MATH 2420 Discrete Mathematics
Lecture notes

Graphs

Objectives

1. Identify loops, parallel edges, etc. in a graph.
2. Draw the complete graph on n vertices, and the complete bipartite graph on (m,n) vertices.
3. Determine whether a graph is bipartite or not.
4. List all the subgraphs of a given graph.
5. Determine the degree of a vertex in a graph.
6. Prove that the sum of the degrees of the vertices is equal to twice the number of edges.
7. Show that in any graph there is an even number of vertices of odd degree.
8. Apply these results.
9. Determine the complement of a simple graph.
10. Determine whether a walk is a path, simple path, closed walk, circuit or a simple circuit.
11. Determine whether a graph is connected or not.
12. Prove that a graph has an Euler circuit if and only if the graph is connected and every vertex of the graph has even degree.
13. Determine whether a given graph has an Euler circuit and, if so, indicate one.
14. Prove that a graph has an Euler path if and only if the graph is connected and has exactly two vertices of odd degree.
15. Determine whether a given graph has an Euler path and, if so, indicate one.
16. Determine whether a graph has a Hamiltonian circuit and, if so, indicate one.

Graphs

Graphs are drawn with dots and lines.

- The dots are **vertices** and the lines that connect the vertices are called **edges**;
- In drawing a graph, the only information of importance is which vertices are connected by which edges;
- In general, a graph consists of a set of vertices and a set of edges connecting various vertices;

- The edges may be straight or curved and should either connect one vertex to another or a vertex to itself.

Definitions

- A **graph** G consists of two finite sets: V(G) of vertices and E(G) of edges.
- Each edge is associated with a set consisting of either one or two vertices called **endpoints**; an edge is said to be **incident** on each of its endpoints.
- The correspondence from edges to endpoints is called the **edge-endpoint function**.
- An edge that connects a vertex to itself is called a **loop**.
- Two edges that connect the same pair of vertices are said to be **parallel**.
- Two vertices connected by an edge are said to be **adjacent**.
- A vertex that is unconnected to any other vertex is **isolated**.

Graphs can be **undirected** or **directed** depending upon whether the pair of vertices is ordered or not. If the graph is directed, the vertices are an ordered pair as opposed to a set.

If a graph has neither loops nor parallel edges it is said to be **simple**.

Graphs can have several representations, depending upon how the vertices are connected

Complete Graphs

A special case of graphs is a **complete** graph:

> The complete graph on n vertices, denoted K_n, is the simple graph with n vertices in which there is an edge between every pair of distinct vertices.

Bipartite Graphs

A second special case is the **bipartite** graph:

A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into two subsets $V1$ and $V2$ such that each edge in E is incident on one vertex in $V1$ and one vertex in $V2$. note, this can leave out some vertex pairs. This definition states that if e is an edge in a bipartite graph, then e is incident on one vertex in $V1$ and one vertex in $V2$. It does **not** state that if $v1$ is in $V1$ and $v2$ is in $V2$ that there is an edge between $v1$ and $v2$.

The **complete bipartite graph** on m and n vertices, denoted $K_{m,n}$, is the simple graph whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which there is an edge between each pair of vertices v_1 and v_2 where v_1 is in V_1 and v_2 is in V_2.

Examples: $K_{3,2}$, $K_{3,3}$, $K_{2,4}$

Degree of Vertices and Graphs

Vertices have degree. The degree of a vertex is the number of edges that originate from it. The degree of a vertex is equal to the number of edges that are incident upon it with a loop counted twice. The total degree of a graph is the sum of the degrees of all vertices of that graph. (Example 11.1.11 pg. 658 Epps)

A graph has a total degree. The total degree of a graph is defined by

> If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G.

If G is a graph with m edges and vertices $\{v_1, v_2, \ldots, v_n\}$, then

$$\sum_{i=1}^{N} d(v_i) = 2m$$

By this we also know

> The total degree of a graph is even.

We also know

> In any graph, there are an even number of vertices of odd degree.

Paths, Circuits, and Cycles

Now let us treat the vertices as points of interest and the edges as routes to get there. We can then think of traversing the graph as walking or tracing a path. Travel is accomplished by starting at one vertex and moving to another vertex along an edge. When we traverse a graph we can describe the result as either a walk, a path, a simple path, a closed walk, a circuit (or cycle), or a simple circuit. Each of these is defined by certain characteristics, which we summarize in the following table:

	Repeated edge?	Repeated vertex?	Same start/end point?
Walk	Allowed	Allowed	Allowed
Path	No	Allowed	Allowed
Simple path	No	No	No
Closed walk	Allowed	Allowed	Yes
Circuit	No	Allowed	Yes
Simple circuit	No	First and last only	Yes

If G is a graph with v and w, vertices in G, then:

- a walk from v to w is a finite alternating sequence of adjacent vertices and edges of G
- the trivial walk from v to v consists of the single vertex v

- a path from v to w is a walk from v to w that does not contain a repeated edge
- a simple path from v to w is a path that does not contain a repeated vertex (all edges and vertices are distinct)
- a closed walk is a walk that starts and ends at the same vertex
- a circuit, or cycle, is a closed walk that does not contain a repeated edge (start = end and all the edges are distinct)
- a simple circuit is a circuit that does not have any other repeated vertex except the first and last (all edges are distinct and all vertices are distinct except for the start and end points)

Graphs can consist of subgraphs, pieces of a graph. To be a subgraph, all the vertices must be in the larger set of vertices for the larger graph, and all edges must be in the larger set of edges for the larger graph. Formally,

Let $G = (V, E)$ be a graph. We call (V', E') a subgraph of G if

1. V' is a subset of V, and E' is a subset of E
2. For every edge e' in E', if e' is incident on v' and w', then v' and w' are in V'

Subgraphs do not have to be connected, since we have seen graphs that are not all one "piece". If, however, all the vertices in a graph are connected by edges, then the graph can be said to be connected. Generically, if it is possible to travel from any vertex to any other vertex in the graph on a path, then the graph is connected. Formally,

> A graph G is connected if given any vertices v and w in G, there is a path from v to w.

Following this, if a graph has a circuit that contains every vertex and every edge, then the graph is an Euler circuit. That is, an Euler circuit is a sequence of adjacent vertices and edges in G that

1. starts and ends at the same vertex
2. uses every vertex of G at least once
3. uses every edge of G exactly once

Related to this is that if a graph G has an Euler circuit, then G is connected and every vertex has even degree. If some vertex of a graph has odd degree, then that graph cannot be an Euler circuit.

A corollary to this is the definition of an Euler path from v to w. An Euler path is a sequence of adjacent edges and vertices that

1. starts at v,
2. ends at w,
3. passes through every vertex of G at least once, and

4. traverses every edge of G exactly once.

To determine if an Euler path from v to w exists in a graph, we must ascertain that

1. G is connected;
2. v an w have odd degree;
3. all other vertices of G have even degree.

Let's increase the difficulty a bit.

If a graph G has a circuit(cycle) that contains each every vertex once (except for the starting and ending vertices) then that graph contains a Hamiltonian circuit. Although a Euler circuit must include every vertex of G, it may visit some vertices more than once and hence cannot be a Hamiltonian circuit. A Hamiltonian circuit does not need to include all the edges of G and hence may not be an Euler circuit.

We cannot prove a graph is Hamiltonian, as we can for Eulerian graphs. But we can detect graphs which should be Hamiltonian if we apply the theorem by Dirac, which states:

> *If a graph G has $N \geq 3$ vertices and every vertex has degree at least $n/2$, then G is Hamiltonian.*