Unit 5: Fractional Factorial Experiments at Two Levels

Source : Chapter 5 (sections 5.1 - 5.5, part of section 5.7).

- Leaf Spring Experiment (Section 5.1)
- Effect aliasing, resolution, minimum aberration criteria (Section 5.2).
- Analysis of Fractional Factorials (Section 5.3).
- Conditional Main Effects Analysis (Section 5.4)
- Techniques for resolving ambiguities in aliased effects (Section 5.5).
- Choice of designs, use of design tables (Section 5.6).
- Blocking in 2^{k-p} designs (Section 5.7).

Leaf Spring Experiment

- y = free height of spring, target = 8.0 inches.
  Goal : get y as close to 8.0 as possible (nominal-the-best problem).
- Five factors at two levels, use a 16-run design with three replicates for each run. It is a 2^{5-1} design, 1/2 fraction of the 2^5 design.

Table 1: Factors and Levels, Leaf Spring Experiment

	Factor	Level	
		−	+
B.	high heat temperature (°F)	1840	1880
C.	heating time (seconds)	23	25
D.	transfer time (seconds)	10	12
E.	hold down time (seconds)	2	3
Q.	quench oil temperature (°F)	130-150	150-170

Leaf Spring Experiment: Design Matrix and Data

Table 2: Design Matrix and Free Height Data, Leaf Spring Experiment

Factor					Free Height			$\bar{y}_i$	s_i^2	$\ln s_i^2$
B	C	D	E	Q						
−	+	+	−	−	7.78	7.78	7.81	7.7900	0.0003	-8.1117
+	+	+	+	−	8.15	8.18	7.88	8.0700	0.0273	-3.6009
−	−	+	+	−	7.50	7.56	7.50	7.5200	0.0012	-6.7254
+	−	+	−	−	7.59	7.56	7.75	7.6333	0.0104	-4.5627
−	+	−	+	−	7.94	8.00	7.88	7.9400	0.0036	-5.6268
+	+	−	−	−	7.69	8.09	8.06	7.9467	0.0496	-3.0031
−	−	−	−	−	7.56	7.62	7.44	7.5400	0.0084	-4.7795
+	−	−	+	−	7.56	7.81	7.69	7.6867	0.0156	-4.1583
−	+	+	−	+	7.50	7.25	7.12	7.2900	0.0373	-3.2888
+	+	+	+	+	7.88	7.88	7.44	7.7333	0.0645	-2.7406
−	−	+	+	+	7.50	7.56	7.50	7.5200	0.0012	-6.7254
+	−	+	−	+	7.63	7.75	7.56	7.6467	0.0092	-4.6849
−	+	−	+	+	7.32	7.44	7.44	7.4000	0.0048	-5.3391
+	+	−	−	+	7.56	7.69	7.62	7.6233	0.0042	-5.4648
−	−	−	−	+	7.18	7.18	7.25	7.2033	0.0016	-6.4171
+	−	−	+	+	7.81	7.50	7.59	7.6333	0.0254	-3.6717

Why Use Fractional Factorial Designs?

- If a 2^5 design is used for the experiment, its 31 degrees of freedom would be allocated as follows:

	Main Effects	Interactions 2-Factor	3-Factor	4-Factor	5-Factor
#	5	10	10	5	1

- Using effect hierarchy principle, one would argue that 4fi's, 5fi and even 3fi's are not likely to be important. There are $10+5+1 = 16$ such effects, half of the total runs! Using a 2^5 design can be wasteful (unless 32 runs cost about the same as 16 runs.)

- Use of a FF design instead of full factorial design is usually done for economic reasons. Since there is no free lunch , what price to pay? See next.

Effect Aliasing and Defining Relation

- In the design matrix, col E = col $B\times$ col $C\times$ col D. That means,

$$\bar{y}(E+) - \bar{y}(E-) = \bar{y}(BCD+) - \bar{y}(BCD-).$$

Therefore the design is not capable of distinguishing E from BCD. The main effect E is aliased with the interaction BCD. Notationally,

$$E = BCD \quad \text{or} \quad \mathbf{I} = BCDE,$$

$\mathbf{I}$ = column of +'s is the identity element in the group of multiplications. (Notice the mathematical similarity between aliasing and confounding. What is the difference?)

- $\mathbf{I} = BCDE$ is the defining relation for the 2^{5-1} design. It implies all the 15 effect aliasing relations :

$$\begin{aligned}
&B = CDE,\ C = BDE,\ D = BCE,\ E = BCD,\\
&BC = DE,\ BD = CE,\ BE = CD,\\
&Q = BCDEQ,\ BQ = CDEQ,\ CQ = BDEQ,\ DQ = BCEQ,\\
&EQ = BCDQ,\ BCQ = DEQ,\ BDQ = CEQ,\ BEQ = CDQ.
\end{aligned}$$

Clear Effects

- A main effect or two-factor interaction (2fi) is called clear if it is not aliased with any other m.e.'s or 2fi's and strongly clear if it is not aliased with any other m.e.'s, 2fi's or 3fi's. Therefore a clear effect is estimable under the assumption of negligible 3-factor and higher interactions and a strongly clear effect is estimable under the weaker assumption of negligible 4-factor and higher interactions.

- In the 2^{5-1} design with $\mathbf{I} = BCDE$, which effects are clear and strongly clear?

  Ans: B, C, D, E are clear, Q, BQ, CQ, DQ, EQ are strongly clear.

- Consider the alternative plan 2^{5-1} design with $\mathbf{I} = BCDEQ$. (It is said to have resolution V because the length of the defining word is 5 while the previous plan has resolution IV.) It can be verified that all five main effects are strongly clear and all 10 2fi's are clear. (Do the derivations). This is a very good plan because each of the 15 degrees of freedom is either clear or strongly clear.

Defining Contrast Subgroup for 2^{k-p} Designs

- A 2^{k-p} design has k factors, 2^{k-p} runs, and it is a 2^{-p}th fraction of the 2^k design. The fraction is defined by p independent defining words. The group formed by these p words is called the defining contrast subgroup. It has $2^p - 1$ words plus the identity element $\mathbf{I}$.

- Resolution = shortest wordlength among the $2^p - 1$ words.

- Example: A 2^{6-2} design with $\mathbf{5} = \mathbf{12}$ and $\mathbf{6} = \mathbf{134}$. The two independent defining words are $\mathbf{I} = \mathbf{125}$ and $\mathbf{I} = \mathbf{1346}$. Then $\mathbf{I} = \mathbf{125} \times \mathbf{1346} = \mathbf{23456}$. The defining contrast subgroup = $\{\mathbf{I}, \mathbf{125}, \mathbf{1346}, \mathbf{23456}\}$. The design has resolution III.

Deriving Aliasing Relations for the 2^{6-2} design

- For the same 2^{6-2} design, the defining contrast subgroup is

$$\mathbf{I} = \mathbf{125} = \mathbf{1346} = \mathbf{23456}.$$

All the 15 degrees of freedom (each is a coset in group theory) are identified.

$$\begin{aligned}
\mathbf{I} &= \mathbf{125} = \mathbf{1346} = \mathbf{23456},\\
\mathbf{1} &= \mathbf{25} = \mathbf{346} = \mathbf{123456},\\
\mathbf{2} &= \mathbf{15} = \mathbf{12346} = \mathbf{3456},\\
\mathbf{3} &= \mathbf{1235} = \mathbf{146} = \mathbf{2456},\\
\mathbf{4} &= \mathbf{1245} = \mathbf{136} = \mathbf{2356},\\
\mathbf{5} &= \mathbf{12} = \mathbf{13456} = \mathbf{2346},\\
\mathbf{6} &= \mathbf{1256} = \mathbf{134} = \mathbf{2345},\\
\mathbf{13} &= \mathbf{235} = \mathbf{46} = \mathbf{12456},\\
\mathbf{14} &= \mathbf{245} = \mathbf{36} = \mathbf{12356},\\
\mathbf{16} &= \mathbf{256} = \mathbf{34} = \mathbf{12345},\\
\mathbf{23} &= \mathbf{135} = \mathbf{1246} = \mathbf{456},\\
\mathbf{24} &= \mathbf{145} = \mathbf{1236} = \mathbf{356},\\
\mathbf{26} &= \mathbf{156} = \mathbf{1234} = \mathbf{345},\\
\mathbf{35} &= \mathbf{123} = \mathbf{1456} = \mathbf{246},\\
\mathbf{45} &= \mathbf{124} = \mathbf{1356} = \mathbf{236},\\
\mathbf{56} &= \mathbf{126} = \mathbf{1345} = \mathbf{234}.
\end{aligned} \qquad (1)$$

- It has the clear effects: 3, 4, 6, 23, 24, 26, 35, 45, 56. It has resolution III.

WordLength Pattern and Resolution

- Define A_i = number of defining words of length i. $W = (A_3, A_4, A_5, \ldots)$ is called the wordlength pattern. In this design, $W = (1, 1, 1, 0)$. It is required that $A_2 = 0$. (Why? No main effect is allowed to be aliased with another main effect.)
- Resolution = smallest r such that $A_r \geq 1$.
- Maximum resolution criterion: For fixed k and p, choose a 2^{k-p} design with maximum resolution.
- Rules for Resolution IV and V Designs:

(**i**) *In any resolution IV design, the main effects are clear.*

(**ii**) *In any resolution V design, the main effects are strongly clear and the* two-factor *interactions are clear.* (2)

(**iii**) *Among the resolution IV designs with given k and p, those with the largest number of clear* two-factor *interactions are the best.*

A Projective Rationale for Resolution

- For a resolution R design, its projection onto any R-1 factors is a full factorial in the R-1 factors. This would allow effects of all orders among the R-1 factors to be estimable. (Caveat: it makes the assumption that other factors are inert.)

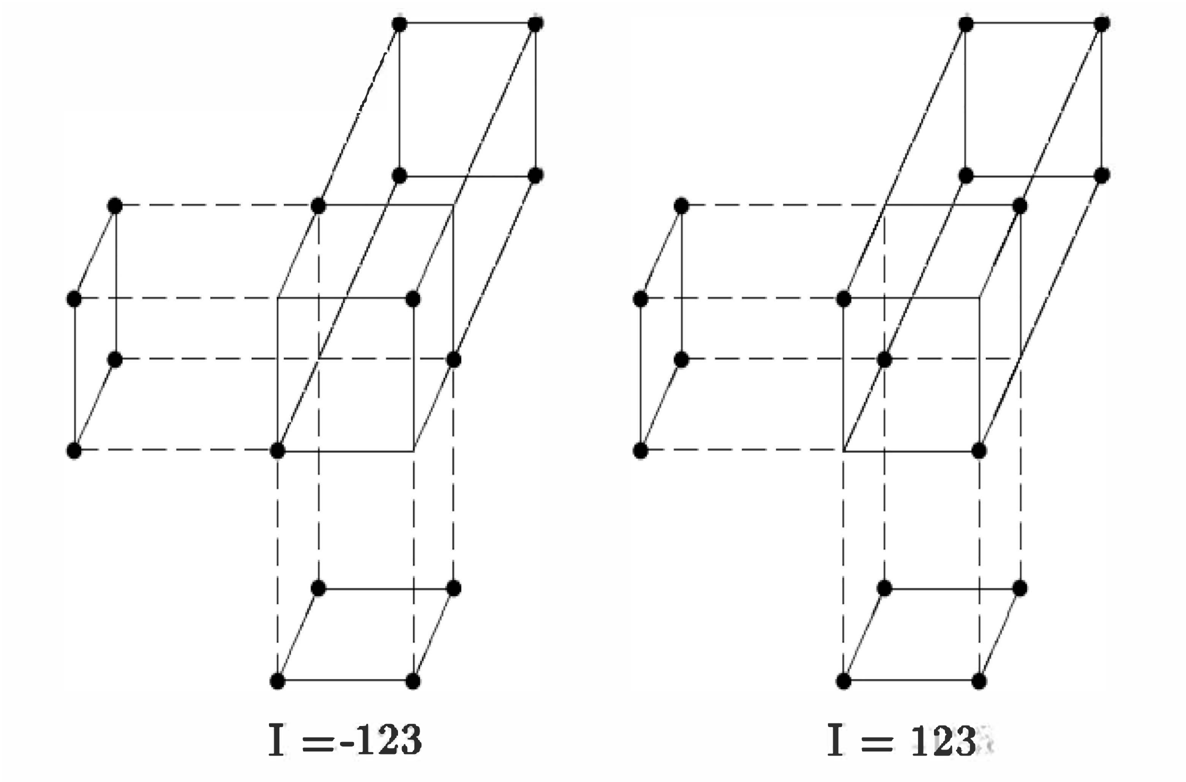


Figure 1: 2^{3-1} Designs Using $\mathbf{I} = \pm\mathbf{123}$ and Their Projections to 2^2 Designs.

Minimum Aberration Criterion

- Motivating example: consider the two 2^{7-2} designs:

$$\begin{aligned} d_1: \quad \mathbf{I} &= \mathbf{4567} = \mathbf{12346} = \mathbf{12357}, \\ d_2: \quad \mathbf{I} &= \mathbf{1236} = \mathbf{1457} = \mathbf{234567}. \end{aligned}$$

  Both have resolution IV, but

$$W(d_1) = (0,1,2,0,0) \text{ and } W(d_2) = (0,2,0,1,0).$$

  Which one is better? Intuitively one would argue that d_1 is better because $A_4(d_1) = 1 < A_4(d_2) = 2$. (Why? Effect hierarchy principle.)

- For any two 2^{k-p} designs d_1 and d_2, let r be the smallest integer such that $A_r(d_1) \neq A_r(d_2)$. Then d_1 is said to have less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. If there is no design with less aberration than d_1, then d_1 has minimum aberration.

- Throughout the book, this is the major criterion used for selecting fractional factorial designs. Its theory is covered in the Mukherjee-Wu (2006) book.

Analysis for Location Effects

- Same strategy as in full factorial experiments except for the interpretation and handling of aliased effects.
- For the location effects (based on $\bar{y}_i$ values), the factorial effects are given in Table 3 and the corresponding half-normal plot in Figure 2. Visually one may judge that Q, B, C, CQ and possibly E, BQ are significant. One can apply the studentized maximum modulus test (see section 4.14, not covered in class) to confirm that Q, B, C, CQ are significant at 0.05 level (see pp. 219 and 221).
- The $B \times Q$ and $C \times Q$ plots (Figure 3) show that they are synergystic.
- For illustration, we use the model

$$\begin{aligned} \hat{y} = \quad & 7.6360 + 0.1106 x_B + 0.0519 x_E + 0.0881 x_C - 0.1298 x_Q \\ & + 0.0423 x_B x_Q - 0.0827 x_C x_Q \end{aligned} \tag{3}$$

Factorial Effects, Leaf Spring Experiment

Table 3: Factorial Effects, Leaf Spring Experiment

Effect	$\bar{y}$	$\ln s^2$
B	0.221	1.891
C	0.176	0.569
D	0.029	-0.247
E	0.104	0.216
Q	-0.260	0.280
BQ	0.085	-0.589
CQ	-0.165	0.598
DQ	0.054	1.111
EQ	0.027	0.129
BC	0.017	-0.002
BD	0.020	0.425
CD	-0.035	0.670
BCQ	0.010	-1.089
BDQ	-0.040	-0.432
BEQ	-0.047	0.854

Half-normal Plot of Location Effects, Leaf Spring Experiment

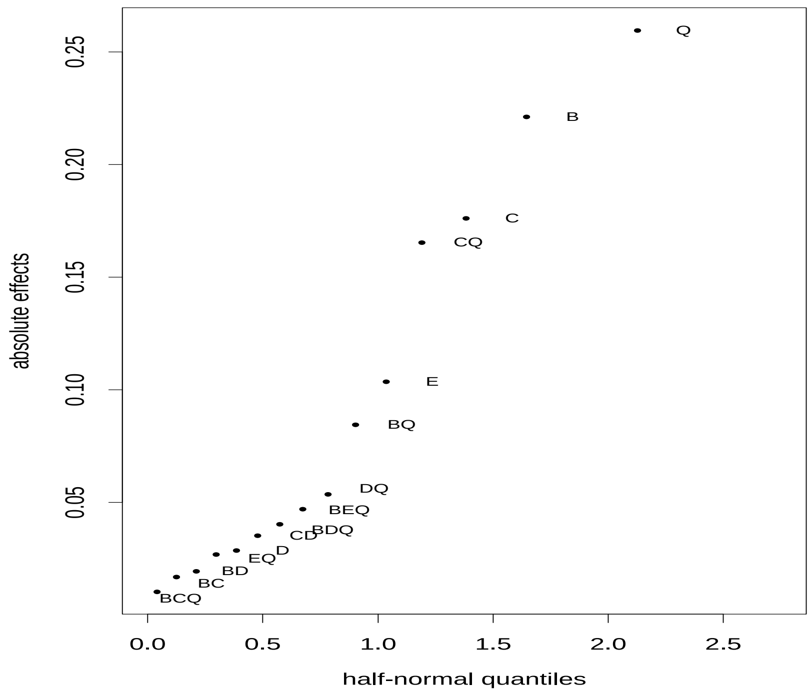


Figure 2: Half-Normal Plot of Location Effects, Leaf Spring Experiment

Interaction Plots

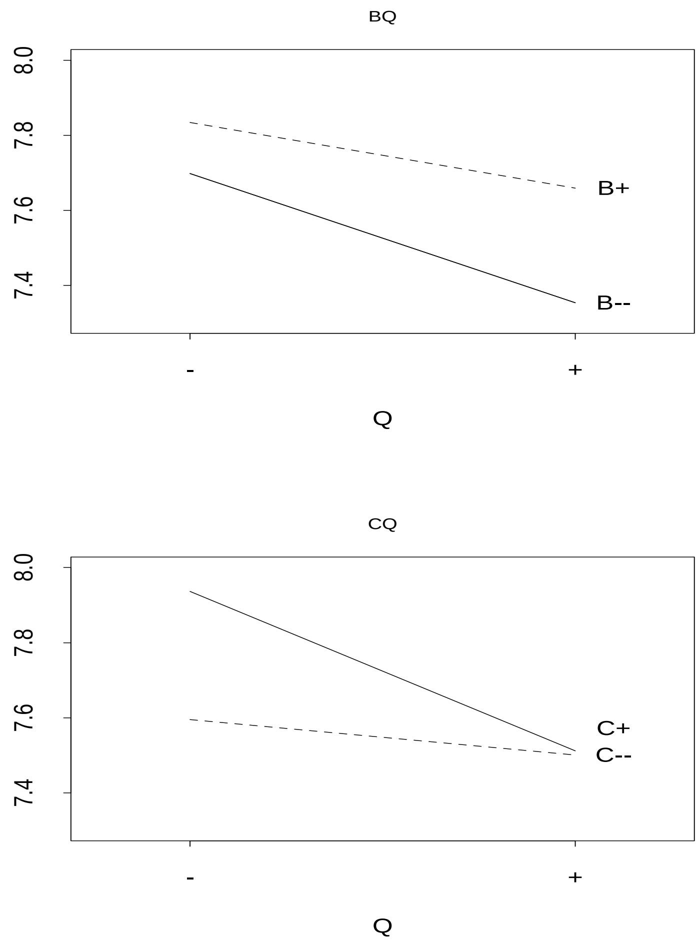


Figure 3: $B \times Q$ and $C \times Q$ interaction plots, Leaf Spring Experiment

Analysis for Dispersion Effects

- For the dispersion effects (based on $z_i = \ln s_i^2$ values), the half-normal plot is given in Figure 4. Visually only effect B stands out. This is confirmed by applying the studentized maximum modulus test (see pp.163-164 of WH, 2000). For illustration, we will include B, DQ, BCQ in the following model,

$$\ln \hat{\sigma}^2 = -4.9313 + 0.9455 x_B + 0.5556 x_D x_Q - 0.5445 x_B x_C x_Q. \tag{4}$$

Half-normal Plot of Dispersion Effects, Leaf Spring Experiment

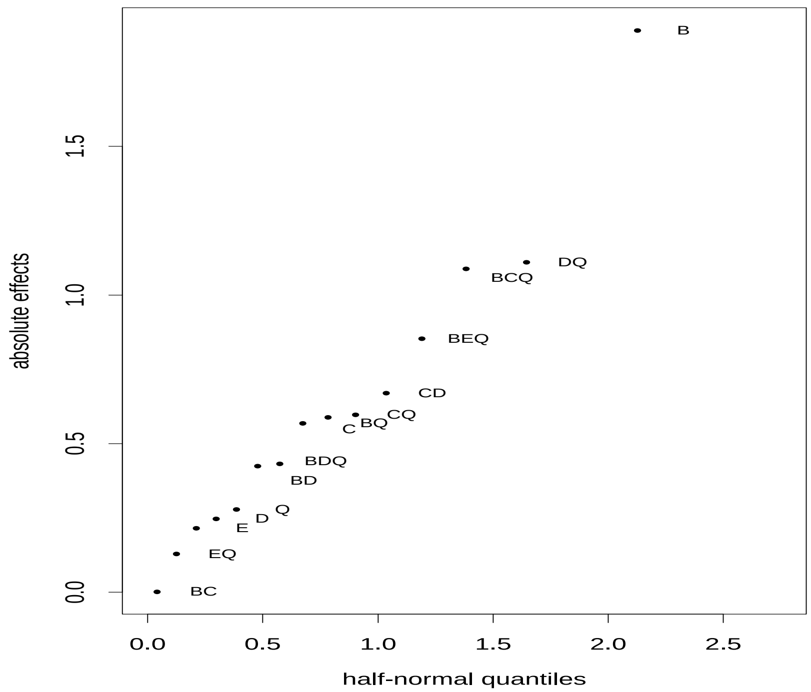


Figure 4: Half-Normal Plot of Dispersion Effects, Leaf Spring Experiment

Two-Step Procedure for Optimization

- Step 1: To minimize s^2 (or $\ln s^2$) based on eq. (4), choose $B = -$. Based on the $D \times Q$ plot (Figure 5), choose the combination with the lowest value, $D = +$, $Q = -$. With $B = -$ and $Q = -$, choose $C = +$ to attain the minimum in the $B \times C \times Q$ interaction plot (Figure 6). Another confirmation: they lead to $x_B = -, x_D x_Q = -$ and $x_B x_C x_Q = +$ in the model (4), which make each of the last three terms negative.

- Step 2: With $BCDQ = (-,+,+,-)$,

$$\begin{aligned}\hat{y} &= 7.6360 + 0.1106(-1) + 0.0519x_E + 0.0881(+1) - 0.1298(-1) \\ &\quad + 0.0423(-1)(-1) - 0.0827(+1)(-1) \\ &= 7.8683 + 0.0519x_E.\end{aligned}$$

  By solving $\hat{y} = 8.0$, $x_E = 2.54$.

  Warning: This is way outside the experimental range for factor E. Such a value may not make physical sense and the predicted variance value for this setting may be too optimistic and not substantiated.

Interaction Plots for Dispersion Effects

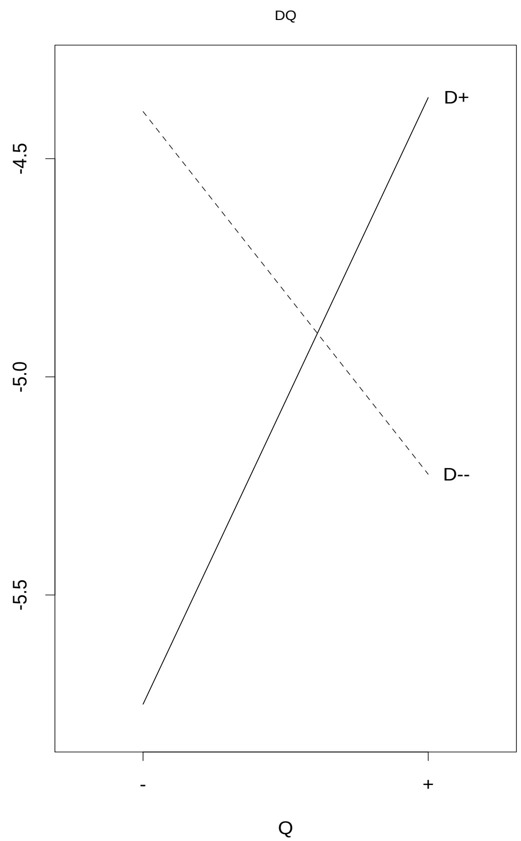


Figure 5 : $D \times Q$ Interaction Plot

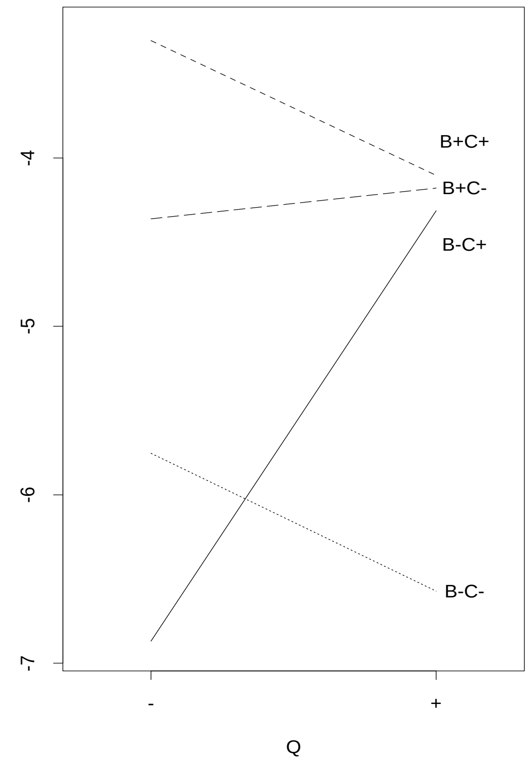


Figure 6 : $B \times C \times Q$ Interaction Plot

Conditional Main Effect

- Consider two factors A and B, each at two levels denoted by $+$ and $-$: Main effects: $ME(A) \; = \; \bar{y}(A+) - \bar{y}(A-) =$

$$\frac{1}{2}[\bar{y}(A+|B+) + \bar{y}(A+|B-)] - \frac{1}{2}[\bar{y}(A-|B+) + \bar{y}(A-|B-)]. \qquad (5)$$

  Interaction effects: $INT(A,B) =$

$$\frac{1}{2}[\bar{y}(A+|B+) - \bar{y}(A+|B-)] - \frac{1}{2}[\bar{y}(A-|B+) - \bar{y}(A-|B-)]. \qquad (6)$$

  Conditional main effect of A given B at $+$:

$$CME(A|B+) \;\; = \;\; \bar{y}(A+|B+) - \bar{y}(A-|B+).$$

  Conditional main effect of A given B at $-$:

$$CME(A|B+) \;\; = \;\; \bar{y}(A+|B-) - \bar{y}(A-|B-).$$

- Switching the roles of A and B, CME($B|A+$) and CME($B|A-$) can be similarly defined.

Rule 1 of CME Analysis

- By adding $ME(A)$ and $INT(A,B)$ (check (5) + (6)),

$$ME(A) + INT(A,B) = CME(A|B+). \tag{7}$$

- By subtracting $ME(A)$ and $INT(A,B)$,

$$ME(A) + INT(A,B) = CME(A|B-). \tag{8}$$

- If $ME(A)$ and $INT(A,B)$ have the same sign and are comparable in magnitude, we can replace $ME(A)$ and $INT(A,B)$ by $CME(A|B+)$.
- Similarly, if $ME(A)$ and $INT(A,B)$ have the opposite sign, they can be replaced by $CME(A|B-)$.

Rule 1:
Substitute a pair of interaction effect and its parental main effect that have similar magnitudes with one of the corresponding two CMEs.

Note: It achieves model parsimony (why?).

A 2^{4-1}_{IV} design and CMEs

For $CME(A|B+)$, we call $ME(A)$ its parent effect and $INT(A,B)$ its interaction effect.

- Use $(A|B+)$, etc. as its shorthand notation.

Table 4: CMEs and Factorial Effects from the $2^{(4-1)}_{IV}$ Design with $I = ABCD$

A	B	C	D	AB	CD	$A\|B+$	$A\|B-$	$A\|C+$	$C\|D-$	$B\|D+$
+	+	+	+	+	+	+	0	+	0	+
+	+	−	−	+	+	+	0	0	−	0
+	−	+	−	−	−	0	+	+	+	0
+	−	−	+	−	−	0	+	0	0	−
−	+	+	−	−	−	−	0	−	+	0
−	+	−	+	−	−	−	0	0	0	+
−	−	+	+	+	+	0	−	−	0	−
−	−	−	−	+	+	0	−	0	−	0

Siblings and Family

- CMEs having the same parent effect and interaction effects are called twin effects, e.g., $(A|B+)$ and $(A|B-)$.
- CMEs having the same parent effect but different interaction effects are called siblings effects, e.g., $(A|B+)$ and $(A|C+)$.
- The group of CMEs having the same or aliased interaction effects belongs to the same family, e.q., $(A|B+)$ and $(C|D-)$ in Table 1.

More Relationship

Summary of the relationships between various CMEs

- CMEs are orthogonal to all the traditional effects except for their parent effects and interaction effects.
- Sibling CMEs are not orthogonal to each other.
- CMEs in the same family are not orthogonal.
- CMEs with different parent effects and different interaction effects are orthogonal. (Example: $(A|B+)$ and $(B|D+)$ in the Table.)

Rules 2 and 3 of CME Analysis

Rule 2:

- Only one CME among its siblings can be included in the model.
- Only one CME from a family can be included in the model.

Rule 3:
CMEs with different parent effects and different interaction effects can be included in the same model.

Justification: In order to avoid generating too many incompatible models, only orthogonal effects are included in the model search.

CME Analysis

i. Use the traditional analysis methods such as ANOVA or half-normal plot, to select significant effects, including aliased pairs of effects. Go to ii.

ii. Among all the significant effects, use Rule 1 to find a pair of interaction effect and its parental main effect, and substitute them with an appropriate CME. Use Rules 2 and 3 to guide the search and substitution of other such pairs until they are exhausted.

In step i, a formal method like Lenth's method (Section 4.9) can be used instead of the half-normal plots.

Illustration with Filtration Experiment

- Four factors:
  - Temperature (A)
  - Pressure (B)
  - Concentration of formaldehyde (C)
  - Stirring rate (D)
- design with $I = ABCD$, aliasing relations like $AB = CD$, $AC = BD$, etc.

Table 5: Design Matrix and Response Data, Filtration Experiment

A	B	C	D	y
-1	-1	-1	-1	45
1	-1	-1	1	100
-1	1	-1	1	45
1	1	-1	-1	65
-1	-1	1	1	75
1	-1	1	-1	60
-1	1	1	-1	80
1	1	1	1	96

Illustration with Filtration Experiment

- 2^{4-1}_{IV} design with $I = ABCD$
- Traditional Analysis:

$$y \sim A + AD + AC + D + C$$

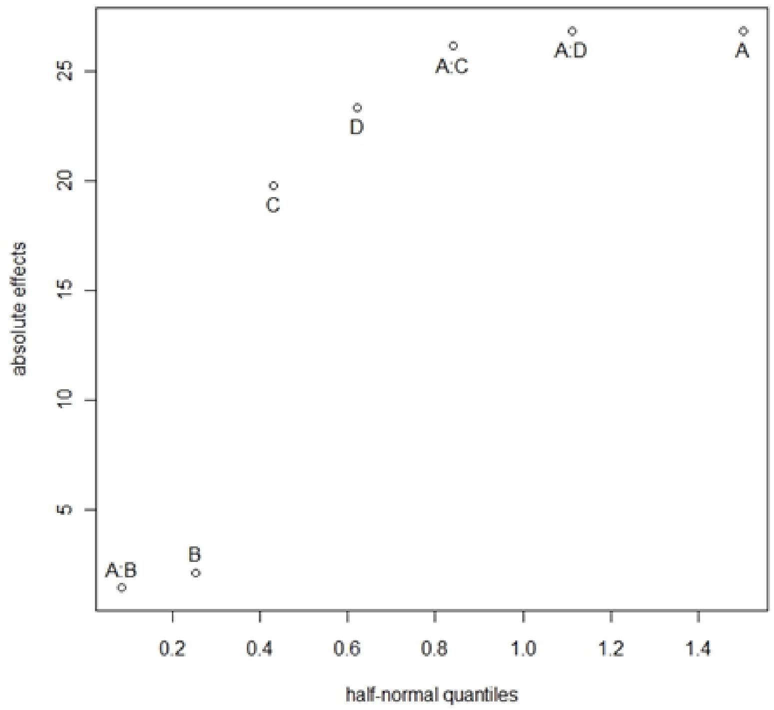


Illustration with Filtration Experiment

- 2^{4-1}_{IV} design with $I = ABCD$
- The CME analysis:

$$y \sim (A|D+) \ + AC + D + C$$

- Step (ii)
  - A and AD are both significant.
  - Consider $A|D+$ since A and D have same sign.

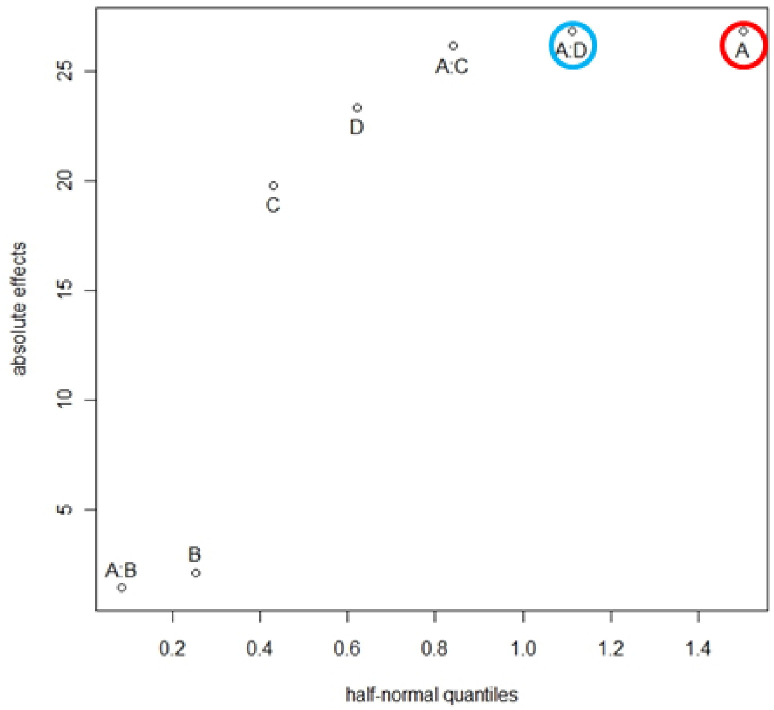
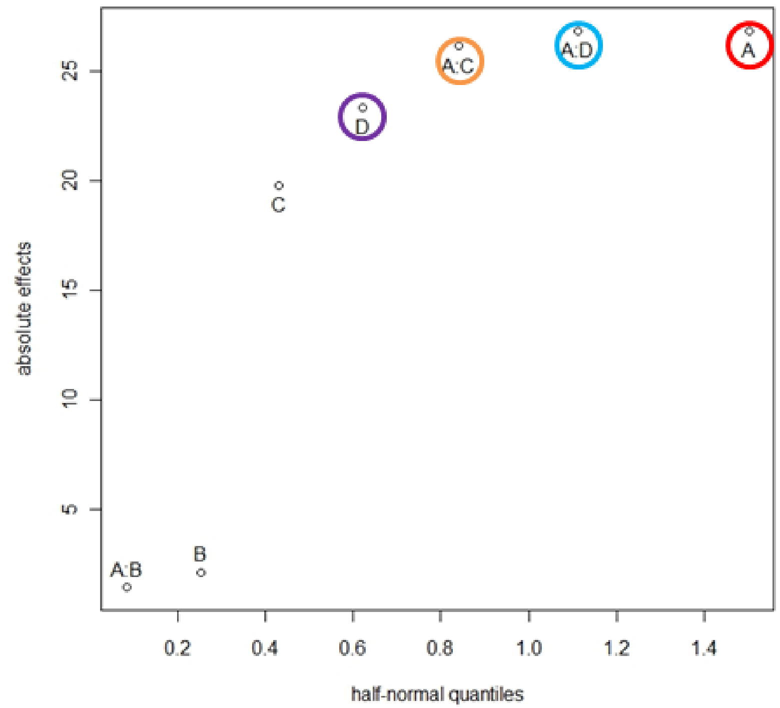


Illustration with Filtration Experiment

- 2^{4-1}_{IV} design with $I = ABCD$
- The CME analysis:

$$y \sim (A|D+)\ + (D|B-)\ + C$$

- Step (ii)
  - A and AD are both significant.
  - Consider $A|D+$ since A and D have same sign.
  - D and AC are both significant.
  - Consider $D|B-$ since D and B have same sign.



Summary of Filtration Experiment

- In the traditional analysis, we have

  $y \sim A(0.45\%) + AD(0.45\%) + AC(0.47\%) + D(0.59\%) + C(0.82\%).(R^2 = 99.79\%)$

- In the CME analysis, we have
  - $y \sim (A|D+)(0.013\%) + AC(0.039\%) + D(0.055\%) + C(0.089\%)$. $(R^2 = 99.79\%)$
  - $y \sim (A|D+)(1.96 \times 10^{-5}) + (D|B-)(2.72 \times 10^{-5}) + C(0.026\%)$. $(R^2 = 99.66\%)$

- The third model is the most parsimonious and best in terms of p values for significant effects. All three models have comparable R^2 values.

- The CMEs $(A|D+)$ and $(D|B-)$ in the last two models have good engineering interpretations.

Techniques for Resolving Ambiguities in Aliased Effects

- Among the three factorial effects that feature in model (4), $\boldsymbol{B}$ is clear and $\boldsymbol{DQ}$ is strongly clear.
- However, the term $x_B x_C x_Q$ is aliased with $x_D x_E x_Q$ (See bottom of page 5). The following three techniques can be used to resolve the ambiguities.
- Subject matter knowledge may suggest some effects in the alias set are not likely to be significant (or does not have a good physical interpretation).
- Or use effect hierarchy principle to assume away some higher order effects.
- Or use a follow-up experiment to de-alias these effects. Two methods are given in section 5.4 of WH.

Fold-over Technique

- Suppose the original experiment is based on a 2^{7-4}_{III} design with generators

$$d_1: \ \mathbf{4} = \mathbf{12}, \ \mathbf{5} = \mathbf{13}, \ \mathbf{6} = \mathbf{23}, \ \mathbf{7} = \mathbf{123}.$$

  None of its main effects are clear.

- To de-alias them, we can choose another 8 runs (no. 9-16 in Table 4) with reversed signs for each of the 7 factors. This follow-up design d_2 has the generators

$$d_2: \mathbf{4} = -\mathbf{12}, \mathbf{5} = -\mathbf{13}, \mathbf{6} = -\mathbf{23}, \mathbf{7} = \mathbf{123}.$$

  With the extra degrees of freedom, we can introduce a new factor 8 (or a blocking variable) for run number 1-8, and -8 for run number 9-16. See Table 4.

- The combined design $d_1 + d_2$ is a 2^{8-4}_{IV} design and thus all main effects are clear. (Its defining contrast subgroup is on p.227 of WH).

Augmented Design Matrix Using Fold-over Technique

Table 6: Augmented Design Matrix Using Fold-Over Technique

	d_1							
Run	1	2	3	4=12	5=13	6=23	7=123	8
1	−	−	−	+	+	+	−	+
2	−	−	+	+	−	−	+	+
3	−	+	−	−	+	−	+	+
4	−	+	+	−	−	+	−	+
5	+	−	−	−	−	+	+	+
6	+	−	+	−	+	−	−	+
7	+	+	−	+	−	−	−	+
8	+	+	+	+	+	+	+	+
	d_2							
Run	-1	-2	-3	-4	-5	-6	-7	-8
9	+	+	+	−	−	−	+	−
10	+	+	−	−	+	+	−	−
11	+	−	+	+	−	+	−	−
12	+	−	−	+	+	−	+	−
13	−	+	+	+	+	−	−	−
14	−	+	−	+	−	+	+	−
15	−	−	+	−	+	+	+	−
16	−	−	−	−	−	−	−	−

Fold-over Technique: Version Two

- Suppose one factor, say 5, is very important. We want to de-alias 5 and all 2fi's involving 5.
- Choose, instead, the following 2_{III}^{7-4} design

$$d_3 : \mathbf{4} = \mathbf{12}, \mathbf{5} = -\mathbf{13}, \mathbf{6} = \mathbf{23}, \mathbf{7} = \mathbf{123}.$$

  Then the combined design $d_1 + d_3$ is a 2_{III}^{7-3} design with the generators

$$d' : \mathbf{4} = \mathbf{12}, \mathbf{6} = \mathbf{23}, \mathbf{7} = \mathbf{123}. \tag{9}$$

  Since 5 does not appear in (5), 5 is strongly clear and all 2fi's involving 5 are clear. However, other main effects are not clear (see Table 5.7 of WH for $d_1 + d_3$).

- Choice between d_2 and d_3 depends on the priority given to the effects (class discussions).

Critique of Fold-over Technique

- Fold-over technique is not an efficient technique. It requires doubling of the run size and can only de-alias a specific set of effects. In practice, after analyzing the first experiment, a set of effects will emerge and need to be de-aliased. It will usually require much fewer runs to de-alias a few effects.
- A more efficient technique that does not have these deficiencies is the optimum design approach given in Section 5.4.2.

Optimal Design Approach for Follow-Up Experiments

- This approach add runs according to a particular optimal design criterion. The D and D_s criteria shall be discussed.

- Optimal design criteria depend on the assumed model. In general, the model should contain:

  1. All effects and their aliases (except those judged unimportant a priori or by the effect hierarchy principle) identified as significant in the initial experiment.

  2. A block variable that accounts for differences in the average value of the response over different time periods.

  3. An intercept.

- In the leaf spring experiment, we specify the model:

  $E(z) = \beta_0 + \beta_{bl} x_{bl} + \beta_B x_B + \beta_{DQ} x_D x_Q + \beta_{BCQ} x_B x_C x_Q + \beta_{DEQ} x_D x_E x_Q,$
  where $z = ln(s^2)$, and β_{bl} is the block effect.

D-Criterion

- In Table 5, the columns B, C, D, E, and Q comprise the design matrix while the columns B, $block$, BCQ, DEQ, DQ comprise the model matrix. Two runs are to be added to the original 16-run experiment. There are $2^{10} = 1024$ possible choices of factor settings for the follow up runs (runs 17 and 18) since each factor can take on either the + or - level in each run.

- For each of the 1024 choices of settings for B, C, D, E, Q for runs 17 and 18, denote the corresponding model matrix by $X_d, d = 1, ..., 1024$. We may choose the factor settings d^* that maximizes the D-criterion, i.e.

$$max_d |X_d^T X_d| = |X_{d^*}^T X_{d^*}|.$$

  Maximizing the D criterion minimizes the volume of the confidence ellipsoid for all model parameters β.

- 64 choices of d attain the maximum D value of 4,194,304. Two are:
  $d_1 : (B,C,D,E,Q) = (+++-+)$ and $(++-+-)$
  $d_2 : (B,C,D,E,Q) = (+++-+)$ and $(-++++)$

Augmented Model Matrix and Design Matrix

Table 7: Augmented Design Matrix and Model Matrix, Leaf Spring Experiment

Run	B	C	D	E	Q	Block	BCQ	DEQ	DQ
1	−	+	+	−	−	−	+	+	−
2	+	+	+	+	−	−	−	−	−
3	−	−	+	+	−	−	−	−	−
4	+	−	+	−	−	−	+	+	−
5	−	+	−	+	−	−	+	+	+
6	+	+	−	−	−	−	−	−	+
7	−	−	−	−	−	−	−	−	+
8	+	−	−	+	−	−	+	+	+
9	−	+	+	−	+	−	−	−	+
10	+	+	+	+	+	−	+	+	+
11	−	−	+	+	+	−	+	+	+
12	+	−	+	−	+	−	−	−	+
13	−	+	−	+	+	−	−	−	−
14	+	+	−	−	+	−	+	+	−
15	−	−	−	−	+	−	+	+	−
16	+	−	−	+	+	−	−	−	−
17						+			
18						+			

D_s-Criterion

- We may primarily be interested in the estimation of BCQ and DEQ. Let $|X_d^T X_d|$ be partitioned as

$$\begin{pmatrix} X_1^T X_1 & X_1^T X_2 \\ X_2^T X_1 & X_2^T X_2 \end{pmatrix},$$

  where $X_d = [X_1, X_2]$, with X_2 corresponding to the variables BCQ and DEQ. Then the lower right 2x2 submatrix of $(X_d^T X_d)^{-1}$ can be shown to be $(X_2^T X_2 - X_2^T X_1 (X_1^T X_1)^{-1} X_1^T X_2)^{-1}$ and the optimal choice of d is one that satisfies

$$max_d |(X_2^T X_2 - X_2^T X_1 (X_1^T X_1)^{-1} X_1^T X_2)|.$$

  This is the D_s-optimal criterion, where s denotes subset.

- For d_1, the lower right 2x2 submatrix of $X_d^T X_d$ have criterion value 128. On the other hand, the D_s-criterion value for d_2 is 113.78. d_1 is D_s-optimal, not d_2.

Use of Design Tables

- Tables are given in Appendix 5A. Minimum aberration (MA) designs are given in the tables. If two designs are given for same k and p, the first is an MA design and the second is better in having a larger number of clear effects. Two tables are given on next pages.
- In Table 7, the first 2^{9-4} design has MA and 8 clear 2fi's. The second 2^{9-4} design is the second best according to the MA criterion but has 15 clear 2fi's. Details on p. 234 of WH. Using Rule (iii) in eq.(2) on page 9, the second design is better because both have resolution IV.
- It is not uncommon to find a design with slightly worse aberration but more clear effects. Thus the number of clear effects should be used as a supplementary criterion to the MA criterion.

Table 6: 16-Run 2^{k-p} FFD $(k-p=4)$

(k is the number of factors and F&R is the fraction and resolution.)

k	F&R	Design Generators	Clear Effects
5	2_{V}^{5-1}	$5=1234$	all five main effects, all 10 2fi's
6	2_{IV}^{6-2}	$5=123,\ 6=124$	all six main effects
6^*	2_{III}^{6-2}	$5=12,\ 6=134$	3, 4, 6, 23, 24, 26, 35, 45, 56
7	2_{IV}^{7-3}	$5=123,\ 6=124,\ 7=134$	all seven main effects
8	2_{IV}^{8-4}	$5=123,\ 6=124,\ 7=134,\ 8=234$	all eight main effects
9	2_{III}^{9-5}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234$	none
10	2_{III}^{10-6}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34$	none
11	2_{III}^{11-7}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24$	none
12	2_{III}^{12-8}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24, t_2=14$	none
13	2_{III}^{13-9}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24, t_2=14,\ t_3=23$	none
14	2_{III}^{14-10}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24, t_2=14,\ t_3=23,\ t_4=13$	none
15	2_{III}^{15-11}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24, t_2=14,\ t_3=23,\ t_4=13,\ t_5=12$	none

Table 7: 32 Run 2^{k-p} FFD ($k-p=5$, $6 \le k \le 11$)

(k is the number of factors and F&R is the fraction and resolution.)

k	F&R	Design Generators	Clear Effects
6	2^{6-1}_{VI}	$6=12345$	all six main effects, all 15 2fi's
7	2^{7-2}_{IV}	$6=123,\ 7=1245$	all seven main effects, 14, 15, 17, 24, 25, 27, 34, 35, 37, 45, 46, 47, 56, 57, 67
8	2^{8-3}_{IV}	$6=123,\ 7=124,\ 8=1345$	all eight main effects, 15, 18, 25, 28, 35, 38, 45, 48, 56, 57, 58, 68, 78
9	2^{9-4}_{IV}	$6=123,\ 7=124,\ 8=125,\ 9=1345$	all nine main effects, 19, 29, 39, 49, 59, 69, 79, 89
9	2^{9-4}_{IV}	$6=123,\ 7=124,\ 8=134,\ 9=2345$	all nine main effects, 15, 19, 25, 29, 35, 39, 45, 49, 56, 57, 58, 59, 69, 79, 89
10	2^{10-5}_{IV}	$6=123,\ 7=124,\ 8=125,\ 9=1345,\ t_0=2345$	all 10 main effects
10	2^{10-5}_{III}	$6=12,\ 7=134,\ 8=135,\ 9=145,\ t_0=345$	3, 4, 5, 7, 8, 9, t_0, 23, 24, 25, 27, 28, 29, $2t_0$, 36, 46, 56, 67, 68, 69, $6t_0$
11	2^{11-6}_{IV}	$6=123,\ 7=124,\ 8=134,\ 9=125,\ t_0=135,\ t_1=145$	all 11 main effects
11	2^{11-6}_{III}	$6=12,\ 7=13,\ 8=234,\ 9=235,\ t_0=245,\ t_1=1345$	4, 5, 8, 9, t_0, t_1, 14, 15, 18, 19, $1t_0$, $1t_1$

Choice of Fractions and Avoidance of Specific Combinations

- A 2^{k-p} design has 2^p choices. In general, use randomization to choose one of them. For example, the 2^{7-3} design has 8 choices $4 = \pm 12, 5 = \pm 13, 6 = \pm 23$. Randomly choose the signs.
- If specific combinations (e.g., $(+++)$ for high pressure, high temperature, high concentration) are deemed undesirable or even disastrous, they can be avoided by choosing a fraction that does not contain them. Example on p.237 of WH.

Blocking in FF Designs

Example: Arrange the 2^{6-2} design in four $(=2^2)$ blocks with

$$\mathbf{I} = \mathbf{1235} = \mathbf{1246} = \mathbf{3456}.$$

Suppose we choose

$$\mathbf{B_1} = \mathbf{134}, \mathbf{B_2} = \mathbf{234}\ , \mathbf{B_1 B_2} = \mathbf{12}.$$

Then

$$\mathbf{B_1} = \mathbf{134} = \mathbf{245} = \mathbf{236} = \mathbf{156},$$
$$\mathbf{B_2} = \mathbf{234} = \mathbf{145} = \mathbf{136} = \mathbf{256},$$
$$\mathbf{B_1 B_2} = \mathbf{12} = \mathbf{35} = \mathbf{46} = \mathbf{123456};$$

i.e., these effects are confounded with block effects and cannot be used for estimation. Among the remaining 12 degrees of freedom, six are main effects and the rest are

$$\begin{aligned}
\mathbf{13} &= \mathbf{25} = \mathbf{2346} = \mathbf{1456},\\
\mathbf{14} &= \mathbf{26} = \mathbf{2345} = \mathbf{1356},\\
\mathbf{15} &= \mathbf{23} = \mathbf{2456} = \mathbf{1346},\\
\mathbf{16} &= \mathbf{24} = \mathbf{2356} = \mathbf{1345},\\
\mathbf{34} &= \mathbf{56} = \mathbf{1245} = \mathbf{1236},\\
\mathbf{36} &= \mathbf{45} = \mathbf{1256} = \mathbf{1234}.
\end{aligned}$$

Use of Design Tables for Blocking

- Among the 15 degrees of freedom for the blocked design on page 33, 3 are allocated for block effects and 6 are for clear main effects (see Table 8). The remaining 6 degrees of freedom are six pairs of aliased two-factor interactions.

- For the 2^{6-2} design with $I = 125 = 1346 = 23456$, if we use the block generators $B_1 = 13, B_2 = 14$, there are a total of 9 clear effects (see Table 8): $3, 4, 6, 23, 24, 26, 35, 45, 56$.

  Thus, the total number of clear effects for this blocked design is 3 more than the total number of clear effects for the blocked design on page 33. However, only the main effects $3, 4, 6$ are clear.

Table 8: Sixteen-Run 2^{k-p} Fractional Factorial Designs in 2^q Blocks

k	p	q	Design Generators	Block Generators	Clear Effects
5	1	1	$5 = 1234$	$B_1 = 12$	all five main effects, all 2fi's except 12
5	1	2	$5 = 1234$	$B_1 = 12$, $B_2 = 13$	all five main effects, 14, 15, 24, 25, 34, 35, 45
5	1	3	$5 = 123$	$B_1 = 14$, $B_2 = 24$, $B_3 = 34$	all five main effects
6	2	1	$5 = 123, 6 = 124$	$B_1 = 134$	all six main effects
6	2	1	$5 = 12, 6 = 134$	$B_1 = 13$	3, 4, 6, 23, 24, 26, 35, 45, 56
6	2	2	$5 = 123, 6 = 124$	$B_1 = 134$, $B_2 = 234$	all six main effects
6	2	2	$5 = 12, 6 = 134$	$B_1 = 13$, $B_2 = 14$	3, 4, 6, 23, 24, 26, 35, 45, 56
6	2	3	$5 = 123, 6 = 124$	$B_1 = 13$, $B_2 = 23$, $B_3 = 14$	all six main effects

Table 8: Sixteen-Run 2^{k-p} Fractional Factorial Designs in 2^q Blocks (Cont.)

k	p	q	Design Generators	Block Generators	Clear Effects
7	3	1	$5 = 123, 6 = 124,$ $7 = 134$	$B_1 = 234$	all seven main effects
7	3	2	$5 = 123, 6 = 124,$ $7 = 134$	$B_1 = 12,$ $B_2 = 13$	all seven main effects
7	3	3	$5 = 123, 6 = 124,$ $7 = 134$	$B_1 = 12,$ $B_2 = 13,$ $B_3 = 14$	all seven main effects
8	4	1	$5 = 123, 6 = 124,$ $7 = 134, 8 = 234$	$B_1 = 12$	all eight main effects
8	4	2	$5 = 123, 6 = 124,$ $7 = 134, 8 = 234$	$B_1 = 12,$ $B_2 = 13$	all eight main effects
8	4	3	$5 = 123, 6 = 124,$ $7 = 134, 8 = 234$	$B_1 = 12,$ $B_2 = 13,$ $B_3 = 14$	all eight main effects
9	5	1	$5 = 12, 6 = 13,$ $7 = 14,\ 8 = 234,$ $9 = 1234$	$B_1 = 23$	none
9	5	2	$5 = 12, 6 = 13,$ $7 = 14, 8 = 234,$ $9 = 1234$	$B_1 = 23,$ $B_2 = 24$	none

Use of Design Tables for Blocking

- More FF designs in blocks are given in Appendix 5B of WH. You only need to learn how to use the tables and interpret the results. Theory or criterion used in choosing designs are not required.

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