Reflected Brownian Motions, Dirichlet Processes and Queueing Networks

K. Ramanan
(Carnegie Mellon University)

includes joint work with
Weining Kang and Martin Reiman

Some Early Influential Papers

1. M. Harrison. The diffusion approximation for tandem queues in heavy traffic. *Adv. Appl. Probab.*, **10** (1978), 886–905.

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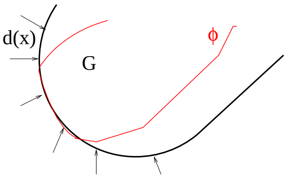
6. M. Harrison and R. Williams Brownian models of open queueing networks with homogeneous customer populations. *Stochastics*, **22** (1987) 77-115.

Reflected Processes – what are they?



G is the closure of some connected domain in $\mathbb{R}^n$

$d(\cdot)$ is a vector field specified on the boundary ∂G

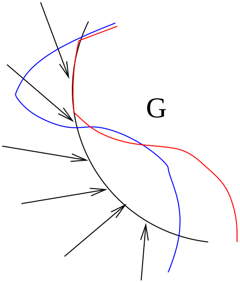
$d(x)$ is a cone for every $x \in \partial G$, graph of $d(\cdot)$ is closed

ϕ satisfies some specified interior dynamics

Want $\phi(t) \in G$ for all $t \in [0, \infty)$

Given $(G, d(\cdot))$, for any continuous ψ, find a continuous ϕ such that

1. $\phi(t) = \psi(t) + \eta(t) \in G$



The Skorokhod Problem – Multidimensional Version

Given $(G, d(\cdot))$, for any continuous ψ, find a continuous ϕ such that

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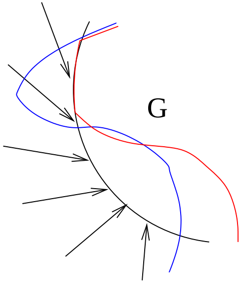

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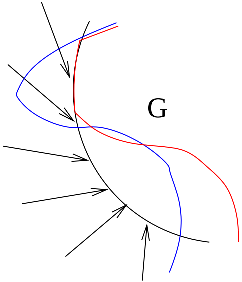
   i.e., total variation $|\eta|$ increases only when ϕ lies in ∂G



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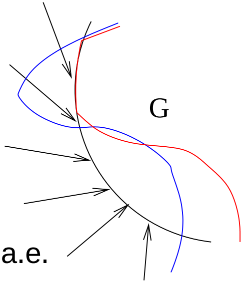

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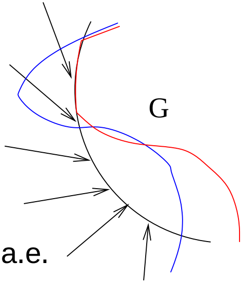

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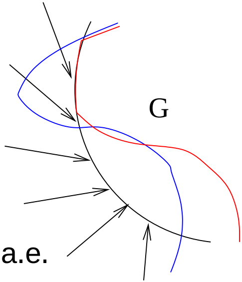

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Note: If X is a martingale, then $Z = \Gamma(X)$ is a semimartingale.

semimartingale $=$ local martingale + bounded variation

The Harrison-Reiman RBM

General Framework

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  All multi-class queueing networks need not be modelled by Γ that are continuous

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- But R completely-$\mathcal{S}$ does not guarantee uniqueness;
- Completely-$\mathcal{S}$ condition is necessary and sufficient for existence and uniqueness (in distribution) of SRBM (Reiman-Williams '88, Taylor-Williams)

Multiclass Queueing Networks to SRBM

The Bramson-Williams Framework ('98)

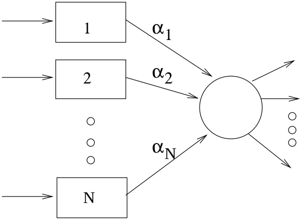
State Space Collapse $+$ Completely-$\mathcal{S}$

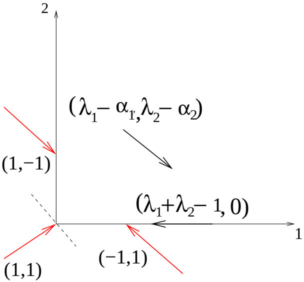
$$\Downarrow$$

Heavy traffic limit theorem for the multi-class queueing network

Examples Outside this Framework

Example 1: Generalized Processor Sharing





ESP describing mapping from inputs to the queue content

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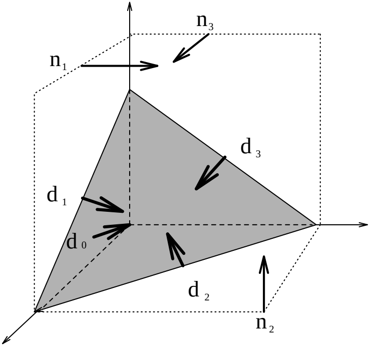
The 3-dimensional GPS Model



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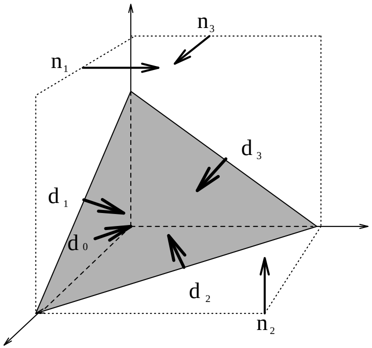

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Does not satisfy Completely-S condition

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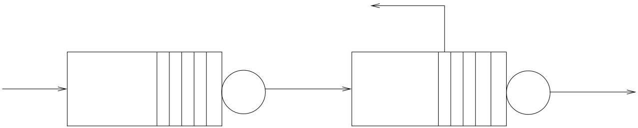
Example 2: FIFO Tandem Queue with Deadlines (Reed)



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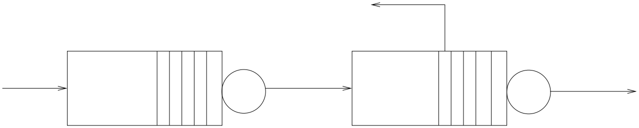

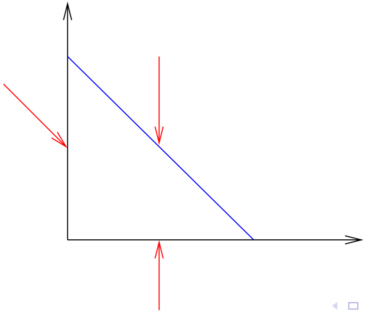

Submartingale Formulation vs. Skorokhod Problem Approach

	Skorokhod Problem	Submartingale Problem
Pros	Constructs strong solutions; yields pathwise uniqueness	Can be used to analyze arbitrary processes
Cons	Can only be used to analyze semimartingales	Provides only weak existence and uniqueness

The Extended Skorokhod Map

Definition of the ESP on $(G, d(\cdot))$ (R '06)

For any continuous ψ find a continuous ϕ such that

1. $\phi(t) = \psi(t) + \eta(t) \in G$;
2. $\eta(t) - \eta(s) \in \overline{co}\left[\bigcup_{u \in (s,t]} d(\phi(u))\right] \quad \forall 0 \leq s < t < \infty$,
   where $\overline{co}(A) =$ closure of the convex hull of A and
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Note: If X is a martingale, then $Z = \bar{\Gamma}(X)$ is not necessarily a semimartingale

Properties of the Extended Skorokhod Problem (ESP)

Theorem
(R. '00, '06)

- If (ϕ, η) solve the SP for ψ, then (ϕ, η) solve the ESP for ψ
- If (ϕ, η) solve the ESP for ψ and $|\eta|(t) < \infty \; \forall t$, then (ϕ, η) solve the SP
- The graph of the ESM $\overline{\Gamma}$ is closed.

Properties of the Limiting Diffusion

Theorem
('R '06 and Kang-'R '08)
The reflected diffusion Z associated with the GPS ESP is a semimartingale on the interval $[0, \tau_0]$, where

$$\tau_0 = \inf\{t \geq 0 : Z(t) = 0\}$$

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2-d $+$ BM case: follows from Williams ('85)

Dirichlet Processes

Definition

A process Z is a Dirichlet process if it admits the decomposition

$$Z = M + A$$

M local martingale and A a continuous process with $A(0) = 0$ that has zero quadratic variation,
i.e., for any sequence of partitions $\{\Pi^n\}$ of $[0, t]$,

$$\lim_{n \to \infty} |\Pi^n| \to 0 \quad \Rightarrow \quad \sum_{t_i \in \Pi^n} |A(t_{i+1}) - A(t_i)|^2 \stackrel{(\mathbb{P})}{\to} 0.$$

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Note:

If A is a process of a.s. finite variation on bounded intervals, Z is a continuous semimartingales.

A Dirichlet Process Characterization of RBMs

Setup

Given $(G, d(\cdot))$, b, σ Lip. cont and σ uniformly elliptic. Suppose there exists a Markov, weak solution $(Z, B), \mathcal{F}_t$ to the associated SDER and let $Y = Z - X$:

$$X(t) = z + \int_0^t b(Z(s))\,ds + \int_0^t \sigma(Z(s))\,dB(s).$$

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Theorem W. Kang and 'R, ('08)

If there exist $p \geq 2$ and $q \geq 2$ such that for every $0 \leq s, t \leq T$,

$$\mathbb{E}\left[|Y(t) - Y(s)|^p | \mathcal{F}_s\right] \leq \mathbb{E}\left[\sup_{u \in s,t]} |X(u) - X(s)|^q \mathcal{F}_s\right],$$

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then Z is a Dirichlet process.

In particular, this holds when the ESM is Hölder continuous or if the directions satisfy the so-called generalized completely-$\mathcal{S}$ condition.

Sketch of the Proof

$$Z(t) = Z(0) + B(t) + Y(t).$$

B standard Brownian motion, Y is the regulating process
Need to show

$$\sum_{t_i \in \Pi^n} |Y(t_i) - Y(t_{i-1})|^p \xrightarrow{\mathbb{P}} 0, \qquad as ||\Pi^n|| \to 0.$$

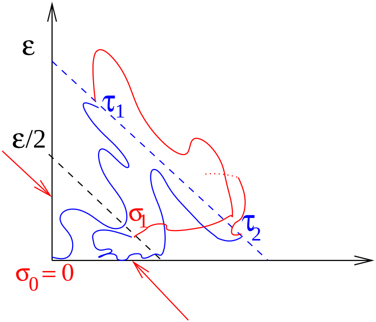
Define

$$\zeta^m = \inf\{t > 0 : |Z(t)| \geq m\}.$$

By localization suffices to show that

$$\sum_{t_i \in \Pi^n} |Y(t_i \wedge \zeta^m) - Y(t_{i-1} \wedge \zeta^m)|^p \xrightarrow{\mathbb{P}} 0, \qquad as ||\Pi^n|| \to 0.$$

Sketch of the Proof (contd.)



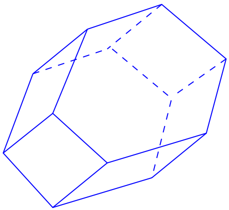
• control p-variation on $[\tau_i, \sigma_i)$ using semimartingale property away from origin; show summable;

• obtain estimates on p-variation on $[\sigma_i, \tau_{i+1})$ in terms of time spent in $\varepsilon/2$-nbhd of 0; show it disappears, on sending $\varepsilon \to 0$, by instantaneous reflection property

Properties of the GPS ESP

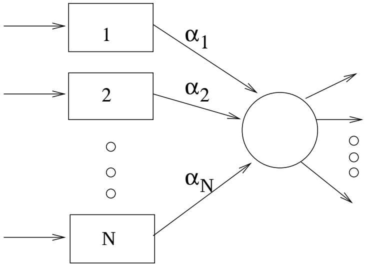
Theorem
(R '06, Dupuis-'R '98)
The GPS ESM is Lipschitz continuous

Proof Involves Constructing an Associated Norm;



Combines convex duality and algebra; vertices of B form the root system for the Lie albegra A_{n-1} of the Lie group $s\ell_n$

Revisiting the GPS Model



Order sources so that

$$\frac{\lambda_1}{\alpha_1} \geq \frac{\lambda_2}{\alpha_2} \geq \cdots \geq \frac{\lambda_N}{\alpha_N},$$

and define

$$J \doteq \max\left\{ j \leq N : \frac{\lambda_j}{\alpha_j} = \frac{\lambda_1}{\alpha_1} \right\}.$$

A Heavy Traffic Limit Theorem for the GPS Model

Theorem
(R.-Reiman '03, R.-Reiman '06)

Suppose the heavy traffic condition holds:

$$\sum_{j=1}^{J} \lambda_j = \sum_{j=1}^{J} \alpha_j = 1.$$

The appropriately scaled workload process in the GPS model converges weakly to the pathwise unique solution of a reflected diffusion in $\mathbb{R}_+^J$ associated with the GPS ESP with weights

$$\tilde{\alpha}_i = \frac{\alpha_i}{\sum_{i \leq J} \alpha_i}.$$

and modified covariance (identified explicitly).

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Lies outside the Bramson+Williams and cont. mapping frameworks

References

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