MATH 2420 Discrete Mathematics
Lecture notes

Sets and Set Theory

Objectives:

1. Determine whether one set is a subset of another
2. Determine whether two sets are equal
3. Determine whether an element is in a set or not
4. Determine the union, intersection, difference, and complement of sets
5. Illustrate sets using Venn diagrams
6. Determine Cartesian product of two or more sets
7. Prove set identities
8. Use set identities to derive new set properties from old set properties
9. Use Venn diagrams to prove set identities
10. Determine whether sets form a partition of a given set
11. Determine the power set of a set

General definitions:

set A collection of discrete items, whether numbers, letters, people, animals, cars, atoms, planets, etc. A set is identified when the items are grouped together between { and }.

element One item in a set, whether a discrete item or a set itself.

cardinality The number of elements in a finite set.

Sets are defined by listing the elements between braces ({ }). Any items listed between the braces are considered elements of the set, even if those items are sets themselves. An item by itself is just an element and not a set.

{a} is a set
a is not a set

If the elements of a set are too numerous to list individually, the set may be defined by a property. We write the set defintion by listing first the membership and then the condition which must be met for membership. As an example

$$\{x \in Z \mid -2 < x < 5\}$$

where **Z** is the set of all integers defines a set which has elements x which are integers between -2 and 5.

Special Sets

Z^+	set of all positive integer numbers
N	set of all natural numbers $(0, 1, 2, 3, 4, \ldots)$
Z	set of all integer numbers
Q	set of all rational numbers
I	set of all irrational numbers
R	set of all real numbers
U	the universal set

Subsets

If A and B are subsets, A is called a **subset** of B if, and only if, every element of A is also an element of B. We write this as $A \subseteq B$. Written symbolically this is:

$$A \subseteq B \Leftrightarrow \forall \text{ x, if x } \in \text{ A then x } \in \text{ B.}$$

The definition of **subset** is rigid and inflexible. If any element in A does not appear in B then A cannot be a subset of B. That is:

$$A \nsubseteq B \Leftrightarrow \exists x \text{ such that } x \in \text{ A and } x \notin \text{ B.}$$

Looking at the special sets above we have

$$N \subseteq Z \subseteq Q \subseteq R$$

A set can be a subset of itself, strange as this may seem. To differentiate subsets by type, we call a subset with *at least one item missing* from the larger set as a **proper subset**. That is, a set may be a **proper subset** if, and only if, every element of A is in B but there is *at least* one element of B that is not in A.

Set Equality

Two sets, A and B, can be said to be equal if every element in set A is also in set B, and vice versa. Stated another way

$$A = B \text{ if } (x \in A \Rightarrow x \in B) \text{ and } (x \in B \Rightarrow x \in A)$$

Operations on Sets

We can operate on sets in a manner similar to other mathematical operations. Our operators are a bit different, though.

union	$\cup$	$A \cup B$	all elements in both sets
intersection	$\cap$	$A \cap B$	all elements common to both sets
difference	$-$	$A - B$	all elements in A but not in B
complement	*various*	A^c	all elements not in A

Procedural definitions:

union	$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
intersection	$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$
difference	$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$
complement	$A^c = \{x \in U \mid x \notin A\}$

Examples:

union	$\{1, 2, 3\} \cup \{2, 4, 6\} = \{1, 2, 3, 4, 6\}$
intersection	$\{1, 2, 3\} \cap \{2, 4, 6\} = \{2\}$
difference	$\{1, 2, 3\} - \{2, 4, 6\} = \{1, 3\}$

Empty Set

A set with no members, however weird, is part of set theory. It is called the *empty set* (or *null set*) and is denoted by either empty brackets () or by the special symbol $\emptyset$.

Partitions and Disjoint Sets

Two sets are called **disjoint** if they have no elements in common. That is

$$A \cap B = \emptyset$$

If we increase the number of sets to more than two we must consider the property known as **mutually disjoint**. Also known as **pairwise disjoint**. This refers to the relationship between any two sets at a time. The definition is

> Sets $A_1, A_2, A - 3, \ldots, A_n$ are mutually disjoint if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, \ldots, n$,
>
> $$A_i \cap A_j = \emptyset \text{ whenever } i \neq j$$

Example: Let A_1 = 1, 2, 3, A_2 = 4, 5 and A_3 = 6, 7, 8. Then

	A_1	A_2	A_3
A_1	na	$\emptyset$	$\emptyset$
A_2	$\emptyset$	na	$\emptyset$
A_3	$\emptyset$	$\emptyset$	na

So what purpose does this serve? It provides the basis for constructing partitions of sets. A **partition** is a collection of mutually disjoint sets that when unioned together form the whole larger set. Each set must be nonempty, that is, must have at least one member and all sets must be pairwise, or mutually, disjoint. We use the following definition:

> A collection of nonempty sets A_1,A_2,...,A_n is a partition of a set A if, and only if,
>
> 1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$;
> 2. A_1,A_2,...,A_n are mutually disjoint.

Power Sets

So far we have seen that any given set can be broken into several different subsets. A subset is simply a set of elements from a (usually) larger set, though the set can be a subset of itself $A \subseteq A$, strange as that may seem. We have also seen that the set of no elements, the empty set, is always a subset of every set. This collection is referred to as the **power set** of a set and is denoted $\mathcal{P}(\S)$. But how many sets are there in the power set? The number of sets is equal to 2^n where n is the number of elements in a set.

For example, if we have a set $A = \{2, 4, 6\}$ then the power set $\mathcal{P}(\mathcal{A})$ consists of

$$\emptyset, \{2\}, \{4\}, \{6\}, \{2, 4\}, \{2, 6\}, \{4, 6\}, \{2, 4, 6\}$$

of which all but $\{2, 4, 6\}$ are proper subsets.

This works even for the empty set $\emptyset$ thusly:

> There are no elements in the empty set since it is "empty". Hence the cardinality is zero. If we compute 2^0 we get an answer of 1, which is the only subset of $\emptyset$ which is itself. $\square$

Cartesian Products

In sets, order and repetition usually don't matter. But what if they do? Then we have an item known as an **ordered n-tuple**. Often seen with only two elements, these are called **ordered pairs**. Ordered n-tuples are unique based on elements and order of elements. Two n-tuples can be equal only if all elements are the same and in the same order.

$$(1, 2) \neq (2, 1)$$
$$(3, (-2)^2, \tfrac{1}{2}) = (\sqrt{9}, 4, \tfrac{3}{6})$$

Now, if X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. This is called the **Cartesian product** of X and Y. Symbolically,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Example:
Let A = 1, 2 and B = 1, 4, 7. Then

$$A \times B = \{(1, 1), (1, 4), (1, 7), (2, 1), (2, 4), (2, 7)\}$$
$$B \times A = \{(1, 1), (1, 2), (4, 1), (4, 2), (7, 1), (7, 2)\}$$

This can be applied to more than two sets at a time.

Properties of Sets

Subset Relations

1. Inclusion of Intersection: For all sets A and B,

$$A \cap B \subseteq A \text{ and } A \cap B \subseteq B$$

2. Inclusion in Union: For all sets A and B,

$$A \subseteq A \cup B \text{ and } B \subseteq A \cup B$$

3. Transitive Property of Subsets: For all sets A, B, and C

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C$$

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1.	$x \in X \cup Y$	$\Leftrightarrow$	$x \in X$ or $x \in Y$
2.	$x \in X \cap Y$	$\Leftrightarrow$	$x \in X$ and $x \in Y$
3.	$x \in X - Y$	$\Leftrightarrow$	$x \in X$ and $x \notin Y$
4.	$x \in X^c$	$\Leftrightarrow$	$x \notin X$
5.	$(x, y) \in X \times Y$	$\Leftrightarrow$	$x \in X$ and $y \in Y$

Element Arguments

Using these subset relations and set definitions, we can prove that sets are subsets of each other. To prove that $X \subseteq Y$ requires two steps:

1. Suppose that x is a particular yet arbitrarily chosen element of X.
2. Show that x is an element of Y.

This is an **element argument** or also known as a "chasing x argument" (in that we "chase" x from one side of the statement to the other).

Using this method of proving subsets we can show that two sets are equal to each other. We know that two sets are equal if they are subsets of each other, so if we can show the subset relation then we have shown the equality. So first we show that $X \subseteq Y$ and then we show that $Y \subseteq X$.

This can best be explained with an example.

Show that $R \cap (S \cup T) \subseteq S \cup (R \cap T)$.

We must show that each x in $R \cap (S \cup T)$ is also in $S \cup (R \cap T)$.

1. Let $x \in R \cap (S \cup T)$.

2. Then x is in both R and $S \cup T$ by definition.
3. Since x is in $S \cup T$, x is in S or x is in T. This gives us two cases.
4. (a) In the case that x is in S, we get that $x \in S \cup (R \cap T)$ by the definition of union.
5. (b) In the case that x is in T, we get that $x \in (R \cap T)$ by the definition of intersection and the observation above that $x \in R$.
6. Then, once again, we get $x \in S \cup (R \cap T)$ by the definition of union.
7. Thus in either case $x \in S \cup (R \cap T)$.
8. Thus each x in $R \cap (S \cup T)$ is in $S \cup (R \cap T)$ as well, so

$$R \cap (S \cup T) \subseteq S \cup (R \cap T)$$

From Bogart, p. 21-22