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Finding defining generators with extreme lengths

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Abstract

In some practical situations the choice of defining generators matters even for the *same* defining contrast subgroup. Two such examples are blocking schemes for full and fractional factorial designs and split-plot fractional factorial designs. We propose an algorithm to find defining generators with extreme lengths for any s^{n-k} designs, s being a prime power. Some illustrations of the method are given. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In certain applications using fractional factorial designs, it is desirable to find defining generators for the designs with extreme (longest or shortest) lengths. Consider the following two examples. Arranging a 2^n design in 2^m blocks is equivalent to selecting a block defining contrast subgroup G_b of the 2^n design, where the 2^m elements of G_b can be generated by taking products of m independent generators $B_1,\ldots,B_m$. Optimal choice of G_b can follow the minimum aberration criterion. For a given G_b, if all the 2^m-1 block effects are thought to be equally important, the choice of the m generators $B_1,\ldots,B_m$ does not matter. In other situations, the generalized block effects $B_{i_1}\cdots B_{i_j}$ which are defined to be the interactions between the blocking variables $B_{i_1},\ldots,B_{i_j}$, do not have a natural interpretation and are therefore likely to be less significant than the block effects $B_1,\ldots,B_m$. For example, B_1 may represent two suppliers and B_2 day

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or night shifts. It would be hard to give a meaning for a $B_1 \times B_2$ interaction and be reasonable to assume that B_1 and B_2 are more significant than $B_1 \times B_2$. If the block effects $B_1, \ldots, B_m$ are more significant than their interactions, longer words should be assigned to $B_1, \ldots, B_m$, which are the block generators. A similar point was made in references like Wu and Zhang (1993) and Sitter et al. (1997). Another example is given in the context of constructing $2^{(n_1+n_2)-(k_1+k_2)}$ designs with n_1 whole-plot factors and 2^{-k_1} fraction in the whole-plots, and n_2 sub-plot factors and $2^{-(k_1+k_2)}$ fraction in the combined whole- and sub-plots. To construct more minimum aberration split-plot $2^{(n_1+n_2)-(k_1+k_2)}$ designs, Huang et al. (1998, p. 319) found it useful to start the search with minimum aberration 2^{n-k} designs, $n=n_1+n_2$ and $k=k_1+k_2$, which have shortest generators.

These examples illustrate the problem of finding defining generators with extreme lengths for a defining contrast subgroup of a design in which all factors are at two levels. In this paper, we consider a more general problem for any defining contrast subgroup of a design with all factors at s levels, s being a prime power. We solve this problem by proposing a simple algorithm and proving its validity. Applications to block designs and split-plot designs are considered.

The rest of the section is devoted to definitions and technical discussion, some of which were referred to in the previous paragraphs. An s^{n-k} fractional factorial design is determined by k independent defining words. A word can be written as an n-dimensional vector with components in a finite field of s elements F_s representing the s levels of the factors. For two words $w_1 = (u_1, \ldots, u_n)$ and $w_2 = (v_1, \ldots, v_n)$, with $u_i, v_i \in F_s$, their addition is a word defined by $w_1+w_2=(u_1+v_1, \ldots, u_n+v_n)$. A subgroup of words generated under this addition is called the *defining contrast subgroup*. A word $w = (a_1, \ldots, a_n)$ and all its constant multiples $\lambda w = (\lambda a_1, \ldots, \lambda a_n)$ for any $\lambda \neq e_0$, $\lambda \in F_s$ are considered to be the same in the subgroup, where e_0 represents the additive identity of F_s. The subgroup is denoted by $\mathscr{G}(w_1, \ldots, w_k)$ if it is generated by words $w_1, \ldots, w_k$. The words $w_1, \ldots, w_k$ are said to be *independent* if $\lambda_1 w_1 + \cdots + \lambda_k w_k = \mathbf{I}$ implies $\lambda_1 = \cdots = \lambda_k = e_0$, where $\lambda_1, \ldots, \lambda_k \in F_s$ and $\mathbf{I} = (e_0, \ldots, e_0)$. Independent words $w_1, \ldots, w_k$ are called *defining generators* if they generate the defining contrast subgroup.

The length of a word (i.e., *wordlength*) is the number of its nonzero components. For a set of words, the vector $W = (A_1, A_2, \ldots)$ is called the *wordlength pattern*, where A_i denotes the number of words of length i in the set. For any two wordlength patterns $W^{(1)}$ and $W^{(2)}$, let r be the smallest value with $A_r^{(1)} \neq A_r^{(2)}$. Then $W^{(1)}$ is said to have *less aberration* than $W^{(2)}$ if $A_r^{(1)} < A_r^{(2)}$. For $W^{(1)}$ and $W^{(2)}$ for the defining contrast subgroups of two s^{n-k} designs $D^{(1)}$ and $D^{(2)}$, $D^{(1)}$ is said to have less aberration than $D^{(2)}$ if $W^{(1)}$ has less aberration than $W^{(2)}$. A design D^* is said to have *minimum aberration* (MA) if no designs have less aberration than D^* (Fries and Hunter, 1980).

For a defining contrast subgroup G_0, suppose there exists a length function $l = (l_1, \ldots, l_d)$ of G_0 into R_+^d, where $d \geqslant 1$ and $R_+ = [0, \infty)$. The ordering of l is defined by lexicographical ordering, that is, for any $g, h \in G_0$, $l(g) = l(h)$ if and only if $l_i(g) = l_i(h)$ for $i = 1, \ldots, d$; and $l(g) > l(h)$ if and only if there exists $1 \leqslant j \leqslant d$ such that $l_i(g) = l_i(h)$ for $i < j$ and $l_j(g) > l_j(h)$. We define $l(g) \geqslant l(h)$, $l(g) < l(h)$ and $l(g) \leqslant l(h)$ in a similar manner. For a 2^n design in 2^m blocks, a scalar length function is provided by the wordlengths of the blocking variables; for a 2^{n-k} design in 2^m blocks, a vector length function is provided by the wordlength pattern vectors of the aliasing sets confounded with the blocking variables. (See more details in Examples 1 and 2 of Section 3.) A set of defining generators $\{g_1,\dots,g_m\}$ of G_0 with $l(g_1) \geqslant \cdots \geqslant l(g_m)$ is said to have the *longest lengths*, if for any defining generators $\{h_1,\dots,h_m\}$ of G_0 with $l(h_1) \geqslant \cdots \geqslant l(h_m)$, we have $l(g_i) \geqslant l(h_i)$ for all i. A set of defining generators with shortest lengths is similarly defined.

2. The algorithm

The following lemma is useful for the algorithm. The proof of the lemma is straightforward and is omitted.

Lemma 1. *A defining contrast subgroup generated by k independent defining words has* $(s^k-1)/(s-1)+1$ *elements.*

Consider any defining contrast subgroup G_0 with m independent defining generators. Then by Lemma 1, G_0 has $\alpha+1$ elements, where $\alpha=(s^m-1)/(s-1)$. To find a set of defining generators for G_0 with *longest* lengths, rearrange the elements of G_0 based on their lengths as $G_0=\{c_1,\dots,c_\alpha,\mathbf{I}\}$, where $l(c_1)\geqslant\cdots\geqslant l(c_\alpha)$. Let $h_1=c_1$. Then select $h_i=c_{k_i}$ inductively for $2\leqslant i\leqslant m$, where

$$k_i=\min\{j\colon c_j\notin\mathscr{G}(h_1,\dots,h_{i-1})\}. \tag{1}$$

That is, h_i is the first c_j with $c_j\notin\mathscr{G}(h_1,\dots,h_{i-1})$. The existence of such a c_j follows from Lemma 1, which implies that for $i\leqslant m$, the number of elements of $\mathscr{G}(h_1,\dots,h_{i-1})$ is at most $(s^{i-1}-1)/(s-1)+1$, which is less than $\alpha+1$.

We first show that $h_1,\dots,h_m$ are defining generators for G_0. Suppose that $\lambda_1h_1+\cdots+\lambda_mh_m=\mathbf{I}$, where $\lambda_1,\dots,\lambda_m\in F_s$. If $\lambda_m\neq e_0$, then $h_m=\lambda_m^{-1}\{(-\lambda_1)h_1+\cdots+(-\lambda_{m-1})h_{m-1}\}\in\mathscr{G}(h_1,\dots,h_{m-1})$, where λ_m^{-1} is the multiplicative inverse of λ_m and $-\lambda_i$ is the additive inverse of λ_i, for $i=1,\dots,m-1$. This contradicts the selection of h_m. Thus, $\lambda_m=e_0$. By induction, we have that $\lambda_i=e_0$ for $i=m,m-1,\dots,1$. Thus, $h_1,\dots,h_m$ are independent words, and by Lemma 1, $\mathscr{G}(h_1,\dots,h_m)$ has $\alpha+1$ elements. This implies that $\mathscr{G}(h_1,\dots,h_m)=G_0$. Hence $h_1,\dots,h_m$ are defining generators for G_0.

To show that $h_1,\dots,h_m$ have the longest lengths, consider any defining generators $q_1,\dots,q_m$ for G_0. Let $(q_{(1)},\dots,q_{(m)})$ be a permutation of $(q_1,\dots,q_m)$ such that $l(q_{(1)})\geqslant\cdots\geqslant l(q_{(m)})$. We will show $l(h_i)\geqslant l(q_{(i)})$ for all i. If this is not true, let r be the first i with $l(h_i)<l(q_{(i)})$. Since $l(h_1)=l(c_1)\geqslant l(q_{(1)})$, then $r>1$. Thus $l(c_1)\geqslant l(q_{(1)})\geqslant\cdots\geqslant l(q_{(r)})>l(h_r)=l(c_{k_r})$. This indicates that the lengths of $q_{(1)},\dots,q_{(r)}$ are between those of c_1 and c_{k_r-1}. From the ordering of $c_1,\dots,c_\alpha$, we have

$$\{q_{(1)},\dots,q_{(r)}\}\subset\{c_1,c_2,\dots,c_{k_r-2},c_{k_r-1}\}. \tag{2}$$

On the other hand, from (1), $c_{k_r}=h_r$ is the first c_j with $c_j\notin\mathscr{G}(h_1,\dots,h_{r-1})$. Together with (2), this implies $\{q_{(1)},\dots,q_{(r)}\}\subset\mathscr{G}(h_1,\dots,h_{r-1})$. Thus, $\mathscr{G}(q_{(1)},\dots,q_{(r)})$

$\subset \mathscr{G}(h_1,\ldots,h_{r-1})$, which is a contradiction because $\mathscr{G}(h_1,\ldots,h_{r-1})$ has $(s^{r-1}-1)/(s-1)+1$ elements while its subset $\mathscr{G}(q_{(1)},\ldots,q_{(r)})$ has $(s^r-1)/(s-1)+1$ elements. Hence the proposed algorithm selects a set of defining generators with longest lengths.

The algorithm for finding a set of defining generators with *shortest* lengths can be similarly defined. The only change is to arrange the c_i with $l(c_1) \leqslant \cdots \leqslant l(c_\alpha)$. Its proof can be obtained in the same manner. If $l(c_i)=l(c_{i+1})$ for some i, with a different ordering of c_i in G_0, the algorithm can select a different set of defining generators, as can be seen in Example 1 of Section 3.

Cheng and Steinberg (1991) proposed a reverse foldover algorithm for constructing run orders of two-level factorial designs with maximum number of level changes. Part of their algorithm (p. 332, steps 1 and 2) is to find a sequence of appropriate generators, which turns out to be the same as our algorithm. There are major differences, however. First, they proved that the algorithm works under the condition that a function δ defined on $G_0 \times G_0$ exists, where $\delta(g,h)=\delta(h,g)$ and $\delta(g,h)=\psi(g-h)$ for $g,h \in G_0$ and a real-valued function ψ defined on G_0 (Cheng, 1985). The function δ is needed for calculating the cost of level changes. In our proof, only an ordering of elements in G_0 is required. For blocked fractional factorial designs (see Example 2), it is not clear how to find δ to be consistent with rank ordering according to the aberration criterion. Second, the focus and applications of our algorithm are different, and our proof is more concise and simpler. Of course, simplification of their proof to the special case could make it as simple and concise as ours. Furthermore, our proof covers all the cases of s-level designs, where s is a prime power, while theirs focused on the two-level cases.

3. Examples

Example 1 (2^n *design in* 2^m *blocks*). Consider a 2^6 design with factors A, B, C, D, E, and F. Suppose that the 64 runs are scheduled to be performed in two days, on two machines, by two operators, and on materials from two different suppliers. It is reasonable to expect significant effects of day (b_1), machine (b_2), operator (b_3), and supplier (b_4). The 64 runs can be grouped into 16 (=2^4) blocks using the optimal blocking scheme from Sun et al. (1997, p. 301) by assigning the four words AB, CD, ACE, and ACF (called Assignment I) to the four blocking variables. This scheme results in the block defining contrast subgroup $G_1=\{\mathrm{ABCDEF},\mathrm{ABCD},\mathrm{ABEF},\mathrm{CDEF},\mathrm{ACE},\mathrm{ACF},\mathrm{ADE},\mathrm{ADF},\mathrm{BCE},\mathrm{BCF},\mathrm{BDE},\mathrm{BDF},\mathrm{AB},\mathrm{CD},\mathrm{EF},\mathbf{I}\}$. Under Assignment I, the four treatment effects AB, CD, ACE, and ACF are confounded with b_1, b_2, b_3, and b_4, while the other treatment effects are confounded with the interactions of b_1, b_2, b_3, and b_4. It is reasonable to assume that the interactions between day, machine, operator, and supplier are less likely to be significant than their main effects. It is then desirable to improve Assignment I by confounding the longer words in G_1 with the four blocking variables. The algorithm in Section 2 can be applied to find a best assignment of words to blocking variables. The first and the second words chosen are $b_1=\mathrm{ABCDEF}$ and $b_2=\mathrm{ABCD}$. They generate the effect $b_1b_2=\mathrm{EF}$. (To follow the standard practice, we use b_1b_2 to denote the "interaction" between b_1 and b_2, where b_1b_2 is defined to be the addition of the words b_1 and b_2 over F_2, a finite field of 2 elements.) The third word chosen is b_3 = ABEF because it is the longest among those in $G_1\backslash\{b_1, b_2, b_1b_2\}$. Then $G_1\backslash\{b_1, b_2, b_1b_2, b_3, b_1b_3, b_2b_3, b_1b_2b_3\}$={ACE, ACF, ADE, ADF, BCE, BCF, BDE, BDF, **I**}, from which we select b_4 = ACE. Call the new assignment Assignment II. Assignments I and II generate the same defining contrast subgroup but Assignment II is preferred because AB and CD are confounded with the main effects of blocking variables in Assignment I while no two-factor interactions are confounded in Assignment II. The proposed algorithm was used in choosing the defining generators in the block design table in Appendix 3A of Wu and Hamada (2000).

Example 2 (2^{n-k} *design in* 2^m *blocks*). The extra complication here is that the block effects are confounded with a set of aliased treatment effects. For example, consider the 2^{8-3} design with the defining relation: **I** = ABCF = ABDG = ACEH = CDFG = BEFH = BCDEGH = ADEFGH. According to Sun et al. (1997, p. 303), an optimal blocking scheme for this design in 8 (=2^3) blocks is to assign AC, BC, and AD to three blocking variables, denoted by b_1, b_2, and b_3. This assignment results in the following aliasing and confounding relations:

$$\begin{aligned} b_3 \leftarrow & \mathrm{AD} = \mathrm{BG} = \mathrm{BCDF} = \mathrm{CDEH} \\ & = \mathrm{ACFG} = \mathrm{EFGH} = \mathrm{ABDEFH} = \mathrm{ABCEGH}, \end{aligned} \tag{3}$$

$$\begin{aligned} b_1b_3 \leftarrow & \mathrm{CD} = \mathrm{FG} = \mathrm{ABDF} = \mathrm{ABCG} \\ & = \mathrm{ADEH} = \mathrm{BEGH} = \mathrm{ACEFGH} = \mathrm{BCDEFH}, \end{aligned} \tag{4}$$

$$\begin{aligned} b_2b_3 \leftarrow & \mathrm{DF} = \mathrm{CG} = \mathrm{ABCD} = \mathrm{BDEH} \\ & = \mathrm{ABFG} = \mathrm{AEGH} = \mathrm{ACDEFH} = \mathrm{BCEFGH}, \end{aligned} \tag{5}$$

$$\begin{aligned} b_1b_2b_3 \leftarrow & \mathrm{AG} = \mathrm{BD} = \mathrm{ACDF} = \mathrm{BCFG} \\ & = \mathrm{DEFH} = \mathrm{CEGH} = \mathrm{ABCDEH} = \mathrm{ABEFGH}, \end{aligned} \tag{6}$$

$$\begin{aligned} b_2 \leftarrow & \mathrm{BC} = \mathrm{AF} = \mathrm{ACDG} = \mathrm{ABEH} \\ & = \mathrm{BDFG} = \mathrm{CEFH} = \mathrm{DEGH} = \mathrm{ABCDEFGH}, \end{aligned} \tag{7}$$

$$\begin{aligned} b_1 \leftarrow & \mathrm{AC} = \mathrm{BF} = \mathrm{EH} = \mathrm{BCDG} \\ & = \mathrm{ADFG} = \mathrm{ABDEGH} = \mathrm{ABCEFH} = \mathrm{CDEFGH}, \end{aligned} \tag{8}$$

$$\begin{aligned} b_1b_2 \leftarrow & \mathrm{AB} = \mathrm{CF} = \mathrm{DG} = \mathrm{BCEH} \\ & = \mathrm{AEFH} = \mathrm{ABCDFG} = \mathrm{ACDEGH} = \mathrm{BDEFGH}, \end{aligned} \tag{9}$$

where "$\leftarrow$" and "=" represent "confounding" and "aliasing", respectively. The right hand side of $\leftarrow$ is an aliasing set which is confounded with the block effect on its left hand side. The aliasing sets in (3)–(9) are rank-ordered according to the aberration criterion on their wordlength pattern vectors and arranged in ascending order of aberration. Each of the sets in (3)–(9) can be regarded as an element and the set G_2 that contains these elements and **I** is a block defining contrast subgroup with defining generators b_1, b_2, and b_3. Therefore the algorithm in Section 2 can be applied to find a best assignment so that the aliasing sets confounded with the main effects of blocking variables have less aberration. According to the algorithm, the first and the second sets to be assigned to blocking variables are (3) and (4). In view of $b_3 \cdot b_1 b_3 = b_1$, these two sets generate the aliasing set (8). Hence the remaining aliasing sets are $\{(5), (6), (7), (9)\}$. Then the algorithm assigns (5) to the remaining blocking variable. In the new assignment, the three sets confounded with the main effects of blocking variables, i.e., $\{(3), (4), (5)\}$, have less aberrations than in the old assignment, i.e., $\{(3), (7), (8)\}$.

Example 3 (*Split-plot design*). Huang et al. (1998) proposed a method to construct MA split-plot designs through the generators of the defining contrast subgroup of MA fractional factorial designs. They noted that generators with the *shortest* wordlength would often lead to more MA split-plot designs. They further noted that "we were not able to find a reference in which such shortest-word generators could be found". The algorithm in Section 2 can be used to solve this problem. Consider an example used in their paper. The defining contrast subgroup of the 2^{9-4} design with $\mathbf{I} = \mathrm{ABCDF} = \mathrm{ABCEG} = \mathrm{BDEH} = \mathrm{CDEI}$ is $G_3 = \{\mathbf{I}, \mathrm{BFGH}, \mathrm{CFGI}, \mathrm{DEFG}, \mathrm{BCHI}, \mathrm{BDEH}, \mathrm{CDEI}, \mathrm{ABCDF}, \mathrm{ABCEG}, \mathrm{ABDGI}, \mathrm{ABEFI}, \mathrm{ACDGH}, \mathrm{ACEFH}, \mathrm{ADFHI}, \mathrm{AEGHI}, \mathrm{BCDEFGHI}\}$, where the words are arranged in ascending order of wordlength. By applying the algorithm to G_3, a set of generators with shortest wordlengths is found: $\{\mathrm{BFGH}, \mathrm{CFGI}, \mathrm{DEFG}, \mathrm{ABCDF}\}$, which can then be used to construct MA split-plot designs. In their method, not every set of generators with shortest wordlengths can be used to construct MA split-plot designs. An example is the set, $\{\mathrm{BFGH}, \mathrm{CFGI}, \mathrm{BDEH}, \mathrm{ABCDF}\}$, which has shortest wordlengths but cannot be used to construct MA split-plot designs. Therefore an algorithm to find *all* sets of generators with shortest wordlengths is needed. The algorithm in Section 2 can be easily modified to achieve this objective.

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References

Cheng, C.S., 1985. Run orders of factorial designs. In: Le Cam, L.M., Olshen, R.A. (Eds.), Proceedings of the Berkeley Conference in Honor of Jerzy Neyman and Jack Kiefer, Vol. 2, Wadsworth, Belmont, CA, pp. 619–633.

Cheng, C.S., Steinberg, D.M., 1991. Trend robust two-level factorial designs. Biometrika 78, 325–336.

Fries, A., Hunter, W.G., 1980. Minimum aberration 2^{k-p} designs. Technometrics 22, 601–608.

Huang, P., Chen, D., Voelkel, J.O., 1998. Minimum-aberration two-level split-plot designs. Technometrics 40, 314–326.

Sitter, R.R., Chen, J., Feder, M., 1997. Fractional resolution and minimum aberration in blocked 2^{n-k} designs. Technometrics 39, 382–390.

Sun, D.X., Wu, C.F.J., Chen, Y., 1997. Optimal blocking schemes for 2^n and 2^{n-p} designs. Technometrics 39, 298–307.

Wu, C.F.J., Hamada, M., 2000. Experiments: Planning, Analysis, and Parameter Design Optimization. Wiley, New York.

Wu, C.F.J., Zhang, R., 1993. Minimum aberration designs with two-level and four-level factors. Biometrika 80, 203–209.