The last inequality is clearly true for $n \geq 3$. Since the sequence is decreasing (a stronger condition than nonincreasing) and is bounded below by zero, the Monotonic Sequence Theorem guarantees that it has a limit.

It would be easy using l'Hôpital's Rule to show that the limit is zero. ■

CONCEPTS REVIEW

1. An arrangement of numbers $a_1, a_2, a_3, \ldots$ is called __________.

2. We say the sequence $\{a_n\}$ converges if __________.

3. An increasing sequence that is also __________ must converge.

4. The sequence $\{r^n\}$ converges if and only if __________ $< r \leq$ __________.

PROBLEM SET 11.1

In Problems 1–20, an explicit formula for a_n is given. Write out the first five terms, determine if the sequence converges or diverges, and if it converges, find $\lim_{n\to\infty} a_n$.

1. $a_n = \dfrac{n}{2n-1}$
2. $a_n = \dfrac{3n+1}{n+2}$
3. $a_n = \dfrac{4n^2+1}{n^2-2n+3}$
4. $a_n = \dfrac{3n^2+2}{n+4}$
5. $a_n = \dfrac{n+4}{2n^2+1}$
6. $a_n = \dfrac{\sqrt{n}}{n+1}$
7. $a_n = (-1)^n \dfrac{n}{n+1}$
8. $a_n = \dfrac{n\sin(n\pi/2)}{2n+1}$
9. $a_n = \dfrac{\sin(n\pi/2)}{n}$
10. $a_n = e^{-n}\cos n$
11. $a_n = \dfrac{e^n}{n^2}$
12. $a_n = \dfrac{e^n}{2^n}$
13. $a_n = \dfrac{(-\pi)^n}{4^n}$
14. $a_n = (\frac{1}{2})^n + 2^n$
15. $a_n = 1 + (0.9)^n$
16. $a_n = \dfrac{n^3}{e^n}$
17. $a_n = \dfrac{\ln n}{n}$
18. $a_n = \dfrac{\ln(1/n)}{\sqrt{n}}$
19. $a_n = \left(1 + \dfrac{1}{n}\right)^n$
20. $a_n = n^{1/n}$

Hint: Theorem 7.5A

In Problems 21–30, find an explicit formula $a_n =$ _____ for each sequence, determine if the sequence converges or diverges, and if it converges, find $\lim_{n\to\infty} a_n$.

21. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
22. $\dfrac{1}{2^2}, \dfrac{2}{2^3}, \dfrac{3}{2^4}, \dfrac{4}{2^5}, \ldots$
23. $-1, \frac{2}{3}, -\frac{3}{5}, \frac{4}{7}, -\frac{5}{9}, \ldots$
24. $1, \dfrac{1}{1-\frac{1}{2}}, \dfrac{1}{1-\frac{2}{3}}, \dfrac{1}{1-\frac{3}{4}}, \ldots$
25. $1, \dfrac{2}{2^2-1^2}, \dfrac{3}{3^2-2^2}, \dfrac{4}{4^2-3^2}, \ldots$
26. $\dfrac{1}{2-\frac{1}{2}}, \dfrac{2}{3-\frac{1}{3}}, \dfrac{3}{4-\frac{1}{4}}, \dfrac{4}{5-\frac{1}{5}}, \ldots$
27. $\sin 1, 2\sin\frac{1}{2}, 3\sin\frac{1}{3}, 4\sin\frac{1}{4}, \ldots$
28. $-\frac{1}{3}, \frac{4}{9}, -\frac{9}{27}, \frac{16}{81}, \ldots$
29. $2, 1, \dfrac{2^3}{3^2}, \dfrac{2^4}{4^2}, \dfrac{2^5}{5^2}, \ldots$
30. $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \ldots$

In Problems 31–36, write the first four terms of the sequence $\{a_n\}$. Then use Theorem D to show that the sequence converges.

31. $a_n = \dfrac{4n-3}{2^n}$
32. $a_n = \dfrac{n}{n+1}\left(2 - \dfrac{1}{n^2}\right)$
33. $a_n = \left(1 - \dfrac{1}{4}\right)\left(1 - \dfrac{1}{9}\right)\cdots\left(1 - \dfrac{1}{n^2}\right), n \geq 2$
34. $a_n = 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$

$(1 + x)^{-1/2} \approx 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$; $|R_3(x)| \le 0.0000023$

$\int_0^{0.5} \sin x\, dx \approx 0.122396$ with an error less than 0.000131.

$-1 - (x - 1)^2 + (x - 1)^3 + (x - 1)^4$; $R_4(x) = 0$ because the fifth derivative is zero.

0.68199832; $|R_3| \le 4.38 \times 10^{-8}$

OBLEM SET 10.3 Page 465

0.50227; 0.50003; 0.5 **3.** 5.2650; 5.3046; 5.3333

1.5708; 1.9541; 1.9886 **7.** 3.1415926

With $|f''(c)| \le 4$, $n = 8$; 0.8057.

With $|f''(c)| \le 0.6$, $n = 3$; 0.9730.

With $|f^{(4)}(c)| \le 48$, $n = 8$; -4.19745.

40,825 **21.** About 3047 ft^2

About 1,074,585,600 ft^3

	$n = 10$	$n = 20$
Trap.	48.7466	48.4867
Mid.	48.2268	48.3567
Simp.	48.4000	48.4000

	$n = 10$	$n = 20$
Trap.	3.13993	3.14118
Mid.	3.14243	3.14180
Simp.	3.14159	3.14159

OBLEM SET 10.4 Page 471

1.29 **3.** 1.29 **5.** -0.26795 **7.** 1.55715

0.73909 **11.** 0.34711; 3.65289 **13.** 0.52658

1.81712 **17.** 4.493409; -0.217234

$m = 2 = M$; $|x_6 - \sqrt{2}| \le 1.0843 \times 10^{-19}$

The small value of $|f'(1.2)|$ sends the intersection of the tangent with the x-axis far to the right.

$i = 0.01513$, $r = 18.16\%$

-1.87939, 0.3473, 1.53209

-2.08204, 0.09251, 0.91314, 1.62015, 1.85411

OBLEM SET 10.5 Page 478

0.08446 **3.** 2.15831

(a) Graph indicates the root is near $\frac{1}{2}$.

(b) 0.49999872 after 4 steps. (c) $\frac{1}{2}$; (d) $g'(\frac{1}{2}) = 0$

(a) Graph indicates the root is near 0.9.

(b) Algorithm behaves very badly.

(c) $|g'(0.9)| \approx 6$, which is larger than 1.

(a) Divide by 6 and write $\frac{1}{6}x = x - \frac{5}{6}x$.

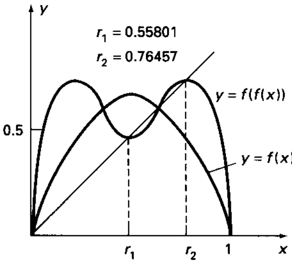
(b) 0.858736;

(c) $|g'(0.858736)| \approx 0.11$, which is much less than 1.

Write $x = \frac{1}{10}(1 + 11x + x^2 - x^3)$; $x = 1.8392868$.

1.7724538; $g'(a) = 0$ **15.** (a) \$293.75; (b) 1.5991%

19. $r = 0.6$ **21.**



23. $s_1 = 0.38282$, $s_2 = 0.82694$, $s_3 = 0.50088$, $s_4 = 0.87500$; $h(x) = f(f(f(f(x)))) = 0$

CHAPTER REVIEW 10.6 Page 479

Concepts Test

1. True. **2.** True. **3.** True. **4.** True. **5.** True.
6. True. **7.** True.
8. False (because of round-off errors).
9. False ($\int e^{x^2}\,dx$ cannot be so expressed).
10. False (it will give a larger value in this case).
11. True. **12.** False (round-off errors) **13.** True.
14. True. **15.** True.
16. False (rather, it converges slowly).
17. False (look at the geometry). **18.** True.
19. False (look at $f'(x)$). **20.** True.

Sample Test Problems

1. 47.26
2. (a) $x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$; 0.110517;
(b) $1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$; 1.0050042
3. $3 + 9(x - 2) + 4(x - 2)^2 + (x - 2)^3$ **4.** 3.941
5. $\frac{1}{2} - \frac{1}{4}(x - 1) + \frac{1}{8}(x - 1)^2 - \frac{1}{16}(x - 1)^3 + \frac{1}{32}(x - 1)^4$
6. $R_4(x) = -(x - 1)^5/(c + 1)^6$, where c is between 1 and x; $|R_4(1.2)| \le 0.000005$.
7. $x^2 - \frac{1}{3}x^4$; $|R_5| \le 0.000003$ **8.** $n = 5$
9. -0.0026987; 0.00002 **10.** -0.002786; 0.00014
11. -0.0026994; 0.0000003 **12.** -0.00269929
13. 0.281785 **14.** 0.281785 **15.** 4.49341
16. $D_x(\tan x) = \sec^2 x \ge 1$ **17.** -3.18306

PROBLEM SET 11.1 Page 489

1. $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}$; converges; $\frac{1}{2}$.
3. $\frac{5}{2}, \frac{17}{3}, \frac{37}{6}, \frac{65}{11}, \frac{101}{18}$; converges; 4.
5. $\frac{5}{3}, \frac{2}{3}, \frac{7}{19}, \frac{8}{33}, \frac{3}{17}$; converges; 0.
7. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}$; diverges.
9. $1, 0, -\frac{1}{3}, 0, \frac{1}{5}$; converges; 0.
11. $e, e^2/4, e^3/9, e^4/16, e^5/25$; diverges.
13. $-\pi/4, \pi^2/16, -\pi^3/64, \pi^4/256, -\pi^5/1024$; converges; 0
15. 1.9, 1.81, 1.729, 1.6561, 1.59049; converges; 1.
17. 0, ln 2/2, ln 3/3, ln 4/4, ln 5/5; converges; 0.
19. 2, 9/4, 64/27, 625/256, 7776/3125; converges; e.
21. $n/(n + 1)$; converges; 1. **23.** $(-1)^n n/(2n - 1)$; diverges.

25. $n/[n^2 - (n-1)^2] = n/(2n-1)$; converges; $\frac{1}{2}$.
27. $n \sin(1/n)$; converges; 1. **29.** $2^n/n^2$; diverges.
31. $\frac{1}{2}, \frac{5}{4}, \frac{9}{8}, \frac{13}{16}$; $0 < a_{n+1}/a_n = (4n+1)/(8n-6) < 1$ for $n \geq 2$.
33. $\frac{3}{4}, \frac{2}{3}, \frac{5}{8}, \frac{3}{5}$; $0 < a_{n+1} = [1 - 1/(n+1)^2]a_n < a_n$
35. $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}$; $0 < a_n = 2 - (\frac{1}{2})^{n-1} < a_{n+1} < 2$ **37.** 2.3028
39. $u = (1 + \sqrt{13})/2 \approx 2.302776$ **41.** 1.1118
43. $\int_0^1 \sin x\, dx \approx 0.4597$ **45.** Choose $N > 1/\varepsilon - 1$.
51. No; let $a_n = n + 1/n$, $b_n = -n$. **53.** $\pi/2\sqrt{3}$
55. 1.64872 **57.** 0.13534 **59.** 0.36794

PROBLEM SET 11.2 Page 497

1. Converges; $\frac{1}{4}$. **3.** Converges; $\frac{33}{5}$.
5. Diverges since $(k-3)/k \to 1$. **7.** Converges; 1.
9. Diverges since $k!/10^k > 1$ for large k.
11. Converges; $e^2/(\pi^2 - e\pi)$. **13.** -3
15. $0.2 + 0.02 + 0.002 + \cdots$; $\frac{2}{9}$
17. $0.013 + 0.000013 + 0.000000013 + \cdots$; $\frac{13}{999}$
19. $0.4 + 0.09 + 0.009 + 0.0009 + \cdots$; $\frac{1}{2}$ **21.** 1
23. $S_n = -\ln(n+1)$ **25.** 500 ft
27. \$4 billion, including the original \$1 billion. **29.** $\frac{1}{4}$
31. $\frac{4}{5}$; no **33.** 1000/9 yd **35.** Try proof by contradiction.
37. Since $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ diverges, the protrusion can be made arbitrarily large.
39. Try proof by contradiction. **41.** (a) 2; (b) 1
43. (a) $A = C/(1 - e^{-kt})$; (b) $\frac{8}{3}$ **45.** 1

PROBLEM SET 11.3 Page 504

1. Diverges. **3.** Diverges. **5.** Diverges. **7.** Diverges.
9. Converges. **11.** Converges.
13. Diverges; nth-Term Test. **15.** Diverges; nth-Term Test.
17. Diverges; nth-Term Test. **19.** Converges; Integral Test.
21. Converges; Integral Test. **23.** $\int_5^\infty xe^{-x}\, dx \approx 0.0404$
25. $\int_5^\infty 1/(1+x^2)\, dx \approx 0.1974$ **27.** $p > 1$ (Integral Test).
31. This is a famous unsolved problem. **33.** 2.724×10^8

PROBLEM SET 11.4 Page 512

1. Diverges. **3.** Converges. **5.** Converges.
7. Diverges. **9.** Converges.
11. Diverges; nth-Term Test.
13. Converges; Limit Comparison Test ($b_n = 1/n^{3/2}$).
15. Converges; Ratio Test.
17. Converges; Limit Comparison Test ($b_n = 4/n^2$).
19. Converges; Ordinary Comparison Test ($b_n = 1/n^2$).
21. Converges; Limit Comparison Test ($b_n = 1/n^2$).
23. Converges; Ratio Test.
25. Converges; p-Series Test ($p = \frac{3}{2}$).
27. Diverges, nth-Term Test.
29. Converges; Ordinary Comparison Test ($b_n = 5/n^3$).
31. Converges; Ratio Test **33.** Converges; Ratio Test.
35. $\sum a_n$ converges $\Rightarrow 0 < a_n < 1$ for large $n \Rightarrow 0 < a_n^2 < a_n$ for large n. Use Ordinary Comparison Test.
37. $a_n/b_n \to 0 \Rightarrow 0 \leq a_n < b_n$ for large n. Use Ordinary Comparison Test.
41. Choose r such that $R < r < 1$. Then choose N so that $n \geq N \Rightarrow a_n^{1/n} \leq r$—that is, $a_n \leq r^n$. Use Ordinary Comparison Test.
43. (a) Diverges; (b) converges; (c) converges; (d) converges; (e) diverges; (f) converges.
45. Converges if $p > 1$, diverges otherwise.

PROBLEM SET 11.5 Page 518

1. $|S - S_9| < \frac{2}{31} \approx 0.0645$ **3.** $|S - S_9| < 1/(\ln 11) \approx 0.417$
5. $|S - S_9| < (\ln 10)/10 \approx 0.230$ **7.** Absolute Ratio Test.
9. Absolute Ratio Test.
11. Ordinary Comparison Test: $|(-1)^{n+1}/[n(n+1)]| \leq 1/n^2$.
13. Converges conditionally. **15.** Diverges.
17. Converges conditionally. **19.** Converges absolutely.
21. Converges conditionally. **23.** Converges conditionally.
25. Converges absolutely. **27.** Converges conditionally.
29. Diverges. **31.** Try proof by contradiction.
33. Key idea: $\sum 1/2n = \sum \frac{1}{2}(1/n)$ diverges by Problem 35 of Section 11.2
45. ln 2

PROBLEM SET 11.6 Page 523

1. $[-1, 1]$ **3.** $\mathbb{R}$ **5.** $(-1, 1)$ **7.** $(-1, 1]$ **9.** $[-1, 1]$
11. $(-2, 2)$ **13.** $\mathbb{R}$ **15.** $[0, 2)$ **17.** $(-3, 1)$
19. $[-6, -4]$ **21.** nth-Term Test. **23.** $\sqrt{2}$
25. $1/(4 - x)$; $2 < x < 4$ **27.** (a) $[-1, \frac{1}{3})$; (b) $(-\frac{1}{2}, \frac{7}{2}]$
29. $S(x) = a_0 + a_1 x + a_2 x^2)/(1 - x^3)$

PROBLEM SET 11.7 Page 529

1. $1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$; 1
3. $x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=2}^{\infty} (-1)^n x^n$; 1
5. $\frac{1}{2}[1 + \frac{3}{2}x + (\frac{3}{2}x)^2 + \cdots] = \sum_{n=0}^{\infty} \frac{1}{2}(\frac{3}{2}x)^n$; $\frac{2}{3}$
7. $x^2 + x^6 + x^{10} + \cdots = \sum_{n=0}^{\infty} x^{4n+2}$; 1
9. $x^2/2 - x^3/3 \cdot 2 + x^4/4 \cdot 3 - x^5/5 \cdot 4 + \cdots = \sum_{n=2}^{\infty} (-1)^n x^n/n(n-1)$; 1
11. $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots = \sum_{n=1}^{\infty} 2x^{2n-1}/(2n-1)$; 1
13. $1 - x + x^2/2! - x^3/3! + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n/n!$
15. $2[1 + x^2/2! + x^4/4! + \cdots] = \sum_{n=0}^{\infty} 2x^{2n}/(2n)!$
17. $1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \cdots$
19. $x - x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{3}{40}x^5 + \cdots$
21. $x + \frac{2}{3}x^3 + \frac{13}{15}x^5 - \frac{29}{105}x^7 + \cdots$
23. $x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5 + \cdots$