On the positive recurrence of semimartingale reflecting Brownian motions in three dimension

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Joint work with Maury Bramson and J. Michael Harrison

References

- A. El Kharroubi, A. Ben Tahar and A. Yaacoubi (2000) Sur la recurrence positive du mouvement Brownien reflechi dans lorthant positif de Rn. Stochastics and Stochastics Reports. 68:229253.
- A. El Kharroubi, A. Ben Tahar and A. Yaacoubi (2002) On the stability of the linear Skorohod problem in an orthant. Math. Meth. Oper. Res. 56:243258.
- M. Bramson, J. G. Dai and J. M. Harrison (2010) Positive recurrence of reflecting Brownian motion in three dimensions, Ann. Appl. Prob., Vol 20:753-783

Outline of the talk

1. Definition of SRBM
2. SRBM as a network model
3. A necessary condition for positive recurrence
4. Positive recurrence in two dimensions
5. Fluid paths and positive recurrence
6. When fluid path spiral on the boundary
7. Linear fluid paths and the LCP
8. Proof sketches
9. Summary for the three-dimensional case
10. Bramson's six-dimensional example

Semimartingale reflecting Brownian motion (SRBM)

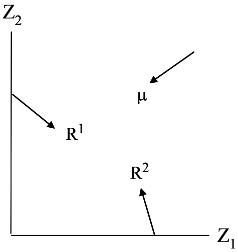
$$Z(t) = X(t) + RY(t) \quad \text{for all } t \geq 0, \tag{1}$$

$$X \text{ is a } (\theta, \Sigma) \text{ Brownian motion}, \tag{2}$$

$$Z(t) \in \mathbb{R}^n_+ \text{ for all } t \geq 0, \tag{3}$$

$$Y(\cdot) \text{ is continuous and nondecreasing with } Y(0) = 0, \tag{4}$$

$$Y_i(\cdot) \text{ only increases when } Z_i(\cdot) = 0, \quad i = 1, \ldots, n. \tag{5}$$



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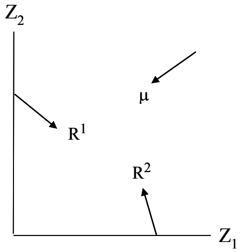
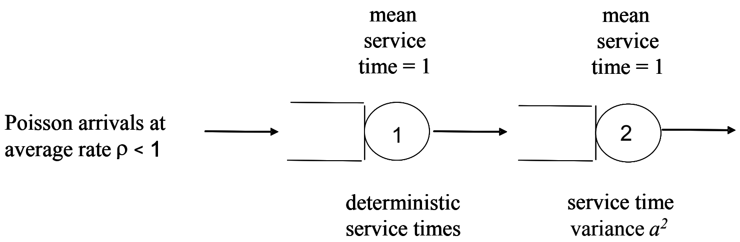
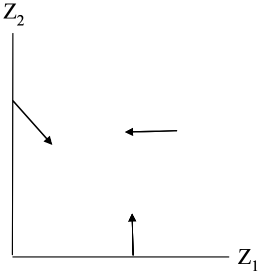


Definition An $n \times n$ matrix R is said to be an $\mathcal{S}$-matrix if $Rv > 0$ for some $v \geq 0$. It is said to be completely-$\mathcal{S}$ if each principal submatrix is an $\mathcal{S}$-matrix.

Taylor and Williams (93): existence and uniqueness (in distribution)

A queueing network





$$Z_1(t) = X_1(t) + Y_1(t),$$
$$Z_2(t) = X_2(t) - Y_1(t) + Y_2(t)$$
drift of X is $\theta = (\rho - 1, 0)$,

covariance of X is $\Sigma = \begin{pmatrix} \rho & 0 \\ 0 & a^2 \end{pmatrix}$

- R. J. Williams (95), Semimartingale reflecting Brownian motions in the orthant, in Stochastic Networks, eds. F. P. Kelly and R. J. Williams, the IMA Volumes in Mathematics and its Applications, Vol. 71 (Springer, New York)
- R. J. Williams (96), On the approximation of queueing networks in heavy traffic, in Stochastic Networks: Theory and Applications, eds. F. P. Kelly, S. Zachary and I. Ziedens, Royal Statistical Society (Oxford Univ. Press, Oxford)

A four-step program

1. Establish foundational properties such as existence and uniqueness

   e.g. Harrison and Reiman (81), Varadhan and R. J. Williams (85), Taylor and Williams (90), Dupuis and Williams (94), Dai and Williams (95), Kang and Williams (07)

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2. Prove a limit theorem connecting a discrete, stochastic network model with a Brownian model

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4. Obtain policies and performance for the stochastic network from the Brownian model.
   e.g. Bell and Williams (05)

Definition Z is said to be *positive recurrent* if the expected time to hit any neighborhood of the origin, starting from any point in the orthant, is finite.

Theorem 1. (EL Kharroubi et 00) Let Z be an n-dimensional SRBM with data (θ, Σ, R). A necessary condition for positive recurrence of Z is that $R^{-1}\theta < 0$, which means that

(a) R is non-singular, and

(b) $0 = \theta + R\beta$ for some $\beta > 0$.

Remark. To understand the intuitive basis for this necessary condition, compare (b) with the basic system equation $Z(t) = X(t) + RY(t)$.

Harrison and Williams (86): sufficient when R is an $\mathcal{M}$-matrix.

Definition. An $n \times n$ matrix R is said to be a $\mathcal{P}$-matrix if its principal submatrices all have positive determinants.

Hobson and Rogers (94) determined necessary and sufficient conditions for positive recurrence in the two-dimensional case. El Kharroubi et al. (00) restated those conditions as follows

Theorem 2 Suppose $n = 2$. Then Z is positive recurrent if and only if

$$R \text{ is a } \mathcal{P}\text{-matrix, and}$$
$$R^{-1}\theta < 0.$$

Definition. A *fluid path* associated with data (θ, R) is a pair of continuous functions $y, z : \mathbb{R}_+ \to \mathbb{R}^n$ that satisfy the following conditions:

$$z(t) = z(0) + \theta t + Ry(t) \text{ for all } t \geq 0, \tag{6}$$

$$z(t) \in \mathbb{R}^n_+ \text{ for all } t \geq 0, \tag{7}$$

$$y(\cdot) \text{ is continuous and nondecreasing with } y(0) = 0, \tag{8}$$

$$y_i(\cdot) \text{ only increases when } z_i(\cdot) = 0, \quad \text{i=1, \ldots, n} \tag{9}$$

Definition

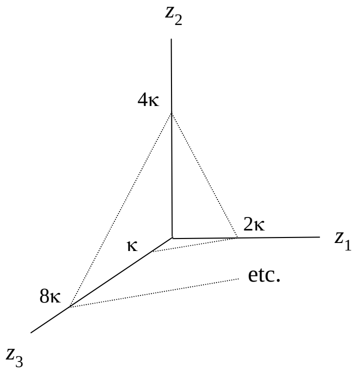
We say that a fluid path (y, z) is attracted to origin if $z(t) \to 0$ as $t \to \infty$. A fluid path is said to be divergent if $|z(t)| \to \infty$ as $t \to \infty$.

Theorem 3 (Dupuis and Williams 94) Let Z be an n-dimensional SRBM with data (θ, Σ, R). If every fluid path associated with (θ, R) is attracted to the origin, then Z is positive recurrent.

Bernard and El Kharroubi (91) devised the following example. Let

$$\theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix}.$$

This reflection matrix R is completely-$\mathcal{S}$, so Z is a well defined SRBM.



There is a unique fluid path starting from $z(0) = (0, 0, \kappa)$.

The path travels in a counter-clockwise and piecewise linear fashion on the boundary, with the first linear segment ending at $(2\kappa, 0, 0)$, the second one ending at $(0, 4\kappa, 0)$, and so forth.

$C_1(R)$ and $C_2(R)$

Let $C_1(R)$ be the set of (θ, R) pairs that satisfy the following inequalities :

$$\theta < 0, \tag{10}$$

$$\theta_1 > \theta_2 R_{12} \quad \text{and} \quad \theta_3 < \theta_2 R_{32}, \tag{11}$$

$$\theta_2 > \theta_3 R_{23} \quad \text{and} \quad \theta_1 < \theta_3 R_{13}, \tag{12}$$

$$\theta_3 > \theta_1 R_{31} \quad \text{and} \quad \theta_2 < \theta_1 R_{21}. \tag{13}$$

Let

$$\beta_1(\theta, R) = \left(\frac{\theta_1 - \theta_2 R_{12}}{\theta_2 R_{32} - \theta_3}\right)\left(\frac{\theta_2 - \theta_3 R_{23}}{\theta_3 R_{13} - \theta_1}\right)\left(\frac{\theta_3 - \theta_1 R_{31}}{\theta_1 R_{21} - \theta_2}\right) > 0. \tag{14}$$

Lemma

If $\theta \in C_1(R)$, then the fluid path starting away from the origin on the boundary behaves as in the B&EK example, spiraling counter-clockwise on the boundary; each such fluid path has a multiplicative gain equal to $\beta_1(\theta)$ per cycle.

Define $C_2(R)$ similarly with $\beta_2(\theta, R)$; clockwise spirals.

First set of results

Definition

Let $C = C_1 \cup C_2$, $\beta(\theta, R) = \beta_1(\theta, R)$ for $(\theta, R) \in C_1$, and $\beta(\theta, R) = \beta_2(\theta, R)$ for $(\theta, R) \in C_2$. $\beta(\theta, R)$ is the single-cycle gain for such a pair.

Theorem 4 (El Kharroubi et al. 02) Suppose that $\theta \in C(R)$ and $\beta(\theta) < 1$. Then every fluid path associated with (θ, R) is attracted to the origin and hence Z is positive recurrent.

Theorem (Bramson-D-Harrison)

Suppose that $\theta \in C(R)$ and $\beta(\theta, R) \geq 1$. Then Z is not positive recurrent.

Proof. For a well-chosen vector $u > 0$, we define $f(t) = u'Z(t)$ and show that $f = M + A$, where M and A continuous, M is a martingale, $A(0) = 0$ and $A(\cdot) \geq 0$.

Equivalence of a linear fluid path and a LCP solution

Recall a fluid path (y, z) satisfies

$$z(t) = z(0) + \theta t + Ry(t) \text{ for all } t \geq 0,$$
$$z(t) \in \mathbb{R}^n_+ \text{ for all } t \geq 0,$$
$$y(\cdot) \text{ is continuous and nondecreasing with } y(0) = 0,$$
$$y_i(\cdot) \text{ only increases when } z_i(\cdot) = 0, \quad \text{i=1, \dots, n}$$

Definition

A fluid path (y, z) is said to be *linear* if it has the form $y(t) = ut$ and $z(t) = vt$, $t \geq 0$, where $u, v \geq 0$.

Linear complementarity problem (LCP): Find vectors $u = (u_i)$ and $v = (v_i)$ in $\mathbb{R}^d_+$ such that

$$v = \theta + Ru \quad \text{and} \quad u'v = 0. \tag{15}$$

LCP solution and divergent linear fluid paths

Lemma

Suppose that $R^{-1}\theta < 0$ holds. Then $(u^*, 0)$ is a proper solution of the LCP, where

$$u^* = -R^{-1}\theta, \tag{16}$$

and any other solution (u, v) of the LCP must be divergent, namely, $v \neq 0$.

If there exists another LCP solution, the corresponding linear fluid path diverges.

Second set of results

Theorem (El Kharroubi et al. 02)

If $\theta \not\in C(R)$ and (u^*0) is the unique solution of the LCP, then all fluid path associated with (θ, R) are attracted to the origin, and hence Z is positive recurrent.

Theorem (Bramson-D-Harrison)

If there exists another solution (u, v) of the LCP, it is necessarily divergent, and Z is not positive recurrent.

Proofs

Proof. Let (u, v) be a LCP solution with $v \neq 0$. We separate into five categories.

Category I: exactly two components of v are positive and the complementary component of u is positive. Using fluid limits

Categories II-V: exactly one component of v is positive.

Assume $v_3 > 0$. Let $\hat{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$.

Categories II to V: $v_3 > 0$

Category II: $\det(\hat{R}) > 0$, $u_1 > 0$, $u_2 \geq 0$.
Using fluid limits

Category III: $\det(\hat{R}) = 0$, $u_1 > 0$, $u_2 \geq 0$.
cannot happen

Category IV: $\det(\hat{R}) < 0$, $u_1 > 0$ and $u_2 > 0$
reduce to either I or II

Category V: $\det(\hat{R}) < 0$, $u_1 > 0$ and $u_2 = 0$.
complicated estimates

Proof sketch for Category I: $v_2 > 0$ and $v_3 > 0$ and $u_1 > 0$

Assume $Z(0) = (0, N, N)'$. Let $\tau = \inf\{t \geq 0 : Z_2(t) = 1 \text{ or } Z_3(t) = 1\}$. For $t < \tau$,

$$Z_1(t) = \theta_1 t + B_1(t) + Y_1(t), \tag{17}$$

$$Z_2(t) = N + \theta_2 t + B_2(t) + R_{21} Y_1(t), \tag{18}$$

$$Z_3(t) = N + \theta_3 t + B_2(t) + R_{31} Y_1(t). \tag{19}$$

We wish to prove that $\mathbb{P}\{\tau = \infty\} > 0$.

For a given Brownian motion B, (17)-(19) uniquely define $(\hat{Y}_1, \hat{Z})$, where $\hat{Z}(t)$ and $\hat{Y}_1(t)$ are defined for all $t \geq 0$. It is sufficient to prove that $\mathbb{P}\{\hat{\tau} = \infty\} > 0$.

Fluid limits

For any solution $(\hat{Y}_1, \hat{Z})$ satisfying (17)-(19), for each $n \geq 1$,

$$\bar{Z}^n(t) = \frac{1}{n}\hat{Z}(nt) \quad \text{and} \quad \bar{Y}_1^n(t) = \frac{1}{n}\hat{Y}_1(nt).$$

Then,

$$\bar{Z}^n \to \bar{Z} \quad \text{and} \quad \bar{Y}^n \to \bar{Y}(t)$$

a.s. as $n \to \infty$, where $\bar{Z}(t) = vt$ and $\bar{Y}(t) = ut$.

Because $v_2 > 0$ and $v_3 > 0$, $\mathbb{P}\{\hat{Z}_2(t) \to \infty, \hat{Z}_3(t) \to \infty\} = 1$, implying that $\mathbb{P}\{\hat{\tau} = \infty\} > 0$.

Proof sketch for Category V: $u_1 > 0$ and $v_3 > 0$

Let $Z(0) = (0, 2, N)'$. Because $Z_2(t) > 0$ and $Z_3(t) > 0$ for $t < \tau$, one has $Y_2(t) = Y_3(t) = 0$ for $t < \tau$. Then, on $t < \tau$,

$$Z_1(t) = -t + B_1(t) + Y_1(t), \tag{20}$$

$$Z_2(t) = 2 - t + B_2(t) + Y_1(t), \tag{21}$$

$$Z_3(t) = N + \theta_3 t + B_3(t) + R_{31} Y_1(t), \tag{22}$$

where we used the fact that $R_{21} = 1$, $\theta_1 = \theta_2 = -1$ because of the scaling convention.

By (20), one has $Y_1(t) = Z_1(t) + t - B_1(t)$ for $t < \tau$.

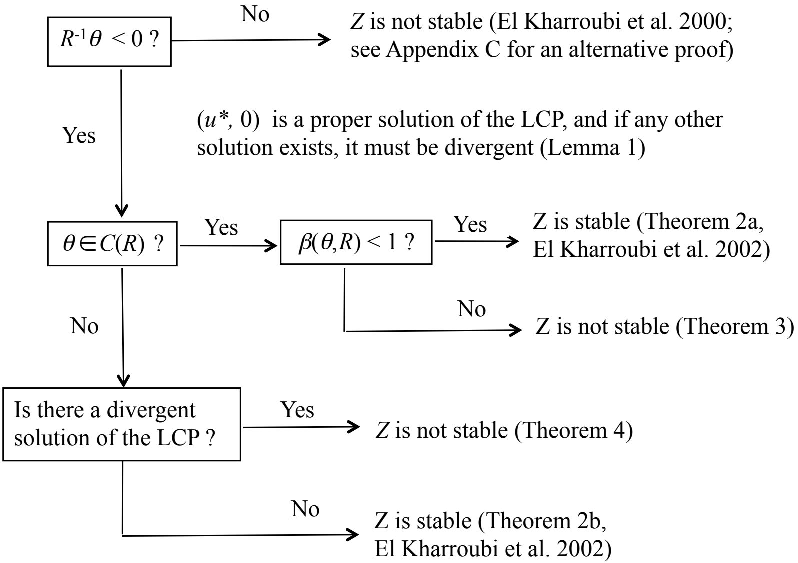
Proof sketch for Category V: $u_1 > 0$ and $v_3 > 0$ (cont')

Substituting $Y_1(t)$ into (21) and (22), one has

$$
\begin{aligned}
Z_1(t) &= -t + B_1(t) + Y_1(t), \\
Z_2(t) &= 2 + B_2(t) - B_1(t) + Z_1(t), \\
Z_3(t) &= N + v_3 t + B_3(t) - R_{31} B_1(t) + R_{31} Z_1(t)
\end{aligned}
$$

on $t < \tau$. For a given Brownian motion B, (23), (23) and (23) defines $(\hat{Y}_1, \hat{Z})$ on $\mathbb{R}_+$. One can prove that $\mathbb{E}\{\hat{\tau}\} = \infty$ because $v_3 > 0$.

Summary of results



Bramson's example

$$\theta = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1.05 & 1.05 & 1.05 & 1.05 & .4 \\ 1 & 1 & .95 & .95 & .95 & .95 \\ 1 & .95 & 1 & .95 & .95 & .95 \\ 1 & .95 & .95 & 1 & .95 & .95 \\ 1 & .95 & .95 & .95 & 1 & .95 \\ 1.05 & .95 & .95 & .95 & .95 & 1 \end{pmatrix},$$

$R^{-1}\theta$ is given by

$$(-0.075472, -0.207547, -0.207547, -0.207547, -0.207547, -0.132075)'$$

Example (Bramson 10)

LCP has a divergent solution (u, v) with $u = e_1$ and $v = .05e_6$. The SRBM is positive recurrent.

M. Bramson (2010) A positive recurrent reflecting Brownian motion with divergent fluid path, Annals of Applied Probability, to appear.

References

- A. Bernard and A. El Kharroubi (1991) Regulation deterministes et Stochastiques dans le premier orthant de Rn. Stochastics and Stochastics Reports. 34:149167
- L. M. Taylor and R. J. Williams (1993) Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. Prob. Th. Rel. Fields. 96: 283-317.
- P. Dupuis and R. J. Williams (1994) Lyapunov functions for semimartingale reflecting Brownian motions. Ann. Probab. 22:680702.
- D. G. Hobson and L. C. G. Rogers (1994) Recurrence and transience of reflecting Brownian motion in the quadrant. Math. Proc. of the Cambridge Phil. Soc. 113: 387-399