, p. 47], Pringle and Rayner [6, p. 33], and Rao and Mitra [7, p. 40].

LEMMA 3. *For any n.n.d. matrix* M *of rank* r *and any vector* x, $M-Mxx'M/(1+x'Mx)$ *has rank* r *and the same null space as* M.

The proof can be obtained simply from the Cauchy-Schwarz inequality. From Lemmas 2 and 3, we have

LEMMA 4.

(i) *For any n.n.d.* $\{1\}$*-inverse* $A^{(1)}$ *of an n.n.d. matrix* A *and any* $x\in\mathfrak{M}(A)$, *there exists an n.n.d.* $\{1\}$*-inverse* $(A+xx')^{(1)}$ *of* $A+xx'$ *such that* $(A+xx')^{(1)}\leqslant A^{(1)}$ *and both have the same rank. Similar result holds for* $\{1,2\}$*- and* $\{1,2,3,4\}$*-inverses.*

(ii) *Without assuming* $x\in\mathfrak{M}(A)$, *the result in* (i) *holds for* $\{2\}$*-inverses.*

LEMMA 5. *Let* B *be an n.n.d. matrix and* $A=B+xx'$ *with* $x\in\mathfrak{M}(B)$. *Then* $x'A^{(1)}x<1$ *and* $x'A^{(2)}x<1$ *hold for any n.n.d. matrices* $A^{(1)}$ *and* $A^{(2)}$.

Proof. From Corollary 1, it suffices to prove $x'A^{(1)}x<1$ for any n.n.d. $A^{(1)}$. Let $B=UU'$ be a rank factorization, where U is an $n\times r$ matrix of rank r, $r=\text{rank } B$. Then $x=Uc$ for some $r\times 1$ vector c, and $A=U(I+cc')U'=UKK'U'$, $K=(I+cc')^{1/2}$. According to Theorem 2, any n.n.d. $A^{(1)}$ of rank $\bar{r}\geqslant r$ is a $\{1,2\}$-inverse of $UKK'U'+WW'$ for some $n\times(\bar{r}-r)$ matrix W with $\text{rank}[UK:W]=\bar{r}$. From Lemma 1, $A^{(1)}=Z'Z$, where Z is an $\bar{r}\times n$ matrix satisfying $Z[UK:W]=I_{\bar{r}}$; this implies

$$ZUK=\begin{pmatrix} I_r \\ 0 \end{pmatrix}.$$

Therefore,

$$x'A^{(1)}x = c'U'Z'ZUc = c'K^{-1}(ZUK)'ZUKK^{-1}c$$
$$= c'K^{-2}c = c'(I+cc')^{-1}c < 1. \qquad \blacksquare$$

LEMMA 6. *Let B be an n.n.d. matrix and $A = B + xx'$.*

(i) *If $x \in \mathfrak{M}(B)$, then for any n.n.d. $\{1\}$-inverse $A^{(1)}$ of A,*

$$A^{(1)} + \frac{A^{(1)}xx'A^{(1)}}{1 - x'A^{(1)}x} \tag{18}$$

is well defined and a $\{1\}$-inverse of B and has the same rank and null space as $A^{(1)}$. Therefore, for any n.n.d. $A^{(1)}$, there exists an n.n.d. $B^{(1)}$ such $B^{(1)} \geqslant A^{(1)}$ and both have the same rank. Similar results holds for $\{2\}$-, $\{1,2\}$-, and $\{1,2,3,4\}$-inverses.

(ii) *For any n.n.d. $\{2\}$-inverse $A^{(2)}$ of A such that $x'A^{(2)}x \neq 1$,*

$$A^{(2)} + \frac{A^{(2)}xx'A^{(2)}}{1 - x'A^{(2)}x} \tag{18$'$}$$

is a $\{2\}$-inverse of B. If it is further assumed that $x'A^{(2)}x < 1$, then all the results in (i) *hold for $\{2\}$-inverses.*

Proof. From Lemma 5, $1 - x'A^{(1)}x > 0$ and the matrix (18) is well defined. All the other assertions in (i) and (ii) can be verified by routine computation.

3. ON THE REVERSE ORDERING PROPERTY: $A \geqslant B$ IMPLIES $A^- \leqslant B^-$

THEOREM 5. *Let A and B be two $n \times n$ n.n.d. matrices with ranks r and s, $A \geqslant B$, and $\bar{r}, \bar{s}$ be two positive integers satisfying $\bar{s} \geqslant \bar{r} \geqslant r \geqslant s$.*

(i) *For any n.n.d $\{1\}$-inverse $A^{(1)}$ of rank $\bar{r}$, there exists an n.n.d. $\{1\}$-inverse $B^{(1)}$ of rank $\bar{s}$ such that $B^{(1)} \geqslant A^{(1)}$.*

(ii) *If $r = s$, then for any n.n.d. $\{1\}$-inverse $B^{(1)}$ of rank $\bar{s}$, there exists an n.n.d. $\{1\}$-inverse $A^{(1)}$ of rank $\bar{r}$ such that $B^{(1)} \geqslant A^{(1)}$.*

Proof. In view of Theorem 3, it suffices to prove the result for $\bar{r}=\bar{s}$.

(i): Let $A-B=\sum_{i=1}^{t}x_ix_i'$ be a rank factorization of $A-B$, where x_i is an $n\times 1$ vector and $\operatorname{rank}(A-B)=t$. Without loss of generality, we can assume that $\operatorname{rank}[B:x_1,\ldots,x_{r-s}]=r$ and $\{x_i\}_{i=r-s+1}^{t}\subset \mathfrak{M}(B+\sum_{i=1}^{r-s}x_ix_i')$. Therefore, for any given $A^{(1)}$, there exists a $\{1\}$-inverse $(B+\sum_{i=1}^{r-s}x_ix_i')^{(1)}$ of $B+\sum_{i=1}^{r-s}x_ix_i'$ such that $(B+\sum_{i=1}^{r-s}x_ix_i')^{(1)}\geqslant A^{(1)}$ and they both have rank $\bar{r}$; this follows by repeatedly applying Lemma 6(i). It remains to show that any n.n.d. $\{1\}$-inverse of $B+\sum_{i=1}^{r-s}x_ix_i'$ is an n.n.d. $\{1\}$-inverse of B. From Theorem 2, any n.n.d. $\{1\}$-inverse of $B+\sum_{i=1}^{r-s}x_ix_i'$ with rank u is an n.n.d. $\{1,2\}$-inverse of $B+\sum_{i=1}^{r-s}x_ix_i'+UU'$, where U is an $n\times(u-r)$ matrix and $\operatorname{rank}[B:x_1\cdots x_{r-s}:U]=u$; this $\{1,2\}$-inverse, again from Theorem 2, is an n.n.d. $\{1\}$-inverse of B.

(ii): If $r=s$, then $A-B=\sum_{i=1}^{t}x_ix_i'$ for some $n\times 1$ vectors $\{x_i\}_{i=1}^{t}$ and $\{x_i\}_{i=1}^{t}\subset\mathfrak{M}(B)$. The desired result follows from a repeated use of Lemma 4(i). ■

For $r>s$, the result in Theorem 5(ii) is not always true. In fact, we can single out a class of n.n.d. $\{1\}$-inverses $B^{(1)}$ of rank $\bar{r}$ such that no n.n.d. $\{1\}$-inverses $A^{(1)}$ of rank $\bar{r}$ will be dominated by $B^{(1)}$.

THEOREM 5A. *Let $A=B+xx'$, where B is n.n.d. of rank s and $x\notin\mathfrak{M}(B)$. (Therefore,* $\operatorname{rank}A=s+1$.) *For any n.n.d. $\{1\}$-inverse $B^{(1)}$ of rank $\bar{s}\geqslant s+1$ with $x'B^{(1)}x<1$, then there exists no n.n.d. $\{1\}$-inverse $A^{(1)}$ of rank $\bar{s}$ such that $B^{(1)}\geqslant a^{(1)}$.*

Proof. From Lemma 1 and Theorem 2, any n.n.d. $A^{(1)}$ of rank $\bar{s}$ can be expressed as $A^{(1)}=XX'$, where X is an $n\times\bar{s}$ matrix of rank $\bar{s}$ and $X'[U:W:x]=I_{\bar{s}}$, $B=UU'$, and W is an $n\times(\bar{s}-s-1)$ matrix. Let $X=[x_1,\ldots,x_{\bar{s}}]$. Then $x_i'x=0$ for $1\leqslant i\leqslant\bar{s}-1$ and $x_{\bar{s}}'x=1$. If $B^{(1)}\geqslant A^{(1)}$ and both have rank $\bar{s}$, then $B^{(1)}=X(I+CC')X'$ for some n.n.d. $\bar{s}\times\bar{s}$ matrix CC'. But then $x'B^{(1)}x=x'X(I+CC')X'x=$[the $(\bar{s},\bar{s})$ entry of $I+CC'$]$\geqslant 1$, a contradiction. ■

The conclusion of Theorem 5A can not be reversed simply by imposing $x'B^{(1)}x\geqslant 1$. This is seen from the following example. Let

$$B=\begin{pmatrix} I_{n-1} & 0\\ \mathbf{0}' & 0\end{pmatrix},\qquad x'=(\mathbf{0}',1),$$

and $\mathbf{0}'$ is the $1\times(n-1)$ vector of zeros. Then $A=B+xx'=I_n=A^{-1}$. Any matrix of the form

$$N=\begin{pmatrix} I_{n-1} & \mathbf{v}\\ \mathbf{v}' & a\end{pmatrix},$$

where $\mathrm{v}'=(v_1,\ldots,v_{n-1})$ and $a>\sum_{i=1}^{n-1}v_i^2$, is a positive definite $\{1\}$-inverse of B. But then

$$N-A^{-1}=N-I_n=\begin{pmatrix} 0 & \mathrm{v} \\ \mathrm{v}' & a-1 \end{pmatrix}$$

is not n.n.d. if $v_i\neq 0$ for some $1\leqslant i\leqslant n-1$, no matter how large we choose $x'Nx=a$.

THEOREM 6. *Let A and B be two $n\times n$ n.n.d. matrices with ranks r and s, $A\geqslant B$, and $\bar r,\bar s$ be two positive integers satisfying $r\geqslant s\geqslant \bar s\geqslant \bar r$.*

(i) *For any n.n.d. $\{2\}$-inverse $B^{(2)}$ of rank $\bar s$, there exists an n.n.d. $\{2\}$-inverse $A^{(2)}$ of rank $\bar r$ such that $B^{(2)}\geqslant A^{(2)}$.*

(ii) *If $r=s$, then for any n.n.d. $\{2\}$-inverse $A^{(2)}$ of rank $\bar r$, there exists an n.n.d. $\{2\}$-inverse $B^{(2)}$ of rank $\bar s$ such that $B^{(2)}\geqslant A^{(2)}$.*

Proof. In view of Theorem 4, it suffices to prove the result for $\bar r=\bar s$.

(i): Since $A-B$ can be expressed as $\sum_{i=1}^{t}x_ix_i'$ for some $n\times 1$ vectors $\{x_i\}_{i=1}^{t}$, the desired result follows from a repeated use of Lemma 4(ii).

(ii): If $r=s$, then $A-B=\sum_{i=1}^{t}x_ix_i'$ for some $n\times 1$ vectors $\{x_i\}_{i=1}^{t}$ and $\{x_i\}_{i=1}^{t}\subset \mathfrak{M}(B)$. The desired result follows from a repeated use of Lemma 6(i). Note that Lemma 6(i) is also valid for $\{2\}$-inverses. ∎

For $r>s$, the assertion in Theorem 6(ii) may not always be true. In fact, for $A=B+xx'$ with $x\notin\mathfrak{M}(B)$ a n.a.s. condition can be found, which is given below.

THEOREM 6A. *Let $A=B+xx'$, where B is n.n.d. of rank s and $x\notin \mathfrak{M}(B)$. (Therefore,* rank $A=s+1$.)

(i) *For any n.n.d. $\{2\}$-inverse $A^{(2)}$ of rank $\bar s\leqslant s$ with $x'A^{(2)}x=1$, then there does not exist any n.n.d. $\{2\}$-inverse $B^{(2)}$ of rank $\bar s$ such that $B^{(2)}\geqslant A^{(2)}$.*

(ii) *If the $A^{(2)}$ in* (i) *satisfies $x'A^{(2)}x<1$, then there exists an n.n.d. $\{2\}$-inverse $B^{(2)}$ of rank $\bar s$ such that $B^{(2)}\geqslant A^{(2)}$.*

Note that (from the proof below) $x'A^{(2)}x\leqslant 1$ is always true.

Proof. (i): Let $A^{(2)}=UU'$ be a rank factorization of A, where U is an $n\times\bar s$ matrix of rank $\bar s$. Then $UU'AUU'=UU'$ implies $U'AU=I_{\bar s}$, which is equivalent to $U'BU=I_{\bar s}-U'xx'U$. Since $U'BU$ is n.n.d., $|x'U|^2=x'UU'x=$

$x'A^{(2)}x \leqslant 1$ is always true. For $x'A^{(2)}x = 1$, $U'BU$ becomes a singular matrix. If there exists a $\{2\}$-inverse $B^{(2)}$ of rank $\bar{s}$ such that $B^{(2)} \geqslant A^{(2)}$, then $B^{(2)} = U(I + CC')U'$ for some $\bar{s} \times \bar{s}$ matrix CC'. From $B^{(2)}BB^{(2)} = B^{(2)}$, we have $\{I + CC')U'BU(I + CC') = I_{\bar{s}}$, which implies that $U'BU$ is a nonsingular matrix, a contradiction.

(ii): This is contained in Lemma 6(ii). ■

The situation for $\{1,2\}$-inverse is more satisfactory. Since rank $A = \text{rank } A^{(1,2)}$, in order that $A \geqslant B$ and $A^{(1,2)} \leqslant B^{(1,2)}$ both hold, it is necessary to have rank $A = \text{rank } B = \text{rank } A^{(1,2)} = \text{rank } B^{(1,2)}$. Now, using the same kind of argument employed in the proof of Theorems 5(ii) and 6(ii), the following theorem can be easily proved. Note that both Lemmas 4(i) and 6(i) hold for $\{1,2\}$-inverses.

THEOREM 7. *Let A and B be two n.n.d. matrices of rank r, $A \geqslant B$. For any n.n.d. $\{1,2\}$-inverse $A^{(1,2)}$ of A with rank r, there exists an n.n.d. $\{1,2\}$-inverse $B^{(1,2)}$ of B with rank r such that $B^{(1,2)} \geqslant A^{(1,2)}$; and vice versa.*

It should also be pointed out that Theorem 7 also holds for $\{1,2,3\}$- and $\{1,2,4\}$-inverses. Both Lemmas 4(i) and 6(i) on which the proof is based can be extended to the $\{1,2,3\}$- and $(1,2,4\}$-inverses. The same kind of argument can also be applied to the case of the $\{1,2,3,4\}$-inverse. Since the inverse is unique, the theorem takes a different form. It is worth noting that A is also the $\{1,2,3,4\}$-inverse of $A^{(1,2,3,4)}$.

THEOREM 8. *Let A and B be two n.n.d. matrices of rank r. Then $A \geqslant B$ iff $B^{(1,2,3,4)} \geqslant A^{(1,2,3,4)}$.*

This result was obtained by Milliken and Akdeniz [4] with a different and longer proof.

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