MATH 2420 Discrete Mathematics
Lecture notes

Series and Sequences

Objectives:

1. Find the explicit formula for a sequence.
2. Be able to do calculations involving factorial, summation and product notations.

Introduction

Patterns exist everywhere. In mathematics, patterns exist in numbers as well as in letters. Examples of patterns might be

- Counting the number of ancestors one has, generation by generation.
- The cost of a taxi ride which increases as the distance traveled increases.
- The time to complete a task where time increases based upon the number of stages to be done.

Some of the patterns are finite and can be grasped easily. Others are infinite, repeating upon themselves endlessly. Identifying those patterns and making them comprehensible, even when infinite, is an important part of discrete mathematics.

Basic Concepts

sequence a list in which order is taken into account

term an individual element in the list, usually denoted a_k

subscript also known as an **index**, this is the k in a_k and denotes which term in the sequence is being examined; this is an integer value but does not have to be explicitly defined, it can be a variable

ellipsis a series of dots (...) which denotes "and so on"; in the middle of a sequence it indicates "in a similar manner to the end" while at the end of a sequence it indicates "in a similar manner forever"

explicit formula a rule that shows how the values of a_k depend on k

Explicit Formula

Given an explicit formula for a sequence one can use a given value for the subscript and obtain the value for the term at that point in the series.

Examples

Give the first five terms for $t_n = n^2 - 1$, $n \geq 1$:

$$0, 3, 8, 15, 24$$

Give the first five terms for

$$a_k = \frac{k}{k+1} \text{ for all integers } k \geq 1$$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$$

What about the first five terms for this one?

$$b_i = \frac{i-1}{i} \text{ for all integers } i \geq 2$$

Interestingly enough

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$$

These two examples demonstrate that an explicit, or general, formula for a sequence may not be unique.

Examine the following sequence and its values:

$$c_j = (-1)^j \text{ for all integers } j \geq 0.$$

$$\begin{aligned} c_0 &= (-1)^0 = 1 \\ c_1 &= (-1)^1 = -1 \\ c_2 &= (-1)^2 = 1 \\ c_3 &= (-1)^3 = -1 \\ c_4 &= (-1)^4 = 1 \\ c_5 &= (-1)^5 = -1 \end{aligned}$$

This is called an **alternating** or **oscillating** sequence. Even though it is an infinite sequence, it only ever generates the values 1 and -1.

Deriving Explicit Formulas

There is, however, no set of rules for figuring out the explicit formula for a series. To find the formula, you must examine the terms in the series and see if you can identify a repeated pattern. This may require that you rewrite some of the terms (often the first and last) to make them have the same form as the other terms. Then you must manipulate the values, replacing them with variables, to see if the pattern emerges. Here are some examples[1]:

a) For the series

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$$

[1]From "Calculus with analytical geometry" by Howard Anton

Begin by comparing terms and term numbers:

term number	1	2	3	4	...
term	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	...

In each term, the numerator is the same as the term number, and the denominator is one greater than the term number. Thus the nth term is $n/(n+1)$ and the sequence may be written

$$a_n = \frac{n}{n+1} \text{ for } n \geq 1$$

Compact Forms

Summation Notation

When we have a sequence that involves adding things together, we write it out term by term, like so:

$$a_1 + a_2 + a_3 + a_4 + a_5 + \dots$$

This is fine for a few terms, but what if the sequence is rather long? Or infinite? That presents a problem. Luckily there is a short-hand method for doing this - **summation notation**. Using the capital Greek letter sigma (Σ) we can write the entire sequence in a handy compact form.

Say we wanted to write the previous sequence in a more compact form. Using the summation notation and the values 1 and 5 for the **lower limit** and **upper limit** we would write

$$\sum_{k=1}^{5} a_k$$

This works even if we don't know the limits, upper or lower. We can use variables in place of known values and the summation notation works the same. For example, if we wanted to make the previous example more general we could write

$$\sum_{k=m}^{n} a_k$$

If given an expanded form, you can reduce it to a summation notation form by recognizing the pattern and reducing. For example:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \dots + \frac{n+1}{2n}$$

Making all the terms look similar makes the process easier. The pattern becomes easier to see that way. First, rewrite the last term to expand the $2n$ in the denominator, like so

$$\frac{n+1}{n+2} \text{ since } 2n = n + n$$

Rewrite the denominator of the first term to match the other terms

$$\frac{1}{n} = \frac{1}{n+0}$$

Write the numerators in a form that matches the last term. See if a pattern emerges there.

$$\frac{0+1}{n+0} + \frac{1+1}{n+1} + \frac{2+1}{n+2} + \cdots + \frac{n+1}{n+n}$$

The pattern which emerges is

$$\frac{k+1}{n+k}$$

where k varies from 0 to n. So our summation form would look like this

$$\sum_{k=0}^{n} \frac{k+1}{n+k}$$

Product Notation

We have a similar form for sequences involving the product of terms. Product, remember, is the result of a multiplication operation. We use the capital Greek letter pi (Π) to represent this operation.

$$\prod_{k=1}^{5} a_k = a_1 \times a_2 \times a_3 \times a_4 \times a_5$$

Factorial - the Special Case The product of all integers from 1 up to a given value, n, is rather common in mathematics. It has been given a special name (**factorial**) and symbol ($n!$). The factorial only operates on positive integers and a special exception is made for zero factorial (0!). We assign that special case a value of 1, for reasons that will become obvious later.

The factorial can be defined upon itself in a manner known as *recursion.* We use the factorial to define itself, in violation of the usual norms. We do so thusly

$$n! = n \times (n-1)!$$

We can use this feature repeatedly, as so

$$n! = n \times (n-1)!(n-1)! = (n-1) \times (n-2)!(n-2)! = (n-2) \times (n-3)!(n-3)! = (n-3) \times (n-4)! \vdots$$

But how does this work? Why do we keep subtracting larger numbers?

$$n! = n \times (n-1)!(n-1)! = (n-1) \times ((n-1)-1)!(n-2)! = (n-2) \times ((n-2)-1)!(n-3)! = (n-3) \times ((n$$

To use a numerical example, 5!, we can express this value as

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

or we could write

$$5! = 5 \times 4! = 120$$

since we know that 5! can be written as $5 \times 4!$ by the recursive definition of $n!$.

This also gives us a chance to look at the values from a factorial and realize that the resulting values get large, exceedingly large, quickly. 4! is 24, but 5! is 120. Quite quickly this number becomes too large for us to handle.

The recursive definition also explains why we need to assign the value 1 to the zero factorial. If $n!$ can be written as $n \times (n-1)!$ and $n = 1$ then we **have** to have a value for 0!. The value 1 solves many problems. The recursive nature of the factorial also makes it possible to easily simplify terms involving factorials.

$$\frac{n!}{(n-3)!} = \frac{n \times (n-1) \times (n-2) \times (n-3)!}{(n-3)!}$$

Cancelling common terms in the numerator and denominator leaves us with

$$n \times (n-1) \times (n-2) = (n^2 - n) \times (n-2) = n^3 - 3n^2 + 2n$$

Properties of Summations and Products

If $a_m, a_{m+1}, a_{m+2}, \ldots,$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

$$c \times \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \times a_k)$$

$$\left(\prod_{k=m}^{n} a_k\right) \times \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \times b_k)$$

Combinations and Permutations

We can use the factorial to determine the number of ways items can be grouped or counted. This is useful when deciding how many license plates can be minted or how many committees can be chosen. The factorial allows us to collect a large “chunk” of multiplications and remove them all at once. So rather than have to write out the full multiplicative series we can stop at a point that allows us to cancel out common terms.

- Determine how many different words of length 5 can be formed from the letters a, f, g, e, t, k, l if no letter can be repeated. *Answer: Start with any of the letters, that gives you 7 choices. After using one, the choices are reduced to 6, which then leaves 5 choices, and so on. The answer is* $7\times\ 6\times\ 5\times\ 4\times\ 3 = 2520$.
- Determine in how many ways first and second place prizes can be awarded in a contest with 10 competitors. *Answer:* 10×9.

The answer to these kinds of problems involves counting options, and then counting options again after an item has been removed. Let's look at two definitions which we can use in solving these problems:

permutation Suppose the $0 \leq r \leq n$. The number of permutations of r elements chosen from n elements is denoted by $P(n, r)$. The number $P(n, r) = n \times (n-1) \times (n-2) \cdots (n-r+1) = n!/(n-r)!$.

combination For any pair of integers n and r such that $0 \leq r \leq n$ we define the number $C(n, r)$ to be $P(n, r)/r! = n!/[(n-r)!r!]$. We call $C(n, r)$ the number of **combinations of r objects chosen from n**, or more simply, "n choose r".

The difference between permuations and combinations is whether order matters or not. In a permutation, order matters. Any change in the arrangement of the items is a new permutation. In a combination, order **does not** matter. Any group of items is the same combination no matter what the arrangment of the items within the group.

"I don't understand why order matters." OK, let's look at it using people on committees. We have a group of people, say n, and we want to make committees of size m, where $m < n$. We have more than enough people to make up committees. If we choose four people and make a committee, that's a combination. If we order them by height, does that change the committee? No. If we order them by age, does that change the committee? No. If we order them by last name, does that change the committee? No.

Any way we order the people on the committee, provided it's the same people, the committee is the same. A change in the order only makes the list of names change, not the names themselves. Any given order is a permutation, a different way of listing the members. The group itself is a combination.

Here's another example of why order matters, or doesn't. Use the letters 'e', 'v', 'i', 'l', and 's'. Now, if I order these letters as "l-i-v-e-s" then that's one particular word. Another arrangement is "e-l-v-i-s" which is an entirely different word. The original order, "e-v-i-l-s" is yet a third word. Each word is different (radically so, you might say), but the letters themselves didn't change. It's still the same five letters just arranged differently. This is a word "game" called an *anagram*.

Numeric Examples

Here are a few examples of permutations and combinations with actual numbers.

1. How many different ways can a salesman visit 8 cities?

$$n = 8, r = 8 \quad \text{so} \quad P(8,8) = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8! = 40320$$

2. How many different ways can 10 horses in a race win, place and show?

$$\begin{aligned} n = 10, r = 3 \quad \text{so} \quad P(10,3) &= \frac{10!}{(10-7)!} \\ &= \frac{10 \times 9 \times 8 \times 7!}{7!} \\ &= 10 \times 9 \times 8 \\ &= 720 \end{aligned}$$

3. In a game of poker, you are dealt a hand of 5 cards from a pack of 52. How many possible hands are there?

$$\begin{aligned} n = 52, r = 5 \quad \text{so} \quad C(52,5) &= \frac{52!}{(52-5)!5!} = \frac{52!}{47!5!} \\ &= \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{47! \times 5!} \\ &= \frac{52 \times 51 \times 50 \times 49 \times 48}{5!} \\ &= \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} \\ &= 2,598,960 \end{aligned}$$