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SOME RESTRICTED RANDOMIZATION RULES IN SEQUENTIAL DESIGNS

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Key Words and Phrases: clinical trial; biased coin design; urn design; big stick design; accidental bias; selection bias; two coin design; square root design.

ABSTRACT

This paper presents a new class of designs (Big Stick Designs) for sequentially assigning experimental units to treatments, when only the time covariate is considered. By prescribing the degree of imbalance which the experimenters can tolerate, complete randomization is used as long as the imbalance of the treatment allocation does not exceed the prescribed value. Once it reaches the value, a deterministic assignment is made to lower the imbalance. Such designs can be easily implemented with no programming and little personnel support. They compare favorably with the Biased Coin Designs, the Permuted Black Designs, and the Urn Designs, as far as the accidental bias and selection bias are concerned. Generalizations of these designs are considered to achieve various purposes, e.g., avoidance of deterministic assignments, early balance, etc.

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1. INTRODUCTION

We consider the situation where eligible subjects arrive sequentially and must be assigned immediately to one of two treatments. In this context, a statistical design is simply an allocation rule, which determines how the treatment for the (m+1)th subject is chosen, given the first m assignments. That the subjects should be assigned somewhat randomly would be agreed upon among Fisherian statisticians. Randomization is regarded as fundamental since it provides protection against bias that unknown covariates may introduce into the experiment. However pure randomization may produce extremely unbalanced allocations across some known prognostic factors. There is some controversy on the best way of dealing with these imbalances. To fix the idea, we will consider clinical trial, where eligible patients arrive sequentially and must receive one of the treatments immediately. The results obtained are equally applicable to other sequential experiments, e.g. cloud seeding experiments. Simon (1979) among others argues that one should avoid them through design techniques. He advocates designs (Pocock and Simon, 1974) that force balance across the prognostic factors. These designs however involve a fair amount of computation and may not be used for this reason. On the other hand, Peto (1978) considers the patient stratification at design stage a nuisance, arguing that any imbalance problem on the known covariates can be dealt with through analysis of covariance technique.

A compromise that is being used in some cancer research centers is to stratify patients according to a small number of covariates and then use a simple randomization rule inside each stratum.

From now on we assume that this compromise is being used and therefore we drop all references to patient stratification. Ideally an allocation procedure would be random and would produce balanced allocations. The main reason for balanced allocation is statistical precision. The test and estimation procedures to be

used in analysing the data are usually more efficient if the allocation is balanced. Also a balanced design is an important factor in convincing others of the validity of results of the trial. These two goals, balance and randomization, are to some extent conflicting and compromise is needed. Since the assignment is frequently handled without computers and/or by non-technical staff, a third desirable property in patient allocation rules is simplicity. See Bailar (1977).

In order to work out the compromise mentioned above, some design criteria should be defined. The criteria being used in the literature are selection bias, which reflects how predictable the assignments are, and accidental bias, which measures the influence of the neglected covariates on statistical analysis. (See Sections 3.2 and 3.3 for precise definitions).

In a pioneering paper Efron (1971) advocates the use of the Biased Coin Design (BCD). Under BCD the assignments are made using a biased coin whenever the allocation is unbalanced. This procedure forces the allocation towards balance in all steps where some imbalance is present. Wei (1977) proposes the Urn Design (UD) as a modification arguing that the probabilities in the biased coin should depend on the size of the imbalance and on the number of patients already in the trial. While these allocation rules are not too hard to implement and enjoy good theoretical properties, the Permuted Block Design (PBD) is probably the most commonly used in practice, mainly due to the simplicity of implementation. For two treatments this design divides the experiment into blocks of even length, say 2b, and within each block assigns b subjects to each treatment in such a way that all $\binom{2b}{b}$ combinations are equally likely. Its main weakness is its high selection bias. An improved version is considered at the end of Section 4.

The Big Stick Design (BSD) to be proposed resembles the PBD, but has low selection bias and low accidental bias. More importantly it can be easily implemented without much personnel

and/or programming support. Since the preference for treatment allocation rules varies greatly, our proposal provides an alternative, which some practitioners may find useful, To avoid the occasional deterministic assignments in the BSD, we consider a general class of designs in Section 4. If $p < 1$ in (4.1), no deterministic assignments are made. Special cases of (4.1) include the BCD, the BSD and other new designs like the Two Coin Design and the Square Root Design. In the next section the Big Stick design is defined. It properties are studied in Section 3. Some generalizations are suggested in Section 4. In Section 5 the proposed designs are compared with other existing designs according to several criteria.

2. BIG STICK DESIGN

This design is based on the intuitive notion that for practical purposes a moderately unbalanced allocation is almost as efficient as the perfectly balanced allocation. This is shown for different models by Pocock (1979) and Soares and Wu (1982). The Big Stick Design (BSD) is such that a completely random assignment is made as long as the imbalance is within a prespecified barrier $\underline{a}$. When the imbalance is about to cross the barrier $\underline{a}$, a deterministic assignment is made to lower the imbalance to an acceptable level. Such an assignment has also been recommended by Zelen (1974) in different context. The term "Big Stick" reflects the forcefulness of the deterministic assignment when imbalance reaches an intolerable limit.

Let $\tilde{D}_m$ be the difference between the number of patients assigned to treatment A and treatment B after the mth patient has been assigned. The treatment to the (m+1)th patient is assigned according to the following rule:

If $|\tilde{D}_m| < a$, use completely random allocation.
If $\tilde{D}_m = a$, assign treatment B.
If $\tilde{D}_m = -a$, assign treatment A.

This design is very simple to implement in practice. It requires only a fair coin and a control chart of how many patients have been assigned to each treatment.

While N is usually unknown, a lower bound for it can always be obtained from budget or time constraints, previous experiments, etc. This value and the power computations as presented by Pocock (1979) and Soares and Wu (1982) are good guides in selecting the barrier a.

The BSD and the PBD are similar. In fact for a = 1, the BSD reduces to the PBD with block size 2. The BSD, while keeping the nice features of the PBD, i.e. simplicity and low accidental bias, has a much lower selection bias. This is due to the reduction on the average number of deterministic assignments. The PBD has at least one deterministic assignment in each block, while in BSD they occur randomly and less often. Deterministic assignments are present in many designs used in practice, Zelen (1974), Pocock and Simon (1975). In multi-institutional trials they do not cause much concern. In single-institutional trials the experimenter's ability to guess correctly the next assignment can be greatly hampered by simple bureaucratic devices.

Some situations require a more strict balance at the early stage of the experiment. This can be readily achieved within the 'Big Stick' scheme by choosing a barrier which is a function of the number of assignments already made. To avoid deterministic assignments when $\tilde{D}_m$ reaches the barrier, a biased coin with $\frac{1}{2} < p < 1$ can be used. These and other extensions of BSD(a) are discussed in Section 4.

3. SOME PROPERTIES OF BSD(a)

3.1 Final Imbalance Distribution

Let $D_m = |\tilde{D}_m|$ be the absolute imbalance at step m. Both $\{D_m\}_1^\infty$ and $\{\tilde{D}_m\}_1^\infty$ under BSD form finite Markov chains. $\{D_m\}_1^\infty$ has states 0,1,2,...a and transition probabilities $P[D_{m+1} = j+1 \mid D_m =$

TABLE I. Final Imbalance Distribution for BSD(6): $100P[D_m = j]$

j	m = 6	10	20	30	40	50	∞
0	31.250	24.609	18.543	17.121	16.772	16.692	16.666
2	46.875	41.211	35.210	33.778	33.439	33.359	33.333
4	18.750	25.391	31.456	32.888	33.228	33.309	33.333
6	3.125	8.789	14.790	16.221	16.561	16.642	16.666

$j] = 1/2$ for $0 < j < a$, $P[D_{m+1} = a-1 \mid D_m = a] = 1$ and $P[D_{m+1} = 1 \mid D_m = 0] = 1$.

At the beginning of the trial there are no patients, i.e., $P[D_0 = 0] = 1$. As a consequence it takes some assignments before the limiting distribution can be used as a good approximation to the distribution of D_m. See Table I. The limiting distribution γ_j is then given by (Cox and Miller, 1965)

(3.1)

i) a even, $\gamma_0 = \gamma_a = \frac{1}{a}$,
$\gamma_j = 2/a$ if j even and $0 < j < a$,
0 otherwise;

ii) a odd, $\gamma_j = \frac{2}{a+1}$ if j odd and $0 < j < a$,
0 otherwise.

From now on we take a as an even number.

3.2 Selection Bias

If the experimenter knows or attempts to guess which treatment a patient will receive before this patient is selected, then he can consciously or unconsciously bias the experiment by his decision of whether or not the individual is a suitable experimental subject. Blackwell and Hodges (1957) define selection bias of a patient allocation rule as the expected number E(G) of correct guesses of treatment assignments which the experimenter can make if he guesses optimally.

Since for the completely random allocation rule E(G) = N/2

and $E(G) \geq \frac{N}{2}$ for any design and any reasonable guessing strategy we will use $B_s = E(G) - N/2$, the excess selection bias, for measuring this selection bias of a design.

BSD(a) has only two types of assignments, the deterministic ones which are completely predictable and the pure random ones which are completely unpredictable. Therefore the best strategy against it is to guess treatment A if $\tilde{D}_m = -a$ and treatment B if $\tilde{D}_m = a$, with no preferred guess if $-a < \tilde{D}_m < a$. In order to compute the selection bias call V_N the number of times the imbalance is equal to the barrier a. Given V_N there exists $N-V_N$ assignments which are completely unpredictable and V_N deterministic. Therefore

$$(3.2) \qquad E(G) = E(E(G|V_N)) = E(\tfrac{1}{2}(N-V_N) + V_N) = \frac{N}{2} + \frac{1}{2}E(V_N)$$

The distribution of V_N can be determined using the theory of delayed recurrent events (Feller, 1977, p. 316). Its expectation is $E(V_N) = \sum_{m=1}^{N} P[D_m = a]$. Karlin (1968) gives explicit formulae for $P[D_m = a]$ which can be used to get exact results. This is done in Table II.

Note that for N = 50 BSD(10) has a very small excess selection bias 0.5270, which corresponds to one deterministic assignment on the average. The most extreme allocation this design can produce is (20,30) or (30,20) which, as shown by Pocock

TABLE II. Expectation and Standard Deviation of V_N and Excess Selection Bias for BSD(a)

	N = 10	20	30	40	50
			a = 6		
$E(V_N)$	0.182	0.834	1.624	2.448	3.278
s.d. (V_N)	0.516	1.339	2.004	2.543	2.998
B_s	0.091	0.417	0.812	1.224	1.639
			a = 10		
$E(V_N)$	0.002	0.089	0.320	0.654	1.054
s.d. (V_N)	0.044	0.430	0.953	1.491	2.004
B_s	0.001	0.045	0.160	0.327	0.527

(1979), Soares and Wu (1982), induces only a small loss of precision.

3.3 Exact and Asymptotic Accidental Bias.

Let $T_m = 1$ or -1 according as the mth patient is assigned to treatment A or B. The vector $\underset{\sim}{T} = (T_1,\ldots,T_N)$ has mean 0 and covariance matrix Z. The accidental bias of the design is defined, Efron (1971), to be the maximum eigenvalue of Z as a measure of vulnerability to unknown covariates.

For BSD(a) $T_m = \tilde{D}_m - \tilde{D}_{m-1}$ and a typical element of Z is given by $\mathrm{Cov}(T_m,T_\ell) = E(T_mT_\ell) = E(\tilde{D}_m\tilde{D}_\ell) + E(\tilde{D}_{m-1}\tilde{D}_{\ell-1}) - E(\tilde{D}_m\tilde{D}_{\ell-1}) - E(\tilde{D}_{m-1}\tilde{D}_\ell)$. Since $\{\tilde{D}_m\}_1^\infty$ forms a finite Markov chain, it is possible to assess the joint distribution of $\tilde{D}_m$ and $\tilde{D}_\ell$ for all m,ℓ and through it to compute $E(\tilde{D}_m\tilde{D}_\ell)$. The results for a = 6,10 are presented in Table III.

TABLE III. Accidental Bias for BSD(a)

	N				
a	10	20	30	40	50
6	1.137	1.367	1.509	1.606	1.676
10	1.000	1.065	1.163	1.251	1.376

For BSD(a) it is clear that as a increases the variables T_i become less correlated, implying that the accidental bias of BSD(a) decreases as a increases. Since BSD(1) is identical to PBD(2), which has accidental bias 2 as shown by Efron (1971), intuitively 2 should be an upper bound for the accidental bias of BSD(a) for all a. This is proved below. Since the accidental bias of a design can be as big as N, all designs in the BSD(a) family have reasonably small accidental bias.

Efron (1971) suggests that the accidental bias of a design should be computed as if $T_1,\ldots,T_N$ were derived from a stationary process. Specifically one should consider the sequence $T_{h+1},\ldots,T_{h+N}$ with both h and N approaching infinity. Making these assumptions he then shows that, instead of computing the

accidental bias through the definition, one can use the following lemma. See also Steele (1980).

Lemma 1 (Efron, 1971) Consider the stationary process $T_{h+1},\ldots,T_{h+N}$. Let $\rho_k = \lim_{h\to\infty} E(T_h T_{h+k})$ be the asymptotic autocovariance function. The associated spectral density is given by $f(w) = \sum_{-\infty}^{\infty} \rho_k e^{-iwk}$. If $\sum|\rho_k| < \infty$, then $f(w)$ exists as a continuous even function,

$$f(w) = 1 + 2\sum_{k=1}^{\infty} \rho_k \cos(wk) \tag{3.3}$$

and the asymptotic accidental bias of the design which generates T_i is given by the maximum value of $f(w)$.

Note that the definition of $f(w)$ differs by a factor of 2π from the usual formula for a spectral density. In order to use Lemma 1 for BSD(a) some additional lemmas are needed.

Lemma 2 The limiting autocovariance ρ_j for BSD(a) are

$\rho_j = 0$ if j is even

$= \frac{1}{2a}\{\Pr[\tilde{D}_j = -a|\tilde{D}_1 = a] - \Pr[\tilde{D}_j = a|\tilde{D}_1 = a]\}$ if j is odd.

Lemma 3

$$\sum_{j=1}^{\infty} |\rho_j| = 1/2.$$

Both proofs are given in the appendix.

Proposition 1. The asymptotic accidental bias of BSD(a) is 2.

Proof. By Lemma 1 we need only to compute the maximum of $f(w)$ as defined in (3.3). Note

$$f(w) = 1 + 2\sum_{k=1}^{\infty} \rho_k \cos(wk) \leq 1 + 2\sum_{k=1}^{\infty} |\rho_k \cos(wk)|$$

Since $\rho_k = 0$ if k is even and $\rho_k < 0$ if k is odd (lemma 2),

$$f(w) \leq 1 + 2 \cdot \sum_{k=1}^{\infty} |\rho_k| = f(\pi) = 2 \quad \text{by Lemma 3.} \qquad \square$$

One should not use this asymptotic accidental bias as an approximation to the exact one for small N. It takes a large number of assignments before the process generated by BSD(a) can be considered stationary. Since by the nature of BSD(a) these early assignments are not very correlated among themselves, neglecting them increases the accidental bias. The difference between the two accidental biases - the asymptotic and the exact - increases as a increases. The asymptotic accidental bias is fixed at 2 as shown above. Since as a increases BSD(a) approaches the complete random allocation design, the exact accidental bias converges to 1.

4. GENERALIZATIONS

The only disadvantage of the complete random allocation is that, mainly in small experiments, very unbalanced allocation can occur. However, as long as the imbalance is not too large, the flipping of a fair coin is the best allocation rule. If one decides to interfere with the complete random allocation only when the imbalance is outside some predetermined limits the following family of designs is very convenient. The treatment for the (m+1)th subject is chosen as follows.

(4.1)
- If $|\tilde{D}_m| < a(m)$, assign treatments by using a fair coin.
- If $\tilde{D}_m \geq a(m)$, assign treatment B with probability $p \geq 1/2$ and A with probability $1 - p$.
- If $\tilde{D}_m \leq -a(m)$, assign treatment B with probability $1 - p$ and A with probability $p \geq 1/2$.

Note that both BSD(a) and BCD(p) are members of the family (4.1). This family of designs is very flexible. It can accommodate many

used in analysing the data are usually more efficient if the allocation is balanced. Also a balanced design is an important factor in convincing others of the validity of results of the trial. These two goals, balance and randomization, are to some extent conflicting and compromise is needed. Since the assignment is frequently handled without computers and/or by non-technical staff, a third desirable property in patient allocation rules is simplicity. See Bailar (1977).

In order to work out the compromise mentioned above, some design criteria should be defined. The criteria being used in the literature are selection bias, which reflects how predictable the assignments are, and accidental bias, which measures the influence of the neglected covariates on statistical analysis. (See Sections 3.2 and 3.3 for precise definitions).

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The Big Stick Design (BSD) to be proposed resembles the PBD, but has low selection bias and low accidental bias. More importantly it can be easily implemented without much personnel

(iii) BSD(6): $a(m) = 6$, $p = 1.0$.

(iv) BCD(2/3): $a(m) = 0$, $p = 2/3$.

(v) TCD(6,2/3): $a(m) = 6$, $p = 2/3$.

(vi) SRD: $a(m) = \sqrt{m}$, $p = 1.0$.

Wei's Urn Design assigns treatments as follows. Start with an urn containing α "A" marked balls and α "B" marked balls. For the first assignment take a ball randomly from this urn and assign the treatment according to the ball drawn. Then add β balls of the opposite type to the urn. The next assignments are made analogously, by drawing a ball from the urn and adding β balls of the opposite type drawn. Wei (1977) recommends $\alpha = 0$. In UD(0,β) the first assignment is made using a fair coin. Its properties do not depend on the value of β.

5.1 Final Imbalance Distribution

For statistical purposes only the final imbalance is important. However, since the number of patients to enter the trial is unknown, it is impossible to control only the final imbalance. The SRD and UD(0,β) are such that the imbalance is controlled more vigorously at the beginning of the experiment. As argued before, only very extreme imbalance, say imbalance bigger than N/3, has to be avoided. Therefore a proper way of comparing final imbalance distributions of different designs is to look at $\Pr[D_n \geq N/3]$. This is the probability that the design produces an unacceptable imbalance.

All designs being compared are practically equivalent in this respect, since they have almost zero probability of producing an extreme imbalance even for N as small as 12.

Note that in using BSD(a) the experimenter is always sure of how big the worst imbalance can be. To know that the design has a small probability of producing extreme imbalances is of little consolation for the experimenter who happens to get it in a trial that can not be repeated. This is why it is desierable to have an imbalance distribution that does not put any mass beyond a

predetermined value a. This also illustrates why the expected final imbalance is a misleading way of summarizing the final imbalance distribution.

5.2 Selection Bias and Accidental Bias

Table IV gives the values of selection bias and accidental bias for the six designs being considered. The selection bias results are exact, and the accidental bias results presented were computed by simulation. 600 sequences of 30 (or 50) assignments were generated and used to estimate the covariance matrix of T, T' = $(T_1,\ldots,T_{30}$(or $T_{50}))$. The maximum eigenvalue of $\mathrm{Cov}(\tilde{T})$ was computed. The process was repeated 3 times and the average value for the maximum eigenvalue of $\mathrm{Cov}(\tilde{T})$ is reported.

The maximum value for accidental bias is N. Hence all designs being considered are very good as far as this criterion is concerned. It is worth mentioning the remarkable difference between the exact accidental bias [we use the simulation results as proxy for the exact ones] and the asymptotic one. For instance, the asymptotic accidental bias for UD(0,β) is 1, Wei (1977). The exact value however is bigger than 2 as shown in Table IV.

TABLE IV. Comparison of Six Designs

N	Design	Excess Selection Bias	Accidental Bias	Simplicity
30	i) BSD(6)	0.812	1.604	yes
	ii) BCD(2/3)	3.506	1.508	yes/no
	iii) TCD(6,2/3)	0.483	1.557	yes/no
	iv) SRD	3.400	2.141	no
	v) UD(0,β)	2.259	2.057	no
	vi) PBD(6)	5.250	1.200	yes
50	i) BSD(6)	1.556	1.738	
	ii) BCD(2/3)	6.000	1.700	
	iii) TCD(6,2/3)	1.167	1.717	
	iv) SRD	4.570	2.218	
	v) UD(0,β)	2.994	2.126	
	vi) PBD(6)	8.500	1.200	

The selection bias values vary greatly. Both BSD(6) and TCD(6,2/3) have very low values. The PBD(6) is inadequate if selection bias is an issue. The selection bias of BCD(2/3) can be reduced if p is descresed. However this has an adverse effect on the final imbalance distribution.

The UD(0,β) and the SRD are very similar, but the latter is simpler to implement. Both control the imbalance very forcefully at the beginning of the experiment and loosely as N increases. The other four designs do not have this property.

The BSD(a) is extremely simple to implement and, as shown in Table IV, has low values for both accidental and selection bias. As a good overall compromise, it could be used profitably in the actual design of clinical trials. The essential idea of BSD(a) must have been used in practice. The present paper provides a theoretical justification for such practices.

APPENDIX

Proof of Lemma 2.

We present the proof for j odd only. For j even, the proof proceeds along the same line. Let

$$\rho_j = E(T_1 T_{j+1}) = \sum_{k=-a}^{a} \pi_k E(T_1 T_{j+1} | \tilde{D}_0 = k), \tag{A.1}$$

where π_K is the stationary distribution of the Markov chain $(\tilde{D}_j)_1^\infty$. Since, by the definition of BSD(a), $\tilde{D}_0 = a$ implies $T_1 = -1$ we have

$$\begin{aligned} E(T_1 T_{j+1} | \tilde{D}_0 = a) &= Pr[T_{j+1} = -1 | \tilde{D}_0 = a] - Pr[T_{j+1} = 1 | \tilde{D}_0 = a] \\ &= - E(T_{j+1} | \tilde{D}_0 = a) = 0. \end{aligned}$$

The last equality is intuitive. Since a is even, all even assignments are random. Therefore since j+1 is even, $E(T_{j+1} | \tilde{D}_0 = a) = 0$. Similarly it can be shown that

$$E(T_1 T_{j+1} | \tilde{D}_0 = -a) = 0.$$

For $-a < k < a$ we proceed as follows. Using the symmetry of $[\tilde{D}_j]_1^\infty$,

$$E(T_1 T_{j+1} | \tilde{D}_0 = k) = 2\{\Pr[T_1=1, T_{j+1}=1 | \tilde{D}_0=k] - \Pr[T_1=1, T_{j+1}=-1 | \tilde{D}_0=k]\}. \tag{A.3}$$

However,

$$\Pr[T_1=1, T_{j+1}=1 | \tilde{D}_0=k] = \Pr[\tilde{D}_1 - \tilde{D}_0 = 1, \tilde{D}_{j+1} - \tilde{D}_j = 1 | \tilde{D}_0 = k]$$
$$= \sum_{-a \leq i < a} \Pr[\tilde{D}_{j+1} = i+1 | \tilde{D}_j = i] \Pr[\tilde{D}_j = i | \tilde{D}_1 = k+1] \Pr[\tilde{D}_1 = k+1 | \tilde{D}_0 = k].$$

Since j is odd, $\tilde{D}_j$ assumes only odd(even) values for even(odd) values of k. Therefore $\Pr[T_1=1, T_{j+1}=1 | \tilde{D}_0=k]$ equals

$$\Pr[\tilde{D}_1 = k+1 | \tilde{D}_0 = k] \cdot \tfrac{1}{2} \cdot \sum_{\substack{-a \leq i < a \\ i \text{ odd}}} \Pr[\tilde{D}_j = i | \tilde{D}_1 = k+1] \text{ for } k \text{ even}, \tag{A.4}$$

and

$$\Pr[\tilde{D}_1 = k+1 | \tilde{D}_0 = k]\{\sum_{\substack{-a < i < a \\ i \text{ even}}} \tfrac{1}{2} \Pr[\tilde{D}_j = i | \tilde{D}_1 = k+1] + \Pr[\tilde{D}_j = -a | \tilde{D}_1 = k+1]\} \text{ for } k \text{ odd}. \tag{A.5}$$

Similarly $\Pr[T_1=1, T_{j+1}=-1 | \tilde{D}_0=k]$ equals

$$\Pr[\tilde{D}_1 = k+1 | \tilde{D}_0 = k] \tfrac{1}{2} \sum_{\substack{-a \leq i < a \\ i \text{ odd}}} \Pr[\tilde{D}_j = 1 | \tilde{D}_1 = k+1] \text{ for } k \text{ even} \tag{A.6}$$

and

$$\Pr[\tilde{D}_1 = k+1 | \tilde{D}_0 = k]\{\sum_{\substack{-a < i < a \\ i \text{ even}}} \tfrac{1}{2} \Pr[\tilde{D}_j = i | \tilde{D}_1 = k+1] + \Pr[\tilde{D}_j = a | \tilde{D}_1 = k+1]\} \tag{A.7}$$

for k odd.

Substituting for $E(T_1T_{j+1}|\tilde{D}_0=K)$ in (A.1) we get

$$\rho_j = 2\sum_{-a<K<a}[(A.4) - (A.6) + (A.5) - (A.7)]$$

$$= 2\sum_{\substack{-a<K<a \\ K \text{ odd}}} \pi_K \Pr[\tilde{D}_1=k+1|\tilde{D}_0=K]\{\Pr[\tilde{D}_j=-a|\tilde{D}_1=k+1]-\Pr[\tilde{D}_j=a|\tilde{D}_1=k+1]\}$$

$$= 2\tfrac{1}{2a}\cdot\tfrac{1}{2}\cdot\sum_{\substack{-a<K<a \\ K \text{ odd}}}\{\Pr[\tilde{D}_j=-a|\tilde{D}_1=k+1]-\Pr[\tilde{D}_j=a|\tilde{D}_1=k+1]\}$$

$$= \tfrac{1}{2a}\{\Pr[\tilde{D}_j=-a|\tilde{D}_1=a]-\Pr[\tilde{D}_j=a|\tilde{D}_1=a]\}. \quad \square$$

Proof of Lemma 3

Karlin (1968, p. 117) gives explicit formulae for the m step probabilities of the Markov chain $\{\tilde{D}_j\}_1^\infty$. In our notation he proves

$$\Pr[\tilde{D}_j=-a|\tilde{D}_1=a] = \tfrac{1}{2a}\sum_{k=0}^{4a-1}(\cos\tfrac{k\pi}{2a})^{j-1}\cos k\pi$$

and

$$\Pr[\tilde{D}_j=a|\tilde{D}_1=a] = \tfrac{1}{2a}\sum_{k=0}^{4a-1}(\cos\tfrac{k\pi}{2a})^{j-1}(\cos k\pi)^2.$$

From Lemma 2,

$$\sum_{j=1}^{\infty}|\rho_j| = \frac{1}{(2a)^2}\sum_{j=1}^{\infty}\sum_{k=0}^{4a-1}(\cos\tfrac{k\pi}{2a})^{j-1}[(\cos k\pi)^2-\cos k\pi]$$

$$= \frac{1}{2a^2}\sum_{r=0}^{\infty}\sum_{k=1}^{2a}(\cos(2k-1)\tfrac{\pi}{2a})^r.$$

The series $\sum_{r=0}^{\infty}[\cos(2k-1)\frac{\pi}{2a}]^r$ converges absolutely for $1 \leq k \leq 2a$. By interchanging the two summation signs it follows that

$$\sum_{j=1}^{\infty}|\rho_j| = \frac{1}{2a^2}\sum_{k=1}^{2a}[1-\cos\tfrac{(2k-1)\pi}{2a}]^{-1}$$

$$= \frac{1}{4a^2}\sum_{k=1}^{2a}[\sin\tfrac{(2k-1)\pi}{4a}]^{-2}.$$

The value of the last summation is $2a^2$, Jolley (1961, formula 442), thus completing the proof.

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