MATH 3070 Introduction to Probability and Statistics
Lecture notes
Concepts of Counting

Objectives:

1. Learn the basic concepts of counting
2. What is the factorial and how to compute it
3. Learn the difference between

   - permutations
   - combinations

How many outcomes are there?

In probability, and some times just in everyday life, you often want to know how many possibilities are there for a certain arrangement of things. This is especially important in probability since the total number of possibilities affects the computation of the probability of a single event.

Circumstances that can occur in series, one after another, can be multiplied together to get the total number of possibilities. This can be summarized in two multiplication "rules":

Multiplication Rule 1 If each event in a sequence of n events has k different possibilities, then the total number of possibilities of the sequence will be

$$k \times k \times k \times \ldots \times k = k^n$$

Example: Three dice are rolled with $6 \times 6 \times 6 = 6^3 = 216$ possibilities.

Example: ID badges are created with two letters followed by two digits. There are $26 \times 26 \times 10 = 67,600$ possibilities.

Multiplication Rule 2 In a sequence of n events in which the first one has k_1 possibilities, the second event has k_2, the third has k_3, etc., then the total number of possibilities of the sequence will be

$$k_1 \times k_2 \times k_3 \times \ldots \times k_n$$

Example: Paint can be mixed with a choice of seven colors, two types, three textures, and two uses. The total number of possible different paints is $7 \times 2 \times 3 \times 2 = 84$.

Factorial - the Special Case

The product of all integers from 1 up to a given value, n, is rather common in mathematics. It has been given a special name (**factorial**) and symbol ($n!$). The factorial only operates on positive integers and a special exception is made for zero factorial ($0!$). We assign that special case a value of 1, for reasons that will become obvious later.

The factorial can be defined upon itself in a manner known as *recursion*. We use the factorial to define itself, in violation of the usual norms. We do so thusly

$$n! = n \times (n-1)!$$

We can use this feature repeatedly, as so

$$n! = n \times (n-1)!(n-1)! = (n-1) \times (n-2)!(n-2)! = (n-2) \times (n-3)!(n-3)! = (n-3) \times (n-4)! \vdots$$

But how does this work? Why do we keep subtracting larger numbers?

$$n! = n \times (n-1)!(n-1)! = (n-1) \times ((n-1)-1)!(n-2)! = (n-2) \times ((n-2)-1)!(n-3)! = (n-3) \times ((n-3)-1$$

To use a numerical example, 5!, we can express this value as

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

or we could write

$$5! = 5 \times 4! = 120$$

since we know that 5! can be written as $5 \times 4!$ by the recursive definition of $n!$.

This also gives us a chance to look at the values from a factorial and realize that the resulting values get large, exceedingly large, quickly. 4! is 24, but 5! is 120. Quite quickly this number becomes too large for us to handle.

The recursive definition also explains why we need to assign the value 1 to the zero factorial. If $n!$ can be written as $n \times (n-1)!$ and $n = 1$ then we **have** to have a value for 0!. The value 1 solves many problems. The recursive nature of the factorial also makes it possible to easily simplify terms involving factorials.

$$\frac{n!}{(n-3)!} = \frac{n \times (n-1) \times (n-2) \times (n-3)!}{(n-3)!}$$

Cancelling common terms in the numerator and denominator leaves us with

$$n \times (n-1) \times (n-2) = (n^2 - n) \times (n-2) = n^3 - 3n^2 + 2n$$

Combinations and Permutations

We can use the factorial to determine the number of ways items can be grouped or counted. This is useful when deciding how many license plates can be minted or how many committees can be chosen. The factorial allows us to collect a large "chunk" of multiplications and remove them all at once. So rather than have to write out the full multiplicative series we can stop at a point that allows us to cancel out common terms.

- Determine how many different words of length 5 can be formed from the letters a, f, g, e, t, k, l if no letter can be repeated. *Answer: Start with any of the letters, that gives you 7 choices. After using one, the choices are reduced to 6, which then leaves 5 choices, and so on. The answer is $7 \times 6 \times 5 \times 4 \times 3 = 2520$.*

- Determine in how many ways first and second place prizes can be awarded in a contest with 10 competitors. *Answer: 10×9.*

The answer to these kinds of problems involves counting options, and then counting options again after an item has been removed. Let's look at two definitions which we can use in solving these problems:

permutation An arrangement of n objects in a specific order is called a permutation of the objects. Suppose that $0 \leq r \leq n$. The number of permutations of r elements chosen from n elements is denoted by $P(n, r)$. The number

$$P(n, r) = n \times (n-1) \times (n-2) \cdots (n - r + 1) = n!/(n-r)!.$$

combination A selection of objects without regard to order is called a combination. For any pair of integers n and r such that $0 \leq r \leq n$ we define the number $C(n, r)$ to be

$$P(n, r)/r! = n!/[(n-r)!r!]$$

We call $C(n, r)$ the number of **combinations of r objects chosen from n**, or more simply, "n choose r".

The difference between permuations and combinations is whether order matters or not. In a permutation, order matters. Any change in the arrangement of the items is a new permutation. In a combination, order **does not** matter. Any group of items is the same combination no matter what the arrangment of the items within the group.

"I don't understand why order matters." OK, let's look at it using people on committees. We have a group of people, say n, and we want to make committees of size m, where $m < n$. We have more than enough people to make up committees. If we choose four people and make a committee, that's a combination. If we order them by height, does that change the committee? No. If we order them by age, does that change the committee? No. If we order them by last name, does that change the committee? No.

Any way we order the people on the committee, provided it's the same people, the committee is the same. A change in the order only makes the list of names change, not the names themselves. Any given order is a permutation, a different way of listing the members. The group itself is a combination.

Here's another example of why order matters, or doesn't. Use the letters 'e', 'v', 'i', 'l', and 's'. Now, if I order these letters as "l-i-v-e-s" then that's one particular word. Another arrangement is "e-l-v-i-s" which is an entirely different word. The original order, "e-v-i-l-s" is yet a third word. Each word is different (radically so, you might say), but the letters themselves didn't change. It's still the same five letters just arranged differently. This is a word "game" called an *anagram*.

Numeric Examples

Here are a few examples of permutations and combinations with actual numbers.

1. How many different ways can a salesman visit 8 cities?

$$n = 8, r = 8 \quad \text{so} \quad P(8, 8) = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8! = 40320$$

2. How many different ways can 10 horses in a race win, place and show?

$$n = 10, r = 3 \quad \text{so} \quad P(10, 3) = \frac{10!}{(10-7)!}$$
$$= \frac{10 \times 9 \times 8 \times 7!}{7!}$$
$$= 10 \times 9 \times 8$$
$$= 720$$

3. In a game of poker, you are dealt a hand of 5 cards from a pack of 52. How many possible hands are there?

$$n = 52, r = 5 \quad \text{so} \quad C(52,5) = \frac{52!}{(52-5)!5!} = \frac{52!}{47!5!}$$

$$= \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{47! \times 5!}$$

$$= \frac{52 \times 51 \times 50 \times 49 \times 48}{5!}$$

$$= \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 2,598,960$$