MATH 2420 Discrete Mathematics
Proof: An Inequality for Harmonic Numbers

Definition

The *harmonic numbers*, denoted $H_1, H_2, H_3, \ldots$, are a special sequence of numbers. The sequence begins at one and continues as an infinite sum, like so

$$\begin{aligned} H_1 &= 1 \\ H_2 &= 1 + \frac{1}{2} \\ H_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ H_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ H_k &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \; \forall n \in Z^+ \end{aligned}$$

Proposal

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a nonnegative integer.

From Rosen, 4th ed, pg. 193

Notice that this only applies to harmonic numbers at powers of 2.

Proof

To carry out the proof, let $P(n)$ be the proposition that

$$H_{2^n} \geq 1 + \frac{n}{2}.$$

Basis Step

Let $n = 0$. Then $P(0)$ is

$$H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}.$$

Inductive Step

Assume that $P(n)$ is true, so that

$$H_{2^n} \geq 1 + \frac{n}{2}.$$

It must be shown that $P(n+1)$, which states

$$H_{2^{n+1}} \geq 1 + \frac{n+1}{2},$$

must also be true under this assumption. This is done as follows:

$$
\begin{aligned}
H_{2^{n+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} && (1)\\
&= \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n}}_{H_{2^n}} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} && (2)\\
&= H_{2^n} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} && (3)\\
&\geq \left(1 + \frac{n}{2}\right) + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} && (4)\\
&\geq \left(1 + \frac{n}{2}\right) + 2^n \times \frac{1}{2^{n+1}} && (5)\\
&\geq \left(1 + \frac{n}{2}\right) + \frac{1}{2} && (6)\\
&= 1 + \frac{n+1}{2}. && (7)
\end{aligned}
$$

Thus, by the Principle of Mathematical Induction, the inequality for the harmonic numbers is valid for all nonnegative integers n.

Discussion

Line 1

This is just the equation for $P(n)$ with $n+1$ substituted for n and then the sequence expanded. Note that the term $\frac{1}{2^n}$ is followed by $\frac{1}{2^n+1}$ and *not* $\frac{1}{2^{n+1}}$. This is because consecutive powers of 2 are not consecutive numbers on the number line ($2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$). In fact, the gap between consecutive powers increases as the power increases.

Line 2

In this line we recognize the part of the expanded series that we can replace, namely, all terms from 1 up to $\frac{1}{2^n}$. This is from the definition of the harmonic numbers.

Line 3

Here the "known" portion of the sequence is replaced by H_{2^n}.

Line 4

We now replace H_{2^n} with the inductive hypothesis which we have already proven.

Line 5

This is the most complex line in the proof. We have a problem in that we have the terms from $\frac{1}{2^n+1}$ to $\frac{1}{2^{n+1}}$ to deal with, and we don't know how many of them there are. *Or do we?* Let's look at the powers of 2 as they increase:

$$
\begin{array}{lllll}
2^0 & = & 1 & & \\
2^1 & = & 2 & = & 1+1 \\
2^2 & = & 4 & = & 2+2 \\
2^3 & = & 8 & = & 4+4 \\
2^4 & = & 16 & = & 8+8 \\
2^5 & = & 32 & = & 16+16
\end{array}
$$

The “distance” on the numberline from a power of 2 to the next power is always the same as the previous power of 2. That is, to get from 2^k to 2^{k+1} we need 2^k terms. We can write this as $2^k + 2^k$ or even as 2×2^k since adding a term to itself is the same as multiplying by 2. If we look at 2 as really 2^1 we then have $2^1 \times 2^k$, which can be rewritten as 2^{k+1}. If we replace k with n we have 2^{n+1} which is the denominator in the last term of the sequence. So we can reliably say that there are 2^n terms in that part of the sequence remaining after we replace the first half with H_{2^n}.

So that answers the first question, but why multiply by $\frac{1}{2^{n+1}}$?

Line 6

In this line we see the result $(\frac{1}{2})$ of the multiplication in the previous line. This results because of cancellation of common terms. The demoninator can be written as $\frac{1}{2^n \times 2}$ which allows us to cancel the 2^n leaving only $\frac{1}{2}$.

Line 7

Here the fraction $\frac{1}{2}$ is added to the fraction $\frac{n}{2}$ to simplify the terms and produce the final form, which is what was to be shown.