MATH 2420 Discrete Mathematics
Lecture notes

Functions

Objectives:

1. Be able to prove statements using mathematical induction.

Proof by Induction

Until now everything we've proven has been a direct proof, whether a proof of a function being one-to-one or onto, or an element argument, or an algebraic proof of a set property.

But in mathematics, not everything can be proven directly, especially when dealing with sequences and series. In that case, we use the technique of proof by induction.

{from MATtours -SciMathMN}

Inductive proofs are based on the idea that you want to prove an infinite sequence of statements:

- property p is true for number 1
- property p is true for number 2
- property p is true for number 3
- etc.

The method of proof is based on adding to the list infinitely many implications:

- p is true for 1
- If p is true for 1 then p is true for 2.
- P is true for 2.
- If p is true for 2 then p is true for 3.
- P is true for 3.
- If p is true for 3 then p is true for 4.
- Etc

We always want to show that our property is true for one more instance of n. That is, $n + 1$.

That is, if you can prove p is true for 1, and all of the "if p is true for ... then p is true for ..." then you've proved all of the "p is true for..." statements. In other words, if you can prove "p is true for 1" and "If p is true for n, then p is true for $n + 1$ for all natural numbers n" then you've proved "p is true for n for all natural numbers n."

This is kind of like thinking about falling dominos. If you choose a domino in a line at random and push it over, the next domino in line falls. That is, if you push domino n over, domino $n+1$ falls, too.

A proof by induction consists of two specific parts:

1. P is true for some particular integer a.
2. If P is true for some particular integer $k \geq a$, then it is true for the next integer $k+1$.

Then P is true for all integers $n \geq a$.

Step 1 of the proof is the basis case, you prove that $P(a)$ is true for a particular integer a. Step 2 is the inductive step, where you prove that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

To generalize, to prove the inductive step you suppose that $P(k)$ is true (where k is some arbitrary integer greater than the basis case) then you show that $P(k+1)$ is true.

This can be done in a variety of ways - algebraically, graphically, or verbally.

The secret to inductive proofs, if there is one, is to recognize the part of the proof that you already know and have stated is proven. When proving for the $k+1$ term, replace the variable with the $k+1$ value, then expand and simplify. Once you expand the form you should recognize the part which you already know, namely, that part which is the answer to $P(k)$.

An example:

Statement Use mathematical induction to prove that the sum of the first n odd positive integers is n^2.

Solution Let $P(n)$ denote the proposition that the sum of the first n odd positive integers is n^2. We must first complete the basis step; that is, we must show that $P(1)$ is true. Then we must carry out the inductive step; that is, we must show that $P(n+1)$ is true when $P(n)$ is assumed to be true.

Basis Step $P(1)$ states that the sum of the first one odd positive integers is 1^2. This is true since the sum of the first odd positive iteger is 1.

Inductive Step To complete the inductive step we must show that the proposition $P(n) \rightarrow P(n+1)$ is true for every positive integer n. To do this, suppose that $P(n)$ is true for a positive integer n; that is,

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

We must show that $P(n+1)$ is true, assuming that $P(n)$ is true. Note that $P(n+1)$ is the statement that

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = (n+1)^2$$

So, assuming that $P(n)$ is true, it follows that

$$\begin{aligned} 1+3+5+\cdots+(2n-1)+(2n+1) &= [1+3+5+\cdots+(2n-1)]+(2n+1) \\ &= n^2+(2n+1) \\ &= n^2+2n-1 \\ &= (n+1)^2 \end{aligned}$$

Note how we replaced the majority of the equation with the simpler form n^2 because we *recognized the pattern we already knew!*

Conclusion Since $P(1)$ is true and the implication $P(n) \to P(n+1)$ is true for all positive integers n, the principle of mathematical induction shows that $P(n)$ is true for all positive integers n.

Verbal Form of Proof

Theorem 1. *Any set of size n has exactly 2^n subsets.*

Proof. For any $n \geq 0$, let $P(n)$ be the proposition that any set of size n has exactly 2^n subsets. Then $P(0)$ is the proposition that any set of size 0 has exactly $2^0 = 1$ subset. But the only set of size 0 is the empty set $\emptyset$, which does indeed have exactly 1 subset, namely, the empty set itself. Hence, $P(0)$ is true.

Now we must show that if $P(k)$ is true, then so is $P(k+1)$. That is, we must show that if any set of size k has exactly 2^k subsets, then any set of size $k+1$ has exactly 2^{k+1} subsets.

So let $s = \{x_1, x_2, x_3, \ldots, x_k, x_{k+1}\}$ be any set of size $k+1$. Then we may classify all of the subsets of S into two groups - those subsets that contain the element x_{k+1}, and those that do not. Now, the subsets of S that do not contain the element x_{k+1} are precisely the subsets of the set $\{x_1, x_2, \ldots, x_k\}$. Furthermore, since this set has size k, we may use the fact that $P(k)$ is true to deduce that there are 2^k such subsets. In other words, there are 2^k subsets of S that do not contain the element x_{k+1}.

On the other hand, the number of subsets of S that contain the element x_{k+1} is the same as the number of subsets of S that do not contain x_{k+1}. One way to see this is to observe that if we take all of the subsets of S that do not contain x_{k+1} and add x_{k+1} to each of these subsets, then we will get a complete list of all of the subsets of S that do contain the element x_{k+1}.

Hence, there are also 2^k subsets of S that do contain the element x_{k+1}, and so the total number of subsets of S is $2^k + 2^k = 2^{k+1}$. Thus, $P(k+1)$ is true, and so the theorem is proved. □

Corrected proof from Roman, 2nd ed, p. 52