CME Analysis: A New Method for Unraveling Aliased Effects in Two-Level Fractional Factorial Experiments

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Ever since the founding work by Finney (1945), it has been widely known and accepted that aliased effects in two-level regular designs cannot be "de- aliased" without adding more runs. A result by Wu in his 2011 Fisher Lecture showed that aliased effects can sometimes be "de-aliased" using a new framework based on the concept of conditional main effects (CMEs). This idea is further developed in this paper into a methodology that can be readily used. Some key properties are derived that govern the relationships among CMEs or between them and related effects. As a consequence, some rules for data analysis are developed. Based on these rules, a new CME-based methodology is proposed. Three real examples are used to illustrate the methodology. The CME analysis can often lead to models with fewer effect terms and smaller p values for the selected effects. Moreover, the selected CME effects are often more interpretable.

Key Words: Conditional Main Effect; Fractional Factorial Design; Orthogonal Modeling.

1. Introduction

It is known in the literature on experimental design that aliased effects cannot be disentangled without adding more runs, where two effects are said to be *aliased* if they represent the same contrast. This concept has been widely adopted since its inception by Finney (1945). After nearly 70 years, this belief and practice was shown to be violable in the 2011 Fisher Lecture by Jeff Wu, which will appear as a paper in Wu (2015). He employed a concept called conditional main effect (CME) to reparameterize the space of aliased effects and used variable selection to identify significant effects among the candidate set consisting of main effects, interactions, and selective conditional main effects. An example from GM of Canada (Brajac and Morey (1987)) was used to illustrate the new idea with promising results. The goal of this paper is to further explore the concept in Wu (2015) and develop a systematic analysis strategy to de-alias aliased effects in two-level fractional factorial designs.

In this work, we consider only the 2^{k-q} designs, where k factors, each at two levels, denoted by $+$ and $-$, are being studied. It is a faction of the 2^k full factorial design. The effects, such as main effects, two-factor and higher order interactions, considered in traditional analysis are referred to collectively as the *traditional effects* in contrast with the *conditional main effects* discussed here. Two effects that are neither orthogonal nor aliased are said to be *partially aliased*. To distinguish the concept of partial aliasing, we shall call the aliasing relationship in traditional 2^{k-q} designs *fully aliased*, which is in line with the terminology in Wu and Hamada (2009, p. 363). For definitions of and discussions on the traditional effects and full and partial aliasing, the readers should consult Wu and Hamada (2009).

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The organization of the paper is as follows. In Section 2, important definitions and properties related to the conditional main effects are given. Rules of analysis are developed from these definitions and some key properties. These rules form the basis for the method of analysis, called the CME Analysis, in Section 3. Three real examples based on resolution IV designs are used to illustrate the analysis strategy in Section 4. In each of the examples, the CME analysis leads to a model that contains fewer effect terms and some conditional main effect term(s), which are more interpretable than traditional two-factor interactions because the latter are fully aliased with other two-factor interactions in resolution IV designs. In two of the three examples, the new models found by CME have smaller p values of the effect terms. Concluding remarks are given in the last section.

2. Properties of CME

Let us start by reviewing the definition of conditional main effects. Consider the first four columns of Table 1. It is a 2^{4-1}_{VI} design with eight runs and four factors; A, B, C, and D. The defining relation of this design is $I = ABCD$.

Suppose we consider only the first two factors A and B. The standard definition of main effects and two-factor interactions (abbreviated as 2FIs hereafter) in texts on design of experiments (Box et al. (2005), Wu and Hamada (2009)) is given by

$$\mathrm{ME}(A) = \bar{y}(A+) - \bar{y}(A-), \quad (1)$$

$$\mathrm{ME}(B) = \bar{y}(B+) - \bar{y}(B-), \quad (2)$$

$$\mathrm{INT}(A,B) = \frac{1}{2}(\bar{y}(A+|B+) + \bar{y}(A-|B-)) - \frac{1}{2}(\bar{y}(A+|B-) + \bar{y}(A-|B+)), \quad (3)$$

$$\mathrm{INT}(A,B) = \frac{1}{2}(\bar{y}(B+|A+) + \bar{y}(B-|A-)) - \frac{1}{2}(\bar{y}(B+|A-) + \bar{y}(B-|A+)), \quad (4)$$

where $\bar{y}(A+)$, $\bar{y}(A-)$, $\bar{y}(B+)$, $\bar{y}(B-)$ are the averages of the responses y at the level settings $A+$, $A-$, $B+$, and $B-$, respectively, and $\bar{y}\bar{y}(A+|B+)$, $\bar{y}(A-|B-)$, $\bar{y}(A+|B-)$, and $\bar{y}(A-|B+)$ are the averages of y at the level settings $A+B+$, $A-B-$, $A+B-$, and $A-B+$, respectively.

Notice that $\bar{y}(A+)$ can also be expressed as $\bar{y}(A+) = (1/2)(\bar{y}(A+|B+) + \bar{y}(A+|B-))$ and, similarly, for $\bar{y}(A-)$, $\bar{y}(B+)$, and $\bar{y}(B-)$. Then Equations (1) and (2) can be written as

$$\mathrm{ME}(A) = \frac{1}{2}(\bar{y}(A+|B+) + \bar{y}(A+|B-)) - \frac{1}{2}(\bar{y}(A-|B+) + \bar{y}(A-|B-)), \quad (5)$$

$$\mathrm{ME}(B) = \frac{1}{2}(\bar{y}(B+|A+) + \bar{y}(B+|A-)) - \frac{1}{2}(\bar{y}(B-|A+) + \bar{y}(B-|A-)). \quad (6)$$

In Wu and Hamada (2009), the conditional main effect, *which is henceforth abbreviated as CME*, of A given B at level $+$ is defined as

$$\mathrm{CME}(A|B+) = \bar{y}(A+|B+) - \bar{y}(A-|B+). \quad (7)$$

Its contrast vector is given by the column $A|B+$ in Table 1. In this vector, two entries have $+$, two entries have $-$, and the four entries corresponding to $B-$ are denoted by 0, meaning that they are not in the contrast vector for $A|B+$. Similarly, the CME of A given B at level $-$ is defined as

$$\mathrm{CME}(A|B-) = \bar{y}(A+|B-) - \bar{y}(A-|B-). \quad (8)$$

Its contrast vector is given by the column $A|B-$ in Table 1 with a similar interpretation to the above. By rearranging terms in Equation (5), it is easy to

TABLE 1. 2^{4-1}_{VI} Design with $I = ABCD$ and Some CMEs from the Design

A	B	C	D	AB	CD	$A\|B+$	$A\|B-$	$B\|A+$	$B\|A-$	$A\|C+$	$C\|D-$	$D\|C-$	$A\|D-$
+	+	+	+	+	+	+	0	+	0	+	0	0	+
+	+	−	−	+	+	+	0	+	0	0	−	−	0
+	−	+	−	−	−	0	+	−	0	+	+	0	0
+	−	−	+	−	−	0	+	−	0	0	0	+	+
−	+	+	−	−	−	−	0	0	+	−	+	0	0
−	+	−	+	−	−	−	0	0	+	0	0	+	−
−	−	+	+	+	+	0	−	0	−	−	0	0	−
−	−	−	−	+	+	0	−	0	−	0	−	−	0

show $\mathrm{ME}(A) = (1/2)(\mathrm{CME}(A|B+) + \mathrm{CME}(A|B-))$. Based on this observation, David Woods calls each CME a "half main effect" (in a personal communication). By interchanging the roles of A and B, we also have

$$\mathrm{CME}(B|A+) = \bar{y}(B+|A+) - \bar{y}(B-|A+) \quad (9)$$

and

$$\mathrm{CME}(B|A-) = \bar{y}(B+|A-) - \bar{y}(B-|A-). \quad (10)$$

Thus far, we have defined the main effects, 2FIs, and CMEs all in terms of the average y values at specific level settings of A and B in Equations (3)–(10). Now we will link these three types of effects through some algebraic relationships.

By adding Equations (3) and (5), we get

$$\mathrm{ME}(A) + \mathrm{INT}(A, B) = \bar{y}(A+|B+) - \bar{y}(A-|B+) = \mathrm{CME}(A|B+). \quad (11)$$

By subtracting Equation (3) from Equation (5), we get

$$\mathrm{ME}(A) - \mathrm{INT}(A, B) = \bar{y}(A+|B-) - \bar{y}(A-|B-) = \mathrm{CME}(A|B-). \quad (12)$$

By adding Equations (4) and (6), we get

$$\mathrm{ME}(B) + \mathrm{INT}(A, B) = \bar{y}(B+|A+) - \bar{y}(B-|A+) = \mathrm{CME}(B|A+). \quad (13)$$

And finally, by subtracting Equation (4) from Equation (6), we get

$$\mathrm{ME}(B) - \mathrm{INT}(A, B) = \bar{y}(B+|A-) - \bar{y}(B-|A-) = \mathrm{CME}(B|A-). \quad (14)$$

From Equations (11)–(14), we can see that each CME is related to a main effect and a 2FI. We call this main effect its *parent effect* and the 2FI its *interaction effect*. The main effect being conditioned on is called the *conditioning effect* and its corresponding level setting is called the *conditioning level*. The relationships in Equations (11)–(14) can be summarized as the first property.

Property 1

A conditional main effect is equal to the sum (and, respectively, the difference) of its parent effect and its interaction effect, if its conditioning level is + (and, respectively, −).

Now let us go back to the design and see if we can derive similar relationships between the columns representing these three types of effects. From now on, we will use the shorthand notation $(A|B+)$, $(A|B-)$, AB, and A to represent $\mathrm{CME}(A|B+)$, $\mathrm{CME}(A|B-)$, $\mathrm{INT}(AB)$, and $\mathrm{ME}(A)$, respectively.

Again, take the first two factors A and B for illustration. We will show how to write down the column of $(A|B+)$ in a standard way. By the definition of CME, $(A|B+)$ is the effect of A given B at level +. For the rows with B at level +, the entries of $(A|B+)$ are the same as the entries of A. On the other hand, at $B-$, the entries of $(A|B+)$ should be empty because the conditioning is on B being +. For completeness of presentation, we use zero to represent the emptiness of entries. This CME is represented in column 7 of Table 1. Similarly, we have $(A|B-)$ in column 8. By interchanging the roles of A and B, we have $(B|A+)$ and $(B|A-)$ in columns 9 and 10, respectively.

However, the above procedure is very tedious. To construct a CME, one has to go through the columns of its parent effect and conditioning effect entry by entry. As inspired by Property 1, we will try to find a simple relationship between these three columns.

Take $(A|B+)$ for example. Its parent effect is the main effect A and its interaction effect is the 2FI AB. By the definition of 2FI, the column of AB (column 5) is constructed by multiplying columns 1 and 2 element by element. Thus, for rows with B at level +, the entries of AB are the same as the entries of A. On the other hand, for $B-$, the entries of AB have the opposite sign of the entries of A. So if we add A and AB in columns 1 and 5, and divide by 2, we get $(A|B+)$ in column 7. Similarly, if we subtract AB from A and divide by 2, we get $(A|B-)$ in column 8. Therefore, we have the following algebraic relationships:

$$(A|B+) = \frac{1}{2}(A + AB), \quad (15)$$

$$(A|B-) = \frac{1}{2}(A - AB). \quad (16)$$

We call them the *construction definition* of CME.

To model a 2^{k-q} design with k factors, the set of candidate effects consists of $4 \times \binom{k}{2}$ CMEs, k main effects, and $\binom{k}{2}$ 2FIs. The set of candidate models is even larger, which consists of certain subsets of the previous set. Without any restriction, it would be hard to find a good model from such a large candidate set. In analyzing experiments with complex aliasing, Hamada and Wu (1992) have encountered similar situations, where they used the effect sparsity principle and the effect heredity principle to reduce the size of the candidate set and exclude incompatible models. In this work, *we restrict the search to orthogonal models*, where all effects in a candidate model have to be orthogonal to each other. Here we only consider the following notion of orthogonality. Two columns are orthogonal if their inner product is zero. Let $\mathbf{u} = (u_i)_{i=1}^n$ and $\mathbf{v} = (v_i)_{i=1}^n$ be two column vectors of size n. Their inner product is defined as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i. \tag{17}$$

From this definition, it follows immediately that any two traditional effects are orthogonal if they are not fully aliased with each other. Additionally, the inner product of a traditional effect with itself or a fully aliased effect is the squared norm of that effect, i.e., the number of runs of the design. These two properties will be used in Equations (19)–(25) without specific referencing.

First, we explore the orthogonality relationships between CMEs and traditional effects. Let $(A|B+)$ be a CME and TE be a traditional effect. By Equation (15), we can write their inner product as

$$(A|B+)\cdot \mathrm{TE} = \frac{1}{2}(A+AB)\cdot \mathrm{TE} = \frac{1}{2}(A\cdot \mathrm{TE}+AB\cdot \mathrm{TE}). \tag{18}$$

If $\mathrm{TE} = A$, i.e., $(A|B+)$'s parent effect, Equation (18) becomes

$$\begin{aligned}(A|B+) \cdot A &= \frac{1}{2}(A \cdot A + AB \cdot A) = \frac{1}{2}|A|^2 + 0 \\ &= \frac{1}{2}|A|^2 \neq 0. \end{aligned} \tag{19}$$

Similarly, if $\mathrm{TE} = AB$, i.e., $(A|B+)$'s interaction effect, Equation (18) becomes

$$\begin{aligned}(A|B+) \cdot AB &= \frac{1}{2}(A \cdot AB + AB \cdot AB) \\ &= 0 + \frac{1}{2}|AB|^2 = \frac{1}{2}|AB|^2 \neq 0. \end{aligned} \tag{20}$$

Otherwise, we have

$$(A|B+)\cdot \mathrm{TE} = \frac{1}{2}(A\cdot \mathrm{TE}+AB\cdot \mathrm{TE}) = 0+0 = 0. \tag{21}$$

Combining Equations (19)–(21), we have the second property.

Property 2

CMEs are orthogonal to all the traditional effects except for their parent effects and interaction effects.

Now we turn to a review of effect aliasing. Consider the same 2_{VI}^{4-1} design. The 2FIs AB and CD are shown in columns 5 and 6 of Table 1. By comparing these two columns, it is seen that they are exactly the same. This suggests that AB and CD represent the same vector (called contrast in design of experiments). Thus, we are not able to distinguish them in the traditional sense. However, by the construction definition in Equations (15) and (16), we can write $AB(= CD)$ in several ways in terms of the CMEs,

$$\begin{aligned} AB &= (A|B+) - (A|B-) = (B|A+) - (B|A-), \\ CD &= (C|D+) - (C|D-) = (D|C+) - (D|C-). \end{aligned}$$

Though AB and CD cannot be disentangled, as inspired by Wu (2015), we can use the above equations to reparameterize the space that represents AB and CD with CMEs and choose a subset of the CMEs to represent or approximate the fully aliased 2FIs. Recall that we require the selected CMEs in the same model to be orthogonal to each other.

Now let us construct some CMEs and check their orthogonality relationships by computing the pairwise inner products. Columns 11–14 of Table 1 are $(A|C+)$, $(C|D-)$, $(D|C-)$, and $(A|D-)$, respectively. Together with the other four CMEs constructed before in columns 7–10, we take pairwise inner products among these eight columns. The computational results are summarized as

(i) $(A|B+)$ is orthogonal to $(A|B-)$.
(ii) For $(A|B+)$, $(B|A+)$, $(C|D-)$, and $(D|C-)$, none of them are orthogonal to each other.
(iii) For $(A|B+)$, $(A|C+)$, and $(A|D-)$, none of them are orthogonal to each other

These grouping relationships serve as the motivation for the definition of *twins, siblings* and *family* to be given below.

The first group of CMEs differ only in their conditioning levels, such as $(A|B+)$ and $(A|B-)$. We call these two CMEs the *twins*. As proven in Equation (22), the twin CMEs are orthogonal to each other,

$$\begin{aligned}&(A|B+) \cdot (A|B-) \\ &\quad = \frac{1}{2}(A + AB) \cdot \frac{1}{2}(A - AB) = 1/4(A \cdot A - AB \cdot AB) \\ &\quad = \frac{1}{4}(|A|^2 - |AB|^2) = 0. \end{aligned} \tag{22}$$

Note that the two-dimensional (2d) space of the twin CMEs is exactly the same as the 2d space of their parent effect and interaction effect according to Equations (15) and (16). If we include both the twin CMEs in the same model, it is the same as having their parent effect and interaction effect and thus no effect de-aliasing can be achieved. Therefore, *only one* of the twin CMEs can be included in the model. From properties 1 and 2, we can substitute a pair of main effects, say A, and a 2FI involving A, say AK by one of the twin CMEs $(A|K+)$ and $(A|K-)$. Suppose A and AK have the same sign. From Equation (11), their sum (i.e., the CME $(A|K+)$) is larger than both A and AK. Therefore, we should choose $(A|K+)$ according to property 1; otherwise, we choose $(A|K-)$. This strategy is especially effective if the effects A and AK have similar magnitudes because, from Equations (11) and (12), the selected CME is much larger than both A and AK and the ignored CME is much smaller. On the other hand, if one of A and AK dominates the other term, then we will use the dominant term in the model. There would be no need to use the CME term as in rule 1 below. For simplicity of terminology, we will refer to AK as a 2FI and A as its parental main effect. Note that each 2FI has two parental main effects. This strategy can be summarized as the first rule of analysis.

Rule 1

Substitute a pair of 2FI and its parental main effect that have similar magnitudes with one of the corresponding twin CMEs.

This rule is the most important one among the three rules because it enables the replacement of a larger model by smaller ones with comparable effectiveness in explaining the data.

Second, we consider the group of CMEs that have the same parent effect but not the interaction effects. Consider $(A|B+)$ and $(A|C+)$, which have the same parent effect A but different interaction effects AB and AC, respectively. We call these two *siblings*. By Equation (15), their inner product can be written as

$$\begin{aligned}(A|B+)\cdot(A|C+) &= \frac{1}{2}(A+AB)\cdot\frac{1}{2}(A+AC)\\ &= \frac{1}{4}(A\cdot A + A\cdot AC + AB\cdot AC + AB\cdot A)\\ &= \frac{1}{4}(|A|^2+0+0+0) = 1/4|A|^2 \neq 0. \qquad (23)\end{aligned}$$

This is stated as the third property.

Property 3

Sibling CMEs are not orthogonal to each other.

Next, we consider the group of CMEs that have the same or fully aliased interaction effects. These CMEs are said to belong to the same family. In Table 1, $(A|B+)$, $(A|B-)$, $(B|A+)$, and $(B|A-)$ have the same interaction effect AB and thus belong to the same family. Similarly $(C|D+)$, $(C|D-)$, $(D|C+)$, and $(D|C-)$ have the same interaction effect CD, which is fully aliased with AB. Therefore, they belong to the same family as the first four CMEs. Note that, in a family, the twin CMEs are orthogonal. Without loss of generality, assume $(A_1|B_1+)$ and $(A_2|B_2+)$ are two nontwin CMEs from the same family. This means $A_1 \neq A_2$ and $B_1 \neq B_2$, but $A_1B_1 = A_2B_2$. By Equation (15), their inner product can be written as

$$\begin{aligned}(A_1|B_1+)\cdot(A_2|B_2+) &= \frac{1}{2}(A_1+A_1B_1)\cdot\frac{1}{2}(A_2+A_2B_2)\\ &= \frac{1}{4}(A_1\cdot A_2 + A_1\cdot A_2B_2 + A_1B_1\cdot A_2B_2\\ &\quad + A_1B_1\cdot A_2)\\ &= \frac{1}{4}(0+0+0+|A_1B_1|^2) = \frac{1}{4}|A_1B_1|^2 \neq 0. \qquad (24)\end{aligned}$$

This is summarized as the fourth property.

Property 4

Nontwin CMEs in a family are not orthogonal.

Recall that we require the effects in a candidate model to be orthogonal to each other. From properties 3 and 4, we have the second rule of analysis, which excludes nonorthogonal terms in the same model.

Rule 2

Only one CME among its siblings can be included in the model. Only one CME from a family can be included in the model.

Finally, we study CMEs that have different parent effects as well as different interaction effects. Without loss of generality, let them be $(A|B+)$ and $(C|D+)$, where $AB \neq CD$. By Equation (15), their inner product can be written as

$$\begin{aligned}(A|B+)\cdot(C|D+) &= \frac{1}{2}(A+AB)\cdot\frac{1}{2}(C+CD)\\ &= \frac{1}{4}(A\cdot C + A\cdot CD + AB\cdot CD + AB\cdot C)\\ &= \frac{1}{4}(0+0+0+0) = 0. \qquad (25)\end{aligned}$$

This gives us the last property.

Property 5

CMEs with different parent effects and different interaction effects are orthogonal.

Because of the orthogonal modeling requirement, Property 5 leads to the third rule of analysis:

Rule 3

CMEs with different parent effects and different interaction effects can be included in the same model.

These three rules serve as the basis for the method of analysis proposed in the next section.

3. Method of Analysis

Our analysis strategy is based on the following two ideas. First, we consider only orthogonal models. Rules 2 and 3 are used to select orthogonal effects in the model. Second, according to rule 1, a pair of 2FI and its parental main effect with similar magnitudes can be replaced by one of the corresponding twin CMEs. In orthogonal models, this CME is orthogonal to the rest of the effects. Therefore, if this 2FI is aliased with other 2FIs, by substituting it with this CME, the effect aliasing will be unraveled.

Based on the above two ideas, we propose a new method of analysis, called the CME analysis.

CME Analysis

(i) Use the traditional analysis methods such as analysis of variance or half-normal plot, to select significant effects, including aliased pairs of effects. Go to (ii).

(ii) Among all the significant effects, use rule 1 to find a pair of fully aliased 2FI and its parental main effect and substitute them with an appropriate CME. Use rules 2 and 3 to guide the search and substitution of other such pairs until they are exhausted.

In step (i), if the use of half-normal plot is considered too subjective or judgmental, a formal method like the Lenth method can be considered (Wu and Hamada (2009), Ch. 4).

In the next section, three examples will be given to illustrate the analysis strategy.

4. Examples

In this section, we give three examples to illustrate the analysis strategies proposed in Section 3. All the examples are from real physical experiments using 2^{k-q} designs with resolution IV. The CME analysis as applied to the data appears to work very well. In each example, the final chosen model or models are indicated in bold face.

Example 1: Injection Molding Experiment

Shrinkage is a common problem in parts manufactured by injection molding and can reduce the efficiency in the upcoming assembly operations. A team of engineers conducted an experiment on the shrinkage from injection molding using a 2^{5-2}_{IV} design with 16 runs. The defining relations of the design are $I = ABCE = BCDF = ADEF$ and the six factors are mold temperature (A), screw speed (B), holding time (C), cycle time (D), gate size (E), and holding pressure (F). The design matrix and data are given in Table 2, where the response is 10 times the percent shrinkage (Montgomery (1991), p. 352). The goal of this experiment was to minimize the shrinkage. We apply the CME analysis to this data set.

In step (i), we use half-normal plot to identify significant effects. From Figure 1, it is clearly seen that the main effect B (screw speed) is the most significant. It is followed by the main effect A and their 2FI AB and the R^2 value for the three terms is 96.24%. Because the remaining effects are not significant, we include only these three terms in the first model, denoted as model 1.1. The p values for B, A, and AB are 2.39e−09, 5.38e−05, and 0.022%, respectively.

TABLE 2. Design Matrix and Response Data, Injection Molding Experiment

A	B	C	D	E	F	y
−1	−1	−1	−1	−1	−1	6
1	−1	−1	−1	1	−1	10
−1	1	−1	−1	1	1	32
1	1	−1	−1	−1	1	60
−1	−1	1	−1	1	1	4
1	−1	1	−1	−1	1	15
−1	1	1	−1	−1	−1	26
1	1	1	−1	1	−1	60
−1	−1	−1	1	−1	1	8
1	−1	−1	1	1	1	12
−1	1	−1	1	1	−1	34
1	1	−1	1	−1	−1	60
−1	−1	1	1	1	−1	16
1	−1	1	1	−1	−1	5
−1	1	1	1	−1	1	37
1	1	1	1	1	1	52

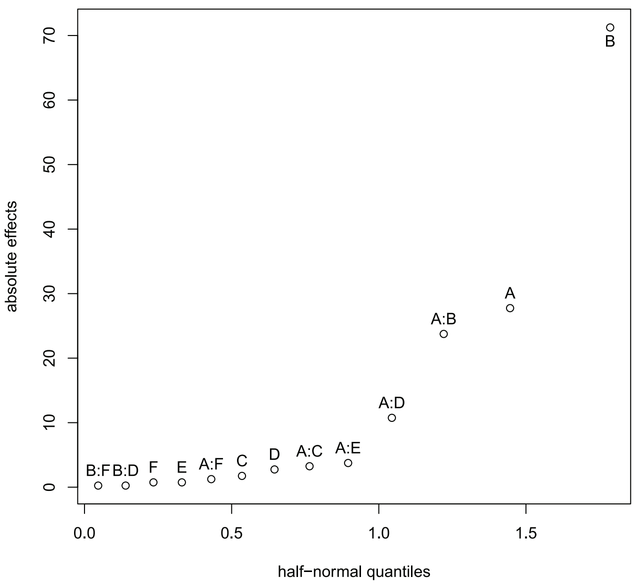

FIGURE 1. Half-Normal Plot, Injection Molding Experiment.

In step (ii), among the significant effects in model 1.1, we use rule 1 to identify a pair of 2FI and its parental main effect with similar magnitudes. From Figure 1, the only such pair is A and AB. Therefore, we consider the twin CMEs $(A|B+)$ and $(A|B-)$. Because A $(= 6.938)$ and AB $(= 5.938)$ have the same sign, by rule 1, we should substitute them with $(A|B+)$. This leads to the model with only two terms, B and $(A|B+)$, denoted as **model 1.2**. Its R^2 value is 96.14% and the p values for these two terms are 6.06e−10 and 1.72e− 06, respectively, each of which is more significant than the corresponding term in model 1.1. Clearly, model 1.2 is better than model 1.1. Moreover, unlike AB in model 1.1, the CME $(A|B+)$ has a good interpretation, i.e., at high screw speed, pressure has a significant effect on shrinkage but not at low speed. For ease of reading, this model is given as

$$\bar{y} = 27.313 + 17.8163B + 12.875(A|B+).$$

Example 2: Filtration Experiment

A team of engineers conducted an experiment on the filtration rate of a chemical product using a 2^{4-1}_{IV} design with eight runs. The defining relation of this design is $I = ABCD$ and the four factors are temperature (A), pressure (B), concentration of formaldehyde (C), and stirring rate (D). The design matrix and data are given in Table 3, where the response is measured in gal/h (Montgomery (1991), p. 342). The

TABLE 3. Design Matrix and Response Data, Filtration Experiment

A	B	C	D	y
−1	−1	−1	−1	45
1	−1	−1	1	100
−1	1	−1	1	45
1	1	−1	−1	65
−1	−1	1	1	75
1	−1	1	−1	60
−1	1	1	−1	80
1	1	1	1	96

goal of this experiment was to maximize the filtration rate. We apply the CME analysis to this data set.

In step (i), we use half-normal plot to identify significant effects. From Figure 2, it is clearly seen that main effects A, D, and C and 2FIs, AD $(= BC)$ and AC $(= BD)$, are significant. There is a huge gap between C and B. Therefore, we include these five terms in the first model, denoted as model 2.1. Its R^2 value is 99.79% and the p values for A, AD, AC, D, and C are 0.45%, 0.45%, 0.47%, 0.59%, and 0.82%, respectively.

In step (ii), among the significant effects in model 2.1, we use rule 1 to identify a pair of 2FI and its

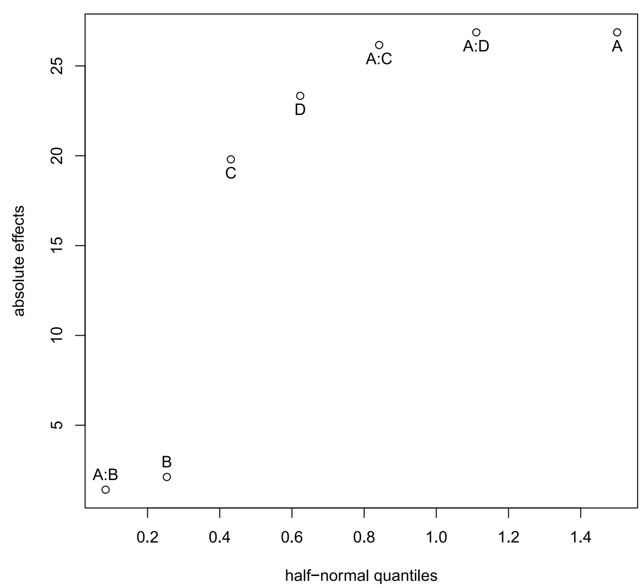

FIGURE 2. Half-Normal Plot, Filtration Experiment.

parental main effect with similar magnitudes. From Figure 2, the first such pair are A and AD. Therefore, we consider the twin CMEs $(A|D+)$ and $(A|D-)$. Because A (= 9.5) and AD (= 9.5) have the same sign, by rule 1, we should substitute them with $(A|D+)$. This leads to a model with four terms; $(A|D+)$, AC, D, and C, denoted as model 2.2. Its R^2 value is 99.79% and the p values for the four terms are 0.013%, 0.039%, 0.055%, and 0.089%, respectively, each of which is more significant than the corresponding term in model 2.1. Clearly, model 2.2 is better than model 2.1. Moreover, unlike AD in model 2.1, the CME $(A|D+)$ has a good engineering interpretation, i.e., at high stirring rate, temperature has a significant effect on filtration rate, but not at low stirring rate.

Next, we search for other such pairs among the rest of the significant effects. In Figure 2, the next such pair is BD (= AC) and D and, thus, we consider the twin CMEs $(D|B+)$ and $(D|B-)$. Because D (= 8.25) and BD (= −9.25) have opposite signs, by rule 1, we should substitute them with $(D|B-)$. The two selected CMEs $(A|D+)$ and $(D|B-)$ are neither siblings nor belong to the same family. Therefore, by rule 3, they are orthogonal to each other and can both be included in the same model. This further reduces the model to three terms; $(A|D+)$, $(D|B-)$, and C, denoted as **model 2.3**. Its R^2 value is 99.66% and the p values for the three terms are 1.96e−05, 2.72e−5 and 0.026%, each of which is more significant than the corresponding term in model 2.1 and model 2.2. Therefore, model 2.3 is the best. Moreover, the CME $(D|B-)$ has a good engineering interpretation, i.e., at low concentration of formaldehyde, the stirring rate has a significant effect on filtration rate but not at high concentration. Because there are no further such pairs among the remaining significant effects, we conclude the CME analysis with model 2.3. For ease of reading, this model is given as

$$\bar{y} = 70.75 + 19(A|D+) + 18(D|B-) + 7C.$$

Example 3: Aluminum Experiment

The Iowa Aluminum Corporation manufactures aluminum sheets from recycled aluminum beverage containers. The molten aluminum was placed onto a continuous strip, and then went through three mills before the final packing. Coolant consisting of oil and water was applied to the metal as it entered the mill each time. The produced aluminum sheets had a rejection rate of 25%, so an experiment was undertaken to improve the quality. Due to the limited time and resource, a 2^{6-2}_{IV} design with 16 runs was used. The defining relations of the design are $I = ABCE = ADEF = BCDF$, and the six factors are coolant temperature (A), oil percentage (B), coolant volume 1 (C), coolant volume 2 (D), coolant volume 3 (E), and strip speed (F). The design matrix and data are shown in Table 4, where the response is the surface impurity score with a scale of 0–10, with 0 being no impurity and 10 high impurity (Neter et al. (1996), p. 1259). The goal of this experiment was to minimize the impurity score. We apply the CME analysis to this data set.

TABLE 4. Design Matrix and Response Data, Aluminum Experiment

A	B	C	D	E	F	y
−1	−1	−1	−1	−1	−1	4
1	−1	−1	−1	1	−1	6
−1	1	−1	−1	1	1	7
1	1	−1	−1	−1	1	2
−1	−1	1	−1	1	1	3
1	−1	1	−1	−1	1	1
−1	1	1	−1	−1	−1	5
1	1	1	−1	1	−1	9
−1	−1	−1	1	−1	1	3
1	−1	−1	1	1	1	2
−1	1	−1	1	1	−1	8
1	1	−1	1	−1	−1	5
−1	−1	1	1	1	−1	4
1	−1	1	1	−1	−1	4
−1	1	1	1	−1	1	4
1	1	1	1	1	1	6

In step (i), we use half-normal plot to identify significant effects. From Figure 3, it is clearly seen that main effects B, F, and E are much more significant than the rest. They are followed by two 2FIs AC (= BE) and AF (= DE), and the R^2 value for the five terms is 96.45%. Because the remaining effects are not significant, we include these five terms in the first model, denoted as model 3.1. The p values for B, F, E, AC, and AF are 3.17e−06, 8.56e−6, 8.56e−06, 0.032%, and 0.135%, respectively.

In step (ii), among the significant effects in model 3.1, we use rule 1 to identify a pair of 2FIs and its parental main effect with similar magnitudes. From Figure 3, the first such pair are E and BE (= AC).

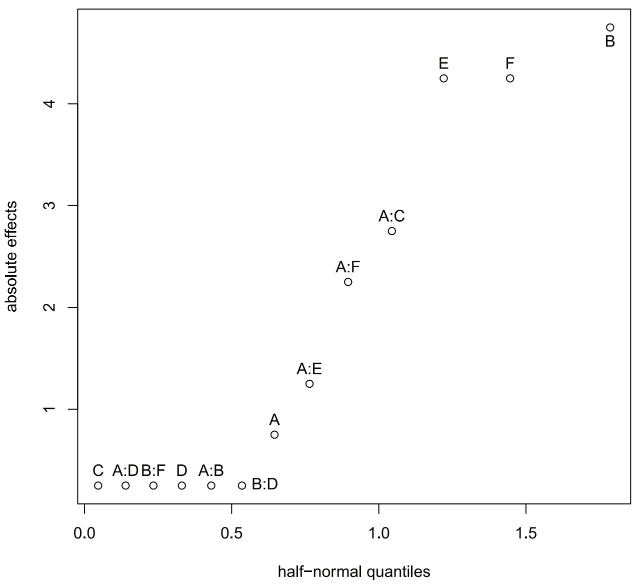

FIGURE 3. Half-Normal Plot, Aluminum Experiment.

Therefore, we consider the twin CMEs $(E|B+)$ and $(E|B-)$. Because E (= 1.0625) and BE (= 0.6875) have the same sign, by rule 1, we should substitute them with $(E|B+)$. This leads to a model with four terms; $(E|B+)$, B, F, and AF, denoted as model 3.2. Its R^2 value is 94.93% and the p values for the four terms are 3.75e−06, 5.57e−06, 1.58e−05, and 2.68%, respectively. By comparing the R^2 values and the p values for the significant effects, we cannot say that model 3.2 is better than model 3.1. However, unlike BE in model 3.1, the CME $(E|B+)$ in model 3.2 has a good engineering interpretation, i.e., at high oil percentage, coolant volume of the third mill has a significant effect on the impurity score but not at low percentage.

Next, we search for other such pairs among the rest of the significant effects. From Figure 3, there are two pairs associated with AF $(= DE)$; AF with F and DE with E. If DE and E are chosen, the corresponding twin CMEs $(E|D+)$ and $(E|D-)$ are siblings of the selected CME $(E|B+)$ in model 3.2. By rule 2, they cannot both be included in the same model. For AF and F, we consider the twin CMEs $(F|A+)$ and $(F|A-)$. Because F (= −1.0625) and AF (= −0.5625) have the same sign, by rule 1, we replace them with $(F|A+)$. The two selected CMEs, $(E|D+)$ and $(F|A+)$, are neither siblings nor belong to the same family. Therefore, by rule 3, they are orthogonal to each other and can both be included in the same model. This further reduces the model to three terms; $(E|B+)$, B, and $(F|A+)$, denoted as **model 3.3**. Its R^2 value is 92.22% and the p values for the three terms are 1.16e−05, 1.75e−05, and 2.40e−05, respectively. By comparing the R^2 values and the p values for significant effects, model 3.3 is not the best among the three models. However, unlike AF in model 3.1 and model 3.2, the CME $(F|A+)$ in model 3.3 has a good engineering interpretation, i.e., at high coolant temperature, strip speed has a significant effect on impurity score but not at low temperature. Therefore, the final model can be written as

$$\hat{y} = 4.563 + 1.188B + 1.75(E|B+) - 1.625(F|A+).$$

5. Conclusions

In this paper, we develop a systematic method of analysis to de-alias aliased 2FIs in 2^{k-q} designs. Properties of conditional main effects (CMEs) are studied. Rules of analysis are developed from these properties. The method of CME analysis is proposed based on three such rules. Three examples are given to illustrate the analysis strategy. The first two examples show dramatic improvement in model fitting and understanding of effects with the CME analysis. For the last example, though the CME analysis does not give better models based on traditional model-selection criteria, the two alternative models are more parsimonious. Furthermore, the CMEs identified in the three models have good engineering interpretations. Thus, they can serve as alternatives to traditional analysis methods to aid the investigators to better understand their experiment. It is not necessary to use the CME analysis for resolution V design because all 2FIs in such designs can be estimated clear of the main effects (Wu and Hamada (2009), p. 217). On the other hand, the CME analysis can be applied to resolution III designs that have some 2FIs aliased with other 2FIs. Finally, note that our method can only condition on the main effects. If we condition on a second-order effect, the corresponding CME will involve three factors, which may become too complicated to be useful.

Another remark concerns the applicability of the three principles (hierarchy, sparsity, and heredity) for factorial designs (Wu and Hamada (2009), Section 4.6) to the current situation. The short answer is no because these principles were developed for traditional factorial effects. Take Example 2. As one referee observed, its model 2.1 has five terms out of seven degrees of freedom. This clearly violates the sparsity principle. But the final model 2.3 has only three terms.

Some questions warrant further study. The first and perhaps most obvious one is whether the proposed method should be extended to 3^{k-q} designs. The answer is no because, for 3^{k-q} designs, the two-factor interactions can be studied unambiguously even for resolution IV designs by using the Hamada–Wu strategy, which exploits the complex aliasing structure of such designs if parametrized by the so-called linear-quadratic system (Wu and Hamada (2009), Chapter 9). The next question concerns the aliasing structure between CMEs and traditional effects in 2^{k-q} designs. For smaller designs, one can do simple algebras to find the relationships. However, for larger designs, this will not be feasible. A recent theory by Sabbaghi (2016) can be used to address this question. He shows that the conditional main effects can be represented by the corresponding indicator functions and that properties 1–5 in Section 2 can be mathematically verified under no or rather weak conditions. For the theory of indicator functions, the readers are referred to Fontana et al. (2000), Ye (2003), and Sabbaghi et al. (2014). Finally, can standard criteria like minimum aberration and minimum moment aberration (Cheng (2013), Mukerjee and Wu (2006)) be extended for selecting optimal fractions of 2^{k-q} designs if the purpose is to use the CME analysis?

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References

Box, G. E. P.; Hunter, W. G.; and Hunter J. S. (2005). *Statistics for Experimenters: Design, Innovation, and Discovery*, 2nd edition. New York, NY: Wiley.

Brajac, M. and Morey, C. (1987). “GM10 Chassis Body Marriage Simulation: A Designed Experiment Approach”. *Journal of Designed Experiment Case Studies* 1(1), pp.4–31.

Cheng, C. S. (2013). *Theory of Factorial Design*. Boca Raton, FL: CRC Press.

Finney, D. J. (1945). “The Fractional Replication of Factorial Arrangements”. *Annals of Eugenics* 12, pp. 291–303.

Fontana, R.; Pistone, G.; and Rogantin, M. P. (2000). “Classification of Two-Level Factorial Fractions”. *Journal of Statistical Planning and Inference* 87(1), pp. 149–172.

Hamada, M. S. and Wu, C. F. J. (1992). “Analysis of Designed Experiments with Complex Aliasing”. *Journal of Quality Technology* 24, pp. 130–137.

Montgomery, D. C. (1991). *Design and Analysis of Experiments*, 3rd edition. New York, NY: Wiley.

Mukerjee, R. and Wu, C. F. J. (2006). *A Modern Theory of Factorial Design*. New York, NY: Springer.

Neter, J.; Kutner, M. H.; Nachtsheim, C. J.; and Wasserman, W. (1996). *Applied Linear Statistical Models*, 4th edition. Chicago, IL: Irwin.

Sabbaghi, A. (2016). “Indicator Functions and Aliasing Relations for Conditional Main Effects”, unpublished.

Sabbaghi, A.; Dasgupta, T.; and Wu, C. F. J. (2014). “Indicator Functions and the Algebra of the Linear-Quadratic Parameterization”. *Biometrika* 101(2), pp. 351–363.

Wu, C. F. J. (2015). “Post-Fisherian Experimentation: From Physical to Virtual”. *Journal of the American Statistical Association* 110, pp. 612–620.

Wu, C. F. J. and Hamada, M. S. (2009). *Experiments: Planning, Analysis, and Optimization*, 2nd edition. New York, NY: Wiley.

Ye, K. Q. (2003). “Indicator Function and Its Application in Two-level Factorial Designs”. *The Annals of Statistics* 31(3), pp. 984–994.